Recap : Fourier Transform

$$f(x) = \int_{-\infty}^{\infty} F(k) e^{ikx} dx$$

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx \quad := \mathcal{F}[f(x)]$$

Properties : • Linearity : $\mathcal{F}[\alpha_1 f_1 + \alpha_2 f_2] = \alpha_1 \mathcal{F}[f_1] + \alpha_2 \mathcal{F}[f_2]$

• Parseval's Identity :

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_{-\infty}^{\infty} |F(k)|^2 dk \quad (L_2\text{-norm})$$

remark: "Conservation of Energy"

Energy of k-th wavelength $= |F(k)|^2$

• Differentiation :

$$\mathcal{F}[f^{(n)}(x)] = (ik)^n \mathcal{F}[f(x)]$$

EX. $\mathcal{F}[f''] = -k^2 F(k)$

Convolution : $G(k) = F(k) H(k)$

$$g(x) := \int_{-\infty}^{\infty} G(k) e^{ikx} dk = \int F(k) H(k) e^{ikx} dk$$

$$= \int \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(y) e^{-iky} dy\right)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} h(z) e^{-ikz} dz\right) e^{ikx} dk$$

$$= \frac{1}{(2\pi)^2} \iint f(y) h(z) \left(\int e^{ik(x-(y+z))} dk\right) dy\, dz$$

$$= \frac{1}{(2\pi)^2} \iint f(y) h(z) \left[2\pi \delta(x-y-z)\right] dy\, dz = \cdots$$

$$= \frac{1}{2\pi} (f * h)(x)$$

Conclusion. $\mathcal{F}(f*h) = 2\pi \mathcal{F}(f)\mathcal{F}(h)$ wow!. 2

Ex: (Correlation function)

$x(t)$: $C(\tau) = \frac{1}{T}\int_0^T (x(t)-\bar{x})(x(t-\tau)-\bar{x})\, dt$

↳ corr. function

$= \frac{1}{T}\int_0^T y(t)\, y(t-\tau)\, dt$

$$\mathcal{F}[c](\omega) = \frac{2\pi}{T}|Y(\omega)|^2 \Rightarrow C(\tau) = \frac{2\pi}{T}\int d\omega\, |Y(\omega)|^2 e^{-i\omega\tau} d\omega$$

To compute the ~~corr~~ correlation function of $x(t)$:

1) $y(t) = x(t) - \bar{x}$

2) F.T. of $y(t) \to Y(\omega)$

3) Inverse F.T. $|Y(\omega)|^2 \to C(\tau)$

Ex: Heat Conduction

$$\begin{cases} \partial_t T = \alpha \partial_x^2 T \\ T(x,0) = \delta(x) \end{cases}$$

$$\theta(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T(x,t)\, e^{-ikx} dx$$

say something about integrating this

$$\partial_t \theta(k,t) = -\alpha k^2 \theta(k,t) \Rightarrow \frac{d\theta}{\theta} = -\alpha k^2 dt$$

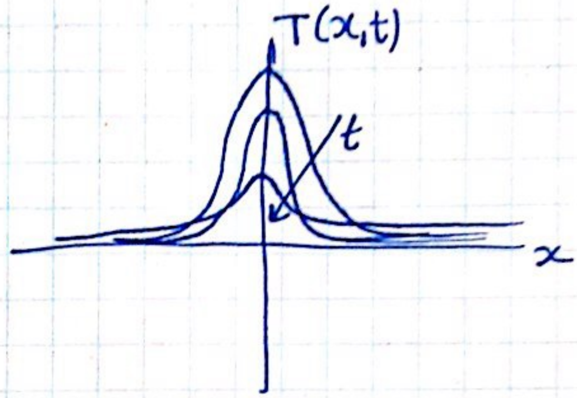
$$\Rightarrow \ln\theta \Big|_0^t = -\alpha k^2 t \Rightarrow \theta(k,t) = \theta_0(k)\, e^{-\alpha k^2 t}$$

What is $\theta_0(k)$? $\theta_0(k) = \theta(k,0) = \frac{1}{2\pi}\int_{\infty}^{\infty} \delta(x) e^{-ikx} dx = \frac{1}{2\pi}$

③

$$\Rightarrow \theta(k,t) = \frac{1}{2\pi} e^{-\alpha k^2 t}$$

$$\Rightarrow T(x,t) = \frac{1}{2\pi}\int e^{-\alpha k^2 t} e^{ikx} dk \sim e^{-x^2/4\alpha t}$$



Ex: (Viscous Burger's Eq.)

$$\begin{cases} \partial_t u + u\,\partial_x u = \nu \partial_x^2 u \\ u(x,0) = f(x) \end{cases} \qquad u(x,t) \longrightarrow \text{1-D version of N-S equation.}$$

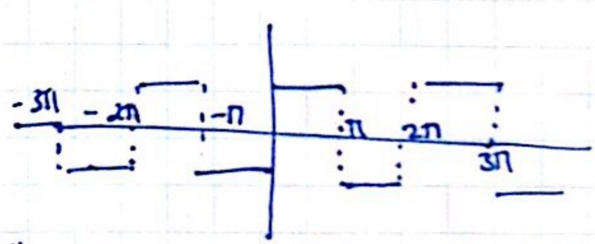
$\hat{u} \equiv \mathcal{F}(u) \Rightarrow$

$$\partial_t \hat{u}(k,t) + \widehat{u\,\partial_x u}(k,t) = -\nu k^2 \hat{u}(k,t)$$

$$\partial_t \hat{u}(k,t) + \underbrace{\int_{-\infty}^{\infty} \hat{u}(k-q)\; iq\, \hat{u}(q)\, dq}_{\text{All other modes affect the mode } k} = -\nu k^2 \hat{u}(k,t)$$

Ex (Square Wave)

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ -1, & -\pi < x < 0 \end{cases}$$

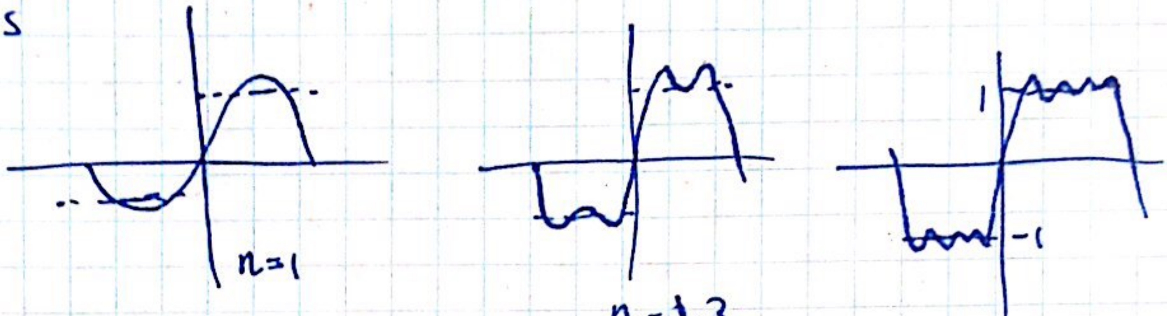


DFT: $f(x) = -f(-x) \Rightarrow \begin{cases} a_n = 0 \\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx = \frac{2}{\pi}\int_0^{2\pi} \sin(nx)\,dx \end{cases}$

$$= \frac{2}{n\pi}\left[1 - (-1)^n\right] = \begin{cases} 0 & : n \text{ even} \\ \frac{4}{n\pi} & : n \text{ odd} \end{cases}$$

$$\Rightarrow f(x) = \sum_{n=1,\,\text{odd}}^{\infty} \frac{4}{\pi n} \sin(nx)$$

Partial Sums



n=1 n=1,3 n=1,3,5

1) $f(x_0) = \frac{1}{2}\left[f(x_0^-) + f(x_0^+)\right]$

2) Overshoot on both sides

(The error does not converge to 0 as $n \to \infty$)

<u>Gibbs Phenomenon</u>

Partial Sum:

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi} f(t)\left[\cos(nt)\cos(nx) + \sin(nt)\sin(nx)\right] dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2}\sum_{n=1}^{N} \cos\{n(t-x)\right] dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin\left(N+\frac{1}{2}\right)(t-x)}{\sin\left(\frac{t-x}{2}\right)}\,dt$$

For Square wave:

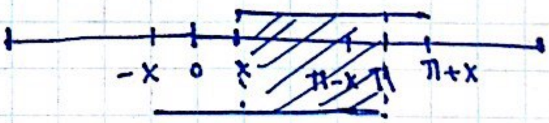
$$f_N(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\sin[(N+\frac{1}{2})(t-x)]}{\sin[\frac{1}{2}(t-x)]}\,dt$$

$$-\frac{1}{2\pi}\int_{-\pi}^{0} \frac{\sin[(N+\frac{1}{2})(t-x)]}{\sin[\frac{1}{2}(t-x)]}\,dt$$

$$= \frac{1}{2\pi}\left\{\int_0^{\pi}\left[\frac{\sin(N+\frac{1}{2})(t-x)}{\sin\frac{1}{2}(t-x)}\right]dt - \int_0^{\pi}\left[\frac{\sin(N+\frac{1}{2})(t+x)}{\sin\frac{1}{2}(t+x)}\right]dt\right\}$$

$$= \frac{1}{2\pi}\left[\int_{-x}^{\pi-x}\frac{\sin(N+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds - \int_{x}^{\pi+x}\frac{\sin(N+\frac{1}{2})s}{\sin\frac{s}{2}}\,ds\right]$$

$$= \frac{1}{2\pi}\left[\int_{-x}^{x}\frac{\sin(N+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds - \int_{\pi-x}^{\pi+x}\frac{\sin(N+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds\right]$$



As $x \to 0$, Second integral is negligible:

$\cancel{\simeq \frac{1}{2\pi}}$

$$f_N(x) \simeq \frac{1}{2\pi}\int_{-x}^{x}\frac{\sin(N+\frac{1}{2})s}{\sin\frac{1}{2}s}\,ds \quad , \quad 0 < x \ll 1$$

Let $N+\frac{1}{2} = p$ and $ps = \xi$ then

$$f_N(x) = \frac{1}{\pi}\int_0^{px}\frac{\sin\xi}{\sin(\frac{\xi}{2p})}\frac{d\xi}{p}$$

x small $\Rightarrow$ $\xi/2p$ small $\Rightarrow$ $\sin(\xi/2p) \simeq \xi/2p$

$$\Rightarrow \quad f_N(x) = \frac{2}{\pi}\int_0^{px} \frac{\sin\xi}{\xi}\, d\xi \simeq 1.17898$$

$\Rightarrow$ Approximately 18% error!