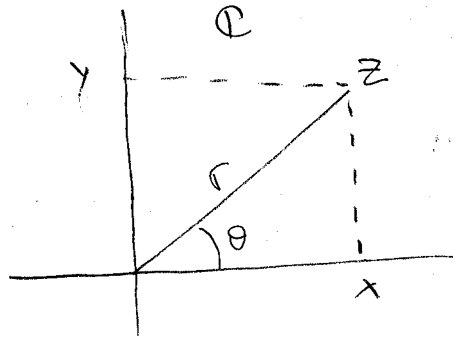
Complex Analysis

Motivation: provides easy way to solve complicated problems in EM and fluid dynamics, useful mathematical technique

Complex number: $z = x + iy = (x, y)$

$$i^2 = -1$$



Polar representation:

$$z = re^{i\theta} = r\cos\theta + ir\sin\theta$$

Functions of complex variable:

$$f(z) = u(x+iy) + iv(x+iy) = (u(z), v(z))$$

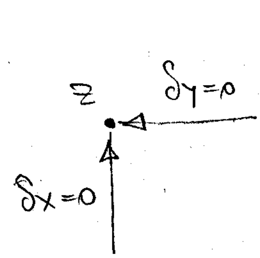
Example: $z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 = \underbrace{x^2 - y^2}_{u} + i\,\underbrace{2xy}_{v}$

Complex conjugate: $z^* = x - iy$

$$\Rightarrow zz^* = (x+iy)(x-iy) = x^2 + y^2 = r^2 = |z|^2$$

Cauchy-Riemann condition

Derivative: $\dfrac{df}{dz} = \lim\limits_{\delta z \to 0} \dfrac{f(z+\delta z) - f(z)}{\delta z}$



Differentiable: lim – the same for ∀ direction.

$$\lim_{\delta x \to 0} \frac{\delta f}{\delta x} = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x} =$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\lim_{\delta y\to 0}\frac{\delta f}{\delta i y}=\lim_{\delta y\to 0}\frac{u(x,y+\delta y)-u(x,y)}{i\delta y}+i\lim_{\delta y\to 0}\frac{v(x,y+\delta y)-v(x,y)}{i\delta y}$$

$$=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{i}{i}\frac{\partial v}{\partial y}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Same limit: $\quad \dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}=\dfrac{\partial v}{\partial y}-i\dfrac{\partial u}{\partial y}$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\,,\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}}$$ – Cauchy-Riemann condition

<u>Definition</u>. $f(z)$ is analytic at $z=z_0$ if $f(z)$ is differentiable in some small neighborhood of z_0.

<u>Properties of analytic functions:</u>

Let $f(z)=(u,v)$ – analytic.

a) u,v satisfy $\vec{\nabla}^2 u=\vec{\nabla}^2 v=0$:

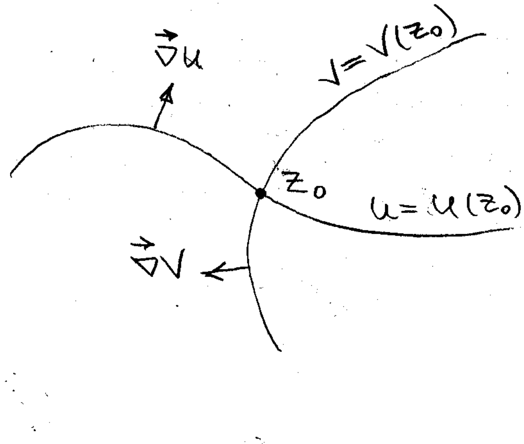
$$\partial_x^2 u+\partial_y^2 u=\partial_x(\partial_x u)+\partial_y(\partial_y u)=\partial_x(\partial_y v)-\partial_y(\partial_x v)=0$$
$$\partial_x^2 v+\partial_y^2 v=\partial_x(\partial_x v)+\partial_y(\partial_y v)=-\partial_x(\partial_y u)+\partial_y(\partial_x u)=0.$$

b) Level sets of u,v are $\perp$:

level set of $u(x,y)$: $\{(x,y)\,|\,u(x,y)=\text{const}\}$

$\vec{n}_u\sim\vec{\nabla}u=(\partial_x u,\partial_y u)$

$\vec{n}_v\sim\vec{\nabla}v=(\partial_x v,\partial_y v)$



$(\vec{n}_u\cdot\vec{n}_v)=\partial_x u\,\partial_x v+\partial_y u\,\partial_y v=$

$=\partial_x u(-\partial_y u)+\partial_y u(\partial_x u)=0$

$\Rightarrow \vec{n}_u\perp\vec{n}_v$, level sets $\perp$

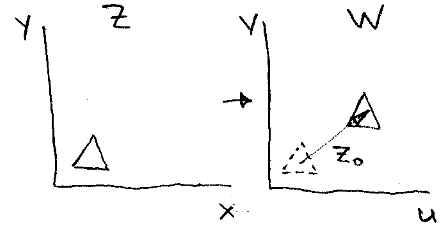
Mappings

function: $w = f(z) = u(x,y) + iv(x,y)$

mapping: $f: z \to w$ $\quad (f: \mathbb{C} \to \mathbb{C})$

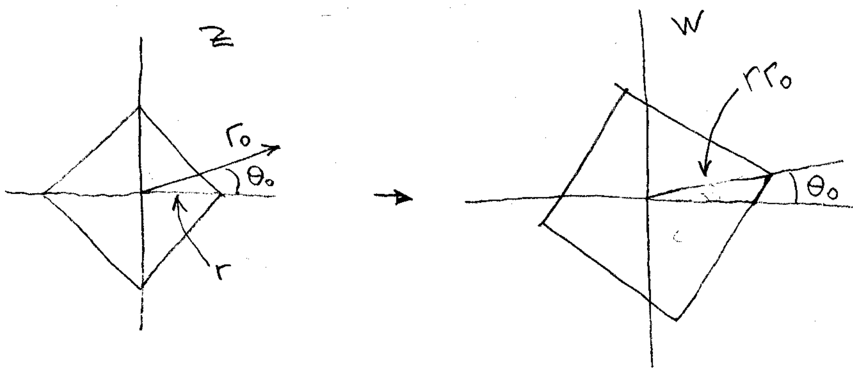
Translation:

$$w = z + z_0 \Rightarrow \begin{cases} u = x + x_0 \\ v = y + y_0 \end{cases}$$



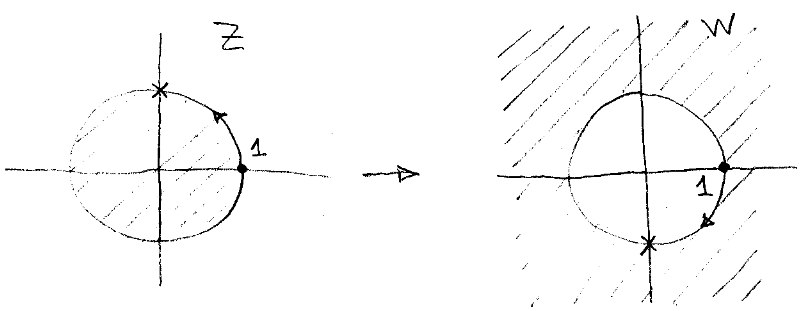
Rotation:
(+ rescaling)

$$w = z z_0 \Rightarrow \rho e^{i\varphi} = r r_0 e^{i(\theta + \theta_0)} \Rightarrow \begin{cases} \rho = r r_0 \\ \varphi = \theta + \theta_0 \end{cases}$$



Inversion:

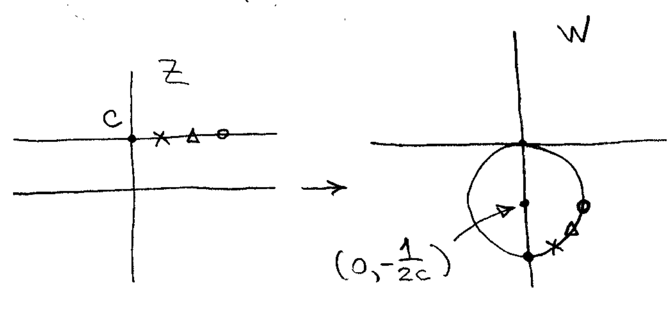
$$w = \frac{1}{z} \Rightarrow \rho e^{i\varphi} = \frac{1}{r} e^{-i\theta} \Rightarrow \begin{cases} \rho = \frac{1}{r} \\ \varphi = -\theta \end{cases}$$



Alternatively: $\quad u + iv = \dfrac{1}{x+iy} = \dfrac{x - iy}{x^2 + y^2} \Rightarrow \begin{cases} u = \dfrac{x}{x^2+y^2} \\ v = \dfrac{-y}{x^2+y^2} \end{cases}$

$$z = \frac{1}{w} \Rightarrow \begin{cases} x = \dfrac{u}{u^2+v^2} \\ y = -\dfrac{v}{u^2+v^2} \end{cases} \Rightarrow y = c \to -\frac{v}{u^2+v^2} = c$$

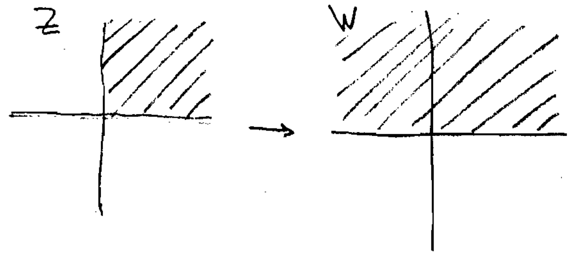
$$\Rightarrow \underbrace{u^2 + \left[v^2 + \frac{v}{c} + \frac{1}{(2c)^2}\right]}_{0} = u^2 + \left(v + \frac{1}{2c}\right)^2 = \frac{1}{(2c)^2}$$

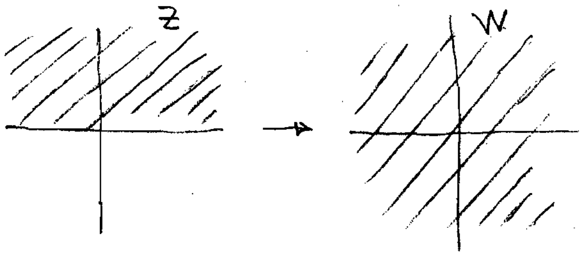


Branch Cuts

So far only looked at 1-to-1 mappings.
What happens if $f: z \to w$ is <u>not</u> 1-to-1?

$2 \to 1$: $w = z^2 \Rightarrow \begin{cases} \rho = r^2 \\ \varphi = 2\theta \end{cases}$





and so on...

$\theta \in [0, 2\pi] \to \varphi = 2\theta \in [0, 4\pi]$

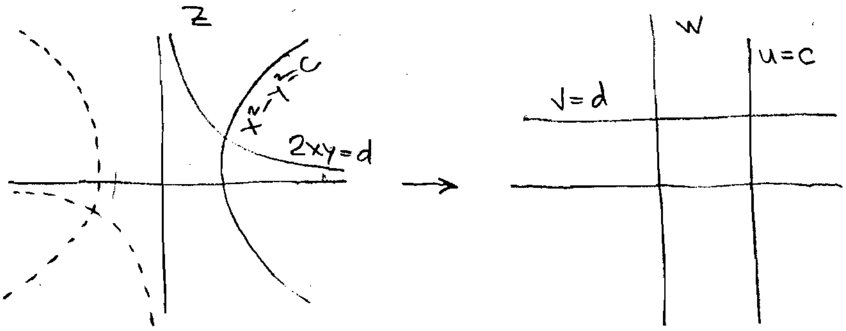
$z_0 \to w_0,\ -z_0 \to w_0 \Rightarrow$ 2-to-1

alternatively: $u + iv = (x+iy)^2 = x^2 - y^2 + 2ixy \to \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$

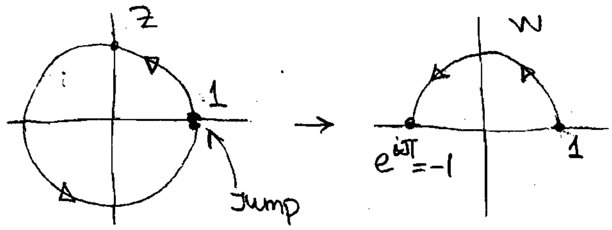
$x^2 - y^2 = c \to u = c$
$2xy = d \to v = d$

hyperbolas

straight lines

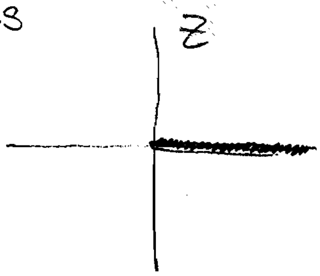


$1 \to 2$: $w = z^{1/2} \Rightarrow \begin{cases} \rho = \sqrt{r} \\ \varphi = \frac{1}{2}\theta\,(+\pi) \end{cases}$
(not single-valued!)

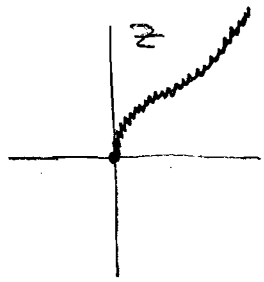


Cut lines

non-analytic!



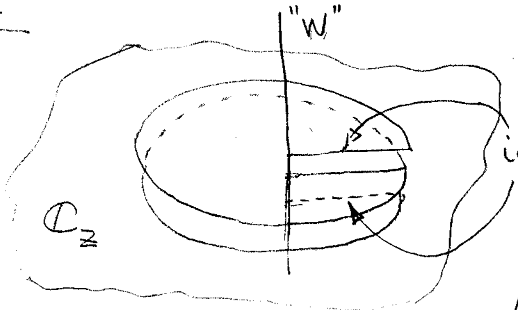
or



or ...

Riemann Surface

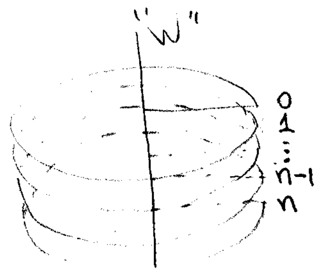
$W = z^{1/2}$



identified $\left(e^{\frac{1}{2}4\pi i} = e^{2\pi i} = 1\right)$

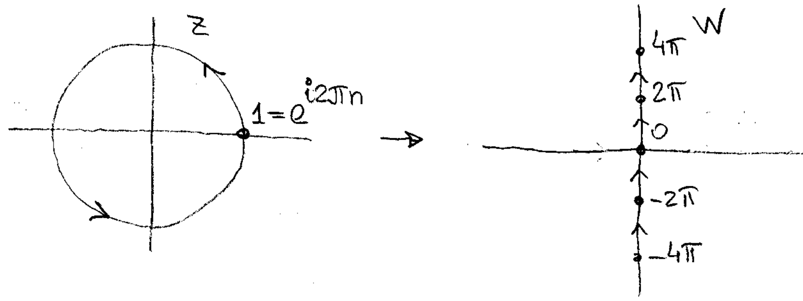
(2D surface in 4D space)

$1 \to n$: $W = z^{1/n}$



$1 \to \infty$: $W = \ln z \Rightarrow u + iv = \ln(re^{i\theta}) = \ln r + i(\theta + 2\pi n)$

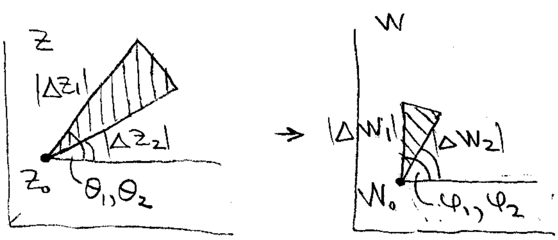
$$\Rightarrow \begin{cases} u = \ln r \\ v = \theta + 2\pi n \end{cases}$$



Conformal mappings

$f: z \to w$ — analytic:

$$ae^{i\alpha} = \frac{df}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \Rightarrow \begin{cases} \left|\frac{df}{dz}\right| = \lim_{\Delta z \to 0} \left|\frac{\Delta w}{\Delta z}\right| = a \\ \arg\left(\frac{df}{dz}\right) = \lim_{\Delta z \to 0} \arg\left(\frac{\Delta w}{\Delta z}\right) = \alpha \end{cases}$$



$$\longrightarrow \begin{cases} |\Delta w_i| = a|\Delta z_i| \\ \varphi_i = \theta_i + \alpha \end{cases}$$

For a pair of lines: $\varphi_2 - \varphi_1 = (\theta_2 + \alpha) - (\theta_1 + \alpha) = \theta_2 - \theta_1$

$\Rightarrow$ conformal maps preserve angles!

Laplace's Equation in EM (Also in Hydrodynamics!)

In steady state: $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \vec{E} = \vec{\nabla}\varphi$ (omit '−' sign)

$$\vec{\nabla}\cdot\vec{D} = \varepsilon_0 \vec{\nabla}\cdot\vec{E} = \varepsilon_0 \underbrace{\vec{\nabla}^2\varphi = 0}$$

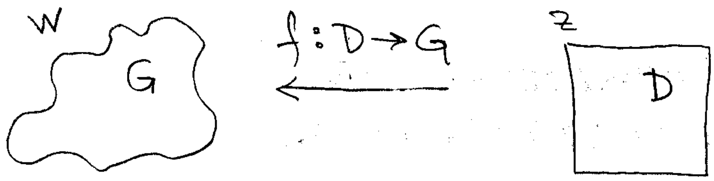
φ satisfies $\nabla^2\varphi = 0 \Rightarrow$ either real or imag. part of analytic func.

$g(z) = \varphi(z) + i\psi(z)$, $z = x+iy$

φ ↑ potential, ψ ↑ stream function (Physical meaning EM, FD)

Cauchy-Riemann relation:
$$\begin{cases} E_x = \dfrac{\partial \varphi}{\partial x} = \dfrac{\partial \psi}{\partial y} \\ E_y = \dfrac{\partial \varphi}{\partial y} = -\dfrac{\partial \psi}{\partial x} \end{cases}$$

Finding Electric field in a complicated domain:

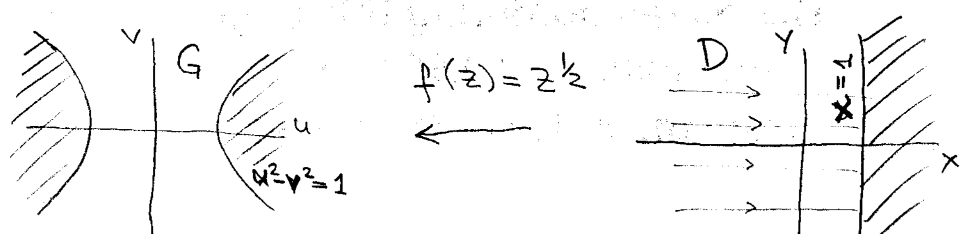


potential $h(w) = ?$ potential $g(z)$ − easily found

On the boundary we have $h|_{\partial G} = g|_{\partial D}$

$\Rightarrow$ In the interior: $h(w) = g(z) = g(f^{-1}(w))$

Example:

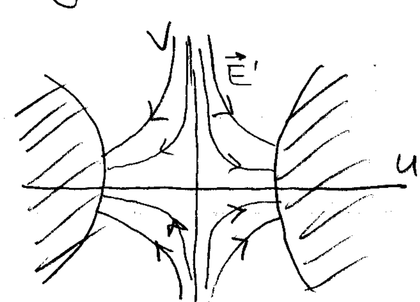


$E = (1,0) \Rightarrow \varphi = x$, $\psi = y$

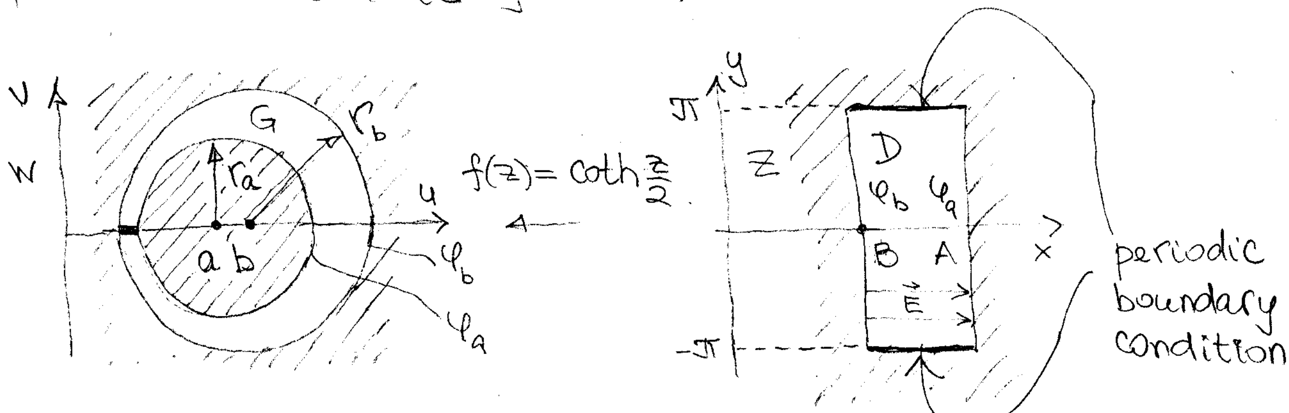
$\Rightarrow g(z) = z$

$$h(w) = g(f^{-1}(w)) = f^{-1}(w) = w^2 = \underbrace{u^2 - v^2}_{\varphi'(u,v)} + i\underbrace{2uv}_{\psi'(u,v)}$$

$$\vec{E}' = \left(\frac{\partial \varphi'}{\partial u}, \frac{\partial \varphi'}{\partial v}\right) = (2u, -2v)$$



Example: (nonconcentric cylinders)



$$a = \coth A, \quad r_a = 1/\sinh A$$
$$b = \coth B, \quad r_b = 1/\sinh B$$

(Check as homework)

Field in D: $\varphi = \varphi_a \frac{x-B}{A-B} + \varphi_b \frac{x-A}{B-A} = \frac{A\varphi_b - B\varphi_a}{A-B} + \frac{\varphi_a - \varphi_b}{A-B} x$

$$E_x = \frac{\partial \varphi}{\partial x} = +\frac{\partial \psi}{\partial y} = \frac{\varphi_a - \varphi_b}{A-B} \quad \Rightarrow \quad \psi = \frac{\varphi_a - \varphi_b}{A-B} y$$

$$E_y = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} = 0$$

$$\Rightarrow g(z) = \varphi + i\psi = \frac{\varphi_a - \varphi_b}{A-B}(x+iy) + \frac{A\varphi_b - B\varphi_a}{A-B} =$$

$$= \frac{\varphi_a - \varphi_b}{A-B} z + \frac{A\varphi_b - B\varphi_a}{A-B} \equiv \alpha z + \beta$$

$$w = f(z) = \coth\frac{z}{2} = \frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = \frac{e^z + 1}{e^z - 1}$$

$$\Rightarrow e^z + 1 = (e^z - 1) w \Rightarrow e^z(-w+1) = -(1+w) \Rightarrow z = \ln\frac{w+1}{w-1}$$

$$h(w) = g(z(w)) = \alpha z + \beta = \alpha \ln\frac{w+1}{w-1} + \beta = \underbrace{\alpha \ln\left|\frac{w+1}{w-1}\right| + \beta}_{\varphi'} + i\underbrace{\alpha \arg\left(\frac{w+1}{w-1}\right)}_{\psi'}$$

Potential in G:

$$\varphi'(u,v) = \beta + \alpha \ln|w+1| - \alpha \ln|w-1| =$$

$$= \beta + \alpha \ln|u+1+iv| - \alpha \ln|u-1+iv| =$$

$$= \beta + \frac{\alpha}{2}\ln\left[(u+1)^2 + v^2\right] - \frac{\alpha}{2}\ln\left[(u-1)^2 + v^2\right]$$

Electric field in G:

$$E_u = \frac{\partial \varphi'}{\partial u} = \dots$$

$$E_v = \frac{\partial \varphi'}{\partial v} = \dots$$

(compute as homework)

Applications to Fluid Dynamics

Incompressible fluids satisfy $\vec{\nabla}\cdot\vec{V}=0$

In 2D this eq. reduces to $\partial_x V_x + \partial_y V_y = 0$

Any velocity field can be written via some stream function, $\Psi(x,y)$

$$V_x = \frac{\partial\Psi}{\partial y}, \quad V_y = -\frac{\partial\Psi}{\partial x} \qquad \text{(check!)}$$

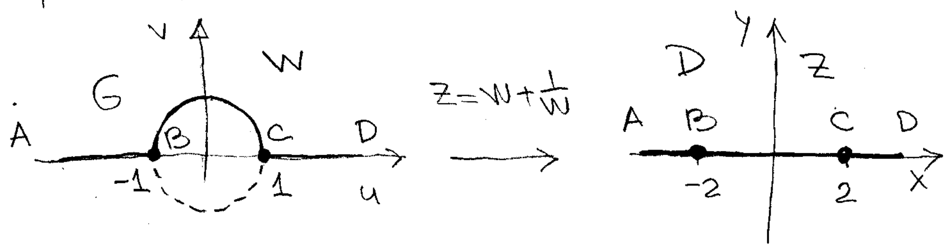
Irrotational velocity field $(\vec{\nabla}\times\vec{V}=0)$ can also be written as

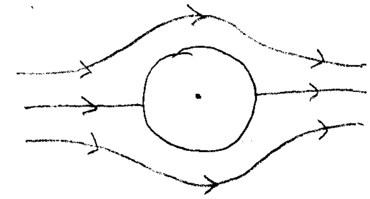
$$\vec{V} = \vec{\nabla}\varphi \quad \text{or} \quad V_x = \frac{\partial\varphi}{\partial x}, \quad V_y = \frac{\partial\varphi}{\partial y}$$

$$\vec{\nabla}\cdot\vec{V} = 0 \Rightarrow \vec{\nabla}^2\varphi = 0 \quad \text{— again Laplace's equation!}$$

$$\Rightarrow g(z) = \varphi(z) + i\Psi(z) \quad \text{- analytic function.}$$

Example (flow past a cylinder)





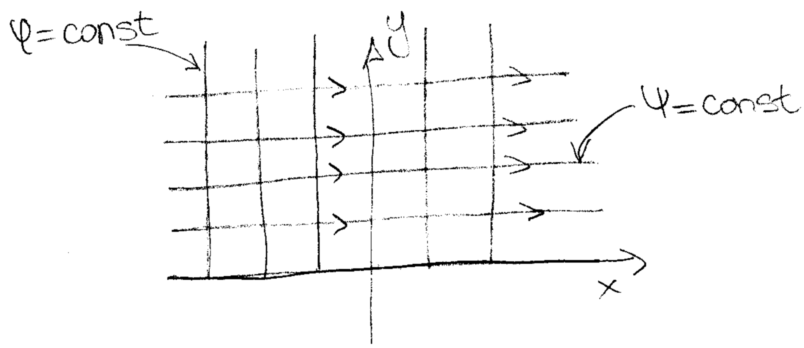
$$\text{A-B:} \quad w = u, \quad -\infty < u < -1 \quad \Rightarrow z = u + \frac{1}{u} \quad \Rightarrow \begin{cases} -\infty < x < -2 \\ y = 0 \end{cases}$$

$$\text{B-C:} \quad w = e^{i\theta}, \quad 0 < \theta < \pi \quad \Rightarrow z = e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad \Rightarrow \begin{cases} -2 < x < 2 \\ y = 0 \end{cases}$$

$$\text{C-D:} \quad w = u, \quad 1 < u < \infty \quad \Rightarrow z = u + \frac{1}{u} \quad \Rightarrow \begin{cases} 2 < x < \infty \\ y = 0 \end{cases}$$

For a cylinder of radius a: $z = w + \frac{a^2}{w}$ $\quad$ (or $\frac{w}{a} + \frac{a}{w}$)

Unperturbed flow (in D):



$$\begin{cases} V_x = V_0 = \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \varphi}{\partial x} \\ V_y = 0 = -\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \varphi}{\partial y} \end{cases}$$

$$\Rightarrow \varphi = V_0 x \,, \quad \psi = V_0 y$$

$$\Rightarrow g(z) = \varphi + i\psi = V_0(x+iy) = V_0 z$$

Perturbed flow (in G):

Change notations: $w = X + iY$ (v denotes velocity field now)

$$h(w) = g(z(w)) = V_0 z(w) = V_0\left(w + \frac{a^2}{w}\right) = V_0\left(X + iY + \frac{a^2}{X+iY}\right)$$

$$= V_0\left(X + iY + a^2\frac{X - iY}{X^2+Y^2}\right) = \underbrace{V_0 X\left(1 + \frac{a^2}{X^2+Y^2}\right)}_{\varphi'} + i\underbrace{V_0 Y\left(1 - \frac{a^2}{X^2+Y^2}\right)}_{\psi'}$$

$$\begin{cases} V_X = \dfrac{\partial \varphi}{\partial X} = V_0\left[1 - a^2\dfrac{X^2-Y^2}{(X^2+Y^2)^2}\right] & \rightarrow V_0 \quad \text{for } X^2+Y^2 \gg a^2 \\ V_Y = \dfrac{\partial \varphi}{\partial Y} = V_0\left[-a^2\dfrac{2XY}{(X^2+Y^2)^2}\right] & \rightarrow 0 \quad \text{for } X^2+Y^2 \gg a^2 \end{cases}$$