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ASYMPTOTIC ESTIMATES AND GAUGE INVARIANCE

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Known low-order contributions suggest that the entire QED perturbation series for the electron magnetic moment is convergent. This behaviour, which differs markedly from the recent asymptotic estimates in various model quantum field theories, might be of practical importance for high accuracy experimental tests of QED and of theoretical importance for study of Yang-Mills theories.

1. Introduction

Reliable estimates of high-order radiative corrections in gauge theories are of great importance, both practical (for example, for the evaluation of the electron magnetic moment) and theoretical (for example, for the study of the renormalization group function $\beta(g)$ in QCD confinement theories). I shall argue that on the basis of the known low-order QED results the asymptotic estimates in gauge theories should be radically different from those obtained in scalar theories [1–5]. The crucial difference is that while in scalar theories individual Feynman *diagrams* can be bounded, in gauge theories only gauge-invariant *sets* of diagrams can have bounds, and the size of a high-order contribution should be controlled by the slowly growing number of gauge-invariant sets, rather than by the combinatorially exploding number of Feynman diagrams.

While it is commonly assumed [6,7] that in QED gauge-invariant sets are bounded, no specific mechanism for this bound has been proposed *. This reflects the long standing frustration [8,9] associated with QED Feynman integrals; due to the gauge dependence of individual diagrams, infinite renormalizations and infrared divergences, we have no idea what size or even what sign to expect for a given radiative correction. Explicit calculations are marked by strong cancellations among diagrams related by gauge transformations, but we know of no way of exploiting the underlying gauge symmetry to effect the cancellations prior to an explicit evaluation of all contributing integrals.

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* The attempt made by Drell and Pagels [8] was not successful in predicting the magnitude of the sixth-order magnetic moment.

Even though such method is lacking, we can use the experience from low-order QED calculations to conjecture that *the size of the 2nth order contribution in a gauge theory is bounded by the number of gauge-invariant subsets in 2nth order.* A detailed analysis of the electromagnetic moment will also suggest a stronger hypothesis that in a "gauge set approximation" the magnetic moment is *convergent* and summable to all orders:

$$\tfrac{1}{2}(g-2) \equiv a \simeq \frac{\alpha}{2\pi}\,\frac{1}{(1-(\alpha/\pi)^2)^2}\;. \tag{1}$$

This is not meant to be an accurate numerical approximation, but only a guide to the magnitude of the 2*nth* term, to be contrasted with the combinatorially growing estimates of scalar field theory type [1–5]. The prediction for the eighth order in this approximation is 0, with an accuracy guess of ±0.3. These error brackets are not to be taken too seriously, but merely compared with a combinatorial estimate [10] of ±25. The combinatorial estimate assumes that individual diagrams are uncorrelated, with contributions statistically distributed around zero with standard deviation ±1.

I shall first review the results of magnetic moment calculations and then contrast bounds based on diagram counting with those based on gauge-set counting. A proof of gauge invariance of gauge sets is outlined, as it might suggest a method for establishing bounds on gauge sets. The paper closes with a brief discussion of problems raised by the conjecture.

2. Numerical results

All contributions to the electron magnetic moment anomaly a up to sixth order are known numerically [11–14] and most also analytically [15]. They are written as

$$a = a^{(2)}\,\frac{\alpha}{\pi} + a^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + a^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + \dots, \tag{2}$$

where

$$a^{(2n)} = \sum_G a_G \tag{3}$$

is computed from all n-loop proper vertex diagrams. The second-order contribution was first computed by Schwinger [16]

$$a^{(2)} = \tfrac{1}{2}\,. \tag{4}$$

Analytic expressions for higher orders consist of many fractions and transcedentals, and display no obvious systematics. The numerical results for individual Feynman diagrams are likewise confusing: they are dependent on the gauge and the method of subtracting infrared divergences. As illustrated in fig. 2, in practice they fall with-

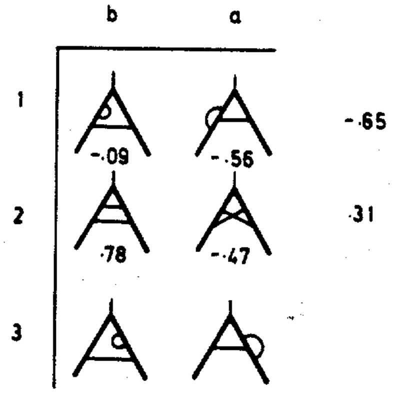

Fig. 1. The fourth-order gauge sets. For diagrams related by time reversal the value listed under the first diagram of the pair is the total contribution of the pair.

in ±10. Only the internally gauge-invariant sets [17] are separately infrared finite and gauge independent. For them we observe

(i) *Sets with photon self-energy subdiagrams* are typically ±0.02 to ±0.1, i.e. 3 to 15% of the corresponding sets without photon self-energy insertions; creation of a virtual e^+e^- pair suppresses the low-energy end of the virtual photon integrations, leading to comparatively small contributions.

(ii) *Sets with electron loop subdiagrams with 4, 6, ... legs.* In spite of the above statement about e^+e^- pairs the only such set computed so far, photon-photon scattering set, yields a comparatively large contribution to $a^{(6)}$: 0.37. If this is caused by the nearly ultraviolet divergent nature of photon-photon scattering diagrams, we can speculate that subdiagrams with 6, 8, ... photon legs will again lead to small contributions, and that only some [6,18] of 4-photon leg subdiagram sets will be of a size comparable to sets with no electron loops.

(iii) *Sets without electron loops*, hereafter referred to as *gauge sets*, are listed in figs. 1 and 2. A gauge set (k, m, m') consists of all proper vertex diagrams without electron loops with k photons crossing the external vertex (cross-photons) and m $[m']$ photons originating and terminating on the incoming [outgoing] electron leg (leg-photons), where $m \geqslant m'$, and for asymmetric sets $(m \neq m')$, *each diagram and its mirror image belong to the same set.* The gauge sets of figs. 1 and 2 are all roughly ±0.5, with the sign given by a simple empirical rule

$$a_{kmm'} = (-1)^{m+m'} \tfrac{1}{2} . \qquad (5)$$

The sign rule is further corroborated by sets with photon self-energy insertions (but with the absolute size scaled down to 3–15% of (5), as mentioned above). In fig. 3 I compare this rule with the actual numbers and make an eighth-order prediction.

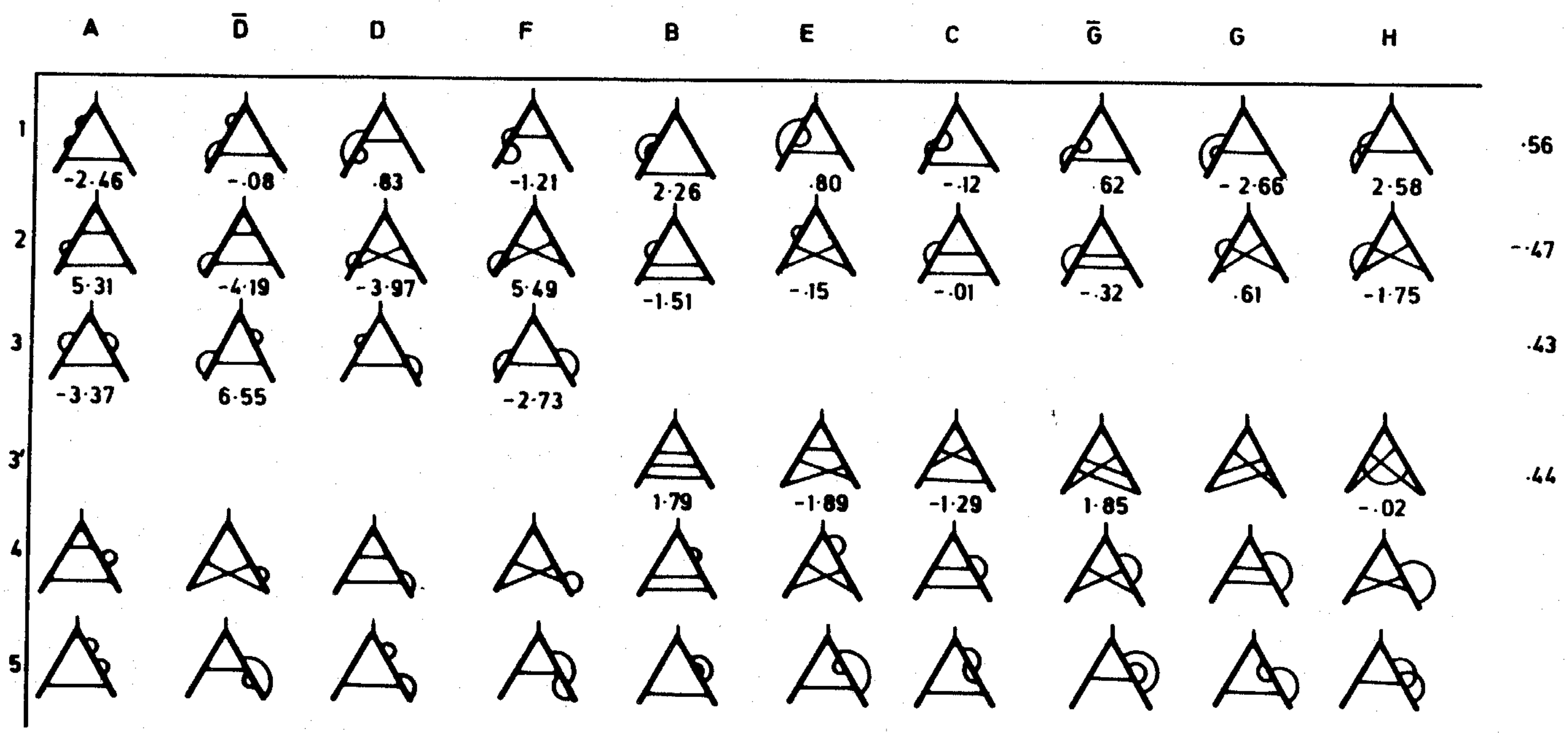

Fig. 2. The sixth-order gauge sets arranged in the rows and the externally gauge-invariant sets in the columns, with the numeration of ref. [12]. The values are finite parts in the ln λ IR cut-off approach, such as those listed in ref. [11]. For different IR separation methods (such as in ref. [12]) and different gauges, individual diagrams have different values.

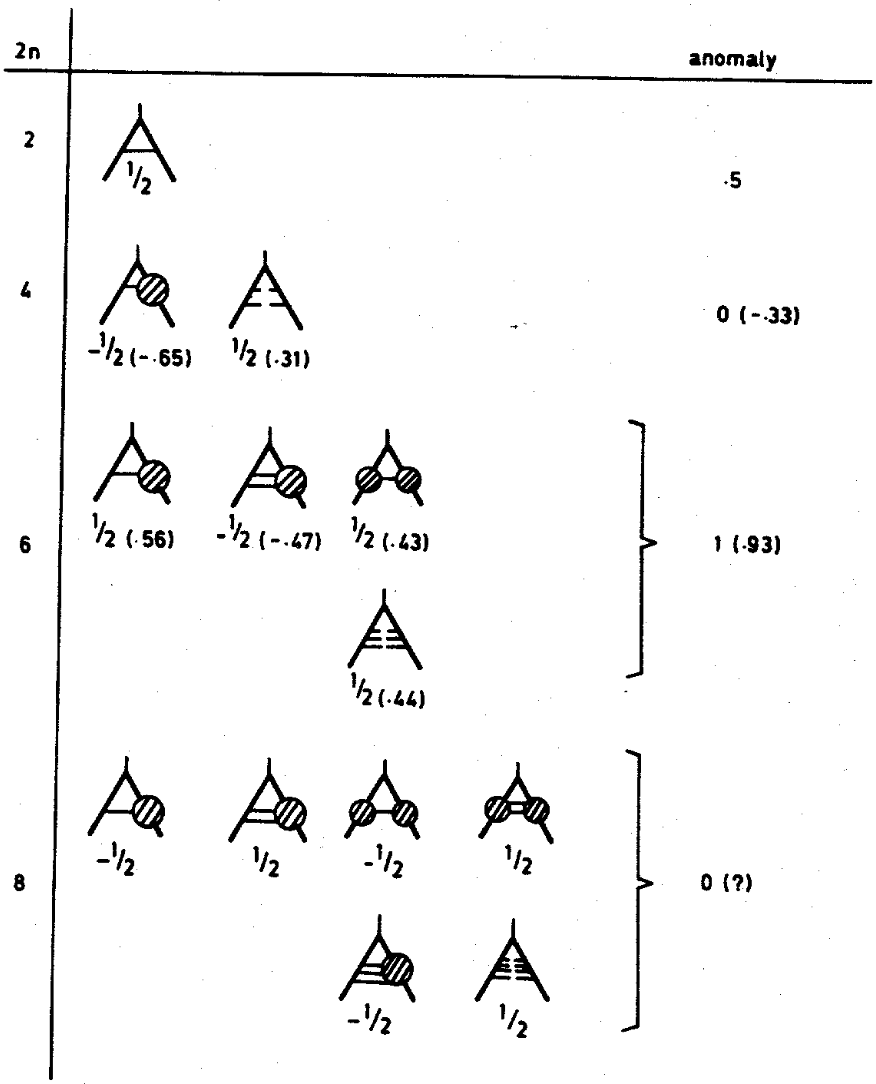

Fig. 3. Comparison of the "gauge-set approximation" and the actual numerical values of corresponding gauge sets, together with an eighth-order prediction.

3. Counting of gauge sets and Feynman diagrams

The extension of fig. 3 to arbitrary order is a matter of simple combinatorics *; in the "gauge-set approximation" the size of $2n$th order term is growing linearly

$$a \simeq \frac{\alpha}{2\pi} \sum_{m=1} \left(\frac{\alpha}{\pi}\right)^{2(m-1)} m \tag{6}$$

with the first few terms listed in table 1. This series is summable, with the sum given

* An invaluable aid in identifying such combinatorial series is Sloane [19].

Table 1
Comparison of the number of vertex diagrams, gauge sets and the "gauge-set approximation" for the magnetic moment in 2nth order.

Order $2n$	Vertex graphs Γ_{2n}	Gauge sets G_{2n}	Anomaly $a(2n)$
2	1	1	1/2
4	6	2	0
6	50	4	1
8	518	6	0
10	6354	9	3/2
12	89 782	12	0
14	1 429 480	16	2

by (1). The estimate can be improved by inclusion of sets with electron loops, but the numerical results quoted above suggest that they are non-leading or at worst comparable to the gauge-set contributions, so I shall *ignore all diagrams with electron loops* in the present paper. (My discussion applies only to electron QED; the muon magnetic moment is characterized by the additional parameter $\ln (m_\mu/m_e)$, and sets with electron loops are actually leading [18]).

Even if the empirical rule (5) should turn out to be too naive, *any reasonable bound on the size of gauge sets will lead to a convergent bound on the magnetic moment.* This is clear from the growth of G_{2n}, the number of gauge sets in $2n$th order, which is easily calculated by continuing fig. 3 to higher orders. (The first few terms are given in table 1). G_{2n} to all orders can be written in terms of a generating function

$$\sum_{n=1}^{\infty} G_{2n}\chi^n = \frac{\chi}{(1+\chi)(1-\chi)^3} \,. \qquad (7)$$

n odd, $G_{2n} = (\frac{n+1}{2})^2$

n even, $G_{2n} = \frac{n}{2}(\frac{n}{2}+1)$

For example, if a bound on each gauge set is +1, with the substitution $\chi = \alpha/\pi$, (7) becomes a convergent bound on a.

Let me now contrast the above estimates with bounds based on the counting of individual Feynman diagrams. Bender and Wu [20] have established bounds on the size of the $2n$th order contribution to the anaharmonic oscillator energy levels by establishing bounds on individual Feynman diagrams and then estimating their number in the $2n$th order. Even though in QED individual Feynman diagrams are clearly not bounded (because of infrared divergences) and not independent (because gauge transformations mix them), the early studies [21,22] of convergence properties of QED were also based on diagram counting. I shall illustrate this type of a bound by counting all diagrams (without electron loops) contributing to the proper vertex Γ^ν I start by counting electron self-energy graphs; there are

$$S_{2n} = \frac{(2n)!}{2^n n!} = 1.3.5 \,..\, (2n-1) \equiv (2n-1)!! \qquad (8)$$

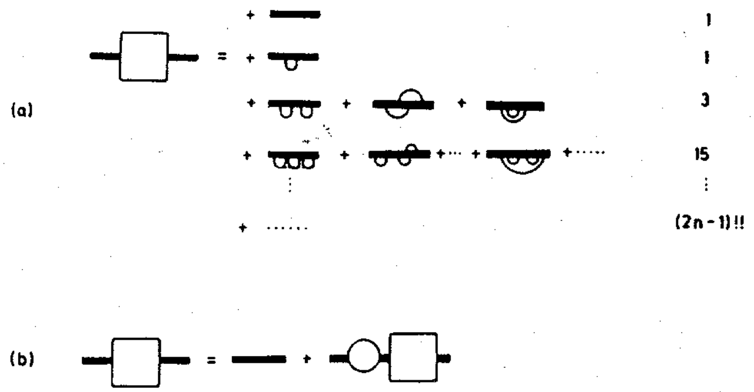

Fig. 4. (a) Counting of the number of diagrams (without electron loops) contributing to the complete electron propagator. (b) Definition of σ, the generating function for the number of one-particle-irreducible self-energy diagrams.

diagrams contributing to the complete electron propagator in $2n$th order (see fig. 4a). σ_{2n}, the number of one-particle-irreducible self-energy diagrams, can be computed from the recursion relation

$$S_{2n} = \sum_{k=1}^{n} \sigma_{2k} S_{2(n-k)}, \tag{9}$$

which follows from the perturbative expansion of fig. 4b. All proper vertex diagrams are obtained by insertions of the external photon into internal electron lines, hence their number is

$$\Gamma_{2n} = (2n-1)\sigma_{2n} \tag{10}$$

with the first few orders listed in table 1.

Even though the number of independent diagrams is somewhat smaller (I have not taken into account the symmetry under time reversal), the combinatorial explosion in the number of diagrams is amply evident. Such combinatorial growth in a number of diagrams is characteristic of all quantum field theories, and any bound on the size of individual diagrams (such as the assumption that their values are statistically distributed around some average value) leads to a perturbation series divergent for any $\alpha \neq 0$.

4. Finiteness and invariance of gauge sets

In perturbation theory the magnetic moment anomaly and the renormalized proper vertex are computed from

$$a = \frac{M}{1+L}, \tag{11}$$

$$\widetilde{\Gamma}^{\mu}(p,q) = \frac{\Gamma^{\mu}(p,q)}{1+L} \,, \tag{12}$$

where $M = F_2(0)$, $1 + L = F_1(0)$ are evaluated from the unrenormalized proper vertex

$$\Gamma^{\nu}(p,q) = F_1(q^2)\gamma^{\nu} + F_2(q^2)\,(i\sigma^{\nu\mu}q_{\mu}/2m)\,,$$

and here Γ^{ν} is restricted to the sum of all one-particle irreducible vertex diagrams *without electron loops*. For this subset of diagrams $Z_3 = 1$, and by the Ward identity $(Z_1 = Z_2)$ $\alpha = \alpha_0$, so (11) and (12) are UV finite without charge renormalization. The factor $(1+L)^{-1} = Z_1$ arises from the charge renormalization for the external vertex (or, equivalently, from the renormalization of external electron lines). (The diagrammatic expansion for Γ^{ν} already includes the electron mass counterterms.)

To expand the above quantities in terms of gauge sets, assume temporarily that cross-photons, in-leg-photons and out-leg-photons couple with different strengths.

$$a = \sum_{\substack{k \geqslant 1 \\ m \geqslant m' \geqslant 0}} \left(\frac{\alpha}{\pi}\right)^{k} \left(\frac{\alpha'}{\pi}\right)^{m} \left(\frac{\alpha''}{\pi}\right)^{m'} a_{kmm'} \,. \tag{13}$$

Expanding M and L the same way we find from (11)

$$a_{kmm'} = M_{kmm'} + \sum M_{k_1 m_1 m_1'}(-L_{k_2 m_2 m_2'}) + \sum M_{k_1 m_1 m_2'}(-L_{k_2 m_2 m_2'})\,(-L_{k_3 m_3 m_3'}) + \ldots, \tag{14}$$

where the sums go over all gauge sets for which $k = k_1 + k_2 + \ldots + k_n$, and so on. Gauge sets have the following properties:

(i) $a_{kmm'}$ *(and generally* $\widetilde{\Gamma}^{\mu}_{kmm'}$*) is ultraviolet (UV) finite*

This follows from the standard UV analysis of each diagram G contributing to the set. UV singularities of G are of two types:

(a) The divergent subdiagram S is an external photon vertex diagram. The UV divergence is cancelled by the $L_S\Gamma^{\mu}_{G/S}$ counterterm present in the expansion (14). (The notation is explained in refs. [23,24].)

(b) The divergent vertex (self-energy) subdiagram lies entirely on the in- or out-electron leg. The gauge set $\Gamma^{\mu}_{kmm'}$ always includes the corresponding diagram with the divergent self-energy (vertex) subdiagram that cancels the first divergence by the Ward identity.

(ii) $a_{kmm'}$ *(but generally not* $\widetilde{\Gamma}^{\mu}_{kmm'}$*) is infrared (IR) finite*

M_G is IR divergent whenever it can be separated into an external photon vertex subdiagram and a cloud of soft photons attached to the external lines. This IR divergence is cancelled by the counterterm $M_S L_{G/S}$ present [24] in expansion (14). (Incidentally, this similarity of UV and IR counterterms is an illustration of the connection between the two types of divergences emphasized elsewhere [25].)

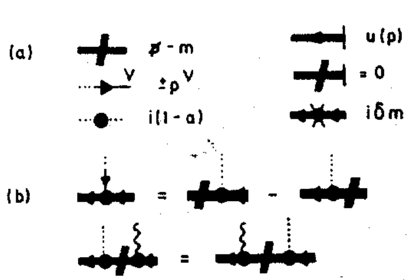

Fig. 5. (a) Diagrammatic notation for QED Feynman rules. (b) Feynman identity.

(iii) $a_{kmm'}$ *(and generally* $\widetilde{\Gamma}^{\mu}_{kmm'}$*) is internally gauge invariant* [17], i.e. invariant under the photon propagator transformation $g^{\mu\nu} \to g^{\mu\nu} + k^{\mu}k^{\nu}f(k^2)$, where $f(k^2)$ can be different for cross-photons, in-leg-photons and out-leg-photons [26]. Convenient diagrammatic notation is introduced in fig. 5, and interested readers can derive the linear gauge dependence of the $\Gamma^{\mu}_{kmm'}$ sets of fig. 2 by inserting $k^{\mu}k^{\nu}$ in all ways in the contributing diagrams and using the Feynman identity fig. 5b. The cancellations occur between diagrams related by symmetric insertion of the gauge photon around a photon vertex as illustrated in fig. 6, and the non-cancelled terms arise from two sources of asymmetry under photon interchange.

(a) *One-particle-irreducibility*. Within a general gauge set there appears a sequence of diagrams of fig. 7. Two terms survive because the diagrams that should cancel them belong to the excluded set of self-energy corrections on the external line. However, the first term yields upon evaluation an explicit factor δm and is cancelled by the other (mass counterterm) term.

(b) *Separation into gauge sets*. If photon i in fig. 6 is external, the cancelling diagrams are in different gauge sets. Uncancelled terms arise from diagrams of the form of fig. 8a, with two possibilities.

(i) The gauge photon is a *leg-photon*, leading to a surviving term of the form fig. 8b. This has precisely the same gauge dependence as the mass counterterm fig. 8c, and gets cancelled (besides, any reasonable regularization sets this equal to zero).

(ii) The gauge photon is a *cross-photon*, leading to a surviving term of the form fig. 8d. This is cancelled by the gauge dependence of the L_{100} counterterm in the expansion (14).

The same cancellations operate between various counterterms in (14), leading to

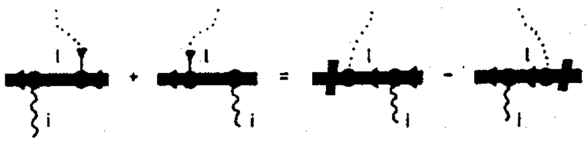

Fig. 6. Symmetric insertion of a gauge photon around a photon i vertex leads to the cancellation of terms for which the propagator of the internal electron line l had been removed.

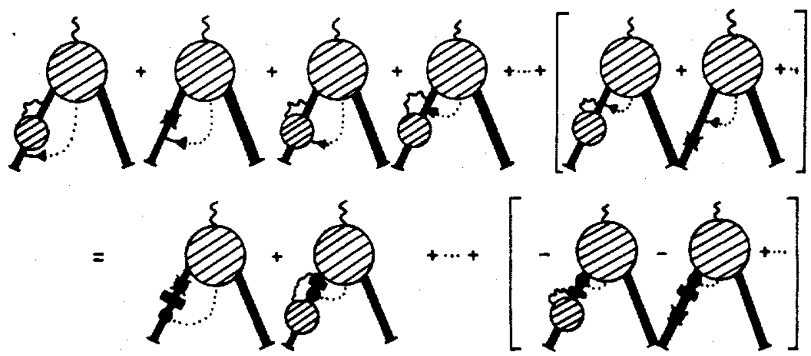

Fig. 7. The cancellation of gauge dependence of one-particle-irreducible vertex diagrams (terms in the brackets correspond to the excluded terms contributing to the full electron-electron-photon Green function). Note that the unrenormalized gauge set is gauge dependent; this is compensated by the gauge dependence of mass counterterms.

the gauge invariance of *mass-shell renormalized* gauge sets $\Gamma^{\mu}_{kmm'}$ and $a_{kmm'}$.

(iv) *A gauge set has no further gauge invariant subsets.* This is clear from the above proof of gauge invariance, which relied on the factorization of gauge dependence. The factorization does not go through if one attempts to further sub-divide

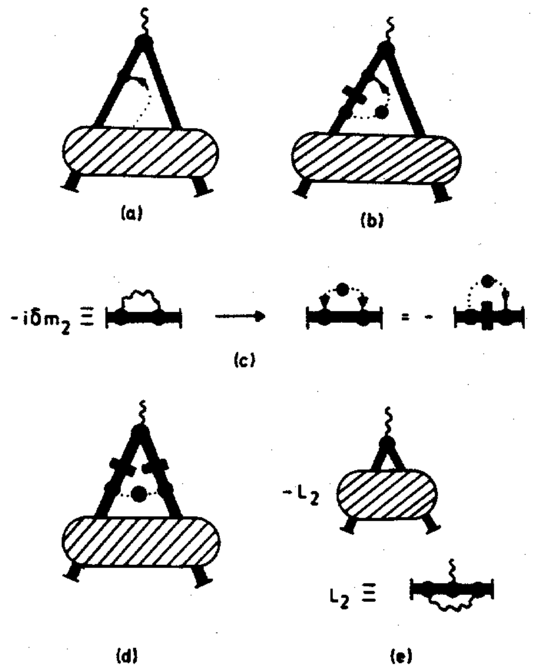

Fig. 8. (a) The general form of gauge dependence introduced by the separation of gauge sets. (b) If the gauge photon is a leg-photon, the gauge dependence is cancelled by the gauge dependence of the mass counterterm (c). (d) If the gauge photon is a cross-photon, the gauge dependence is cancelled by the gauge dependence of the vertex counterterm (e).

the photon types. Another way of seeing this is by taking any diagram and generating the entire gauge set from it by requiring that its gauge dependence be cancelled by the gauge dependence of diagrams related to it by the rule of fig. 6.

5. Discussion

The "gauge-set hypothesis" and the rule (5) are merely attempts to extract some suggestive information from the tremendous effort that has gone into computing low-order radiative corrections in QED. Some of the weaknesses of above conjectures are:

(i) The most glaring pitfall is the lack of a computational method for implementing them. Of presently available techniques, the approach of refs. [26,27] seems most suggestive; one can envisage decompositions of cross- and leg-photon propagators in which the leading terms would add up to rule (5), and the remainder be shown to be non-leading. Alternatively, rule (5) could be a consequence of some semiclassical solution similar to those employed in scalar theories [1–5].

(ii) As far as the gauge invariance is concerned, gauge sets (k, m, m') and (k, m', m), $m \neq m'$ could be counted as distinct.

(iii) In the light of the rather large photon-photon scattering contributions, it is not certain that diagrams with electron loops will continue giving non-leading contributions.

(iv) Why is α/π and not $\alpha/4\pi$ or something else the expansion parameter? Scalar theory estimates suggest a dependence on $2n$ of a much more complicated form than (1).

(v) While no way of combining internally gauge invariant sets is known, an *externally* gauge-invariant set [17] can be combined into a single integral [12]. Mass-operator formalism yields the same result [13]. Inspection of fig. 2 shows that externally gauge-invariant sets mix up gauge sets, and the origin of rule (5) would seem very obscure in such an approach.

The convergence of the perturbation series (1) might also seem surprising in the light of Dyson's [28] argument about the non-analyticity of QED perturbation expansions. However, it is very hard to see what bearing his argument (about the breakdown of the physical vacuum) has on a, which is defined as a formal Feynman diagram series indifferent to the "true vacuum" of the theory. Convergence of a is also not necessarily in contradiction with the combinatorial growth of off-mass-shell Green functions suggested by the work of Itzykson, Parisi and Zuber [4] on scalar QED. The essential ingredients of a convergence are gauge invariance and mass-shell renormalization, and off-mass-shell Green functions are both gauge and renormalization method dependent.

In the absence of a method for estimating the size of gauge set contributions it is impossible to say whether the success of rule (5) is fortuitous, but the importance of establishing a such or similar rule cannot be overemphasized. The increase in the experimental accuracy of the electron magnetic moment measurements [29] will soon confront us with the necessity of computing $a^{(8)}$ if we wish to determine the fine structure constant α from QED. It is doubtful whether this can be carried out within the Feynman diagram approach; the sixth-order calculation already strains the limits of the computationally possible. Numerical calculations [11–14] require large amounts of computing time, and the analytic techniques [15], though very impressive, are not yet sufficiently developed to cover even all sixth-order contributions. The eighth-order calculation would entail 891 diagrams [6] (and some 100 6th order counterterms), most of which are as lengthy to compute as the entire sixth order!

If it is the gauge invariance that controls the asymptotic estimates in QED, one would expect that it plays a similar role in QCD. There, an asymptotic estimate of the infrared renormalization group function β might resolve the important theoretical problem of whether the low-energy effective coupling has any IR fixed points, or if indeed it diverges as hypothesized by the advocates of "infrared slavery".

I would like to thank R.P. Feynman for a stimulating discussion.

Note added in proof

Recently Lautrup [30] has shown that in the $2n$th order the contribution of the diagram with $n - 1$ electron loops grows as $(n - 1)!$. This is a general feature of renormalizable theories [31]. My "gauge-set hypothesis" still applies to diagrams without electron loops, but the full QED perturbation series for the magnetic moment will be divergent unless there occur further cancellations among gauge invariant contributions to the photon propagator.

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