Feynman Propagator for the Dirac Spinor Field

In the momentum space the scalar propagator is

$$G^F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i0} \times e^{-ip(x-y)}, \tag{25}$$

where the $i0$ in the denominator means the limit of $i\epsilon$ for $\epsilon \to +0$, and its purpose is to regularizes the integration over the poles along the mass shells $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$. There are different ways to regularize the poles, but this particular regulator corresponds to the Feynman propagator.

The Dirac propagator is

$$\begin{aligned} S^F_{\alpha\beta}(x-y) &= +(i\partial\!\!\!/ + m)_{\alpha\beta} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i0} \times e^{-ip(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)_{\alpha\beta}}{p^2 - m^2 + i0} \times e^{-ip(x-y)}. \end{aligned} \tag{26}$$

Furthermore, using

$$(p\!\!\!/ + m) \times (p\!\!\!/ - m) = p\!\!\!/\,p\!\!\!/ - m^2 = (p^2 - m^2) \times \mathbf{1}_{4\times 4}\,, \tag{27}$$

we may rewrite

$$\frac{(p\!\!\!/ + m)_{\alpha\beta}}{p^2 - m^2 + i0} = \frac{(p\!\!\!/ + m - i0)_{\alpha\beta}}{p^2 - (m - i0)^2} = \left(\frac{1}{p\!\!\!/ - m + i0}\right)_{\alpha\beta}, \tag{28}$$

so the Feynman propagator for the Dirac field becomes

$$S^F_{\alpha\beta}(x-y) = \int \frac{d^4p}{(2\pi)^4} \left(\frac{i}{p\!\!\!/ - m + i0}\right)_{\alpha\beta} \times e^{-ip(x-y)}\,. \tag{29}$$

This propagator is a Green's function of the Dirac equation,

$$(i\partial\!\!\!/ - m)_{\alpha\beta} S^F_{\beta\gamma}(x-y) = i\delta^{(4)}(x-y) \times \delta_{\alpha\gamma}\,. \tag{30}$$

Indeed,

$$\begin{aligned}
(i\,\partial\!\!\!/ - m)_{\alpha\beta}\, S^F_{\beta\gamma}(x-y) &= (i\,\partial\!\!\!/ - m)_{\alpha\beta} \times (i\,\partial\!\!\!/ + m)_{\beta\gamma}\, G^F(x-y) \\
&= ((\partial^2 + m^2) \times \mathbf{1})_{\alpha\gamma}\, G^F(x-y) \\
&= \delta_{\alpha\gamma} \times (\partial^2 + m^2)\, G^F(x-y) \\
&= \delta_{\alpha\gamma} \times i\delta^{(4)}(x-y).
\end{aligned} \tag{31}$$