Renormalization, Unstable Manifolds, and the Fractal Structure of Mode Locking

Predrag Cvitanović[(a)]
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14853

and

Mogens H. Jensen and Leo P. Kadanoff
The James Franck and Enrico Fermi Institutes, University of Chicago, Chicago, Illinois 60637

and

Itamar Procaccia[(b)]
Department of Chemistry and The James Franck Institute, University of Chicago, Chicago, Illinois 60637
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The apparent universality of the fractal dimension of the set of quasiperiodic windings at the onset of chaos in a wide class of circle maps is described by construction of a universal one-parameter family of maps which lies along the unstable manifold of the renormalization group. The manifold generates a universal "devil's staircase" whose dimension agrees with direct numerical calculations. Applications to experiments are discussed.

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In the context of the transition to chaos via quasiperiodicity, most attention has been paid to the *local* scaling behavior at a particular irrational winding number.[1] Although universal behavior has been theoretically predicted,[1b,c] its experimental verification has not followed, simply because minute changes in winding numbers lead to large changes in scaling behavior.[2] It appears that of greater interest and experimental accessability are those universal properties that are *global* in the sense of pertaining to a *range* of winding numbers. Indeed, such a property has been found and reported by Jensen, Bak, and Bohr,[3] and has to do with the set complementary to the "tongues"[4] on which the dynamical system is mode locked. This set of unlocked or irrational windings has at the onset of chaos Lebesgue measure zero, and apparent universal fractal dimension D. Recent experiments on Josephson junction simulators[5,6] and charge density waves[7] have indicated the existence of this phenomenon and revealed results in agreement with the findings in Ref. 3.

For the simple circle map $\theta_{n+1} = \theta_n + \Omega - (K/2\pi)\sin(2\pi\theta_n)$ this transition occurs at $K=1$; see Fig. 1. On the plotted intervals the winding number W is locked on a rational value as shown. The gaps between the locked states are "full" of locked states that add up to Lebesgue measure 1. The set of irrational winding numbers is the complement of the locked intervals. We calculated the dimension D of this set in a way slightly different from Ref. 3. We believe that D is the same for all regions of gaps. Thus we can start with any pair of locked intervals P/Q and P'/Q'. The length of the gap between them is denoted by $\bar{s}$. Next the locked interval $(P+P')/(Q+Q')$ is found, and the gaps of length s_1 and s_2 between the new interval and the preceding one are found. This "Farey tree"[8] construction is continued until a large number of gap sizes s_i are found. The fractal dimension D is then estimated from the formula[9] $\sum_i R_i^D = 1$, where $R_i = s_i/\bar{s}$. Denoting the result from the nth Farey level as D_n, and the quantity $\min_i\{R_i^n\}$ as R^n, we fitted a power law $D_n = D^* + a(R^n)^x$. An excellent fit with eleven Farey levels $(n=1,\ldots,11)$ starting with $P/Q = \frac{5}{13}$ and $P'/Q' = \frac{8}{21}$ was obtained. The number D^*, which is our direct numerical estimate of the dimension of the set, was found to be 0.868 ± 0.002, in agreement with Ref. 3. Surprisingly, the value of D_1, an estimate based on only two gaps, was always very close to D^* (the deviation less than 1%). The result was invariant to the choice of P/Q and P'/Q' and can be applied to any interval of the staircase on Fig. 1. Moreover, the result is invariant to the choice of dynamical system $\theta_{n+1} = f(\theta_n)$ as long

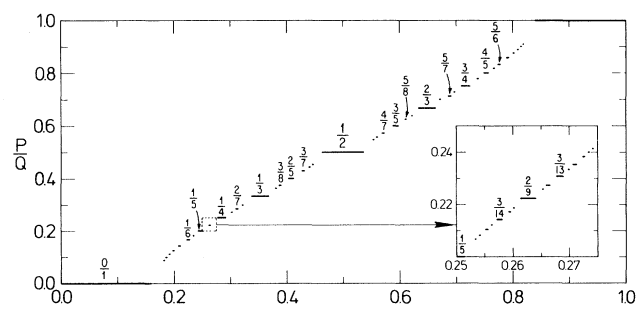

FIG. 1. The mode-locking structure at $K=1$ for the map (1). The "devil's staircase" is complete, and the complement of the mode-locked windings is of Lebesgue measure zero and universal fractal dimension D (Ref. 3).

as $f(\theta)$ has a cubic inflection point.

To understand the apparent universality we turned to a renormalization-group formulation. The analysis given below will provide convincing evidence that all the D's constructed in a small neighborhood of any "golden" winding number—one whose continued fraction ends in an infinite string of ones—will have the very same value of D. Since these numbers are dense in the interval [0, 1], one has the first step in an argument that D is truly universal.

Previously,[1b,c] the renormalization-group formulation has been used in this context to study the local scaling properties of golden winding numbers like $w^* = (\sqrt{5}-1)/2$.[10] A series of rational approximants $w_n = F_n/F_{n+1}$ was constructed by using Fibonacci numbers $F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$. Defining

$$f_n(x) = f^{F_{n+1}}(x) - F_n, \quad f^{(n)} = \alpha^n f_n(\alpha^{-n} x),$$

one finds

$$f^{(n+1)} = \alpha f^{(n)}(\alpha f^{(n-1)}(\alpha^{-2} x)).$$

In the limit $n \to \infty$ one obtains the fixed-point equation $f^*(x) = \alpha f^*(\alpha f^*(\alpha^{-2} x))$. The solution to this equation and its linearized version yields the relevant scaling parameters α and δ, which are the exponents in x space and in parameter space, respectively.[1b,c] Unfortunately in this formulation the dependence on the parameter Ω is lost, and the universal mode-locking structure cannot be investigated. What is needed is a formulation that maintains the dependence on a parameter. Such a formulation is achieved by parametrization of the unstable manifold.[11]

We construct the unstable manifold by starting with any given one-parameter family of functions $f_\Omega(x)$ which have a cubic inflection point at $x = 0$. Since we found empirically that the set of interest was invariant to the choice of initial P/Q and P'/Q' of the Farey-tree construction, we can as well pick values according to two Fibonacci-number ratios. In this way we shall make full use of the work that has been done on the *local* scaling properties near the golden mean. Define now

$$f_\Omega^{(n)} = f_\Omega^{F_{n+1}}(x) - F_n.$$

Denoting by Ω_n the value of Ω for which $f_\Omega^{(n)}(x)$ has a superstable cycle with winding number F_{n-1}/F_n, we define

$$g_0^{(n)}(x) = c_{n+1} f_{\Omega_{n+1}}^{(n)}(x/c_{n+1}).$$

By construction $x = 0$ is a superstable fixed point of $g_0^{(n)}(x)$. We want now the parameter range that spans the distance between the superstable cycle F_{n-1}/F_n and the next one F_n/F_{n+1} to be rescaled to the interval [0, 1]. We do so by turning $g^{(n)}(x)$ into a one-parameter family by defining

$$g_p^{(n)}(x) = c_{n+1} f_{\Omega_{n+1}+p\Delta_{n+1}}^{(n)}(x/c_{n+1}), \quad (1)$$

where Δ_n is picked such that $\Omega_{n+1} = \Omega_n + \Delta_n$. Accordingly, for $p = 0$ $g_p^{(n)}$ has a superstable fixed point. The value $p = 1$ corresponds to the next Fibonacci level (F_{n+2}) superstable cycle of the original map. Notice that in Eq. (1) c_{n+1} is an arbitrary scale factor. We fix it by picking the normalization $g_1^{(n)}(0) = 1$. Writing now the composition

$$f_{\Omega_{n+1}+p\Delta_{n+1}}^{(n)}(x/c_{n+1}) = f_{\Omega_{n+1}+p\Delta_{n+1}}^{(n-1)}(f_{\Omega_{n+1}+p\Delta_{n+1}}^{(n-2)}(x/c_{n+1})), \quad (2)$$

we use Eq. (1) to obtain the exact result

$$g_p^{(n)}(x) = \alpha_n g_{1+p/\delta_n}^{(n-1)}(\alpha_{n-1} g_{1+1/\delta_{n-1}+p/\delta_{n-1}\delta_n}^{(n-2)}(x/\alpha_n\alpha_{n-1})), \quad (3)$$

where $\delta_n = \Delta_n/\Delta_{n+1}$ and $\alpha_n = c_{n+1}/c_n$. After infinitely many α rescalings of x space around the inflection point $x = 0$, and infinitely many δ shifts and rescalings in parameter space, we reach the universal one-parameter family of maps $g_p(x)$ which lies on the unstable manifold and is invariant under rescaling and two-cycle composition. From Eq. (3) we get the exact result

$$g_p(x) = \alpha g_{1+p/\delta}(\alpha g_{1+1/\delta+p/\delta^2}(x/\alpha^2)). \quad (4)$$

The normalization conditions are $g_0(0) = 0$, $g_1(0) = 1$. We use now the universal object $g_p(x)$ to investigate the structure of mode lockings. As noted before, for $p = 0$ $g_p(x)$ has a superstable fixed point at $x = 0$. The range of p around zero for which $g_p(x)$ still has a fixed point is the range of parameters for which the original map is locked on some ("infinitely" high) winding ratio w_n. However, around $p = 1$ there is another locked state which corresponds to the next locked region in the sequence and the width of this region is scaled down by δ compared to the first (we remember[1a] that the meaning of δ is that $\Omega_n = \Omega_{GM} + a/\delta^n$). Around $p = 1 + 1/\delta$ there is another, scaled down by δ^2 compared to the first, etc. Thus, by studying the stability of the fixed point of g_p we can find an infinity of mode-locked states which are universally located. However, these are not *all* the locked ranges. In Fig. 2 we plot the largest locking ranges that can be obtained in the way just described, and also indicate some of those that do not fall into this category, since they correspond to winding numbers that are not F_n/F_{n+1}. These are indeed needed to determine the fractal dimension D. How can we get them from the universal object $g_p(x)$?

Consider for example the range denoted $(F_n + F_{n+2})/(F_{n+1}+F_{n+3})$. There the original map has an $(F_{n+1}+F_{n+3})$-order cycle. The function

$$q_p^{(n+3)}(x) = c f^{(n+2)}_{\Omega_{n+1}+p\Delta_{n+1}}(f^{(n)}_{\Omega_{n+1}+p\Delta_{n+1}}(x/c)) \tag{5}$$

will therefore have, for the range of p which falls in the desired region, a fixed point of order 1. We know from the Farey construction[8] that this p falls somewhere between $p=0$ and $p=1+1/\delta$. The function $q_p^{(n+3)}$ can also be written, using Eq. (1), as

$$q_p^{(n+3)}(x) = (c/c_{n+3}) g^{(n+2)}_{-\delta_{n+2}+\delta_{n+1}\delta_{n+2}(p-1)}((c_{n+3}/c_{n+1}) g_p^{(n)}(c_{n+1}x/c)). \tag{6}$$

The range in p for which (6) has a fixed point is independent of c; picking $c = c_{n+1}$ we see that this range can be obtained from studying, in the limit $n \to \infty$, the universal p range of stability of the fixed point of the function $q_p^{[3]}$,[12] which is obtained exactly from the composition

$$q_p^{[3]}(x) = (1/\alpha^2) g_{-\delta+\delta^2(p-1)}(\alpha^2 g_p(x)). \tag{7}$$

Once this range of p is found, another will occur between $p = 1+1/\delta$ and $p=1$, self-similarly placed but scaled down by δ etc., etc.

Similarly the locking between F_{n+2}/F_{n+3} and $(F_n+F_{n+2})/(F_{n+1}+F_{n+3})$, which on the universal plot appears between [2] and [3],[12] can be found by studying the stability of the fixed point of

$$q_p^{[2,2]} = (1/\alpha^2) g_{-\delta+\delta^2(p-1)}(\alpha^2 q_p^{[3]}(x)),$$

whereas that between $F_n/F_{n+1}([0])$ and $(F_n+F_{n+2})/(F_{n+1}+F_{n+3})([3])$ from $q_p^{[4]}(x) = q_p^{[3]}(g_p(x))$.

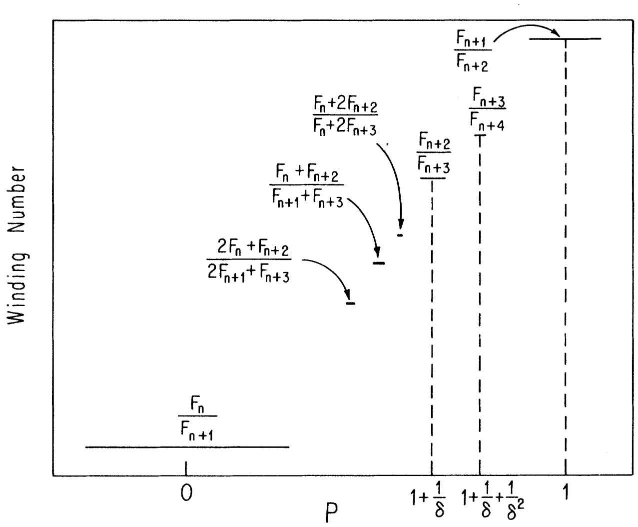

FIG. 2. The universal "devil's staircase" as generated from the unstable manifold. Locked ranges denoted by a thin line can be obtained directly from the stability of $g_p(x)$. The locked ranges shown by thick lines are obtained by composing the universal objects. The notation F_n/F_{n+1} is arbitrary in the sense that n is "very high." In fact, we use the notation [0] for F_n/F_{n+1}, [1] for F_{n+1}/F_{n+2}, etc. (Ref. 11).

Also these lockings will have infinite numbers of counterparts in the scaled-down regions on the right. All the universal locking intervals can be found by composing a pair of universal functions since every rational number can be expressed as a Farey composition of its two parent rationals.

In practice, an approximation to the function $g_p(x)$ can be found relatively easily, straight from the definition (1) and any starting function $f_\Omega(x)$. We picked $f_\Omega(x) = x + \Omega - (1/2\pi)\sin(2\pi x)$ and the value $\Omega_{n+1} = 0.606\,657\,620\,1$, which corresponds to winding number $\frac{144}{233}$. Using the procedure described above we found the universal interval of p with the mode-locked structure shown in Fig. 2. The estimates of the fractal dimension were performed as before. With two gaps we find $\bar{s} = 0.6326$, $s_1 = 0.3425$, $s_2 = 0.2232$, and $D_1 = 0.858$. Continuing this process we divided the universal set into more and more gaps and found D_n for four Farey levels from $\sum_i R_i^D = 1$. Fitting as before D_n by $D_n = D^* + a(R^n)^x$ we obtained $D^* = 0.867$ in excellent agreement with the direct estimate. In this universal construction we know that $\min_i\{R_i^n\}$ always comes from the gap closest to the golden mean and is asymptotically scaled down by δ compared to $\min_i\{R_i^{n-1}\}$. Accordingly this implies $D_n = D^* + a'/\delta^{nx}$. Notice that this amounts to a geometric convergence of D_n. However, the convergence is rather slow ($\delta^x < 1.1$) and therefore a very precise statement about the rate of convergence cannot be made at this point.

A few comments are now in order. (i) More accurate results can probably be obtained by representation of $g_p(x)$ as a double series expansion $\sum_{i,j} a_{ij} \times p^i x^{3j}$.[8b, 11, 13] For our purpose it was sufficient to work with $g_p(x)$ obtained directly from Eq. (6) and use the previously determined[1] values of the exponents α and δ as "input" for the calculation.

(ii) In analyzing experiments we suggest that the same procedure of estimating D would be taken. Two locked states with windings P/Q and P'/Q' should be identified, and the lockings which correspond to a Farey tree should be considered. $\sum_i R_i^D = 1$ and the fit can then be used to estimate the dimension. In fact, preliminary results in the context of convection experiments in an electrically conducting fluid appear to

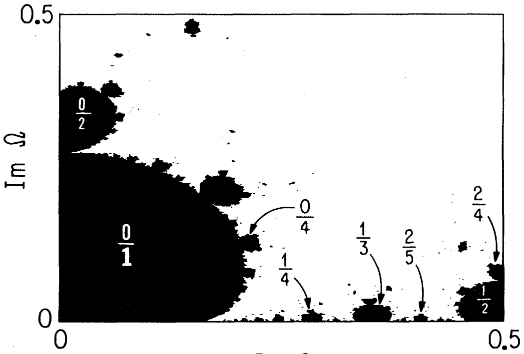

FIG. 3. Mode locking in the complexified circle map. Plotted is the complex Ω plane and in the black regions the iterations are mode locked. The real axis has the same structure as Fig. 1.

agree with our predictions (private communication from A. Libchaber and J. Stavans).

(iii) The universality of the mode-locking structure discussed here can be continued to the complex plane, yielding a set analogous to the Mandelbrot set for the map $z' = z^2 + c$.[14] In Fig. 3 we show the set of complex parameters Ω for which the complexified map $f(z) = z + \Omega - (1/2\pi)\sin(2\pi z)$ has locked solutions. We convinced ourselves numerically that this set has similar universal properties to the ones discussed above. In particular, the regions obtained by stretching up the set contained between winding numbers F_n/F_{n+1} and F_{n+1}/F_{n+2} seem to result in an invariant set. One can also see that the shape of any "egg" has a degree of universality. In particular the ratio of the width on the real axis to the height as measured by the point linking the "egg" to its largest leaf is apparently constant (~ 1.1).

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[(a)]Permanent address: Theoretical Physics, Chalmers, Gøteborg, Sweden.

[(b)]Permanent address: Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel.

[1a]Scott J. Shenker, Physica (Amsterdam) **5D**, 405 (1982).

[1b]M. J. Feigenbaum, L. P. Kadanoff, and Scott J. Shenker, Physica (Amsterdam) **5D**, 370 (1982).

[1c]D. Rand, S. Ostlund, J. Sethna, and E. Siggia, Phys. Rev. Lett. **49**, 132 (1982), and Physica (Amsterdam) **6D**, 303 (1984).

[2]A. P. Fein, M. S. Heutmaker, and J. P. Gollub, Phys. Scr. **T9**, 79 (1985). Recently, however, preliminary experimental results by A. Libchaber and J. Stavans (private communication) seem to support the theoretical predictions in Ref. 1.

[3]M. H. Jensen, P. Bak, and T. Bohr, Phys. Rev. Lett. **50**, 1637 (1983), and Phys. Rev. A **30**, 1960, 1970 (1984).

[4]V. I. Arnold, Am. Math. Soc. Transl. Ser. 2 **46**, 213 (1965).

[5]W. J. Yeh, D.-R. He, and Y. Kao, Phys. Rev. Lett. **52**, 480 (1984).

[6]P. Alstrøm, M. H. Jensen, and M. T. Levinsen, Phys. Lett. **103A**, 171 (1984).

[7]S. E. Brown, G. Mozurkewich, and G. Grüner, Phys. Rev. Lett. **52**, 2272 (1984).

[8a]B. Shraiman (unpublished) introduced the Farey tree in this context.

[8b]P. Cvitanović and J. Myrheim, to be published.

[8c]M. J. Feigenbaum, to be published; P. Cvitanović, Phys. Scr. **T9**, 202 (1985); S.-H. Kim and S. Ostlund, Phys. Scr. **T9**, 193 (1985).

[9]H. G. E. Hentschel and I. Procaccia, Physica (Amsterdam) **8D**, 435 (1983); P. Moran, Proc. Camb. Philos. Soc. **42**, 15 (1946).

[10]A different, ergodic renormalization-group formulation was performed by J. D. Farmer and I. Satija, Phys. Rev. A **31**, 3520 (1985).

[11]A. I. Goldberg, Ya. G. Sinai, and K. M. Khanin, Usp. Mat. Nauk. **38**, 159 (1983); E. B. Vul and K. M. Khanin, Usp. Mat. Nauk. **37**, 173 (1982).

[12]Each mode locking is conveniently labeled by the continued-fraction $[k_1, k_2, k_3, \ldots]$ representation of the winding number in the corresponding Farey tree. F_n/F_{n+1} is the root of the tree [0]; F_{n+1}/F_{n+2} is the next level, [1]; $(F_{n+2}+F_n)/(F_{n+3}+F_{n+1})$ corresponds to [3], F_{n+3}/F_{n+4} to [1,2], and so on.

[13]M. J. Feigenbaum and G. Gunaratne, unpublished.

[14]B. B. Mandelbrot, Ann. N. Y. Acad. Sci. **357**, 249 (1980).