Integral Transforms

Examples:

Hilbert transform: $$u(y) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)}{x-y} dx$$

$$v(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y)}{y-x} dy$$

Mellin transform: $$\varphi(z) = \int_0^{\infty} t^{z-1} f(t) dt$$

$$f(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} t^{-z} \varphi(z) dz$$

Hankel transform: $$g(k) = \int_0^{\infty} f(x) J_m(kx) x dx$$

$$f(x) = \int_0^{\infty} g(k) J_m(kx) k dk$$

Laplace transform: $$F(s) = \int_0^{\infty} f(x) e^{-sx} dx$$

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{sx} ds$$

Fourier transform: $$F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} F(k) dk$$

Fourier transform

Q.M.: $E\psi = -\frac{\hbar^2}{2m} \partial_x^2 \psi$ — independent of $x \Rightarrow p = \text{const}$

$\psi_p = \psi_0 e^{i\frac{1}{\hbar}px} \Rightarrow E = -\frac{\hbar^2}{2m} \cdot \left(-\frac{p^2}{\hbar^2}\right) = \frac{p^2}{2m}$

$\hat{p} = -i\hbar \frac{\partial}{\partial x}$: $\hat{p}\psi_q = q\psi_q$ — eigenfunction with eigenvalue q

General solution (Linear equation!):

$$\psi(x) = \int_{-\infty}^{\infty} \Psi(p)\psi_p(x) dp = \int_{-\infty}^{\infty} \Psi(p) e^{i\frac{px}{\hbar}} dp \overset{p=\hbar k}{=} \underbrace{\hbar \int_{-\infty}^{\infty} \Psi(\hbar k) e^{ikx} dk}_{\text{Fourier transform of } \Psi(\hbar k)}$$

wave function in x-space

wave function in p-space

More generally:

Differential equation is independent of x

$\Rightarrow$ solution is translation invariant

$\Rightarrow$ solution can be represented as a superposition of eigenfunctions of translation operator, $T = \frac{\partial}{\partial x}$
(i.e., Fourier modes)

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right), \quad 0 < x < 2\pi \quad (-\pi < x < \pi)$$

or

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad c_{\pm n} = \frac{1}{2}(a_n \mp i b_n), \; n \geq 0$$

Fourier coefficients:

Cronecker δ-symbol

$$\left[\begin{array}{l} \int_0^{2\pi} \sin nx \sin mx \, dx = \begin{cases} \pi \delta_{m,n}, & m \neq 0 \\ 0, & m = 0 \end{cases} \\ \int_0^{2\pi} \cos nx \cos mx \, dx = \begin{cases} \pi \delta_{m,n}, & m \neq 0 \\ 2\pi, & m = n = 0 \end{cases} \\ \int_0^{2\pi} \cos mx \sin nx \, dx = 0, \quad \forall m, n \in \mathbb{Z} \end{array}\right.$$

completeness: Sturm-Liouville

$$\Rightarrow \begin{cases} a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx, \\ b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx \end{cases}$$

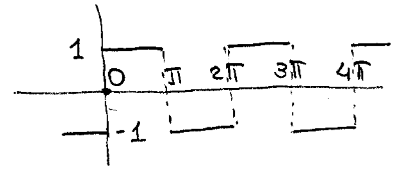
Domain of length L:

$$f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2\pi}{L} nx\right) + B_n \sin\left(\frac{2\pi}{L} nx\right) \right]$$

$$\begin{cases} A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi}{L} nx\right) dx, \\ B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi}{L} nx\right) dx \end{cases}$$

Example: (square wave)

$$f(x) = \begin{cases} 1 \ , & 0 < x < \pi \\ -1 \ , & -\pi < x < 0 \end{cases}$$

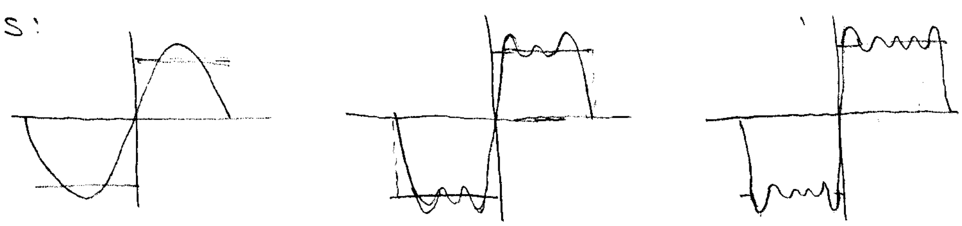


$$f(2\pi - x) = -f(x) \Rightarrow \begin{cases} a_n = 0 \\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \sin nx\, dx = \end{cases}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

$$\Rightarrow f(x) = \sum_{n=1,\text{odd}}^{\infty} \frac{4}{\pi n}\sin(nx).$$

Partial sums:



$n = 1$ $\qquad$ $n = 1, 3$ $\qquad$ $n = 1, 3, 5$

Behavior at Discontinuities:

1) $f(x_0) = \frac{1}{2}\left[f(x_0^-) + f(x_0^+)\right]$

2) Overshoot on both sides $\not\to 0$ as $n \to \infty$

Gibbs phenomenon:

Partial sum:

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\left[a_n\cos nx + b_n\sin nx\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt +$$

$$+ \frac{1}{\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi} f(t)\left[\cos nt\cos nx + \sin nt\sin nx\right]dt =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{n=1}^{N}\cos n(t-x)\right]dt = \frac{1}{\pi}\mathrm{Re}\int_{-\pi}^{\pi} f(t)\Big[\frac{1}{2} + \underbrace{\sum_{n=1}^{N} e^{in(t-x)}}_{\text{Geometric progression}}\Big]dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin\left[(N+\frac{1}{2})(t-x)\right]}{\sin\left[\frac{1}{2}(t-x)\right]}\,dt$$

For square wave:

$$f_N(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin[(N+\frac{1}{2})(t-x)]}{\sin[\frac{1}{2}(t-x)]}dt - \frac{1}{2\pi}\int_{-\pi}^{0} \frac{\sin[(N+\frac{1}{2})(t-x)]}{\sin[\frac{1}{2}(t-x)]}dt =$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} \frac{\sin[(N+\frac{1}{2})(t-x)]}{\sin[\frac{1}{2}(t-x)]} - \int_0^{\pi} \frac{\sin[(N+\frac{1}{2})(t+x)]}{\sin[\frac{1}{2}(t+x)]}dt\right] =$$

quickly oscillates $\Rightarrow$ small.

$$= \frac{1}{2\pi}\left[\int_{-x}^{\pi-x} \frac{\sin(N+\frac{1}{2})s}{\sin\frac{1}{2}s}ds - \int_{x}^{\pi+x} \frac{\sin(N+\frac{1}{2})s}{\sin\frac{1}{2}s}ds\right] = \frac{1}{2\pi}\left\{\int_{-x}^{x} - \int_{\pi-x}^{\pi+x}\right\}$$

<u>for $N\to\infty$:</u> $$f_N(x) \approx \frac{1}{2\pi}\int_{-x}^{x} \frac{\sin(N+\frac{1}{2})s}{\sin\frac{1}{2}s}ds = \frac{1}{\pi}\int_0^{xp} \frac{\sin\xi}{\sin(\xi/2p)}\cdot\frac{d\xi}{p}, \quad p = N+\frac{1}{2}, \quad \xi = ps$$

$$f_N(x)_{max} \approx \frac{1}{\pi}\int_0^{\pi} \frac{\sin\xi}{\sin\xi/2p}\frac{d\xi}{p} \to \underbrace{\frac{2}{\pi}\int_0^{\pi}\frac{\sin\xi}{\xi}d\xi}_{1.17897 \Rightarrow \text{overshoot by } 18\%!}, \quad p\to\infty \; (N\to\infty)$$

Convergence Speed

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)(\cos nx - i\sin nx)\,dx = \frac{1}{2\pi}\int_{x_0}^{x_0+2\pi} f(x)e^{-inx}dx$$

Integrate by parts:

$$c_n = \frac{i}{2\pi n}\int_{x_0}^{x_0+2\pi} f(x)\,d e^{-inx} = \frac{i}{2\pi n}\underbrace{\left[e^{-inx}f(x)\right]_{x_0}^{x_0+2\pi}}_{0,\text{ iff } f(x)\text{ - continuous at } x_0,\,0} - \frac{i}{2\pi n}\int_{x_0}^{x_0+2\pi} f'(x)e^{-inx}dx$$

- $f(x)$ - discontinuous at $x_0 \Rightarrow c_n = o(\frac{1}{n})$

$f(x)$ - continuous $\to$ integrate by parts:

$$c_n = \frac{1}{2\pi n^2}\int_{x_0}^{x_0+2\pi} f'(x)\,d e^{-inx} = \frac{1}{2\pi n^2}\underbrace{\left[e^{-inx}f'(x)\right]_{x_0}^{x_0+2\pi}}_{0,\text{ iff } f'(x)\text{ - continuous at } x_0,\,0} - \frac{1}{2\pi n^2}\int_{x_0}^{x_0+2\pi} f''(x)e^{-inx}dx$$

- $f'(x)$ - discontinuous at $x_0 \Rightarrow c_n = o(\frac{1}{n^2})$

...

- $f^{(k)}(x)$ - discontinuous at $x_0 \Rightarrow c_n = o(\frac{1}{n^{k+1}})$

Properties:

Integration: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$

$$\int_{x_0}^{x} f(t)\,dt = \frac{a_0 t}{2}\Big|_{x_0}^{x} + \underbrace{\sum_{n=1}^{\infty} \frac{a_n}{n} \sin nt \Big|_{x_0}^{x} - \sum_{n=1}^{\infty} \frac{b_n}{n} \cos nt \Big|_{x_0}^{x}}$$

a) also Fourier series! b) better convergence!

Differentiation:

$$f'(x) = \underbrace{\sum_{n=1}^{\infty} \left[-a_n \cdot n \sin nx + b_n \cdot n \cos nx\right]}$$

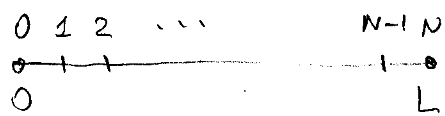
a) also Fourier series b) might not converge

Superposition: $f = \alpha_1 f_1 + \alpha_2 f_2 \Rightarrow \begin{cases} a_n = \alpha_1 a_n^{(1)} + \alpha_2 a_n^{(2)} \\ b_n = \dots \end{cases}$

Parseval's identity $\frac{1}{\pi} \int_0^{2\pi} f(x)^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$

Discrete Fourier Transform

$x_k = L \frac{k}{N}$



$$f_k \equiv f(x_k) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(2\pi \frac{kn}{N}\right) + b_n \sin\left(2\pi \frac{kn}{N}\right)\right]$$

$$= \sum_{n=0}^{N-1} c_n e^{i 2\pi \frac{kn}{N}}$$

$$\sum_{n=0}^{N-1} e^{i2\pi \frac{k}{N} n} = \sum_{n=0}^{N-1} \left(e^{i2\pi \frac{k}{N}}\right)^n = \begin{cases} \frac{1 - e^{i2\pi k}}{1 - e^{i2\pi \frac{k}{N}}} = 0, & k \neq 0 \\ N, & k = 0 \end{cases}$$

$$\Rightarrow \sum_{k=0}^{N-1} f_k e^{-i2\pi \frac{k}{N} m} = \sum_{n=0}^{N-1} c_n \underbrace{\sum_{k=0}^{N-1} e^{i2\pi \frac{k}{N}(n-m)}}_{N\delta_{n,m}} = N c_m$$

$$\Rightarrow c_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-i2\pi \frac{km}{N}}$$