Fourier Transform

$$f(x) = \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk \quad (\text{"forward"})$$

Dirac's δ-function

$$\int_{-\infty}^{\infty} f(x)\, e^{-iqx}\, dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} F(k)\, e^{i(k-q)x}\, dk$$

$$= \int_{-\infty}^{\infty} dk\, F(k) \int_{-\infty}^{\infty} e^{i(k-q)x}\, dx = \int_{-\infty}^{\infty} dk\, F(k)\, 2\pi\, \delta(k-q) = 2\pi F(q)$$

$$\Rightarrow F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx \equiv \mathcal{F}[f(x)] \quad (\text{"backward"})$$

integral operator

Properties

Linearity: $\mathcal{F}[\alpha_1 f_1(x) + \alpha_2 f_2(x)] = \alpha_1 \mathcal{F}[f_1(x)] + \alpha_2 \mathcal{F}[f_2(x)]$

Parseval's identity:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} f^*(x) f(x)\, dx =$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk'\, F^*(k')\, e^{-ik'x} \int_{-\infty}^{\infty} dk\, F(k)\, e^{ikx} =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk\, dk'\, F^*(k') F(k) \int_{-\infty}^{\infty} e^{i(k-k')x}\, dx =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk\, dk'\, F^*(k) F(k)\, 2\pi\, \delta(k-k') =$$

$$= 2\pi \int_{-\infty}^{\infty} F^*(k) F(k)\, dk = 2\pi \int_{-\infty}^{\infty} |F(k)|^2 dk \quad (L_2 \text{ norm } \|\cdot\|_2)$$

Differentiation:

$$f'(x) = \frac{d}{dx} \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk = \int_{-\infty}^{\infty} F(k)\, (ik)\, e^{ikx}\, dk$$

$$\Rightarrow \mathcal{F}[f^{(n)}(x)] = (ik)^n \mathcal{F}[f(x)]$$

Integration: $g(x) = \int_{x_0}^{x} f(t)\,dt$

$$g'(x) = f(x) \Rightarrow ik\mathcal{F}[g(x)] = \mathcal{F}[f(x)]$$

$$\Rightarrow \mathcal{F}[g(x)] = \frac{1}{ik}\mathcal{F}[f(x)], \quad k \neq 0$$

What happens when $k=0$?

$$\mathcal{F}[g(x)] = c\,\delta(k) \text{ is a solution of } ik\mathcal{F}[g(x)] = 0 \Rightarrow$$

$$\mathcal{F}\left[\int f(x)\,dx\right] = \frac{1}{ik}\mathcal{F}[f(x)] + c\,\delta(k)$$

↖ Integration constant!

Convolution: $G(k) = F(k)\,H(k)$

$$g(x) = \int G(k)e^{ikx}dk = \int dk\, F(k)H(k)e^{ikx}dk =$$

$$= \int dk \frac{e^{ikx}}{(2\pi)^2}\int e^{-iky}f(y)\,dy \int e^{-ikz}h(z)\,dz =$$

$$= \frac{1}{(2\pi)^2}\iint dy\,dz\, f(y)h(z)\int e^{-ik(y+z-x)}dk =$$

$$= \frac{1}{2\pi}\int dy\,dz\, f(y)h(z)\,\delta(y+z-x) = \frac{1}{2\pi}\int dy\, f(y)\,h(x-y)$$

Applications of Fourier Transform

- Low-pass filter — Data smoothing
- High-pass filter — Background removal
- Deconvolution — Image analysis
- Normal mode analysis — Diff. equations

Example (Heisenberg uncertainty principle)

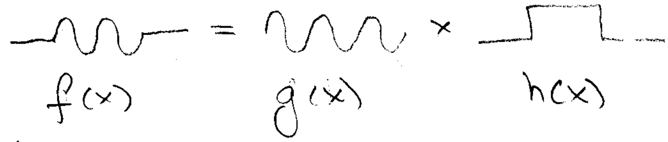
$$\int e^{-\frac{ip}{\hbar}x}\, e^{-\frac{x^2}{2a^2}}dx = \int e^{-\frac{1}{2}\left(\frac{x}{a}+\frac{ipa}{\hbar}\right)^2} e^{-\frac{p^2a^2}{2\hbar^2}}dx = a\sqrt{\pi}\, e^{-\frac{p^2}{2(\hbar/a)^2}}$$

x-localization: a

p-localization: $\hbar/a$

$$\Rightarrow \Delta x \cdot \Delta p = \hbar$$

Example: (Finite wavetrain)

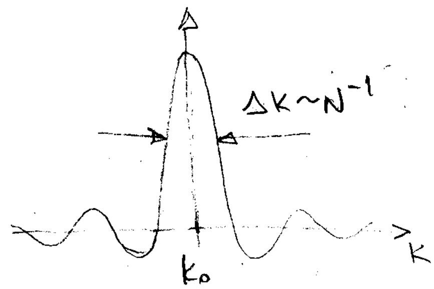


$G(k) = G_0\,\delta(k-k_0)$ – monochromatic wave

$$2\pi H(k) = \int_{-a}^{a} e^{-ikx}\,dx = -\frac{1}{ik}\,e^{-ikx}\Big|_{-a}^{a} = \frac{2}{k}\,\frac{1}{2i}\left(e^{ika}-e^{-ika}\right) = \frac{2\sin ka}{k}$$

$$F(k) \sim \int G(k-q)\,H(q)\,dq = \int \delta(k-k_0-q)\,\frac{\sin qa}{\pi q}\,dq = \frac{\sin(k-k_0)a}{\pi(k-k_0)}$$

width of F: $\Delta k a = \pi \Rightarrow \Delta k = \frac{\pi}{a} = \frac{2\pi}{\lambda N}$



Example: (Harmonic oscillator)

$$H(p,x) = \frac{p^2}{2m} + \frac{1}{2}kx^2 = E$$

$$-\frac{\hbar^2}{2m}\underbrace{\partial_x^2}_{-p^2}\psi + \frac{1}{2}k\underbrace{x^2}_{-\partial_p^2}\psi = E\psi \quad , \quad \psi(x) \sim e^{-\frac{\sqrt{mk}}{2\hbar}x^2} \text{ – ground state}$$

$$\psi(x) = \int \varphi(p)\,e^{i\frac{1}{\hbar}px}\,dp$$

$$\Rightarrow \frac{\hbar^2}{2m}p^2\varphi - \frac{1}{2}k\,\partial_p^2\varphi = E\varphi \Rightarrow \varphi(p) \sim e^{-\frac{p^2}{2\hbar\sqrt{mk}}} \text{ – momentum representation}$$

Example: (Airy Function)

$$f'' - xf = 0 \quad , \quad f = \int_{-\infty}^{\infty} F(k)\,e^{ikx}\,dk$$

$$-k^2F(k) - i\frac{d}{dk}F(k) = 0 \Rightarrow \frac{dF}{F} = d\ln F = ik^2\,dk$$

$$\ln F = +\frac{i}{3}k^3 + C \Rightarrow F = F_0\,e^{\frac{i}{3}k^3}$$

$$f(x) = F_0\int_{-\infty}^{\infty} e^{+\frac{i}{3}k^3 + ikx}\,dk$$

calculate using contour integration

Example (Heat conduction)

$$\begin{cases} \partial_t T = \varkappa \partial_x^2 T & \leftarrow \text{doesn't depend on either } x \text{ or } t \\ T(x,0) = \delta(x) \end{cases}$$

$$T(x,t) = \iint \Theta(k,\omega)\, e^{ikx + i\omega t}\, dk\, d\omega$$

$$\Rightarrow \; i\omega\Theta = -\varkappa k^2 \Theta \;\Rightarrow\; \omega = i\varkappa k^2 \quad (\text{unless } \Theta = 0)$$

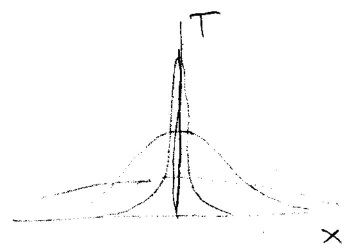
$$T(x,t) = \int \Theta(k)\, e^{ikx - \varkappa k^2 t}\, dk$$, but what is $\Theta(k)$?

$$T(x,0) = \int \Theta(k)\, e^{ikx}\, dk = \delta(x) = \frac{1}{2\pi}\int e^{ikx}\, dk$$

$$\Rightarrow \Theta(k) = \frac{1}{2\pi} ,$$

$$T(x,t) = \frac{1}{2\pi}\int e^{-\varkappa k^2 t + ikx}\, dk = \frac{1}{2\pi}\int e^{-\left(\sqrt{\varkappa t}\,k - \frac{ix}{2\sqrt{\varkappa t}}\right)^2 - \frac{x^2}{4\varkappa t}}\, dk$$

$$\sim e^{-\frac{x^2}{4\varkappa t}}$$



Example: Correlation function

$x(t)$: $C(\tau) = \langle \underbrace{(x(t)-\bar{x})}_{y(t)} \underbrace{(x(t+\tau)-\bar{x})}_{y(t+\tau)} \rangle_t$ ← too long a computation (N^2 operations)

$$= \frac{1}{T}\int_0^T y(t)y(t+\tau)dt = \frac{1}{T}\int y(t-\tau)\,y^*(t)dt =$$

$$= \frac{1}{T}\int dt \int d\omega\, Y(\omega) e^{i\omega t} e^{-i\omega\tau} \int d\omega' Y^*(\omega') e^{-i\omega' t} =$$

$$= \frac{1}{T}\int d\omega \int d\omega'\, Y(\omega) Y^*(\omega') e^{-i\omega\tau} \int dt\, e^{i(\omega-\omega')t} =$$

$$= \frac{2\pi}{T}\int d\omega d\omega' Y(\omega) Y^*(\omega') e^{-i\omega\tau}\delta(\omega-\omega') =$$

$Y(-\omega) = Y^*(\omega)$

$$= \frac{2\pi}{T}\int d\omega |Y(\omega)|^2 e^{-i\omega\tau} = \frac{4\pi}{T}\int_0^\infty d\omega |Y(\omega)|^2 \cos\omega\tau$$ – real function

1) Get the signal $x(t)$ $\sim N$
2) Subtract average $x(t) \to y(t)$ $\sim N$
3) Calculate FFT: $Y(\omega)$ $\sim N\log N$
4) Inverse FFT $|Y(\omega)|^2 \to C(\tau)$ $\sim N\log N$

} → $N\log N$ operations

Hilbert Transform & Complex Demodulation

$$u(y) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)}{x-y}\, dx = \mathcal{H}^{-1}[v]$$

$$v(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y)}{y-x}\, dy = \mathcal{H}[u]$$

Analytic signal:

$$w(t) = \underset{\substack{\uparrow \\ \text{actually measure}}}{u(t)} + i v(t) = \int_0^{\infty} F(\omega) e^{i\omega t}\, d\omega$$

Titchmarsh thm: $F(\omega) = 0,\ \omega < 0 \Rightarrow \begin{cases} v = \mathcal{H}[u] \\ (u = \mathcal{H}^{-1}[v]) \end{cases}$

Complex Demodulation:

$$w(t) = u(t) + i v(t) = \underset{\substack{\uparrow \\ \text{amplitude}}}{r(t)}\, e^{i\theta(t)}$$

phase

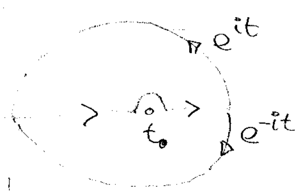
Instanteneous phase: $\theta(t) = \arg w(t)$

& amplitude: $r(t) = |w(t)|$

Example: (Harmonic signal)

$$u(t) = \sin t \ : \quad v(t_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin t}{t - t_0}\, dt = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{e^{it}}{t-t_0} - \frac{e^{-it}}{t-t_0} \right] dt$$

$$= \frac{1}{2\pi i}\left(\left(\frac{1}{2}\right) 2\pi i \operatorname{res} \frac{e^{it}}{t-t_0}\Big|_{t=t_0} - \left(\frac{1}{2}\right) 2\pi i \operatorname{res} \frac{e^{-it}}{t-t_0}\Big|_{t=t_0} \right) =$$



$$= -\frac{1}{2}\left(e^{it_0} + e^{-it_0}\right) = -\cos t_0$$

$$\Rightarrow w = u + iv = \sin t - i\cos t = -i(\cos t_0 + i \sin t_0) = -i e^{it_0} = e^{i(t_0 - \frac{\pi}{2})}$$

Phase: $\theta(t) = \arg w(t) = t_0 - \frac{\pi}{2}$ (exactly what we expected!)