Local stability

4

(R. Mainieri and P. Cvitanović)

So far we have concentrated on description of the trajectory of a single initial point. Our next task is to define and determine the size of a *neighborhood* of $x(t)$. We shall do this by assuming that the flow is locally smooth, and describe the local geometry of the neighborhood by studying the flow linearized around $x(t)$. Nearby points aligned along the stable (contracting) directions remain in the neighborhood of the trajectory $x(t) = f^t(x_0)$; the ones to keep an eye on are the points which leave the neighborhood along the unstable directions. As we shall demonstrate in Chapter **??**, in hyperbolic systems what matters are the expanding directions. The repercussion are far-reaching: As long as the number of unstable directions is finite, the same theory applies to finite-dimensional ODEs, state space volume preserving Hamiltonian flows, and dissipative, volume contracting infinite-dimensional PDEs.

4.1 Flows transport neighborhoods

Major combat operations in Iraq have ended.
President G. W. Bush, May 1, 2003

As a swarm of representative points moves along, it carries along and distorts neighborhoods. The deformation of an infinitesimal neighborhood is best understood by considering a trajectory originating near $x_0 = x(0)$ with an initial infinitesimal displacement $\delta x(0)$, and letting the flow transport the displacement $\delta x(t)$ along the trajectory $x(x_0, t) = f^t(x_0)$.

4.1.1 Instantaneous shear

The system of linear *equations of variations* for the displacement of the infinitesimally close neighbor $x + \delta x$ follows from the flow equations (2.5) by Taylor expanding to linear order

$$\dot{x}_i + \dot{\delta x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j} \delta x_j \, .$$

The infinitesimal displacement δx is thus transported along the trajectory $x(x_0, t)$, with time variation given by

$$\frac{d}{dt}\delta x_i(x_0,t) = \sum_j \left.\frac{\partial v_i(x)}{\partial x_j}\right|_{x=x(x_0,t)} \delta x_j(x_0,t)\,. \tag{4.1}$$

As both the displacement and the trajectory depend on the initial point x_0 and the time t, we shall often abbreviate the notation to $x(x_0,t) \to x(t) \to x$, $\delta x_i(x_0,t) \to \delta x_i(t) \to \delta x$ in what follows. Taken together, the set of equations

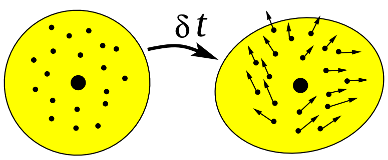

Fig. 4.1 A swarm of neighboring points of $x(t)$ is instantaneously sheared by the action of the stability matrix A - a bit hard to draw.

$$\dot{x}_i = v_i(x)\,, \qquad \dot{\delta x}_i = \sum_j A_{ij}(x)\delta x_j \tag{4.2}$$

governs the dynamics in the tangent bundle $(x, \delta x) \in \mathbf{T}\mathcal{M}$ obtained by adjoining the d-dimensional tangent space $\delta x \in \mathbf{T}_x\mathcal{M}$ to every point $x \in \mathcal{M}$ in the d-dimensional state space $\mathcal{M} \subset \mathbb{R}^d$. The *stability matrix*

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j} \tag{4.3}$$

describes the instantaneous rate of shearing of the infinitesimal neighborhood of $x(t)$ by the flow, Fig. 4.1.

Example 4.1 Rössler flow, linearized:
For the Rössler flow (2.14) the stability matrix is

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{pmatrix}. \tag{4.4}$$

4.1.2 Roll your own cigar

Taylor expanding a *finite time* flow to linear order,

$$f_i^t(x_0 + \delta x) = f_i^t(x_0) + \sum_j \frac{\partial f_i^t(x_0)}{\partial x_{0j}}\delta x_j + \cdots\,, \tag{4.5}$$

one finds that the linearized neighborhood is transported by

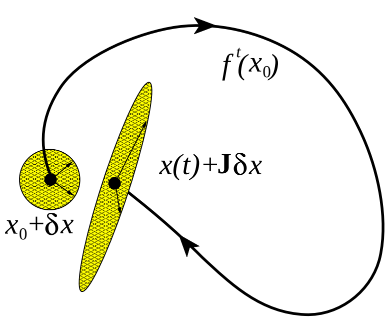

Fig. 4.2 The fundamental matrix J^t maps an infinitesimal spherical neighborhood of x_0 into a cigar-shaped neighborhood finite time t later.

$$\delta x(t) = J^t(x_0)\delta x(0)\,, \qquad J^t_{ij}(x_0) = \left.\frac{\partial x_i(t)}{\partial x_j}\right|_{x=x_0}. \tag{4.6}$$

This Jacobian matrix has inherited the name *fundamental solution matrix* or simply *fundamental matrix* from the theory of linear ODEs. It describes the deformation of an infinitesimal neighborhood at finite time t in the co-moving frame of $x(t)$.

As this is a deformation in the linear approximation, one can think of it as a linear deformation of an infinitesimal sphere enveloping x_0 into an ellipsoid around $x(t)$, described by the eigenvectors and eigenvalues of the fundamental matrix of the linearized flow, Fig. 4.2. Nearby trajectories separate along the *unstable directions*, approach each other along the *stable directions*, and change their distance along the *marginal directions* at a rate slower than exponential, corresponding to the eigenvalues of the fundamental matrix with magnitude larger than, smaller than, or equal 1. In the literature adjectives *neutral* or *indifferent* are often used instead of 'marginal,' (attracting) stable directions are sometimes called 'asymptotically stable,' and so on.

One of the preferred directions is what one might expect, the direction of the flow itself. To see that, consider two initial points along a trajectory separated by infinitesimal flight time δt: $\delta x(0) = f^{\delta t}(x_0) - x_0 = v(x_0)\delta t$. By the semigroup property of the flow, $f^{t+\delta t} = f^{\delta t+t}$, where

$$f^{\delta t+t}(x_0) = \int_0^{\delta t+t} d\tau\, v(x(\tau)) = \delta t\, v(x(t)) + f^t(x_0)\,.$$

Expanding both sides of $f^t(f^{\delta t}(x_0)) = f^{\delta t}(f^t(x_0))$, keeping the leading term in δt, and using the definition of the fundamental matrix (4.6), we observe that $J^t(x_0)$ transports the velocity vector at x_0 to the velocity vector at $x(t)$ at time t:

$$v(x(t)) = J^t(x_0)\, v(x_0)\,. \tag{4.7}$$

In nomenclature of page 54, the fundamental matrix maps the initial, Lagrangian coordinate frame into the current, Eulerian coordinate frame.

The velocity at point $x(t)$ in general does not point in the same direction as the velocity at point x_0, so this is not an eigenvalue condition for J^t; the fundamental matrix computed for an arbitrary segment of an arbitrary trajectory has no invariant meaning.

As the eigenvalues of finite time J^t have invariant meaning only for periodic orbits, we postpone their interpretation to Chapter 8. However, already at this stage we see that if the orbit is periodic, $x(T_p) = x(0)$, at any point along cycle p the velocity v is an eigenvector of the fundamental matrix $J_p = J^{T_p}$ with a unit eigenvalue,

$$J_p(x)\, v(x) = v(x)\,, \qquad x \in p\,. \tag{4.8}$$

Two successive points along the cycle separated by $\delta x(0)$ have the same separation after a completed period $\delta x(T_p) = \delta x(0)$, see Fig. 4.3, hence eigenvalue 1.

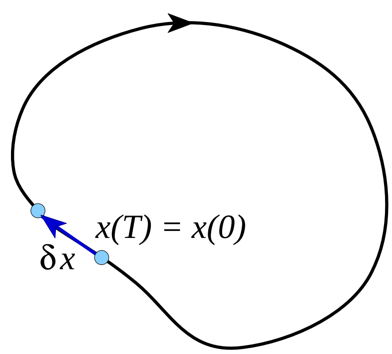

Fig. 4.3 For a periodic orbit p, any two points along the cycle are mapped into themselves after one cycle period T, hence $\delta x = v(x_0)\delta t$ is mapped into itself by the cycle fundamental matrix J_p.

As we started by assuming that we know the equations of motion, from (4.3) we also know stability matrix A, the instantaneous rate of shear of an infinitesimal neighborhood $\delta x_i(t)$ of the trajectory $x(t)$. What we do not know is the finite time deformation (4.6).

Our next task is to relate the stability matrix A to fundamental matrix J^t. On the level of differential equations the relation follows by taking the time derivative of (4.6) and replacing $\dot{\delta x}$ by (4.2)

$$\begin{aligned} \dot{\delta x}(t) &= \dot{J}^t\, \delta x(0)\,, \\ &= A\delta x(t) = AJ^t\, \delta x(0)\,. \end{aligned}$$

Hence the d^2 matrix elements of fundamental matrix satisfy the linearized equation (4.1)

$$\frac{d}{dt}J^t(x) = A(x)\, J^t(x)\,, \quad \text{initial condition } \; J^0(x) = \mathbf{1}\,. \tag{4.9}$$

Given a numerical routine for integrating the equations of motion, evaluation of the fundamental matrix requires minimal additional programming effort; one simply extends the d-dimensional integration routine and integrates concurrently with $f^t(x)$ the d^2 elements of $J^t(x)$.

The qualifier 'simply' is perhaps too glib. Integration will work for short finite times, but for exponentially unstable flows one quickly runs into numerical over- and / or underflow problems, so further thought will have to go into implementation this calculation.

So now we know how to compute fundamental matrix J^t given the stability matrix A, at least when the d^2 extra equations are not too expensive to compute. Mission accomplished.

fast track
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And yet... there are mopping up operations left to do. We persist until we derive the integral formula (4.32) for the fundamental matrix, an analogue of the finite-time "Green function" or "path integral" solutions of other linear problems.

We are interested in smooth, differentiable flows. If a flow is smooth, in a sufficiently small neighborhood it is essentially linear. Hence the next section, which might seem an embarrassment (what is a section on *linear* flows doing in a book on *non*linear dynamics?), offers a firm stepping stone on the way to understanding nonlinear flows. If you know your eigenvalues and eigenvectors, you may prefer to fast forward here.

fast track
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4.2 Linear flows

> Diagonalizing the matrix: that's the key to the whole thing.
> Governor Arnold Schwarzenegger

Linear fields are the simplest vector fields. Described by linear differential equations which can be solved explicitly, with solutions that are good for all times. The state space for linear differential equations is $\mathcal{M} = \mathbb{R}^d$, and the equations of motion (2.5) are written in terms of a vector x and a constant stability matrix A as

$$\dot{x} = v(x) = Ax\,. \tag{4.10}$$

Solving this equation means finding the state space trajectory

$$x(t) = (x_1(t), x_2(t), \ldots, x_d(t))$$

passing through the point x_0.

If $x(t)$ is a solution with $x(0) = x_0$ and $y(t)$ another solution with $y(0) = y_0$, then the linear combination $ax(t) + by(t)$ with $a, b \in \mathbb{R}$ is also a solution, but now starting at the point $ax_0 + by_0$. At any instant in time, the space of solutions is a d-dimensional vector space, which means that one can find a basis of d linearly independent solutions. How do we solve the linear differential equation (4.10)? If instead of a matrix equation we have a scalar one, $\dot{x} = \lambda x$, with a a real number, then the solution is

$$x(t) = e^{t\lambda} x_0 \,. \tag{4.11}$$

In order to solve the d-dimensional matrix case, it is helpful to rederive the solution (4.11) by studying what happens for a short time step δt. If at time $t = 0$ the position is $x(0)$, then

$$\frac{x(\delta t) - x(0)}{\delta t} = \lambda x(0)\,, \tag{4.12}$$

which we iterate m times to obtain Euler's formula for compounding interest

$$x(t) \approx \left(1 + \frac{t}{m}\lambda\right)^m x(0)\,. \tag{4.13}$$

The term in parentheses acts on the initial condition $x(0)$ and evolves it to $x(t)$ by taking m small time steps $\delta t = t/m$. As $m \to \infty$, the term in parentheses converges to $e^{t\lambda}$. Consider now the matrix version of equation (4.12):

$$\frac{x(\delta t) - x(0)}{\delta t} = A x(0)\,. \tag{4.14}$$

A representative point x is now a vector in $\mathbb{R}^d$ acted on by the matrix A, as in (4.10). Denoting by $\mathbf{1}$ the identity matrix, and repeating the steps (4.12) and (4.13) we obtain Euler's formula for the exponential of a matrix:

$$x(t) = \lim_{m\to\infty} \left(\mathbf{1} + \frac{t}{m}A\right)^m x(0) = e^{tA} x(0)\,. \tag{4.15}$$

We will use this expression as the definition of the exponential of a matrix.

How do we compute the exponential (4.15)?


in depth:
Appendix **??**, p. **??**


fast track
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Example 4.2 Fundamental matrix eigenvalues, diagonalizable case:

Should we be so lucky that A happens to be a diagonal matrix with real eigenvalues $(\mu_1, \mu_2, \ldots, \mu_d)$, the exponential is simply

$$J^t = e^{tA} = \begin{pmatrix} e^{t\mu_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{t\mu_d} \end{pmatrix} . \tag{4.16}$$

Next, suppose that A is diagonalizable and that U is a nonsingular matrix that brings it to a diagonal form $A_D = U^{-1}AU$. The transformation U is a

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linear coordinate transformation which rotates, skews, and possibly flips the coordinate axis of the vector space. Then J can also be brought to a diagonal form (insert factors $\mathbf{1} = UU^{-1}$ between the steps of the product (4.15)):

$$J^t = e^{tA} = U e^{tA_D} U^{-1} \,. \tag{4.17}$$

The action of both A and J is very simple; the axes of orthogonal coordinate system where A is diagonal are also the eigen-directions of both A and J^t, and under the flow the neighborhood is deformed by a multiplication by an eigenvalue factor for each coordinate axis.

Throughout this text the symbol Λ_k will always denote the kth *eigenvalue* (in literature sometimes referred to as the *multiplier*) of the finite time fundamental matrix J^t. Symbol λ_k will be reserved for the kth *characteristic exponent* or *characteristic value*, with real part μ_k and ν_k the kth phase

$$\Lambda_k = e^{t\lambda_k} = e^{t(\mu_k + i\nu_k)} \,. \tag{4.18}$$

The $J^t(x_0)$ depends on the initial point x_0 and the elapsed time t. For notational brevity we tend to omit this dependence, but in general

$$\Lambda_k = \Lambda_k(x_0, t)\,,\ \lambda_k = \lambda_k(x_0, t)\,,\ \nu_k = \nu_k(x_0, t)\,, \cdots\,,$$

depend on both the trajectory traversed and the choice of coordinates. Conventionally we label eigenvalues Λ in decreasing order

$$|\Lambda_1| \geq |\Lambda_2| \geq \ldots \geq |\Lambda_d|$$

Since $|\Lambda_j| = e^{t\mu_j}$, this is the same as

$$\mu_1 \geq \mu_2 \geq \ldots \geq \mu_d$$

Section 8.1.1 ⟵

As A has only real entries, it will in general have either real eigenvalues, or complex conjugate pairs of eigenvalues. That is not surprising, but also the corresponding eigenvectors can be either real or complex. All coordinates used in defining the flow are real numbers, so what is the meaning of a *complex* eigenvector?

Example 4.3 Complex eigenvalues:

To develop some intuition about that, let us work out the behavior for the simplest nontrivial case, the 2–d/ case

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \,. \tag{4.19}$$

The eigenvalues λ_1, λ_2 of A are the roots

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{tr}\, A \pm \sqrt{(\mathrm{tr}\, A)^2 - 4 \det A}\right) \tag{4.20}$$

of the characteristic equation

$$\det(A - z\mathbf{1}) = (\lambda_1 - z)(\lambda_2 - z) \;=\; 0\,, \tag{4.21}$$

$$\left\| \begin{matrix} A_{11} - z & A_{12} \\ A_{21} & A_{22} - z \end{matrix} \right\| = z^2 - (A_{11} + A_{22})\, z + (A_{11}A_{22} - A_{12}A_{21}) \,.$$

For the distinct eigenvalues case $\lambda_1 \neq \lambda_2$, he eigenvectors are obtained by applying to an arbitrary vector $x \in \mathbb{R}^2$ the projection operators

$$P_1 = \frac{A - \lambda_2 \mathbf{1}}{\lambda_1 - \lambda_2}, \qquad P_2 = \frac{A - \lambda_1 \mathbf{1}}{\lambda_2 - \lambda_1}. \tag{4.22}$$

The qualitative behavior of e^A for real eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ differs from the case that they form a complex conjugate pair,

$$\lambda_1 = \mu_1 + i\nu_1, \quad \lambda_2 = \lambda_1^* = \mu_1 - i\nu_1.$$

These two possibilities are refined further into sub-cases depending on the signs of the real part. The matrix might have only one eigenvector, or two linearly independent eigenvectors, which may or may not be orthogonal. Along each of these directions the motion is of the form $x_k \exp(t\lambda_k)$. If the exponent λ_k is positive, then the component x_k will grow; if the exponent λ_k is negative, it will shrink.

We sketch the full set of possibilities in Fig. 4.2 (a), and work out in detail the case when A can be brought to the diagonal form, and the case of degenerate eigenvalues.

Example 4.4 Complex eigenvalues, diagonal:

If A can be brought to the diagonal form, the solution (4.15) to the differential equation (4.10) can be written either as

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} e^{t\mu_1} & 0 \\ 0 & e^{t\mu_2} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}, \tag{4.23}$$

or

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{t\mu} \begin{pmatrix} e^{it\nu} & 0 \\ 0 & e^{-it\nu} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}. \tag{4.24}$$

In the case $\mu_1 > 0$, $\mu_2 < 0$, x_1 grows exponentially with time, and x_2 contracts exponentially. This behavior, called a *saddle*, is sketched in Fig. 4.2 (b), as are the remaining possibilities: in / out nodes, inward / outward spirals, and the center. Spirals arise from taking a real part of the action of J^t on a complex eigenvector. The magnitude of $|x(t)|$ diverges exponentially when $\mu > 0$, and contracts toward 0 when the $\mu < 0$, whereas the imaginary phase ν controls its oscillations.

In general J^t is neither diagonal, nor diagonalizable, nor constant along the trajectory. As any matrix, J^t can also be expressed in the singular value decomposition form

$$J = UDV^T$$

where D is diagonal, and U, V are orthogonal matrices. The diagonal elements $\Lambda_1, \Lambda_2, \ldots, \Lambda_d$ of D are the eigenvalues.

Under the action of the flow an infinitesimally small ball of initial points is deformed into an ellipsoid: $|\Lambda_i|$ is the relative stretching of the ith principal axis of the ellipsoid, the columns of the matrix V are the principal axes e_i of stretching in the Lagrangian coordinate frame, and the orthogonal matrix U gives the orientation of the ellipse in the Eulerian coordinates.

Now that we have some feeling for the qualitative behavior of eigenvectors and eigenvalues of linear flows, we are ready to return to the nonlinear case.

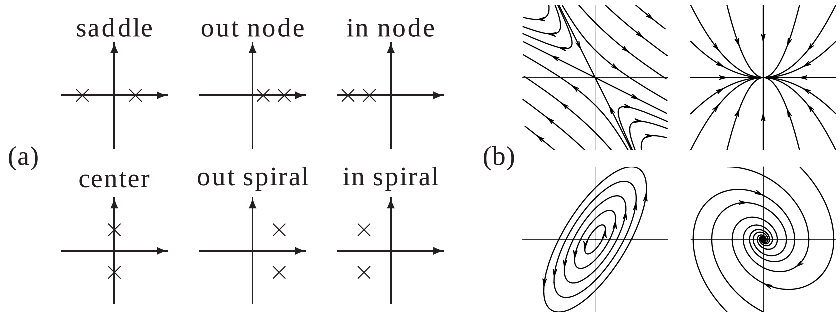

Fig. 4.4 (a) Qualitatively distinct types of exponents of a $[2\times 2]$ fundamental matrix. (b) Streamlines for several typical 2-dimensional flows: saddle (hyperbolic), in node (attracting), center (elliptic), in spiral.

4.3 Stability of flows

> Mopping up operations are the activities that engage most scientists throughout their careers.
> Thomas Kuhn, *The Structure of Scientific Revolutions*

How do you determine the eigenvalues of the finite time local deformation J^t for a general nonlinear smooth flow? The fundamental matrix is computed by integrating the equations of variations (4.2)

$$x(t) = f^t(x_0)\,, \quad \delta x(x_0,t) \;=\; J^t(x_0)\delta x(x_0,0)\,. \tag{4.25}$$

The equations are linear, so we should be able to integrate them–but in order to make sense of the answer, we derive it step by step.

4.3.1 Stability of equilibria

For a start, consider the case where x is an equilibrium point (2.7). Expanding around the equilibrium point x_q, using the fact that the stability matrix $A = A(x_q)$ in (4.2) is constant, and integrating,

$$f^t(x) = x_q + e^{At}(x - x_q) + \cdots\,, \tag{4.26}$$

we verify that the simple formula (4.15) applies also to the fundamental matrix of an equilibrium point,

$$J^t(x_q) = e^{At}\,, \qquad A = A(x_q)\,. \tag{4.27}$$

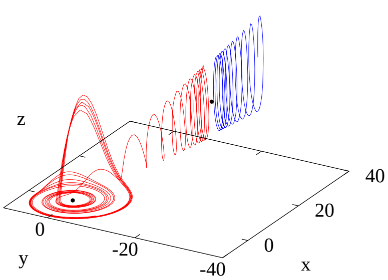

Fig. 4.5 Two trajectories of the Rössler flow initiated in the neighborhood of the '+' or 'outer' equilibrium point (2.15). (R. Paškauskas)

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Example 4.5 Stability of equilibria of the Rössler flow.
The Rösler system (2.14) has two equilibrium points (2.15), the inner equilibrium (x^-, y^-, z^-), and the outer equilibrium point (x^+, y^+, z^+). Together with their exponents (eigenvalues of the stability matrix) the two equilibria now yield quite detailed information about the flow. Figure 4.5 shows two trajectories which start in the neighborhood of the '+' equilibrium point. Trajectories to the right of the outer equilibrium point '+' escape, and those to

the left spiral toward the inner equilibrium point '$-$', where they seem to wander chaotically for all times. The stable manifold of outer equilibrium point thus serves as a attraction basin boundary. Consider now the eigenvalues of the two equilibria

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$$
\begin{array}{rlll}
(\mu_1^-, \mu_2^- \pm i\,\nu_2^-) = & (\,-5.686, & 0.0970 \pm i\,0.9951\,) \\
(\mu_1^+, \mu_2^+ \pm i\,\nu_2^+) = & (\,0.1929, & -4.596 \times 10^{-6} \pm i\,5.428\,)
\end{array}
\tag{4.28}
$$

Outer equilibrium: The $\mu_2^+ \pm i\,\nu_2^+$ complex eigenvalue pair implies that that neighborhood of the outer equilibrium point rotates with angular period $T_+ \approx \left|2\pi/\nu_2^+\right| = 1.1575$. The multiplier by which a trajectory that starts near the '$+$' equilibrium point contracts in the stable manifold plane is the excrutiatingly slow $\Lambda_2^+ \approx \exp(\mu_2^+ T_+) = 0.9999947$ per rotation. For each period the point of the stable manifold moves away along the unstable eigen-direction by factor $\Lambda_1^+ \approx \exp(\mu_1^+ T_+) = 1.2497$. Hence the slow spiraling on both sides of the '$+$' equilibrium point.

Inner equilibrium: The $\mu_2^- \pm i\,\nu_2^-$ complex eigenvalue pair tells us that neighborhood of the '$-$' equilibrium point rotates with angular period $T_- \approx \left|2\pi/\nu_2^-\right| =$ 6.313, slightly faster than the harmonic oscillator estimate in (2.11). The multiplier by which a trajectory that starts near the '$-$' equilibrium point spirals away per one rotation is $\Lambda_{\text{radial}} \approx \exp(\mu_2^- T_-) = 1.84$. The μ_1^- eigenvalue is essentially the z expansion correcting parameter c introduced in (2.13). For each Poincaré section return, the trajectory is contracted into the stable manifold by the amazing factor of $\Lambda_1 \approx \exp(\mu_1^- T_-) = 10^{-15.6}$ (!).
Suppose you start with a 1 *mm* interval pointing in the Λ_1 eigen-direction. After one Poincaré return the interval is of order of 10^{-4} *fermi*, the furthest we will get into subnuclear structure in this book. Of course, from the mathematical point of view, the flow is reversible, and the Poincaré return map is invertible. (Rytis Paškauskas)

4.3.2 Stability of trajectories

Next, consider the case of a general, non-stationary trajectory $x(t)$. The exponential of a constant matrix can be defined either by its Taylor series expansion, or in terms of the Euler limit (4.15):

Appendix ??

$$
\begin{aligned}
e^{tA} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k && (4.29)\\
&= \lim_{m\to\infty} \left(\mathbf{1} + \frac{t}{m} A\right)^m . && (4.30)
\end{aligned}
$$

Taylor expanding is fine if A is a constant matrix. However, only the second, tax-accountant's discrete step definition of an exponential is appropriate for the task at hand, as for a dynamical system the local rate of neighborhood distortion $A(x)$ depends on where we are along the trajectory. The linearized neighborhood is multiplicatively deformed along the flow, and the m discrete time step approximation to J^t is therefore given by a generalization of the Euler product (4.30):

$$
J^t = \lim_{m\to\infty} \prod_{n=m}^{1} (\mathbf{1} + \delta t A(x_n)) = \lim_{m\to\infty} \prod_{n=m}^{1} e^{\delta t\, A(x_n)} \tag{4.31}
$$

$$= \lim_{m\to\infty} e^{\delta t\, A(x_n)} e^{\delta t\, A(x_{m-1})} \cdots e^{\delta t\, A(x_2)} e^{\delta t\, A(x_1)} \,,$$

where $\delta t = (t - t_0)/m$, and $x_n = x(t_0 + n\delta t)$. Slightly perverse indexing of the products indicates that in our convention the successive infinitesimal deformation are applied by multiplying from the left. The two formulas for J^t agree to leading order in δt, and the $m \to \infty$ limit of this procedure is the integral

Appendix ??

$$J^t_{ij}(x_0) = \left[\mathbf{T} e^{\int_0^t d\tau A(x(\tau))} \right]_{ij} \,, \tag{4.32}$$

where $\mathbf{T}$ stands for time-ordered integration, *defined* as the continuum limit of the successive left multiplications (4.31). This formula for J is the main result of this chapter.

Appendix ??

It is evident from the time-ordered product structure (4.31) that the fundamental matrices are multiplicative along the flow,

$$J^{t+t'}(x) = J^{t'}(x')\, J^t(x), \qquad \text{where } x' = f^t(x) \,. \tag{4.33}$$

in depth:
Section ??, p. ??

4.4 Neighborhood volume

Section ??
Remark ??

Consider a small state space volume $\Delta V = d^d x$ centered around the point x_0 at time $t = 0$. The volume $\Delta V' = \Delta V(t)$ around the point $x' = x(t)$ time t later is

$$\Delta V' = \frac{\Delta V'}{\Delta V} \Delta V = \left| \det \frac{\partial x'}{\partial x} \right| \Delta V = \left| \det\, J(x_0)^t \right| \Delta V \,, \tag{4.34}$$

so the $|\det J|$ is the ratio of the initial and the final volumes. The determinant $\det J^t(x_0) = \prod_{i=1}^d \Lambda_i(x_0, t)$ is the product of the Floquet multipliers. We shall refer to this determinant as the *Jacobian* of the flow. This Jacobian is easily evaluated. Take the time derivative and use the matrix identity $\ln \det J = \operatorname{tr} \ln J$:

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$$\frac{d}{dt} \ln \Delta V(t) = \frac{d}{dt} \ln \det J = \operatorname{tr} \frac{d}{dt} \ln J = \operatorname{tr} \frac{1}{J} \dot{J} = \operatorname{tr} A = \partial_i v_i \,.$$

(Here, as elsewhere in this book, a repeated index implies summation.) As the divergence $\partial_i v_i$ is a scalar quantity, the integral in the exponent needs *no time ordering*. Integrate both sides to obtain the time evolution of an infinitesimal volume

$$\det J^t(x_0) = \exp \left[\int_0^t d\tau \operatorname{tr} \mathbf{A}(x(\tau)) \right] = \exp \left[\int_0^t d\tau\, \partial_i v_i(x(\tau)) \right] \,. \tag{4.35}$$

All we need to do is evaluate the time average

$$\overline{\partial_i v_i} \;=\; \lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau \sum_{i=1}^d A_{ii}(x(\tau))$$

$$= \quad \frac{1}{t} \ln \left| \prod_{i=1}^{d} \Lambda_i(x_0, t) \right| = \sum_{i=1}^{d} \mu_i(x_0, t) \qquad (4.36)$$

along the trajectory. If the flow is not singular (for example, the trajectory does not run head-on into the Coulomb $1/r$ singularity), the stability matrix elements are bounded everywhere, $|A_{ij}| < M$, and so is the trace $\sum_i A_{ii}$. The time integral in (4.35) grows at most linearly with t, hence $\overline{\partial_i v_i}$ is bounded for all times, and numerical estimates of the $t \to \infty$ limit in (4.36) are not marred by any blowups.

Even if we were to insist on extracting $\overline{\partial_i v_i}$ from (4.31) by first multiplying fundamental matrices along the flow, and then taking the logarithm, we can avoid exponential blowups in J^t by using the multiplicative structure (4.33), $\det J^{t'+t}(x_0) = \det J^{t'}(x') \det J^t(x_0)$ to restart with $J^0(x') = \mathbf{1}$ whenever the eigenvalues of $J^t(x_0)$ start getting out of hand. In numerical evaluations of Lyapunov exponents, $\lambda_i = \lim_{t\to\infty} \mu_i(x_0, t)$, the sum rule (4.36) can serve as a helpful check on the accuracy of the computation.

⇒ Section **??**

The divergence $\partial_i v_i$ is an important characterization of the flow - it describes the behavior of a state space volume in the infinitesimal neighborhood of the trajectory. If $\partial_i v_i < 0$, the flow is *locally contracting*, and the trajectory might be falling into an attractor. If $\partial_i v_i(x) < 0$, for all $x \in \mathcal{M}$, the flow is *globally* contracting, and the dimension of the attractor is necessarily smaller than the dimension of state space $\mathcal{M}$. If $\partial_i v_i = 0$, the flow preserves state space volume and $\det J^t = \mathbf{1}$. A flow with this property is called *incompressible*. An important class of such flows are the Hamiltonian flows considered in Section 5.2.

But before we can get to that, Henri Roux, the perfect student always on alert, pipes up. He does not like our definition of the fundamental matrix in terms of the time-ordered exponential (4.32). Depending on the signs of Floquet exponents, the left hand side of (4.35) can be either positive or negative. But the right hand side is an exponential of a real number, and that can only be positive. What gives? As we shall see much later on in this text, in discussion of topological indices arising in semiclassical quantization, this is not at all a dumb question.

⇒ Section **??**

in depth:
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4.5 Stability of maps

The transformation of an infinitesimal neighborhood of a trajectory under the iteration of a map follows from Taylor expanding the iterated mapping at *discrete* time n to linear order, as in (4.5). The linearized neighborhood is transported by the fundamental matrix evaluated at a discrete set of times $n = 1, 2, \ldots$,

$$M^n_{ij}(x_0) = \left. \frac{\partial f^n_i(x)}{\partial x_j} \right|_{x=x_0} . \qquad (4.37)$$

We shall refer to this Jacobian matrix as the *monodromy* matrix, in order to include the case where the map is a Poincaré return map for a flow. Derivative notation $J^t(x_0) \to Df^t(x_0)$ is frequently employed in the literature. In this book Λ_k denotes the kth *eigenvalue* of the finite time monodromy matrix $M^n(x_0)$, and μ_k the real part of kth *eigen-exponent*

$$|\Lambda| = e^{n\lambda}\,, \qquad \Lambda_\pm = e^{n(\mu \pm i\nu)}\,.$$

For complex eigenvalue pairs the phase ν describes rotational motion in the plane defined by the corresponding pair of eigenvectors.

Example 4.6 Stability of a 1-dimensional map:

Consider a 1-d map $f(x)$. The chain rule yields the stability of the nth iterate

$$\Lambda(x_0, n) = \frac{d}{dx} f^n(x_0) = \prod_{m=0}^{n-1} f'(x_m)\,, \quad x_m = f^m(x_0)\,. \tag{4.38}$$

The 1-step product formula for the stability of the nth iterate of a d-dimensional map

$$\begin{aligned} M^n(x_0) &= M(x_{n-1}) \cdots M(x_1) M(x_0)\,, \\ M(x)_{kl} &= \frac{\partial}{\partial x_l} f_k(x)\,, \quad x_m = f^m(x_0) \end{aligned} \tag{4.39}$$

follows from the chain rule for matrix derivatives

$$\frac{\partial}{\partial x_i} f_j(f(x)) = \sum_{k=1}^{d} \frac{\partial}{\partial y_k} f_j(y)\bigg|_{y=f(x)} \frac{\partial}{\partial x_i} f_k(x)\,.$$

If you prefer to think of a discrete time dynamics as a sequence of Poincaré section returns, then (4.39) follows from (4.33): fundamental matrices are multiplicative along the flow.

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Example 4.7 Hénon map monodromy matrix:

For the Hénon map (3.15) the monodromy matrix for the nth iterate of the map is

$$M^n(x_0) = \prod_{m=n}^{1} \begin{pmatrix} -2ax_m & b \\ 1 & 0 \end{pmatrix}\,, \qquad x_m = f_1^m(x_0, y_0)\,. \tag{4.40}$$

The determinant of the Hénon one time step monodromy matrix (4.40) is constant,

$$\det M = \Lambda_1 \Lambda_2 = -b \tag{4.41}$$

so in this case only one eigenvalue $\Lambda_1 = -b/\Lambda_2$ needs to be determined. This is not an accident; a constant Jacobian was one of desiderata that led Hénon to construct a map of this particular form.

fast track
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4.5.1 Stability of Poincaré return maps

(R. Paškauskas and P. Cvitanović)

We now relate the linear stability of the Poincaré return map $P : \mathcal{P} \to \mathcal{P}$ defined in Section 3.1 to the stability of the continuous time flow in the full state space.

The hypersurface $\mathcal{P}$ can be specified implicitly through a function $U(x)$ that is zero whenever a point x is on the Poincaré section. A nearby point $x + \delta x$ is in the hypersurface $\mathcal{P}$ if $U(x + \delta x) = 0$, and the same is true for variations around the first return point $x' = x(\tau)$, so expanding $U(x')$ to linear order in δx leads to the condition

$$\sum_{i=1}^{d+1} \frac{\partial U(x')}{\partial x_i} \left. \frac{dx'_i}{dx_j} \right|_{\mathcal{P}} = 0 . \tag{4.42}$$

In what follows U_i is the gradient of U defined in (3.3), unprimed quantities refer to the starting point $x = x_0 \in \mathcal{P}$, $v = v(x_0)$, and the primed quantities to the first return: $x' = x(\tau)$, $v' = v(x')$, $U' = U(x')$. For brevity we shall also denote the full state space fundamental matrix at the first return by $J = J^\tau(x_0)$. Both the first return x' and the time of flight to the next Poincaré section $\tau(x)$ depend on the starting point x, so the fundamental matrix

$$\hat{J}(x)_{ij} = \left. \frac{dx'_i}{dx_j} \right|_{\mathcal{P}} \tag{4.43}$$

with both initial and the final variation constrained to the Poincaré section hypersurface $\mathcal{P}$ is related to the continuous flow fundamental matrix by

$$\left. \frac{dx'_i}{dx_j} \right|_{\mathcal{P}} = \frac{\partial x'_i}{\partial x_j} + \frac{dx'_i}{d\tau} \frac{d\tau}{dx_j} = J_{ij} + v'_i \frac{d\tau}{dx_j} .$$

The return time variation $d\tau/dx$, Fig. 4.6, is eliminated by substituting this expression into the constraint (4.42),

$$0 = \partial_i U' \, J_{ij} + (v' \cdot \partial U') \frac{d\tau}{dx_j} ,$$

yielding the projection of the full space $(d+1)$-dimensional fundamental matrix to the Poincaré map d-dimensional fundamental matrix:

$$\hat{J}_{ij} = \left(\delta_{ik} - \frac{v'_i \, \partial_k U'}{(v' \cdot \partial U')} \right) J_{kj} . \tag{4.44}$$

Substituting (4.7) we verify that the initial velocity $v(x)$ is a zero-eigenvector of $\hat{J}$

$$\hat{J} v = 0 , \tag{4.45}$$

so the Poincaré section eliminates variations parallel to v, and $\hat{J}$ is a rank d matrix, i.e., one less than the dimension of the continuous time flow.

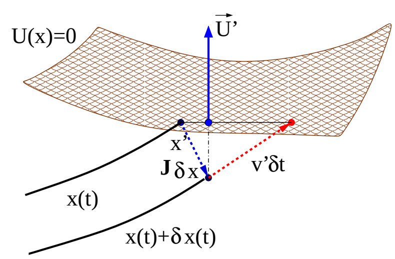

Fig. 4.6 If $x(t)$ intersects the Poincaré section $\mathcal{P}$ at time τ, the nearby $x(t) + \delta x(t)$ trajectory intersects it time $\tau + \delta t$ later. As $(U' \cdot v' \delta t) = -(U' \cdot J\, \delta x)$, the difference in arrival times is given by $\delta t = -(U' \cdot J\, \delta x)/(U' \cdot v')$.

Summary

A neighborhood of a trajectory deforms as it is transported by a flow. In the linear approximation, the stability matrix A describes the shearing / -compression / expansion of an infinitesimal neighborhood in an infinitesimal time step. The deformation after a finite time t is described by the fundamental matrix

$$J^t(x_0) = \mathbf{T}e^{\int_0^t d\tau A(x(\tau))} ,$$

where $\mathbf{T}$ stands for the time-ordered integration, defined multiplicatively along the trajectory. For discrete time maps this is multiplication by time step fundamental matrix M along the n points x_0, x_1, $x_2, \ldots$, x_{n-1} on the trajectory of x_0,

$$M^n(x_0) = M(x_{n-1})M(x_{n-2}) \cdots M(x_1)M(x_0) ,$$

with $M(x)$ the single discrete time step fundamental matrix. In this book Λ_k denotes the kth *eigenvalue* of the finite time fundamental matrix $J^t(x_0)$, and μ_k the real part of kth *eigen-exponent*

$$|\Lambda| = e^{n\lambda} , \qquad \Lambda_\pm = e^{n(\mu \pm i\nu)} .$$

For complex eigenvalue pairs the phase ν describes rotational motion in the plane defined by the corresponding pair of eigenvectors.

The eigenvalues and eigen-directions of the fundamental matrix describe the deformation of an initial infinitesimal sphere of neighboring trajectories into an ellipsoid a finite time t later. Nearby trajectories separate exponentially along unstable directions, approach each other along stable directions, and change slowly (algebraically) their distance along marginal directions. The fundamental matrix J^t is in general neither symmetric, nor diagonalizable by a rotation, nor do its (left or right) eigenvectors define an orthonormal coordinate frame. Furthermore, although the fundamental matrices are multiplicative along the flow, in dimensions higher than one their eigenvalues in general are not. This lack of multiplicativity has important repercussions for both classical and quantum dynamics.

☞ Appendix ??

Further reading

Linear flows. The theory of linear flows and their stability is only sketched in Section 4.2. They are presented at length in many textbooks. We liked the discussion in chapter 1 of Perko [1] and chapters 3 and 5 of Glendinning [2]. The nomenclature is a bit confusing. Sometimes A, the stability matrix (4.3) which describes the instantaneous shear of the trajectory point $x(x_0, t)$ is referred to as the 'Jacobian matrix,' a particularly unfortunate usage when one considers linearized stability of an equilibrium point (4.27). What Jacobi had in mind in his 1841 fundamental paper [3] on the determinants today known as 'jacobians' were transformations between different coordinate frames. More unfortunate still is referring to $J^t = e^{tA}$ as an 'evolution operator,' which here (see Section **??**) refers to something altogether different. In this book fundamental matrix J^t always refers to (4.6), the linearized deformation after a finite time t, either for a continuous time flow, or a discrete time mapping.

Exercises

(4.1) **Trace-log of a matrix.** Prove that

$$\det M = e^{\mathrm{tr} \ln M} .$$

for an arbitrary finite dimensional matrix M.

(4.2) **Stability, diagonal case.** Verify the relation (4.17)

$$J^t = e^{t\mathbf{A}} = \mathbf{U}^{-1} e^{t\mathbf{A}_D} \mathbf{U} , \quad \text{where} \quad \mathbf{A}_D = \mathbf{U}\mathbf{A}\mathbf{U}^{-1} .$$

(4.3) **State space volume contraction in Rössler flow.**

(a) Compute the Rössler flow volume contraction rate at the equilibria.

(b) Study numerically the instantaneous $\partial_i v_i$ along a typical trajectory on the Rössler attractor; color-code the points on the trajectory by the sign (and perhaps the magnitude) of $\partial_i v_i$. If you see regions of local expansion, explain them.

(c) Compute numerically the average contraction rate (4.36) along a typical trajectory on the Rössler attractor.

(d) (optional) color-code the points on the trajectory by the sign (and perhaps the magnitude) of $\partial_i v_i - \overline{\partial_i v_i}$.

(e) Argue on basis of your results that this attractor is of dimension smaller than the state space $d = 3$.

(f) (optional) Start some trajectories on the escape side of the outer equilibrium, color-code the points on the trajectory. Is the flow volume contracting?

(4.4) **Topology of the Rössler flow.** (continuation of Exercise 3.1)

(a) Show that equation $|\det(A - \lambda \mathbf{1})| = 0$ for Rössler system in the notation of Exercise 2.18 can be written as

$$\lambda^3 + \lambda^2 c\,(p^{\mp} - \epsilon) + \lambda(p^{\pm}/\epsilon + 1 - c^2 \epsilon p^{\mp}) \mp c\sqrt{D} = 0 \tag{4.46}$$

(b) Solve (4.46) for eigenvalues $\lambda^{\pm}$ for each equilibrium as an expansion in powers of ϵ. Derive

$$\begin{aligned} \lambda_1^- &= -c + \epsilon c/(c^2+1) + o(\epsilon) \\ \lambda_2^- &= \epsilon c^3/[2(c^2+1)] + o(\epsilon^2) \\ \theta_2^- &= 1 + \epsilon/[2(c^2+1)] + o(\epsilon) \\ \lambda_1^+ &= c\epsilon(1-\epsilon) + o(\epsilon^3) \\ \lambda_2^+ &= -\epsilon^5 c^2/2 + o(\epsilon^6) \\ \theta_2^+ &= \sqrt{1 + 1/\epsilon}\,(1 + o(\epsilon)) \end{aligned} \tag{4.47}$$

Compare with exact eigenvalues. What are dynamical implications of the extravagant value of λ_1^-? (continued as Exercise 11.7)

(Rytis Paškauskas)

(4.5) **A contracting baker's map.** Consider a contracting (or 'dissipative') baker's map, acting on a unit square $[0,1]^2 = [0,1] \times [0,1]$, defined by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/3 \\ 2y_n \end{pmatrix} \quad y_n \le 1/2$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/3 + 1/2 \\ 2y_n - 1 \end{pmatrix} \quad y_n > 1/2 .$$

This map shrinks strips by a factor of 1/3 in the x-direction, and then stretches (and folds) them by a factor of 2 in the y-direction.

By how much does the state space volume contract for one iteration of the map?

References

[1] L. Perko, *Differential Equations and Dynamical Systems* (Springer-Verlag, New York 1991).

[2] P. Glendinning, *Stability, Instability, and Chaos* (Cambridge Univ. Press, Cambridge 1994).

[3] C. G. J. Jacobi, "De functionibus alternantibus earumque divisione per productum e differentiis elementorum conflatum," in *Collected Works*, Vol. 22, 439; *J. Reine Angew. Math. (Crelle)* (1841).

[4] J.-L. Thiffeault, *Physica* **D 172**, 139 (2002); `nlin.CD/0101012`