8: The Path Integral for Free Field Theory

Prerequisite: 3, 7

Our results for the harmonic oscillator can be straightforwardly generalized to a free field theory with hamiltonian density

$$\mathcal{H}_0 = \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}(\nabla\varphi)^2 + \tfrac{1}{2}m^2\varphi^2 \, . \tag{184}$$

The dictionary we need is

$$\begin{aligned} q(t) &\longrightarrow \varphi(\mathbf{x},t) \quad \text{(classical field)} \\ Q(t) &\longrightarrow \varphi(\mathbf{x},t) \quad \text{(operator field)} \\ f(t) &\longrightarrow J(\mathbf{x},t) \quad \text{(classical \textit{source})} \end{aligned} \tag{185}$$

The distinction between the classical field $\varphi(x)$ and the corresponding operator field should be clear from context.

To employ the ϵ trick, we multiply $\mathcal{H}_0$ by $1-i\epsilon$. The results are equivalent to replacing m^2 in $\mathcal{H}_0$ with $m^2 - i\epsilon$. From now on, for notational simplicity, we will write m^2 when we really mean $m^2 - i\epsilon$.

Let us write down the path integral (also called the *functional integral*) for our free field theory:

$$Z_0(J) \equiv \langle 0|0\rangle_J = \int \mathcal{D}\varphi \; e^{i\int d^4x[\mathcal{L}_0 + J\varphi]} \, , \tag{186}$$

where

$$\mathcal{L}_0 = -\tfrac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \tfrac{1}{2}m^2\varphi^2 \, . \tag{187}$$

Note that when we say *path integral*, we now mean a path in the space of field configurations.

We can evaluate $Z_0(J)$ by mimicking what we did for the harmonic oscillator in section 7. We introduce four-dimensional Fourier transforms,

$$\widetilde{\varphi}(k) = \int d^4x \, e^{-ikx} \, \varphi(x) \; , \qquad \varphi(x) = \int \frac{d^4k}{(2\pi)^4} \, e^{ikx} \, \widetilde{\varphi}(k) \; , \tag{188}$$

where $kx = -k^0 t + \mathbf{k}\cdot\mathbf{x}$, and k^0 is an integration variable. Then, starting with $S_0 = \int d^4x \, [\mathcal{L}_0 + J\varphi]$, we get

$$S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \Big[-\widetilde{\varphi}(k)(k^2+m^2)\widetilde{\varphi}(-k) + \widetilde{J}(k)\widetilde{\varphi}(-k) + \widetilde{J}(-k)\widetilde{\varphi}(k) \Big], \tag{189}$$

where $k^2 = \mathbf{k}^2 - (k^0)^2$. We now change path integration variables to

$$\widetilde{\chi}(k) = \widetilde{\varphi}(k) - \frac{\widetilde{J}(k)}{k^2+m^2} \; . \tag{190}$$

Since this is merely a shift by a constant, we have $\mathcal{D}\varphi = \mathcal{D}\chi$. The action becomes

$$S_0 = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[\frac{\widetilde{J}(k)\widetilde{J}(-k)}{k^2+m^2} - \widetilde{\chi}(k)(k^2+m^2)\widetilde{\chi}(-k) \right] . \tag{191}$$

Just as for the harmonic oscillator, the integral over χ simply yields a factor of $Z_0(0) = \langle 0|0\rangle_{J=0} = 1$. Therefore

$$\begin{aligned} Z_0(J) &= \exp\!\left[\frac{i}{2} \int \frac{d^4k}{(2\pi)^4} \, \frac{\widetilde{J}(k)\widetilde{J}(-k)}{k^2+m^2-i\epsilon} \right] \\ &= \exp\!\left[\frac{i}{2} \int d^4x \, d^4x' \, J(x)\Delta(x-x')J(x') \right] . \end{aligned} \tag{192}$$

Here we have defined the *Feynman propagator*,

$$\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} \, \frac{e^{ik(x-x')}}{k^2+m^2-i\epsilon} \; . \tag{193}$$

The Feynman propagator is a Green's function for the Klein-Gordon equation,

$$(-\partial_x^2 + m^2)\Delta(x-x') = \delta^4(x-x') \; . \tag{194}$$

This can be seen directly by plugging eq. (193) into eq. (194) and then taking the $\epsilon \to 0$ limit. We can also evaluate $\Delta(x-x')$ explicitly by treating the k^0 integral on the right-hand side of eq. (193) as a contour integration in the complex k^0 plane, and then evaluating the contour integral via the residue theorem. The result is

$$\begin{aligned}\Delta(x-x') &= \int \widetilde{dk}\; e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-i\omega|t-t'|} \\ &= i\theta(t-t')\int \widetilde{dk}\; e^{ik(x-x')} + i\theta(t'-t)\int \widetilde{dk}\, e^{-ik(x-x')}\;, \end{aligned} \tag{195}$$

where $\theta(t)$ is the unit step function. The integral over $\widetilde{dk}$ can also be performed in terms of Bessel functions; see section 4.

Now, by analogy with the formula for the ground-state expectation value of a time-ordered product of operators for the harmonic oscillator, we have

$$\langle 0|\mathrm{T}\varphi(x_1)\ldots|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\ldots Z_0(J)\Big|_{J=0}\;. \tag{196}$$

Using our explicit formula, eq. (192), we have

$$\begin{aligned}\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle &= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\,Z_0(J)\Big|_{J=0} \\ &= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\Big[\int d^4x'\,\Delta(x_2-x')J(x')\Big]\,Z_0(J)\Big|_{J=0} \\ &= \Big[\tfrac{1}{i}\Delta(x_2-x_1) + (\text{term with } J\text{'s})\Big]\,Z_0(J)\Big|_{J=0} \\ &= \tfrac{1}{i}\Delta(x_2-x_1)\;. \end{aligned} \tag{197}$$

We can continue in this way to compute the ground-state expectation value of the time-ordered product of more φ's. If the number of φ's is odd, then there is always a left-over J in the prefactor, and so the result is zero. If the number of φ's is even, then we must pair up the functional derivatives in an appropriate way to get a nonzero result. Thus, for example,

$$\begin{aligned}\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)|0\rangle = \frac{1}{i^2}\Big[&\Delta(x_1-x_2)\Delta(x_3-x_4) \\ &+\Delta(x_1-x_3)\Delta(x_2-x_4) \\ &+\Delta(x_1-x_4)\Delta(x_2-x_3)\Big]. \end{aligned} \tag{198}$$

More generally,

$$\langle 0|\mathrm{T}\varphi(x_1)\ldots\varphi(x_{2n})|0\rangle = \frac{1}{i^n}\sum_{\text{pairings}}\Delta(x_{i_1}-x_{i_2})\ldots\Delta(x_{i_{2n-1}}-x_{i_{2n}}) \ . \qquad (199)$$

This result is known as *Wick's theorem.*

Problems

8.1) Starting with eq. (193), verify eq. (194).

8.2) Starting with eq. (193), verify eq. (195).

8.3) Use eq. (86), the commutation relations eq. (95), and $a(\mathbf{k})|0\rangle = 0$, $\langle 0|a^\dagger(\mathbf{k}) = 0$ to verify the last line of eq. (197).

8.4) The retarded and advanced Green's functions for the Klein-Gordon wave operator satisfy $\Delta_{\rm ret}(x-y) = 0$ for $x^0 \geq y^0$ and $\Delta_{\rm adv}(x-y) = 0$ for $x^0 \leq y^0$. Find the pole prescriptions on the right-hand side of eq. (193) that yield these Green's functions.

8.5) Let $Z_0(J) = \exp iW_0(J)$, and evaluate the real and imaginary parts of $W_0(J)$.

8.6) Repeat the analysis of this section for the complex scalar field that was introduced in problem 3.3, and further studied in problem 5.1. Write your source term in the form $J^\dagger\varphi + J\varphi^\dagger$, and find an explicit formula, analogous to eq. (192), for $Z_0(J^\dagger, J)$. Write down the appropriate generalization of eq. (196), and use it to compute $\langle 0|\mathrm{T}\varphi(x_1)\varphi(x_2)|0\rangle$, $\langle 0|\mathrm{T}\varphi^\dagger(x_1)\varphi(x_2)|0\rangle$, and $\langle 0|\mathrm{T}\varphi^\dagger(x_1)\varphi^\dagger(x_2)|0\rangle$. Then verify your results by using the method of problem 8.3. Finally, give the appropriate generalization of eq. (199).