Mathematics for Physics

A guided tour for graduate students

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Chapter 1

Calculus of Variations

We begin our tour of useful mathematics with what is called the *calculus of variations*. Many physics problems can be formulated in the language of this calculus, and once they are there are useful tools to hand. In the text and associated exercises we will meet some of the equations whose solution will occupy us for much of our journey.

1.1 What is it good for?

The classical problems that motivated the creators of the calculus of variations include:

i) *Dido's problem*: In Virgil's *Aeneid* we read how Queen Dido of Carthage must find largest area that can be enclosed by a curve (a strip of bull's hide) of fixed length.
ii) *Plateau's problem*: Find the surface of minimum area for a given set of bounding curves. A soap film on a wire frame will adopt this minimal-area configuration.
iii) *Johann Bernoulli's Brachistochrone*: A bead slides down a curve with fixed ends. Assuming that the total energy $\frac{1}{2}mv^2 + V(x)$ is constant, find the curve that gives the most rapid descent.
iv) *Catenary*: Find the form of a hanging heavy chain of fixed length by minimizing its potential energy.

These problems all involve finding maxima or minima, and hence equating some sort of derivative to zero. In the next section we define this derivative, and show how to compute it.

1.2 Functionals

In variational problems we are provided with an expression $J[y]$ that "eats" whole functions $y(x)$ and returns a single number. Such objects are called *functionals* to distinguish them from ordinary functions. An ordinary function is a map $f : \mathbb{R} \to \mathbb{R}$. A functional J is a map $J : C^\infty(\mathbb{R}) \to \mathbb{R}$ where $C^\infty(\mathbb{R})$ is the space of smooth (having derivatives of all orders) functions. To find the function $y(x)$ that maximizes or minimizes a given functional $J[y]$ we need to define, and evaluate, its *functional derivative.*

1.2.1 The functional derivative

We restrict ourselves to expressions of the form

$$J[y] = \int_{x_1}^{x_2} f(x, y, y', y'', \cdots y^{(n)})\, dx, \tag{1.1}$$

where f depends on the value of $y(x)$ and only finitely many of its derivatives. Such functionals are said to be *local* in x.

Consider first a functional $J = \int f dx$ in which f depends only x, y and y'. Make a change $y(x) \to y(x) + \varepsilon\eta(x)$, where ε is a (small) x-independent constant. The resultant change in J is

$$\begin{aligned}
J[y+\varepsilon\eta] - J[y] &= \int_{x_1}^{x_2} \left\{ f(x, y+\varepsilon\eta, y'+\varepsilon\eta') - f(x,y,y') \right\} dx \\
&= \int_{x_1}^{x_2} \left\{ \varepsilon\eta \frac{\partial f}{\partial y} + \varepsilon \frac{d\eta}{dx}\frac{\partial f}{\partial y'} + O(\varepsilon^2) \right\} dx \\
&= \left[\varepsilon\eta \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2} + \int_{x_1}^{x_2} (\varepsilon\eta(x)) \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} dx + O(\varepsilon^2).
\end{aligned}$$

If $\eta(x_1) = \eta(x_2) = 0$, the variation $\delta y(x) \equiv \varepsilon\eta(x)$ in $y(x)$ is said to have "fixed endpoints." For such variations the integrated-out part $[\ldots]_{x_1}^{x_2}$ vanishes. Defining δJ to be the $O(\varepsilon)$ part of $J[y+\varepsilon\eta] - J[y]$, we have

$$\begin{aligned}
\delta J &= \int_{x_1}^{x_2} (\varepsilon\eta(x)) \left\{ \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \right\} dx \\
&= \int_{x_1}^{x_2} \delta y(x) \left(\frac{\delta J}{\delta y(x)} \right) dx.
\end{aligned} \tag{1.2}$$

The function

$$\frac{\delta J}{\delta y(x)} \equiv \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \tag{1.3}$$

is called the *functional* (or *Fréchet*) derivative of J with respect to $y(x)$. We can think of it as a generalization of the partial derivative $\partial J/\partial y_i$, where the discrete subscript "i" on y is replaced by a continuous label "x," and sums over i are replaced by integrals over x:

$$\delta J = \sum_i \frac{\partial J}{\partial y_i}\delta y_i \to \int_{x_1}^{x_2} dx \left(\frac{\delta J}{\delta y(x)}\right)\delta y(x). \tag{1.4}$$

1.2.2 The Euler-Lagrange equation

Suppose that we have a differentiable function $J(y_1, y_2, \ldots, y_n)$ of n variables and seek its *stationary points* — these being the locations at which J has its maxima, minima and saddlepoints. At a stationary point $(y_1, y_2, \ldots, y_n)$ the variation

$$\delta J = \sum_{i=1}^{n} \frac{\partial J}{\partial y_i}\delta y_i \tag{1.5}$$

must be zero for all possible δy_i. The necessary and sufficient condition for this is that all partial derivatives $\partial J/\partial y_i$, $i = 1, \ldots, n$ be zero. By analogy, we expect that a functional $J[y]$ will be stationary under fixed-endpoint variations $y(x) \to y(x)+\delta y(x)$, when the functional derivative $\delta J/\delta y(x)$ vanishes for all x. In other words, when

$$\boxed{\frac{\partial f}{\partial y(x)} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'(x)}\right) = 0, \quad x_1 < x < x_2.} \tag{1.6}$$

The condition (1.6) for $y(x)$ to be a stationary point is usually called the *Euler-Lagrange* equation.

That $\delta J/\delta y(x) \equiv 0$ is a *sufficient* condition for δJ to be zero is clear from its definition in (1.2). To see that it is a *necessary* condition we must appeal to the assumed smoothness of $y(x)$. Consider a function $y(x)$ at which $J[y]$ is stationary but where $\delta J/\delta y(x)$ is non-zero at some $x_0 \in [x_1, x_2]$. Because $f(y, y', x)$ is smooth, the functional derivative $\delta J/\delta y(x)$ is also a smooth function of x. Therefore, by continuity, it will have the same sign throughout some open interval containing x_0. By taking $\delta y(x) = \varepsilon\eta(x)$ to be

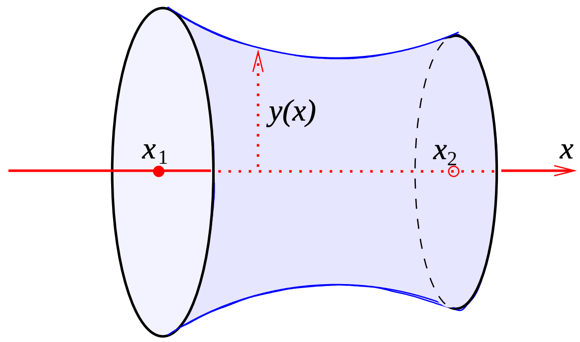

Figure 1.1: *Soap film between two rings.*

zero outside this interval, and of one sign within it, we obtain a non-zero δJ — in contradiction to stationarity. In making this argument, we see why it was essential to integrate by parts so as to take the derivative off δy: when y is fixed at the endpoints, we have $\int \delta y'\, dx = 0$, and so we cannot find a $\delta y'$ that is zero everywhere outside an interval and of one sign within it.

When the functional depends on more than one function y, then stationarity under all possible variations requires one equation

$$\frac{\delta J}{\delta y_i(x)} = \frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \tag{1.7}$$

for each function $y_i(x)$.

If the function f depends on higher derivatives, y'', $y^{(3)}$, *etc.*, then we have to integrate by parts more times, and we end up with

$$0 = \frac{\delta J}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) - \frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y^{(3)}}\right) + \cdots. \tag{1.8}$$

1.2.3 Some applications

Now we use our new functional derivative to address some of the classic problems mentioned in the introduction.

Example: Soap film supported by a pair of coaxial rings (figure 1.1) This a simple case of Plateau's problem. The free energy of the soap film is equal to twice (once for each liquid-air interface) the surface tension σ of the soap solution times the area of the film. The film can therefore minimize its free energy by minimizing its area, and the axial symmetry suggests that the minimal surface will be a surface of revolution about the x axis. We therefore seek the profile $y(x)$ that makes the area

$$J[y] = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx \tag{1.9}$$

of the surface of revolution the least among all such surfaces bounded by the circles of radii $y(x_1) = y_1$ and $y(x_2) = y_2$. Because a minimum is a stationary point, we seek candidates for the minimizing profile $y(x)$ by setting the functional derivative $\delta J/\delta y(x)$ to zero.

We begin by forming the partial derivatives

$$\frac{\partial f}{\partial y} = 4\pi\sigma\sqrt{1+y'^2}, \quad \frac{\partial f}{\partial y'} = \frac{4\pi\sigma y y'}{\sqrt{1+y'^2}} \tag{1.10}$$

and use them to write down the Euler-Lagrange equation

$$\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = 0. \tag{1.11}$$

Performing the indicated derivative with respect to x gives

$$\sqrt{1+y'^2} - \frac{(y')^2}{\sqrt{1+y'^2}} - \frac{yy''}{\sqrt{1+y'^2}} + \frac{y(y')^2y''}{(1+y'^2)^{3/2}} = 0. \tag{1.12}$$

After collecting terms, this simplifies to

$$\frac{1}{\sqrt{1+y'^2}} - \frac{yy''}{(1+y'^2)^{3/2}} = 0. \tag{1.13}$$

The differential equation (1.13) still looks a trifle intimidating. To simplify further, we multiply by y' to get

$$\begin{aligned} 0 &= \frac{y'}{\sqrt{1+y'^2}} - \frac{yy'y''}{(1+y'^2)^{3/2}} \\ &= \frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right). \end{aligned} \tag{1.14}$$

The solution to the minimization problem therefore reduces to solving

$$\frac{y}{\sqrt{1+y'^2}} = \kappa, \tag{1.15}$$

where κ is an as yet undetermined integration constant. Fortunately this non-linear, first order, differential equation is elementary. We recast it as

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{\kappa^2} - 1} \tag{1.16}$$

and separate variables

$$\int dx = \int \frac{dy}{\sqrt{\frac{y^2}{\kappa^2} - 1}}. \tag{1.17}$$

We now make the natural substitution $y = \kappa \cosh t$, whence

$$\int dx = \kappa \int dt. \tag{1.18}$$

Thus we find that $x + a = \kappa t$, leading to

$$y = \kappa \cosh \frac{x + a}{\kappa}. \tag{1.19}$$

We select the constants κ and a to fit the endpoints $y(x_1) = y_1$ and $y(x_2) = y_2$.

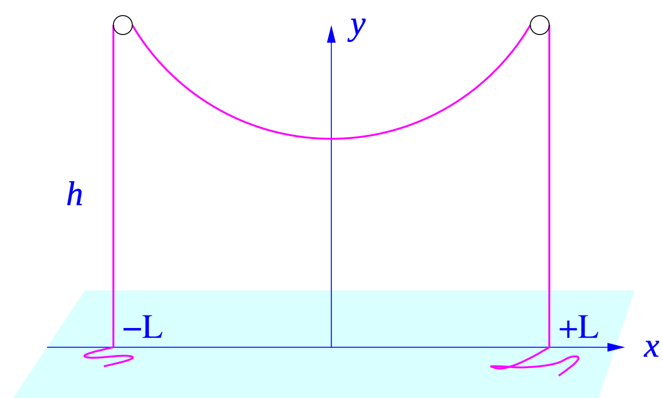

Figure 1.2: *Hanging chain*

Example: Heavy Chain over Pulleys. We cannot yet consider the form of the catenary, a hanging chain of fixed length, but we can solve a simpler problem of a heavy flexible cable draped over a pair of pulleys located at $x = \pm L$, $y = h$, and with the excess cable resting on a horizontal surface as illustrated in figure 1.2.

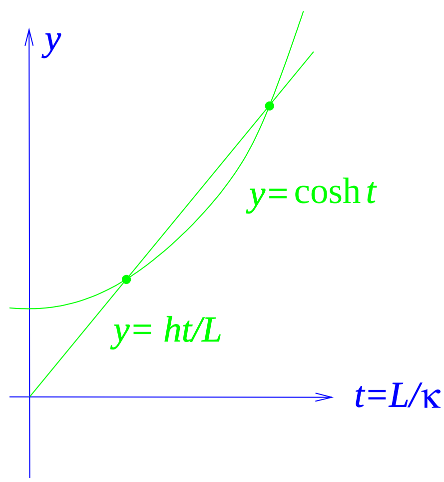

Figure 1.3: *Intersection of* $y = ht/L$ *with* $y = \cosh t$.

The potential energy of the system is

$$P.E. = \sum mgy = \rho g \int_{-L}^{L} y\sqrt{1+(y')^2}dx + \text{const}. \tag{1.20}$$

Here the constant refers to the unchanging potential energy

$$2 \times \int_0^h mgy\, dy = mgh^2 \tag{1.21}$$

of the vertically hanging cable. The potential energy of the cable lying on the horizontal surface is zero because y is zero there. Notice that the tension in the suspended cable is being tacitly determined by the weight of the vertical segments.

The Euler-Lagrange equations coincide with those of the soap film, so

$$y = \kappa \cosh \frac{(x+a)}{\kappa} \tag{1.22}$$

where we have to find κ and a. We have

$$\begin{aligned} h &= \kappa \cosh(-L+a)/\kappa, \\ &= \kappa \cosh(L+a)/\kappa, \end{aligned} \tag{1.23}$$

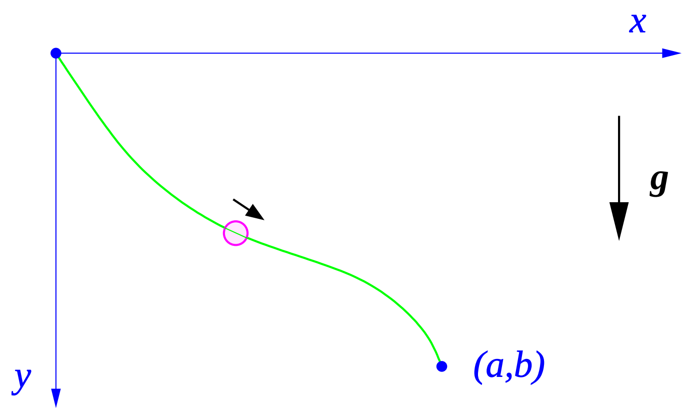

Figure 1.4: *Bead on a wire.*

so $a = 0$ and $h = \kappa \cosh L/\kappa$. Setting $t = L/\kappa$ this reduces to

$$\left(\frac{h}{L}\right) t = \cosh t. \tag{1.24}$$

By considering the intersection of the line $y = ht/L$ with $y = \cosh t$ (figure 1.3) we see that if h/L is too small there is no solution (the weight of the suspended cable is too big for the tension supplied by the dangling ends) and once h/L is large enough there will be two possible solutions. Further investigation will show that the solution with the larger value of κ is a point of stable equilibrium, while the solution with the smaller κ is unstable.

Example: The Brachistochrone. This problem was posed as a challenge by Johann Bernoulli in 1696. He asked what shape should a wire with endpoints $(0,0)$ and (a,b) take in order that a frictionless bead will slide from rest down the wire in the shortest possible time (figure 1.4). The problem's name comes from Greek: $\beta\rho\alpha\chi\iota\sigma\tau o\varsigma$ means shortest and $\chi\rho o\nu o\varsigma$ means time.

When presented with an ostensibly anonymous solution, Johann made his famous remark: *"Tanquam ex unguem leonem"* (I recognize the lion by his clawmark) meaning that he recognized that the author was Isaac Newton.

Johann gave a solution himself, but that of his brother Jacob Bernoulli was superior and Johann tried to pass it off as his. This was not atypical. Johann later misrepresented the publication date of his book on hydraulics to make it seem that he had priority in this field over his own son, Daniel Bernoulli.

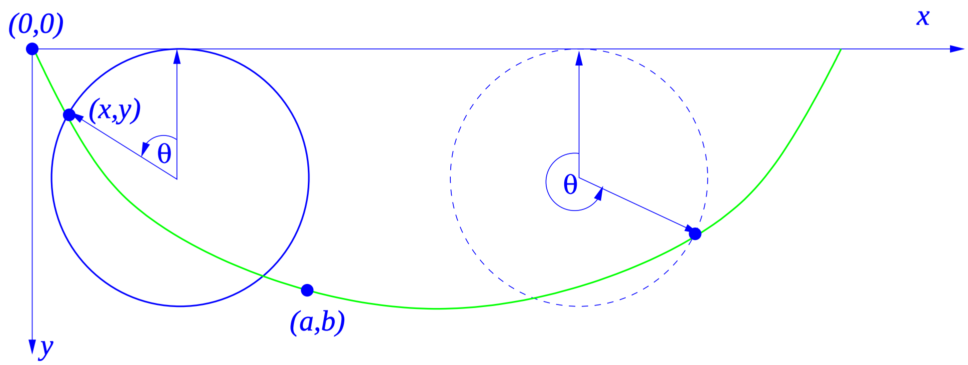

Figure 1.5: *A wheel rolls on the x axis. The dot, which is fixed to the rim of the wheel, traces out a cycloid.*

We begin *our* solution of the problem by observing that the total energy

$$E = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy = \frac{1}{2}m\dot{x}^2(1 + y'^2) - mgy, \tag{1.25}$$

of the bead is constant. From the initial condition we see that this constant is zero. We therefore wish to minimize

$$T = \int_0^T dt = \int_0^a \frac{1}{\dot{x}}\, dx = \int_0^a \sqrt{\frac{1 + y'^2}{2gy}} dx \tag{1.26}$$

so as find $y(x)$, given that $y(0) = 0$ and $y(a) = b$. The Euler-Lagrange equation is

$$yy'' + \frac{1}{2}(1 + y'^2) = 0. \tag{1.27}$$

Again this looks intimidating, but we can use the same trick of multiplying through by y' to get

$$y'\left(yy'' + \frac{1}{2}(1 + y'^2)\right) = \frac{1}{2}\frac{d}{dx}\left\{y(1 + y'^2)\right\} = 0. \tag{1.28}$$

Thus

$$2c = y(1 + y'^2). \tag{1.29}$$

This differential equation has a parametric solution

$$\begin{aligned} x &= c(\theta - \sin\theta), \\ y &= c(1 - \cos\theta), \end{aligned} \tag{1.30}$$

(as you should verify) and the solution is the cycloid shown in figure 1.5. The parameter c is determined by requiring that the curve does in fact pass through the point (a, b).

1.2.4 First integral

How did we know that we could simplify both the soap-film problem and the brachistochrone by multiplying the Euler equation by y'? The answer is that there is a general principle, closely related to energy conservation in mechanics, that tells us when and how we can make such a simplification. The y' trick works when the f in $\int f\,dx$ is of the form $f(y, y')$, *i.e.* has no *explicit* dependence on x. In this case the last term in

$$\frac{df}{dx} = y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + \frac{\partial f}{\partial x} \tag{1.31}$$

is absent. We then have

$$\begin{aligned} \frac{d}{dx}\left(f - y'\frac{\partial f}{\partial y'}\right) &= y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} - y''\frac{\partial f}{\partial y'} - y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) \\ &= y'\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right), \end{aligned} \tag{1.32}$$

and this is zero if the Euler-Lagrange equation is satisfied.

The quantity

$$I = f - y'\frac{\partial f}{\partial y'} \tag{1.33}$$

is called a *first integral* of the Euler-Lagrange equation. In the soap-film case

$$f - y'\frac{\partial f}{\partial y'} = y\sqrt{1 + (y')^2} - \frac{y(y')^2}{\sqrt{1 + (y')^2}} = \frac{y}{\sqrt{1 + (y')^2}}. \tag{1.34}$$

When there are a number of dependent variables y_i, so that we have

$$J[y_1, y_2, \dots y_n] = \int f(y_1, y_2, \dots y_n; y_1', y_2', \dots y_n')\,dx \tag{1.35}$$

then the first integral becomes

$$I = f - \sum_i y_i'\frac{\partial f}{\partial y_i'}. \tag{1.36}$$

Again

$$\begin{aligned}\frac{dI}{dx} &= \frac{d}{dx}\left(f - \sum_i y'_i \frac{\partial f}{\partial y'_i}\right)\\ &= \sum_i \left(y'_i \frac{\partial f}{\partial y_i} + y''_i \frac{\partial f}{\partial y'_i} - y''_i \frac{\partial f}{\partial y'_i} - y'_i \frac{d}{dx}\left(\frac{\partial f}{\partial y'_i}\right)\right)\\ &= \sum_i y'_i \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'_i}\right)\right), \qquad (1.37)\end{aligned}$$

and this zero if the Euler-Lagrange equation is satisfied for each y_i.

Note that there is only *one* first integral, no matter how many y_i's there are.

1.3 Lagrangian mechanics

In his *Mécanique Analytique* (1788) Joseph-Louis de La Grange, following Jean d'Alembert (1742) and Pierre de Maupertuis (1744), showed that most of classical mechanics can be recast as a variational condition: the *principle of least action*. The idea is to introduce the *Lagrangian* function $L = T - V$ where T is the kinetic energy of the system and V the potential energy, both expressed in terms of *generalized co-ordinates* q^i and their time derivatives $\dot{q}^i$. Then, Lagrange showed, the multitude of Newton's $\mathbf{F} = m\mathbf{a}$ equations, one for each particle in the system, can be reduced to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad (1.38)$$

one equation for each generalized coordinate q. Quite remarkably — given that Lagrange's derivation contains no mention of maxima or minima — we recognise that this is precisely the condition that the *action functional*

$$S[q] = \int_{t_{initial}}^{t_{final}} L(t, q^i; {q'}^i)\, dt \qquad (1.39)$$

be stationary with respect to variations of the trajectory $q^i(t)$ that leave the initial and final points fixed. This fact so impressed its discoverers that they believed they had uncovered the unifying principle of the universe. Maupertuis, for one, tried to base a proof of the existence of God on it. Today the action integral, through its starring role in the Feynman path-integral formulation of quantum mechanics, remains at the heart of theoretical physics.

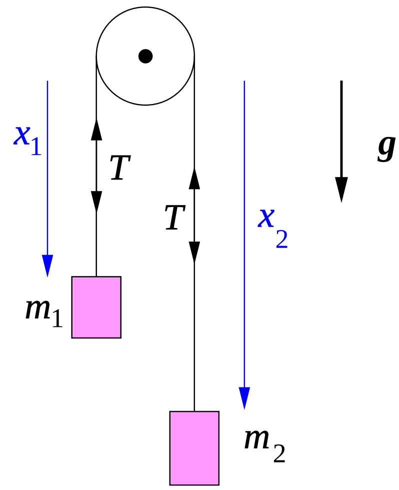

Figure 1.6: *Atwood's machine.*

1.3.1 One degree of freedom

We shall not attempt to derive Lagrange's equations from d'Alembert's extension of the principle of virtual work – leaving this task to a mechanics course — but instead satisfy ourselves with some examples which illustrate the computational advantages of Lagrange's approach, as well as a subtle pitfall.

Consider, for example, *Atwood's Machine* (figure 1.6). This device, invented in 1784 but still a familiar sight in teaching laboratories, is used to demonstrate Newton's laws of motion and to measure g. It consists of two weights connected by a light string of length l which passes over a light and frictionless pulley

The elementary approach is to write an equation of motion for each of the two weights

$$\begin{aligned} m_1\ddot{x}_1 &= m_1 g - T, \\ m_2\ddot{x}_2 &= m_2 g - T. \end{aligned} \tag{1.40}$$

We then take into account the constraint $\dot{x}_1 = -\dot{x}_2$ and eliminate $\ddot{x}_2$ in favour of $\ddot{x}_1$:

$$\begin{aligned} m_1\ddot{x}_1 &= m_1 g - T, \\ -m_2\ddot{x}_1 &= m_2 g - T. \end{aligned} \tag{1.41}$$

Finally we eliminate the constraint force, the tension T, and obtain the acceleration

$$(m_1 + m_2)\ddot{x}_1 = (m_1 - m_2)g. \tag{1.42}$$

Lagrange's solution takes the constraint into account from the very beginning by introducing a single generalized coordinate $q = x_1 = l - x_2$, and writing

$$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{q}^2 - (m_2 - m_1)gq. \tag{1.43}$$

From this we obtain a single equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0 \quad \Rightarrow \quad (m_1 + m_2)\ddot{q} = (m_1 - m_2)g. \tag{1.44}$$

The advantage of the the Lagrangian method is that constraint forces, which do no net work, never appear. The disadvantage is exactly the same: if we need to find the constraint forces – in this case the tension in the string — we cannot use Lagrange alone.

Lagrange provides a convenient way to derive the equations of motion in non-cartesian co-ordinate systems, such as plane polar co-ordinates.

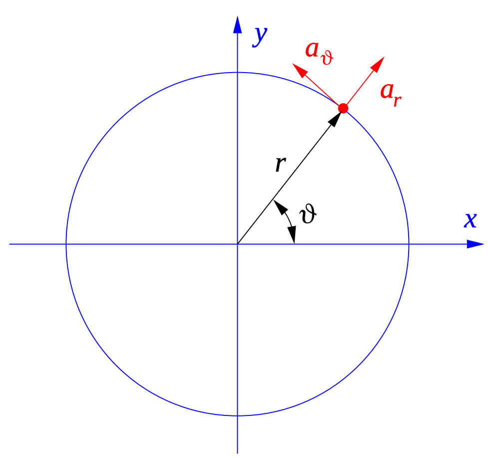

Figure 1.7: *Polar components of acceleration.*

Consider the central force problem with $F_r = -\partial_r V(r)$. Newton's method begins by computing the acceleration in polar coordinates. This is most easily done by setting $z = re^{i\theta}$ and differentiating twice:

$$\begin{aligned}\dot{z} &= (\dot{r} + ir\dot{\theta})e^{i\theta},\\ \ddot{z} &= (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + i(2\dot{r}\dot{\theta} + r\ddot{\theta})e^{i\theta}.\end{aligned} \tag{1.45}$$

Reading off the components parallel and perpendicular to $e^{i\theta}$ gives the radial and angular acceleration

$$\begin{aligned}a_r &= \ddot{r} - r\dot{\theta}^2,\\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta}.\end{aligned} \tag{1.46}$$

Newton's equations therefore become

$$\begin{aligned}m(\ddot{r} - r\dot{\theta}^2) &= -\frac{\partial V}{\partial r}\\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= 0, \quad \Rightarrow \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0.\end{aligned} \tag{1.47}$$

Setting $l = mr^2\dot{\theta}$, the conserved angular momentum, and eliminating $\dot{\theta}$ gives

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{\partial V}{\partial r}. \tag{1.48}$$

(If this were Kepler's problem, where $V = GmM/r$, we would now proceed to simplify this equation by substituting $r = 1/u$, but that is another story.)

Following Lagrange we first compute the kinetic energy in polar coordinates (this requires less thought than computing the acceleration) and set

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{1.49}$$

The Euler-Lagrange equations are now

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0, &\quad \Rightarrow \quad m\ddot{r} - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0,\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0, &\quad \Rightarrow \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0,\end{aligned} \tag{1.50}$$

and coincide with Newton's.

The first integral is

$$\begin{aligned} E &= \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r). \end{aligned} \tag{1.51}$$

which is the total energy. Thus the constancy of the first integral states that

$$\frac{dE}{dt} = 0, \tag{1.52}$$

or that energy is conserved.
Warning: We might realize, without having gone to the trouble of deriving it from the Lagrange equations, that rotational invariance guarantees that the angular momentum $l = mr^2\dot{\theta}$ is constant. Having done so, it is almost irresistible to try to short-circuit some of the labour by plugging this prior knowledge into

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{1.53}$$

so as to eliminate the variable $\dot{\theta}$ in favour of the constant l. If we try this we get

$$L \overset{?}{\rightarrow} \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - V(r). \tag{1.54}$$

We can now directly write down the Lagrange equation r, which is

$$m\ddot{r} + \frac{l^2}{mr^3} \overset{?}{=} -\frac{\partial V}{\partial r}. \tag{1.55}$$

Unfortunately this has the wrong sign before the l^2/mr^3 term! The lesson is that we must be very careful in using consequences of a variational principle to modify the principle. It can be done, and in mechanics it leads to the *Routhian* or, in more modern language to *Hamiltonian reduction*, but it requires using a Legendre transform. The reader should consult a book on mechanics for details.

1.3.2 Noether's theorem

The time-independence of the first integral

$$\frac{d}{dt}\left\{\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right\} = 0, \tag{1.56}$$

and of angular momentum

$$\frac{d}{dt}\{mr^2\dot{\theta}\} = 0, \tag{1.57}$$

are examples of *conservation laws*. We obtained them both by manipulating the Euler-Lagrange equations of motion, but also indicated that they were in some way connected with symmetries. One of the chief advantages of a variational formulation of a physical problem is that this connection

$$\boxed{\text{Symmetry} \Leftrightarrow \text{Conservation Law}}$$

can be made explicit by exploiting a strategy due to Emmy Noether. She showed how to proceed directly from the action integral to the conserved quantity without having to fiddle about with the individual equations of motion. We begin by illustrating her technique in the case of angular momentum, whose conservation is a consequence the rotational symmetry of the central force problem. The action integral for the central force problem is

$$S = \int_0^T \left\{\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)\right\} dt. \tag{1.58}$$

Noether observes that the integrand is left unchanged if we make the variation

$$\theta(t) \to \theta(t) + \varepsilon\alpha \tag{1.59}$$

where α is a fixed angle and ε is a small, time-independent, parameter. This invariance is the symmetry we shall exploit. It is a mathematical identity: it does not require that r and θ obey the equations of motion. She next observes that since the *equations of motion* are equivalent to the statement that S is left stationary under *any* infinitesimal variations in r and θ, they necessarily imply that S is stationary under the specific variation

$$\theta(t) \to \theta(t) + \varepsilon(t)\alpha \tag{1.60}$$

where now ε is allowed to be time-dependent. This stationarity of the action is no longer a mathematical identity, but, because it requires r, θ, to obey the equations of motion, has physical content. Inserting $\delta\theta = \varepsilon(t)\alpha$ into our expression for S gives

$$\delta S = \alpha \int_0^T \left\{mr^2\dot{\theta}\right\} \dot{\varepsilon}\, dt. \tag{1.61}$$

Note that this variation depends only on the time derivative of ε, and not ε itself. This is because of the invariance of S under time-independent rotations. We now assume that $\varepsilon(t) = 0$ at $t = 0$ and $t = T$, and integrate by parts to take the time derivative off ε and put it on the rest of the integrand:

$$\delta S = -\alpha \int \left\{ \frac{d}{dt}(mr^2\dot{\theta}) \right\} \varepsilon(t)\, dt. \tag{1.62}$$

Since the equations of motion say that $\delta S = 0$ under all infinitesimal variations, and in particular those due to any time dependent rotation $\varepsilon(t)\alpha$, we deduce that the equations of motion imply that the coefficient of $\varepsilon(t)$ must be zero, and so, provided $r(t)$, $\theta(t)$, obey the equations of motion, we have

$$0 = \frac{d}{dt}(mr^2\dot{\theta}). \tag{1.63}$$

As a second illustration we derive energy (first integral) conservation for the case that the system is invariant under time translations — meaning that L does not depend explicitly on time. In this case the action integral is invariant under constant time shifts $t \to t + \varepsilon$ in the argument of the dynamical variable:

$$q(t) \to q(t + \varepsilon) \approx q(t) + \varepsilon\dot{q}. \tag{1.64}$$

The equations of motion tell us that that the action will be stationary under the variation

$$\delta q(t) = \varepsilon(t)\dot{q}, \tag{1.65}$$

where again we now permit the parameter ε to depend on t. We insert this variation into

$$S = \int_0^T L\, dt \tag{1.66}$$

and find

$$\delta S = \int_0^T \left\{ \frac{\partial L}{\partial q}\dot{q}\varepsilon + \frac{\partial L}{\partial \dot{q}}(\ddot{q}\varepsilon + \dot{q}\dot{\varepsilon}) \right\} dt. \tag{1.67}$$

This expression contains undotted ε's. Because of this the change in S is not obviously zero when ε is time independent — but the absence of any explicit t dependence in L tells us that

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}. \tag{1.68}$$

As a consequence, for time independent ε, we have

$$\delta S = \int_0^T \left\{ \varepsilon \frac{dL}{dt} \right\} dt = \varepsilon [L]_0^T, \tag{1.69}$$

showing that the change in S comes entirely from the endpoints of the time interval. These fixed endpoints explicitly break time-translation invariance, but in a trivial manner. For general $\varepsilon(t)$ we have

$$\delta S = \int_0^T \left\{ \varepsilon(t) \frac{dL}{dt} + \frac{\partial L}{\partial \dot{q}} \dot{q} \dot{\varepsilon} \right\} dt. \tag{1.70}$$

This equation is an identity. It does not rely on q obeying the equation of motion. After an integration by parts, taking $\varepsilon(t)$ to be zero at $t = 0, T$, it is equivalent to

$$\delta S = \int_0^T \varepsilon(t) \frac{d}{dt} \left\{ L - \frac{\partial L}{\partial \dot{q}} \dot{q} \right\} dt. \tag{1.71}$$

Now we assume that $q(t)$ does obey the equations of motion. The variation principle then says that $\delta S = 0$ for any $\varepsilon(t)$, and we deduce that for $q(t)$ satisfying the equations of motion we have

$$\frac{d}{dt} \left\{ L - \frac{\partial L}{\partial \dot{q}} \dot{q} \right\} = 0. \tag{1.72}$$

The general strategy that constitutes "Noether's theorem" must now be obvious: we look for an invariance of the action under a symmetry transformation with a time-independent parameter. We then observe that if the dynamical variables obey the equations of motion, then the action principle tells us that the action will remain stationary under such a variation of the dynamical variables even after the parameter is promoted to being time dependent. The resultant variation of S can only depend on time derivatives of the parameter. We integrate by parts so as to take all the time derivatives off it, and on to the rest of the integrand. Because the parameter is arbitrary, we deduce that the equations of motion tell us that that its coefficient in the integral must be zero. This coefficient is the time derivative of something, so this something is conserved.

1.3.3 Many degrees of freedom

The extension of the action principle to many degrees of freedom is straightforward. As an example consider the small oscillations about equilibrium of a system with N degrees of freedom. We parametrize the system in terms of deviations from the equilibrium position and expand out to quadratic order. We obtain a Lagrangian

$$L = \sum_{i,j=1}^{N} \left\{ \frac{1}{2} M_{ij} \dot{q}^i \dot{q}^j - \frac{1}{2} V_{ij} q^i q^j \right\}, \tag{1.73}$$

where M_{ij} and V_{ij} are $N \times N$ symmetric matrices encoding the inertial and potential energy properties of the system. Now we have one equation

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \sum_{j=1}^{N} \left(M_{ij} \ddot{q}^j + V_{ij} q^j \right) \tag{1.74}$$

for each i.

1.3.4 Continuous systems

The action principle can be extended to field theories and to continuum mechanics. Here one has a continuous infinity of dynamical degrees of freedom, either one for each point in space and time or one for each point in the material, but the extension of the variational derivative to functions of more than one variable should possess no conceptual difficulties.

Suppose we are given an action functional $S[\varphi]$ depending on a field $\varphi(x^\mu)$ and its first derivatives

$$\varphi_\mu \equiv \frac{\partial \varphi}{\partial x^\mu}. \tag{1.75}$$

Here x^μ, $\mu = 0, 1, \ldots, d$, are the coordinates of $d+1$ dimensional space-time. It is traditional to take $x^0 \equiv t$ and the other coordinates spacelike. Suppose further that

$$S[\varphi] = \int L\, dt = \int \mathcal{L}(x^\mu, \varphi, \varphi_\mu)\, d^{d+1}x, \tag{1.76}$$

where $\mathcal{L}$ is the *Lagrangian density*, in terms of which

$$L = \int \mathcal{L}\, d^d x, \tag{1.77}$$

and the integral is over the space coordinates. Now

$$\begin{aligned} \delta S &= \int \left\{ \delta\varphi(x) \frac{\partial \mathcal{L}}{\partial \varphi(x)} + \delta(\varphi_\mu(x)) \frac{\partial \mathcal{L}}{\partial \varphi_\mu(x)} \right\} d^{d+1}x \\ &= \int \delta\varphi(x) \left\{ \frac{\partial \mathcal{L}}{\partial \varphi(x)} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \varphi_\mu(x)} \right) \right\} d^{d+1}x. \end{aligned} \tag{1.78}$$

In going from the first line to the second, we have observed that

$$\delta(\varphi_\mu(x)) = \frac{\partial}{\partial x^\mu}\delta\varphi(x) \tag{1.79}$$

and used the divergence theorem,

$$\int_\Omega \left(\frac{\partial A^\mu}{\partial x^\mu}\right) d^{n+1}x = \int_{\partial\Omega} A^\mu n_\mu dS, \tag{1.80}$$

where Ω is some space-time region and $\partial\Omega$ its boundary, to integrate by parts. Here dS is the element of area on the boundary, and n_μ the outward normal. As before, we take $\delta\varphi$ to vanish on the boundary, and hence there is no boundary contribution to variation of S. The result is that

$$\frac{\delta S}{\delta\varphi(x)} = \frac{\partial\mathcal{L}}{\partial\varphi(x)} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\varphi_\mu(x)}\right), \tag{1.81}$$

and the equation of motion comes from setting this to zero. Note that a sum over the repeated coordinate index μ is implied. In practice it is easier not to use this formula. Instead, make the variation by hand—as in the following examples.

Example: The Vibrating string. The simplest continuous dynamical system is the transversely vibrating string. We describe the string displacement by $y(x,t)$.

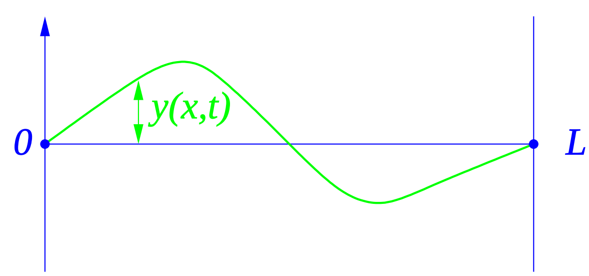

Figure 1.8: *Transversely vibrating string*

Let us suppose that the string has fixed ends, a mass per unit length of ρ, and is under tension T. If we assume only small displacements from equilibrium, the Lagrangian is

$$L = \int_0^L dx \left\{\frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}Ty'^2\right\}. \tag{1.82}$$

The dot denotes a partial derivative with respect to t, and the prime a partial derivative with respect to x. The variation of the action is

$$\begin{aligned} \delta S &= \iint_0^L dtdx\,\{\rho\dot{y}\,\delta\dot{y} - Ty'\delta y'\} \\ &= \iint_0^L dtdx\,\{\delta y(x,t)\,(-\rho\ddot{y} + Ty'')\}\,. \end{aligned} \tag{1.83}$$

To reach the second line we have integrated by parts, and, because the ends are fixed, and therefore $\delta y = 0$ at $x = 0$ and L, there is no boundary term. Requiring that $\delta S = 0$ for all allowed variations δy then gives the equation of motion

$$\rho\ddot{y} - Ty'' = 0 \tag{1.84}$$

This is the wave equation describing transverse waves propagating with speed $c = \sqrt{T/\rho}$. Observe that from (1.83) we can read off the functional derivative of S with respect to the variable $y(x,t)$ as being

$$\frac{\delta S}{\delta y(x,t)} = -\rho\ddot{y}(x,t) + Ty''(x,t). \tag{1.85}$$

In writing down the first integral for this continuous system, we must replace the sum over discrete indices by an integral:

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \to \int dx\left\{\dot{y}(x)\frac{\delta L}{\delta\dot{y}(x)}\right\} - L. \tag{1.86}$$

When computing $\delta L/\delta\dot{y}(x)$ from

$$L = \int_0^L dx\left\{\frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}Ty'^2\right\},$$

we must remember that it is the continuous analogue of $\partial L/\partial\dot{q}_i$, and so, in contrast to what we do when computing $\delta S/\delta y(x)$, we must treat $\dot{y}(x)$ as a variable independent of $y(x)$. We then have

$$\frac{\delta L}{\delta\dot{y}(x)} = \rho\dot{y}(x), \tag{1.87}$$

leading to

$$E = \int_0^L dx\left\{\frac{1}{2}\rho\dot{y}^2 + \frac{1}{2}Ty'^2\right\}. \tag{1.88}$$

This, as expected, is the total energy, kinetic plus potential, of the string.

The energy-momentum tensor

If we consider an action of the form

$$S = \int \mathcal{L}(\varphi, \varphi_\mu)\, d^{d+1}x, \tag{1.89}$$

in which $\mathcal{L}$ does not depend *explicitly* on any of the co-ordinates x^μ, we may refine Noether's derivation of the law of conservation total energy and obtain accounting information about the position-dependent energy *density*. To do this we make a variation of the form

$$\varphi(x) \to \varphi(x^\mu + \varepsilon^\mu(x)) = \varphi(x^\mu) + \varepsilon^\mu(x)\partial_\mu\varphi + O(|\varepsilon|^2), \tag{1.90}$$

where ε depends on $x \equiv (x^0, \ldots, x^d)$. The resulting variation in S is

$$\begin{aligned} \delta S &= \int \left\{ \frac{\partial \mathcal{L}}{\partial \varphi}\varepsilon^\mu \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial \varphi_\nu}\partial_\nu(\varepsilon^\mu \partial_\mu\varphi) \right\} d^{d+1}x \\ &= \int \varepsilon^\mu(x) \frac{\partial}{\partial x^\nu} \left\{ \mathcal{L}\delta^\nu_\mu - \frac{\partial \mathcal{L}}{\partial \varphi_\nu}\partial_\mu \varphi \right\} d^{d+1}x. \end{aligned} \tag{1.91}$$

When φ satisfies the the equations of motion this δS will be zero for arbitrary $\varepsilon^\mu(x)$. We conclude that

$$\frac{\partial}{\partial x^\nu} \left\{ \mathcal{L}\delta^\nu_\mu - \frac{\partial \mathcal{L}}{\partial \varphi_\nu}\partial_\mu \varphi \right\} = 0. \tag{1.92}$$

The $(d+1)$-by-$(d+1)$ array of functions

$$T^\nu{}_\mu \equiv \frac{\partial \mathcal{L}}{\partial \varphi_\nu}\partial_\mu\varphi - \delta^\nu_\mu \mathcal{L} \tag{1.93}$$

is known as the *canonical energy-momentum tensor* because the statement

$$\partial_\nu T^\nu{}_\mu = 0 \tag{1.94}$$

often provides book-keeping for the flow of energy and momentum.

In the case of the vibrating string, the $\mu = 0, 1$ components of $\partial_\nu T^\nu{}_\mu = 0$ become the two following *local* conservation equations:

$$\frac{\partial}{\partial t}\left\{ \frac{\rho}{2}\dot{y}^2 + \frac{T}{2}y'^2 \right\} + \frac{\partial}{\partial x}\left\{ -T\dot{y}y' \right\} = 0, \tag{1.95}$$

and

$$\frac{\partial}{\partial t}\{-\rho\dot{y}y'\} + \frac{\partial}{\partial x}\left\{\frac{\rho}{2}\dot{y}^2 + \frac{T}{2}y'^2\right\} = 0. \tag{1.96}$$

It is easy to verify that these are indeed consequences of the wave equation. They are "local" conservation laws because they are of the form

$$\frac{\partial q}{\partial t} + \operatorname{div}\mathbf{J} = 0, \tag{1.97}$$

where q is the local density, and $\mathbf{J}$ the flux, of the globally conserved quantity $Q = \int q\, d^d x$. In the first case, the local density q is

$$T^0{}_0 = \frac{\rho}{2}\dot{y}^2 + \frac{T}{2}y'^2, \tag{1.98}$$

which is the energy density. The energy flux is given by $T^1{}_0 \equiv -T\dot{y}y'$, which is the rate that a segment of string is doing work on its neighbour to the right. Integrating over x, and observing that the fixed-end boundary conditions are such that

$$\int_0^L \frac{\partial}{\partial x}\{-T\dot{y}y'\}\, dx = [-T\dot{y}y']_0^L = 0, \tag{1.99}$$

gives us

$$\frac{d}{dt}\int_0^L \left\{\frac{\rho}{2}\dot{y}^2 + \frac{T}{2}y'^2\right\} dx = 0, \tag{1.100}$$

which is the global energy conservation law we obtained earlier.

The physical interpretation of $T^0{}_1 = -\rho\dot{y}y'$, the locally conserved quantity appearing in (1.96) is less obvious. If this were a relativistic system, we would immediately identify $\int T^0{}_1\, dx$ as the x-component of the energy-momentum 4-vector, and therefore $T^0{}_1$ as the density of x-momentum. Now any real string will have some motion in the x direction, but the magnitude of this motion will depend on the string's elastic constants and other quantities unknown to our Lagrangian. Because of this, the $T^0{}_1$ derived from L cannot be the string's x-momentum density. Instead, it is the density of something called *pseudo-momentum*. The distinction between true and pseudo-momentum is best appreaciated by considering the corresponding Noether symmetry. The symmetry associated with Newtonian momentum is the invariance of the action integral under an x translation of the entire apparatus: the string, and any wave on it. The symmetry associated with pseudo-momentum is the invariance of the action under a shift

$y(x) \to y(x-a)$ of the location of the wave on the string — the string itself not being translated. Newtonian momentum is conserved if the *ambient space* is translationally invariant. Pseudo-momentum is conserved only if the *string* is translationally invariant — *i.e.* if ρ and T are position independent. A failure to realize that the presence of a medium (here the string) requires us to distinguish between these two symmetries is the origin of much confusion involving "wave momentum."

Maxwell's equations

Michael Faraday and and James Clerk Maxwell's description of electromagnetism in terms of dynamical vector fields gave us the first modern field theory. D'Alembert and Maupertuis would have been delighted to discover that the famous equations of Maxwell's *A Treatise on Electricity and Magnetism* (1873) follow from an action principle. There is a slight complication stemming from gauge invariance but, as long as we are not interested in exhibiting the covariance of Maxwell under Lorentz transformations, we can sweep this under the rug by working in the *axial gauge*, where the scalar electric potential does not appear.

We will start from Maxwell's equations

$$\begin{aligned} \operatorname{div} \mathbf{B} &= 0, \\ \operatorname{curl} \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \operatorname{curl} \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \\ \operatorname{div} \mathbf{D} &= \rho, \end{aligned} \tag{1.101}$$

and show that they can be obtained from an action principle. For convenience we shall use *natural* units in which $\mu_0 = \varepsilon_0 = 1$, and so $c = 1$ and $\mathbf{D} \equiv \mathbf{E}$ and $\mathbf{B} \equiv \mathbf{H}$.

The first equation $\operatorname{div} \mathbf{B} = 0$ contains no time derivatives. It is a constraint which we satisfy by introducing a vector potential $\mathbf{A}$ such that $\mathbf{B} = \operatorname{curl} \mathbf{A}$. If we set

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \tag{1.102}$$

then this automatically implies Faraday's law of induction

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{1.103}$$

We now guess that the Lagrangian is

$$L = \int d^3x \left[\frac{1}{2} \left\{ \mathbf{E}^2 - \mathbf{B}^2 \right\} + \mathbf{J} \cdot \mathbf{A} \right]. \tag{1.104}$$

The motivation is that L looks very like $T - V$ if we regard $\frac{1}{2}\mathbf{E}^2 \equiv \frac{1}{2}\dot{\mathbf{A}}^2$ as being the kinetic energy and $\frac{1}{2}\mathbf{B}^2 = \frac{1}{2}(\operatorname{curl}\mathbf{A})^2$ as being the potential energy. The term in $\mathbf{J}$ represents the interaction of the fields with an external current source. In the axial gauge the electric charge density ρ does not appear in the Lagrangian. The corresponding action is therefore

$$S = \int L\,dt = \iint d^3x \left[\frac{1}{2}\dot{\mathbf{A}}^2 - \frac{1}{2}(\operatorname{curl}\mathbf{A})^2 + \mathbf{J} \cdot \mathbf{A} \right] dt. \tag{1.105}$$

Now vary $\mathbf{A}$ to $\mathbf{A} + \delta\mathbf{A}$, whence

$$\delta S = \iint d^3x \left[-\ddot{\mathbf{A}} \cdot \delta\mathbf{A} - (\operatorname{curl}\mathbf{A}) \cdot (\operatorname{curl}\delta\mathbf{A}) + \mathbf{J} \cdot \delta\mathbf{A} \right] dt. \tag{1.106}$$

Here, we have already removed the time derivative from $\delta\mathbf{A}$ by integrating by parts in the time direction. Now we do the integration by parts in the space directions by using the identity

$$\operatorname{div}(\delta\mathbf{A} \times (\operatorname{curl}\mathbf{A})) = (\operatorname{curl}\mathbf{A}) \cdot (\operatorname{curl}\delta\mathbf{A}) - \delta\mathbf{A} \cdot (\operatorname{curl}(\operatorname{curl}\mathbf{A})) \tag{1.107}$$

and taking $\delta\mathbf{A}$ to vanish at spatial infinity, so the surface term, which would come from the integral of the total divergence, is zero. We end up with

$$\delta S = \iint d^3x \left\{ \delta\mathbf{A} \cdot \left[-\ddot{\mathbf{A}} - \operatorname{curl}(\operatorname{curl}\mathbf{A}) + \mathbf{J} \right] \right\} dt. \tag{1.108}$$

Demanding that the variation of S be zero thus requires

$$\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\operatorname{curl}(\operatorname{curl}\mathbf{A}) + \mathbf{J}, \tag{1.109}$$

or, in terms of the physical fields,

$$\operatorname{curl}\mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \tag{1.110}$$

This is Ampère's law, as modified by Maxwell so as to include the displacement current.

How do we deal with the last Maxwell equation, Gauss' law, which asserts that $\operatorname{div}\mathbf{E} = \rho$? If ρ were equal to zero, this equation would hold if $\operatorname{div}\mathbf{A} = 0$, *i.e.* if $\mathbf{A}$ were solenoidal. In this case we might be tempted to impose the constraint $\operatorname{div}\mathbf{A} = 0$ on the vector potential, but doing so would undo all our good work, as we have been assuming that we can vary $\mathbf{A}$ freely.

We notice, however, that the three Maxwell equations we already possess tell us that

$$\frac{\partial}{\partial t}\left(\operatorname{div}\mathbf{E} - \rho\right) = \operatorname{div}\left(\operatorname{curl}\mathbf{B}\right) - \left(\operatorname{div}\mathbf{J} + \frac{\partial \rho}{\partial t}\right). \tag{1.111}$$

Now $\operatorname{div}(\operatorname{curl}\mathbf{B}) = 0$, so the left-hand side is zero provided charge is conserved, *i.e.* provided

$$\dot{\rho} + \operatorname{div}\mathbf{J} = 0. \tag{1.112}$$

We assume that this is so. Thus, if Gauss' law holds initially, it holds eternally. We arrange for it to hold at $t = 0$ by imposing initial conditions on $\mathbf{A}$. We first choose $\mathbf{A}|_{t=0}$ by requiring it to satisfy

$$\mathbf{B}|_{t=0} = \operatorname{curl}\ \left(\mathbf{A}|_{t=0}\right). \tag{1.113}$$

The solution is not unique, because may we add any $\nabla\phi$ to $\mathbf{A}|_{t=0}$, but this does not affect the physical $\mathbf{E}$ and $\mathbf{B}$ fields. The initial "velocities" $\dot{\mathbf{A}}|_{t=0}$ are then fixed uniquely by $\dot{\mathbf{A}}|_{t=0} = -\mathbf{E}|_{t=0}$, where the initial $\mathbf{E}$ satisfies Gauss' law. The subsequent evolution of $\mathbf{A}$ is then uniquely determined by integrating the second-order equation (1.109).

The first integral for Maxwell is

$$\begin{aligned} E &= \sum_{i=1}^{3}\int d^3x \left\{\dot{A}_i \frac{\delta L}{\delta \dot{A}_i}\right\} - L \\ &= \int d^3x \left[\frac{1}{2}\left\{\mathbf{E}^2 + \mathbf{B}^2\right\} - \mathbf{J}\cdot\mathbf{A}\right]. \end{aligned} \tag{1.114}$$

This will be conserved if $\mathbf{J}$ is time independent. If $\mathbf{J} = 0$, it is the total field energy.

Suppose $\mathbf{J}$ is neither zero nor time independent. Then, looking back at the derivation of the time-independence of the first integral, we see that if L *does* depend on time, we instead have

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t}. \tag{1.115}$$

In the present case we have

$$-\frac{\partial L}{\partial t} = -\int \dot{\mathbf{J}} \cdot \mathbf{A}\, d^3x, \tag{1.116}$$

so that

$$-\int \dot{\mathbf{J}} \cdot \mathbf{A}\, d^3x = \frac{dE}{dt} = \frac{d}{dt}(\text{Field Energy}) - \int \left\{ \mathbf{J} \cdot \dot{\mathbf{A}} + \dot{\mathbf{J}} \cdot \mathbf{A} \right\} d^3x. \tag{1.117}$$

Thus, cancelling the duplicated term and using $\mathbf{E} = -\dot{\mathbf{A}}$, we find

$$\frac{d}{dt}(\text{Field Energy}) = -\int \mathbf{J} \cdot \mathbf{E}\, d^3x. \tag{1.118}$$

Now $\int \mathbf{J} \cdot (-\mathbf{E})\, d^3x$ is the rate at which the power source driving the current is doing work against the field. The result is therefore physically sensible.

Continuum mechanics

Because the mechanics of discrete objects can be derived from an action principle, it seems obvious that so must the mechanics of continua. This is certainly true if we use the *Lagrangian* description where we follow the history of each particle composing the continuous material as it moves through space. In fluid mechanics it is more natural to describe the motion by using the *Eulerian* description in which we focus on what is going on at a particular point in space by introducing a velocity field $\mathbf{v}(\mathbf{r}, t)$. Eulerian action principles can still be found, but they seem to be logically distinct from the Lagrangian mechanics action principle, and mostly were not discovered until the 20th century.

We begin by showing that Euler's equation for the irrotational motion of an inviscid compressible fluid can be obtained by applying the action principle to a functional

$$S[\phi, \rho] = \int dt\, d^3x \left\{ \rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho(\nabla\phi)^2 + u(\rho) \right\}, \tag{1.119}$$

where ρ is the mass density and the flow velocity is determined from the velocity potential ϕ by $\mathbf{v} = \nabla\phi$. The function $u(\rho)$ is the internal energy density.

Varying $S[\phi, \rho]$ with respect to ρ is straightforward, and gives a time dependent generalization of (Daniel) Bernoulli's equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\mathbf{v}^2 + h(\rho) = 0. \tag{1.120}$$

Here $h(\rho) \equiv du/d\rho$, is the specific enthalpy.[1] Varying with respect to ϕ requires an integration by parts, based on

$$\mathrm{div}\,(\rho\,\delta\phi\,\nabla\phi) = \rho(\nabla\delta\phi)\cdot(\nabla\phi) + \delta\phi\,\mathrm{div}\,(\rho\nabla\phi), \tag{1.121}$$

and gives the equation of mass conservation

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho\mathbf{v}) = 0. \tag{1.122}$$

Taking the gradient of Bernoulli's equation, and using the fact that for potential flow the *vorticity* $\boldsymbol{\omega} \equiv \mathrm{curl}\,\mathbf{v}$ is zero and so $\partial_i v_j = \partial_j v_i$, we find that

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla h. \tag{1.123}$$

We now introduce the pressure P, which is related to h by

$$h(P) = \int_0^P \frac{dP}{\rho(P)}. \tag{1.124}$$

We see that $\rho\nabla h = \nabla P$, and so obtain Euler's equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla P. \tag{1.125}$$

For future reference, we observe that combining the mass-conservation equation

$$\partial_t \rho + \partial_j\left\{\rho v_j\right\} = 0 \tag{1.126}$$

with Euler's equation

$$\rho(\partial_t v_i + v_j \partial_j v_i) = -\partial_i P \tag{1.127}$$

[1]The enthalpy $H = U + PV$ per unit mass. In general u and h will be functions of both the density and the specific entropy. By taking u to depend only on ρ we are tacitly assuming that specific entropy is constant. This makes the resultant flow *barotropic*, meaning that the pressure is a function of the density only.

yields

$$\partial_t \{\rho v_i\} + \partial_j \{\rho v_i v_j + \delta_{ij} P\} = 0, \tag{1.128}$$

which expresses the local conservation of momentum. The quantity

$$\Pi_{ij} = \rho v_i v_j + \delta_{ij} P \tag{1.129}$$

is the *momentum-flux tensor*, and is the j-th component of the flux of the i-th component $p_i = \rho v_i$ of momentum density.

The relations $h = du/d\rho$ and $\rho = dP/dh$ show that P and u are related by a Legendre transformation: $P = \rho h - u(\rho)$. From this, and the Bernoulli equation, we see that the integrand in the action (1.119) is equal to minus the pressure:

$$-P = \rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho(\nabla\phi)^2 + u(\rho). \tag{1.130}$$

This Eulerian formulation cannot be a "follow the particle" action principle in a clever disguise. The mass conservation law is only a consequence of the equation of motion, and is not built in from the beginning as a constraint. Our variations in ϕ are therefore conjuring up new matter rather than merely moving it around.

1.4 Variable endpoints

We now relax our previous assumption that all boundary or surface terms arising from integrations by parts may be ignored. We will find that variation principles can be very useful for working out what boundary conditions we should impose on our differential equations.

Consider the problem of building a railway across a parallel sided isthmus.

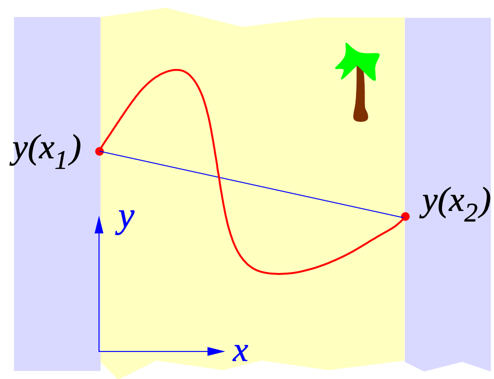

Figure 1.9: *Railway across isthmus.*

Suppose that the cost of construction is proportional to the length of the track, but the cost of sea transport being negligeable, we may locate the terminal seaports wherever we like. We therefore wish to minimize the length

$$L[y] = \int_{x_1}^{x_2} \sqrt{1+(y')^2}dx, \tag{1.131}$$

by allowing both the path $y(x)$ and the endpoints $y(x_1)$ and $y(x_2)$ to vary. Then

$$\begin{aligned} L[y+\delta y]-L[y] &= \int_{x_1}^{x_2} (\delta y')\frac{y'}{\sqrt{1+(y')^2}}dx \\ &= \int_{x_1}^{x_2} \left\{\frac{d}{dx}\left(\delta y\frac{y'}{\sqrt{1+(y')^2}}\right) - \delta y\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right)\right\}dx \\ &= \delta y(x_2)\frac{y'(x_2)}{\sqrt{1+(y')^2}} - \delta y(x_1)\frac{y'(x_1)}{\sqrt{1+(y')^2}} \\ &\qquad - \int_{x_1}^{x_2} \delta y\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right)dx. \end{aligned} \tag{1.132}$$

We have stationarity when both

i) the coefficient of $\delta y(x)$ in the integral,

$$-\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right), \tag{1.133}$$

is zero. This requires that y' =const., *i.e.* the track should be straight.

ii) The coefficients of $\delta y(x_1)$ and $\delta y(x_2)$ vanish. For this we need

$$0 = \frac{y'(x_1)}{\sqrt{1+(y')^2}} = \frac{y'(x_2)}{\sqrt{1+(y')^2}}. \tag{1.134}$$

This in turn requires that $y'(x_1) = y'(x_2) = 0$.

The integrated-out bits have determined the boundary conditions that are to be imposed on the solution of the differential equation. In the present case they require us to build perpendicular to the coastline, and so we go straight across the isthmus. When boundary conditions are obtained from endpoint variations in this way, they are called *natural boundary conditions*.

Example: Sliding String. A massive string of linear density ρ is stretched between two smooth posts separated by distance $2L$. The string is under tension T, and is free to slide up and down the posts. We consider only a small deviations of the string from the horizontal.

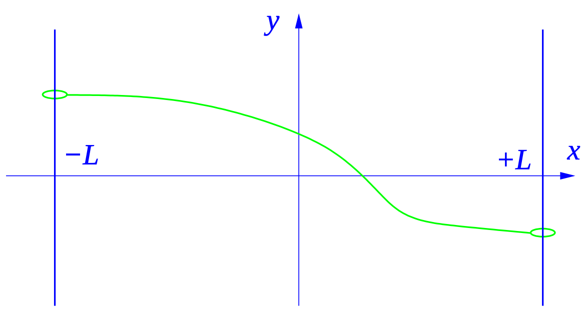

Figure 1.10: *Sliding string.*

As we saw earlier, the Lagrangian for a stretched string is

$$L = \int_{-L}^{L} \left\{ \frac{1}{2}\rho \dot{y}^2 - \frac{1}{2}T(y')^2 \right\} dx. \tag{1.135}$$

Now, Lagrange's principle says that the equation of motion is found by requiring the action

$$S = \int_{t_i}^{t_f} L\, dt \tag{1.136}$$

to be stationary under variations of $y(x,t)$ that vanish at the initial and final times, t_i and t_f. It does *not* demand that δy vanish at ends of the string, $x = \pm L$. So, when we make the variation, we must not assume this. Taking care not to discard the results of the integration by parts in the x direction, we find

$$\begin{aligned}\delta S &= \int_{t_i}^{t_f}\int_{-L}^{L} \delta y(x,t)\left\{-\rho\ddot{y}+Ty''\right\}dxdt - \int_{t_i}^{t_f}\delta y(L,t)Ty'(L)\,dt \\ &\qquad + \int_{t_i}^{t_f}\delta y(-L,t)Ty'(-L)\,dt. \end{aligned} \tag{1.137}$$

The equation of motion, which arises from the variation within the interval, is therefore the wave equation

$$\rho\ddot{y} - Ty'' = 0. \tag{1.138}$$

The boundary conditions, which come from the variations at the endpoints, are

$$y'(L,t) = y'(-L,t) = 0, \tag{1.139}$$

at all times t. These are the physically correct boundary conditions, because any up-or-down component of the tension would provide a finite force on an infinitesimal mass. The string must therefore be horizontal at its endpoints.

Example: Bead and String. Suppose now that a bead of mass M is free to slide up and down the y axis,

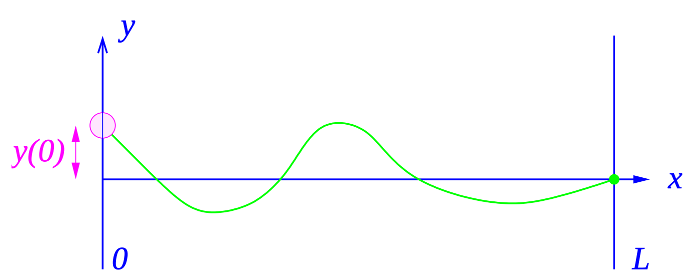

Figure 1.11: *A bead connected to a string.*

and is is attached to the $x = 0$ end of our string. The Lagrangian for the string-bead contraption is

$$L = \frac{1}{2}M[\dot{y}(0)]^2 + \int_0^L \left\{\frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}Ty'^2\right\}dx. \tag{1.140}$$

Here, as before, ρ is the mass per unit length of the string and T is its tension. The end of the string at $x = L$ is fixed. By varying the action $S = \int L dt$, and taking care not to throw away the boundary part at $x = 0$ we find that

$$\delta S = \int_{t_i}^{t_f} [Ty' - M\ddot{y}]_{x=0}\, \delta y(0,t)\, dt + \int_{t_i}^{t_f} \int_0^L \{Ty'' - \rho\ddot{y}\}\, \delta y(x,t)\, dxdt. \tag{1.141}$$

The Euler-Lagrange equations are therefore

$$\begin{aligned} \rho\ddot{y}(x) - Ty''(x) &= 0, \quad 0 < x < L, \\ M\ddot{y}(0) - Ty'(0) &= 0, \quad y(L) = 0. \end{aligned} \tag{1.142}$$

The boundary condition at $x = 0$ is the equation of motion for the bead. It is clearly correct, because $Ty'(0)$ is the vertical component of the force that the string tension exerts on the bead.

These examples led to boundary conditions that we could easily have figured out for ourselves without the variational principle. The next example shows that a variational formulation can be exploited to obtain a set of boundary conditions that might be difficult to write down by purely "physical" reasoning.

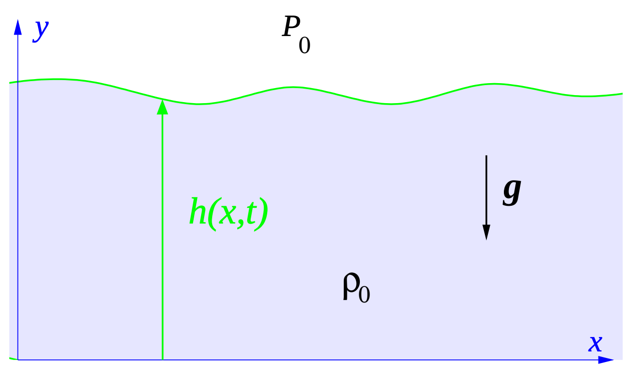

Figure 1.12: *Gravity waves on water.*

Harder example: Gravity waves on the surface of water. An action suitable for describing water waves is given by[2] $S[\phi, h] = \int L\, dt$, where

$$L = \int dx \int_0^{h(x,t)} \rho_0 \left\{ \frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gy \right\} dy. \tag{1.143}$$

[2]J. C. Luke, *J. Fluid Dynamics*, **27** (1967) 395.

Here ϕ is the velocity potential and ρ_0 is the density of the water. The density will not be varied because the water is being treated as incompressible. As before, the flow velocity is given by $\mathbf{v} = \nabla\phi$. By varying $\phi(x, y, t)$ and the depth $h(x, t)$, and taking care not to throw away any integrated-out parts of the variation at the physical boundaries, we obtain:

$$\begin{aligned}
\nabla^2\phi &= 0, \quad \text{within the fluid.} \\
\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gy &= 0, \quad \text{on the free surface.} \\
\frac{\partial\phi}{\partial y} &= 0, \quad \text{on} \quad y = 0. \\
\frac{\partial h}{\partial t} - \frac{\partial\phi}{\partial y} + \frac{\partial h}{\partial x}\frac{\partial\phi}{\partial x} &= 0, \quad \text{on the free surface.}
\end{aligned} \tag{1.144}$$

The first equation comes from varying ϕ within the fluid, and it simply confirms that the flow is incompressible, *i.e.* obeys $\operatorname{div}\mathbf{v} = 0$. The second comes from varying h, and is the Bernoulli equation stating that we have $P = P_0$ (atmospheric pressure) everywhere on the free surface. The third, from the variation of ϕ at $y = 0$, states that no fluid escapes through the lower boundary.

Obtaining and interpreting the last equation, involving $\partial h/\partial t$, is somewhat trickier. It comes from the variation of ϕ on the upper boundary. The variation of S due to $\delta\phi$ is

$$\delta S = \int \rho_0 \left\{ \frac{\partial}{\partial t}\delta\phi + \frac{\partial}{\partial x}\left(\delta\phi\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\delta\phi\frac{\partial\phi}{\partial y}\right) - \delta\phi\,\nabla^2\phi \right\} dtdxdy. \tag{1.145}$$

The first three terms in the integrand constitute the three-dimensional divergence $\operatorname{div}(\delta\phi\,\mathbf{\Phi})$, where, listing components in the order t, x, y,

$$\mathbf{\Phi} = \left[1, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right]. \tag{1.146}$$

The integrated-out part on the upper surface is therefore $\int(\mathbf{\Phi}\cdot\mathbf{n})\delta\phi\, d|S|$. Here, the outward normal is

$$\mathbf{n} = \left(1 + \left(\frac{\partial h}{\partial t}\right)^2 + \left(\frac{\partial h}{\partial x}\right)^2\right)^{-1/2} \left[-\frac{\partial h}{\partial t}, -\frac{\partial h}{\partial x}, 1\right], \tag{1.147}$$

and the element of area

$$d|S| = \left(1 + \left(\frac{\partial h}{\partial t}\right)^2 + \left(\frac{\partial h}{\partial x}\right)^2\right)^{1/2} dtdx. \tag{1.148}$$

The boundary variation is thus

$$\delta S|_{y=h} = -\int \left\{\frac{\partial h}{\partial t} - \frac{\partial \phi}{\partial y} + \frac{\partial h}{\partial x}\frac{\partial \phi}{\partial x}\right\} \delta\phi\Big(x, h(x,t), t\Big)\, dxdt. \tag{1.149}$$

Requiring this variation to be zero for arbitrary $\delta\phi\Big(x, h(x,t), t\Big)$ leads to

$$\frac{\partial h}{\partial t} - \frac{\partial \phi}{\partial y} + \frac{\partial h}{\partial x}\frac{\partial \phi}{\partial x} = 0. \tag{1.150}$$

This last boundary condition expresses the geometrical constraint that the surface moves with the fluid it bounds, or, in other words, that a fluid particle initially on the surface stays on the surface. To see that this is so, define $f(x, y, t) = h(x, t) - y$. The free surface is then determined by $f(x, y, t) = 0$. Because the surface particles are carried with the flow, the convective derivative of f,

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f, \tag{1.151}$$

must vanish on the free surface. Using $\mathbf{v} = \nabla\phi$ and the definition of f, this reduces to

$$\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} - \frac{\partial \phi}{\partial y} = 0, \tag{1.152}$$

which is indeed the last boundary condition.

1.5 Lagrange multipliers

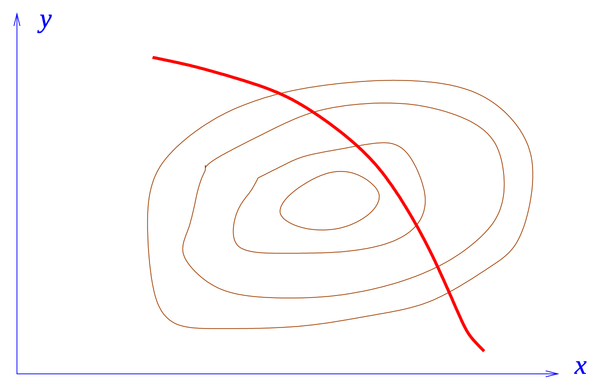

Figure 1.13: *Road on hill.*

Figure 1.13 shows the contour map of a hill of height $h = f(x, y)$. The hill traversed by a road whose points satisfy the equation $g(x, y) = 0$. Our challenge is to use the data $h(x, y)$ and $g(x, y)$ to find the highest point on the road.

When $\mathbf{r}$ changes by $d\mathbf{r} = (dx, dy)$, the height f changes by

$$df = \nabla f \cdot d\mathbf{r}, \tag{1.153}$$

where $\nabla f = (\partial_x f, \partial_y f)$. The highest point, being a stationary point, will have $df = 0$ for all displacements $d\mathbf{r}$ that stay on the road — that is for all $d\mathbf{r}$ such that $dg = 0$. Thus $\nabla f \cdot d\mathbf{r}$ must be zero for those $d\mathbf{r}$ such that $0 = \nabla g \cdot d\mathbf{r}$. In other words, at the highest point ∇f will be orthogonal to all vectors that are orthogonal to ∇g. This is possible only if the vectors ∇f and ∇g are parallel, and so $\nabla f = \lambda \nabla g$ for some λ.

To find the stationary point, therefore, we solve the equations

$$\begin{aligned} \nabla f - \lambda \nabla g &= 0, \\ g(x, y) &= 0, \end{aligned} \tag{1.154}$$

simultaneously.

Example: Let $f = x^2 + y^2$ and $g = x + y - 1$. Then $\nabla f = 2(x, y)$ and $\nabla g = (1, 1)$. So

$$2(x, y) - \lambda(1, 1) = 0, \quad \Rightarrow \quad (x, y) = \frac{\lambda}{2}(1, 1)$$

$$x + y \;=\; 1, \quad \Rightarrow \quad \lambda = 1, \quad \Longrightarrow \quad (x, y) = (\frac{1}{2}, \frac{1}{2}).$$

When there are n constraints, $g_1 = g_2 = \cdots = g_n = 0$, we want ∇f to lie in

$$(< \nabla g_i >^{\perp})^{\perp} = < \nabla g_i >, \tag{1.155}$$

where $< \mathbf{e}_i >$ denotes the space spanned by the vectors $\mathbf{e}_i$ and $< \mathbf{e}_i >^{\perp}$ is the its orthogonal complement. Thus ∇f lies in the space spanned by the vectors ∇g_i, so there must exist n numbers λ_i such that

$$\nabla f = \sum_{i=1}^{n} \lambda_i \nabla g_i. \tag{1.156}$$

The numbers λ_i are called *Lagrange multipliers*. We can therefore regard our problem as one of finding the stationary points of an auxilliary function

$$F = f - \sum_i \lambda_i g_i, \tag{1.157}$$

with the n undetermined multipliers $\lambda_i, i = 1, \ldots, n$, subsequently being fixed by imposing the n requirements that $g_i = 0, i = 1, \ldots, n$.
Example: Find the stationary points of

$$F(\mathbf{x}) = \frac{1}{2}\mathbf{x} \cdot \mathbf{A}\mathbf{x} = \frac{1}{2} x_i A_{ij} x_j \tag{1.158}$$

on the surface $\mathbf{x} \cdot \mathbf{x} = 1$. Here A_{ij} is a symmetric matrix.
Solution: We look for stationary points of

$$G(\mathbf{x}) = F(\mathbf{x}) - \frac{1}{2}\lambda |\mathbf{x}|^2. \tag{1.159}$$

The derivatives we need are

$$\begin{aligned} \frac{\partial F}{\partial x^k} &= \frac{1}{2}\delta_{ki} A_{ij} x_j + \frac{1}{2} x_i A_{ij} \delta_{jk} \\ &= A_{kj} x_j, \end{aligned} \tag{1.160}$$

and

$$\frac{\partial}{\partial x_k}\left(\frac{\lambda}{2} x_j x_j\right) = \lambda x_k. \tag{1.161}$$

Thus, the stationary points must satisfy

$$\begin{aligned} A_{kj}x_j &= \lambda x_k, \\ x^i x^i &= 1, \end{aligned} \tag{1.162}$$

and so are the normalized eigenvectors of the matrix $\mathbf{A}$. The Lagrange multiplier at each stationary point is the corresponding eigenvalue.

Example: Statistical Mechanics. Let Γ denote the classical phase space of a mechanical system of n particles governed by Hamiltonian $H(p,q)$. Let $d\Gamma$ be the Liouville measure $d^{3n}p\, d^{3n}q$. In statistical mechanics we work with a probability density $\rho(p,q)$ such that $\rho(p,q)d\Gamma$ is the probability of the system being in a state in the small region $d\Gamma$. The *entropy* associated with the probability distribution is the functional

$$S[\rho] = -\int_\Gamma \rho \ln \rho \, d\Gamma. \tag{1.163}$$

We wish to find the $\rho(p,q)$ that maximizes the entropy for a given energy

$$\langle E \rangle = \int_\Gamma \rho H \, d\Gamma. \tag{1.164}$$

We cannot vary ρ freely as we should preserve both the energy and the normalization condition

$$\int_\Gamma \rho \, d\Gamma = 1 \tag{1.165}$$

that is required of any probability distribution. We therefore introduce two Lagrange multipliers, $1+\alpha$ and β, to enforce the normalization and energy conditions, and look for stationary points of

$$F[\rho] = \int_\Gamma \left\{-\rho \ln \rho + (\alpha+1)\rho - \beta \rho H\right\} \, d\Gamma. \tag{1.166}$$

Now we can vary ρ freely, and hence find that

$$\delta F = \int_\Gamma \left\{-\ln \rho + \alpha - \beta H\right\} \delta\rho \, d\Gamma. \tag{1.167}$$

Requiring this to be zero gives us

$$\rho(p,q) = e^{\alpha - \beta H(p,q)}, \tag{1.168}$$

where α, β are determined by imposing the normalization and energy constraints. This probability density is known as the *canonical distribution*, and the parameter β is the inverse temperature $\beta = 1/T$.

Example: The Catenary. At last we have the tools to solve the problem of the hanging chain of fixed length. We wish to minimize the potential energy

$$E[y] = \int_{-L}^{L} y\sqrt{1+(y')^2}dx, \tag{1.169}$$

subject to the constraint

$$l[y] = \int_{-L}^{L} \sqrt{1+(y')^2}dx = \text{const.}, \tag{1.170}$$

where the constant is the length of the chain. We introduce a Lagrange multiplier λ and find the stationary points of

$$F[y] = \int_{-L}^{L} (y-\lambda)\sqrt{1+(y')^2}dx, \tag{1.171}$$

so, following our earlier methods, we find

$$y = \lambda + \kappa \cosh \frac{(x+a)}{\kappa}. \tag{1.172}$$

We choose κ, λ, a to fix the two endpoints (two conditions) and the length (one condition).

Example: Sturm-Liouville Problem. We wish to find the stationary points of the quadratic functional

$$J[y] = \int_{x_1}^{x_2} \frac{1}{2}\left\{p(x)(y')^2 + q(x)y^2\right\}\, dx, \tag{1.173}$$

subject to the boundary conditions $y(x) = 0$ at the endpoints x_1, x_2 and the normalization

$$K[y] = \int_{x_1}^{x_2} y^2\, dx = 1. \tag{1.174}$$

Taking the variation of $J - (\lambda/2)K$, we find

$$\delta J = \int_{x_1}^{x_2} \{-(py')' + qy - \lambda y\}\, \delta y\, dx. \tag{1.175}$$

Stationarity therefore requires

$$-(py')' + qy = \lambda y, \quad y(x_1) = y(x_2) = 0. \tag{1.176}$$

This is the *Sturm-Liouville eigenvalue problem*. It is an infinite dimensional analogue of the $F(\mathbf{x}) = \frac{1}{2}\mathbf{x} \cdot \mathbf{A}\mathbf{x}$ problem.

Example: Irrotational Flow Again. Consider the action functional

$$S[\mathbf{v}, \phi, \rho] = \int \left\{ \frac{1}{2}\rho \mathbf{v}^2 - u(\rho) + \phi \left(\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} \right) \right\} dt d^3x \tag{1.177}$$

This is similar to our previous action for the irrotational barotropic flow of an inviscid fluid, but here $\mathbf{v}$ is an independent variable and we have introduced infinitely many Lagrange multipliers $\phi(x, t)$, one for each point of space-time, so as to enforce the equation of mass conservation $\dot{\rho} + \operatorname{div} \rho \mathbf{v} = 0$ everywhere, and at all times. Equating $\delta S/\delta \mathbf{v}$ to zero gives $\mathbf{v} = \nabla \phi$, and so these Lagrange multipliers become the velocity potential as a consequence of the equations of motion. The Bernoulli and Euler equations now follow almost as before. Because the equation $\mathbf{v} = \nabla \phi$ does not involve time derivatives, this is one of the cases where it is legitimate to substitute a consequence of the action principle back into the action. If we do this, we recover our previous formulation.

1.6 Maximum or minimum?

We have provided many examples of stationary points in function space. We have said almost nothing about whether these stationary points are maxima or minima. There is a reason for this: investigating the character of the stationary point requires the computation of the second functional derivative.

$$\frac{\delta^2 J}{\delta y(x_1) \delta y(x_2)}$$

and the use of the functional version of Taylor's theorem to expand about the stationary point $y(x)$:

$$\begin{aligned} J[y + \varepsilon \eta] &= J[y] + \varepsilon \int \eta(x) \left. \frac{\delta J}{\delta y(x)} \right|_y dx \\ &\quad + \frac{\varepsilon^2}{2} \int \eta(x_1)\eta(x_2) \left. \frac{\delta^2 J}{\delta y(x_1) \delta y(x_2)} \right|_y dx_1 dx_2 + \cdots . \end{aligned} \tag{1.178}$$

Since $y(x)$ is a stationary point, the term with $\delta J/\delta y(x)|_y$ vanishes. Whether $y(x)$ is a maximum, a minimum, or a saddle therefore depends on the number of positive and negative eigenvalues of $\delta^2 J/\delta(y(x_1))\delta(y(x_2))$, a matrix with a continuous infinity of rows and columns—these being labeled by x_1 and x_2 repectively. It is not easy to diagonalize a continuously infinite matrix! Consider, for example, the functional

$$J[y] = \int_a^b \frac{1}{2}\left\{p(x)(y')^2 + q(x)y^2\right\} dx, \tag{1.179}$$

with $y(a) = y(b) = 0$. Here, as we already know,

$$\frac{\delta J}{\delta y(x)} = Ly \equiv -\frac{d}{dx}\left(p(x)\frac{d}{dx}y(x)\right) + q(x)y(x), \tag{1.180}$$

and, except in special cases, this will be zero only if $y(x) \equiv 0$. We might reasonably expect the second derivative to be

$$\frac{\delta}{\delta y}(Ly) \stackrel{?}{=} L, \tag{1.181}$$

where L is the Sturm-Liouville differential operator

$$L = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x). \tag{1.182}$$

How can a differential operator be a matrix like $\delta^2 J/\delta(y(x_1))\delta(y(x_2))$?

We can formally compute the second derivative by exploiting the Dirac delta "function" $\delta(x)$ which has the property that

$$y(x_2) = \int \delta(x_2 - x_1)y(x_1)\,dx_1. \tag{1.183}$$

Thus

$$\delta y(x_2) = \int \delta(x_2 - x_1)\delta y(x_1)\,dx_1, \tag{1.184}$$

from which we read off that

$$\frac{\delta y(x_2)}{\delta y(x_1)} = \delta(x_2 - x_1). \tag{1.185}$$

Using (1.185), we find that

$$\frac{\delta}{\delta y(x_1)}\left(\frac{\delta J}{\delta y(x_2)}\right) = -\frac{d}{dx_2}\left(p(x_2)\frac{d}{dx_2}\delta(x_2-x_1)\right)+q(x_2)\delta(x_2-x_1). \quad (1.186)$$

How are we to make sense of this expression? We begin in the next chapter where we explain what it means to differentiate $\delta(x)$, and show that (1.186) does indeed correspond to the differential operator L. In subsequent chapters we explore the manner in which differential operators and matrices are related. We will learn that just as some matrices can be diagonalized so can some differential operators, and that the class of diagonalizable operators includes (1.182).

If all the eigenvalues of L are positive, our stationary point was a minimum. For each negative eigenvalue, there is direction in function space in which $J[y]$ decreases as we move away from the stationary point.

1.7 Further exercises and problems

Here is a collection of problems relating to the calculus of variations. Some date back to the 16th century, others are quite recent in origin.

Exercise 1.1: A smooth path in the x-y plane is given by $\mathbf{r}(t) = (x(t), y(t))$ with $\mathbf{r}(0) = \mathbf{a}$, and $\mathbf{r}(1) = \mathbf{b}$. The length of the path from $\mathbf{a}$ to $\mathbf{b}$ is therefore.

$$S[\mathbf{r}] = \int_0^1 \sqrt{\dot{x}^2+\dot{y}^2}\,dt,$$

where $\dot{x} \equiv dx/dt$, $\dot{y} \equiv dy/dt$. Write down the Euler-Lagrange conditions for $S[\mathbf{r}]$ to be stationary under small variations of the path that keep the endpoints fixed, and hence show that the shortest path between two points is a straight line.

Exercise 1.2: *Fermat's principle*. A medium is characterised optically by its *refractive index* n, such that the speed of light in the medium is c/n. According to Fermat (1657), the path taken by a ray of light between any two points makes stationary the travel time between those points. Assume that the ray propagates in the x, y plane in a layered medium with refractive index $n(x)$. Use Fermat's principle to establish Snell's law in its general form $n(x)\sin\psi = \text{constant}$ by finding the equation giving the stationary paths $y(x)$ for

$$F_1[y] = \int n(x)\sqrt{1+y'^2}dx.$$

(Here the prime denotes differentiation with respect to x.) Repeat this exercise for the case that n depends only on y and find a similar equation for the stationary paths of

$$F_2[y] = \int n(y)\sqrt{1+y'^2}dx.$$

By using suitable definitions of the angle of incidence ψ in each case, show that the two formulations of the problem give physically equivalent answers. In the second formulation you will find it easiest to use the first integral of Euler's equation.

Problem 1.3: *Hyperbolic Geometry.* This problem introduces a version of the Poincaré model for the non-Euclidean geometry of Lobachevski.

a) Show that the stationary paths for the functional

$$F_3[y] = \int \frac{1}{y}\sqrt{1+y'^2}dx,$$

with $y(x)$ restricted to lying in the upper half plane are semi-circles of arbitrary radius and with centres on the x axis. These paths are the *geodesics*, or minimum length paths, in a space with Riemann metric

$$ds^2 = \frac{1}{y^2}(dx^2+dy^2), \qquad y > 0.$$

b) Show that if we call these geodesics "lines", then one and only one line can be drawn though two given points.
c) Two lines are said to be *parallel* if, and only if, they meet at "infinity", *i.e.* on the x axis. (Verify that the x axis is indeed infinitely far from any point with $y > 0$.) Show that given a line q and a point A not lying on that line, that there are *two* lines passing through A that are parallel to q, and that between these two lines lies a pencil of lines passing through A that never meet q.

Problem 1.4: *Elastic Rods.* The elastic energy per unit length of a bent steel rod is given by $\frac{1}{2}YI/R^2$. Here R is the radius of curvature due to the bending, Y is the Young's modulus of the steel and $I = \iint y^2 dxdy$ is the moment of inertia of the rod's cross section about an axis through its centroid and perpendicular to the plane in which the rod is bent. If the rod is only slightly bent into the yz plane and lies close to the z axis, show that this elastic energy can be approximated as

$$U[y] = \int_0^L \frac{1}{2}YI\left(y''\right)^2\, dz,$$

where the prime denotes differentiation with respect to z and L is the length of the rod. We will use this approximate energy functional to discuss two practical problems.

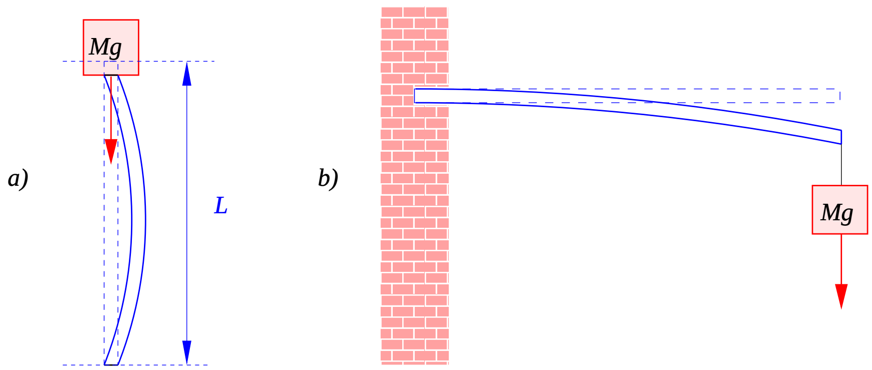

Figure 1.14: *A rod used as: a) a column, b) a cantilever.*

a) *Euler's problem: the buckling of a slender column.* The rod is used as a column which supports a compressive load Mg directed along the z axis (which is vertical). Show that when the rod buckles slighly (*i.e.* deforms with both ends remaining on the z axis) the total energy, including the gravitational potential energy of the loading mass M, can be approximated by

$$U[y] = \int_0^L \left\{ \frac{YI}{2}\left(y''\right)^2 - \frac{Mg}{2}\left(y'\right)^2 \right\} dz.$$

By considering small deformations of the form

$$y(z) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi z}{L}$$

show that the column is unstable to buckling and collapse if $Mg \geq \pi^2 YI/L^2$.

b) *Leonardo da Vinci's problem: the light cantilever.* Here we take the z axis as horizontal and the y axis as being vertical. The rod is used as a beam or cantilever and is fixed into a wall so that $y(0) = 0 = y'(0)$. A weight Mg is hung from the end $z = L$ and the beam sags in the $-y$ direction. We wish to find $y(z)$ for $0 < z < L$. We will ignore the weight of the beam itself.
   - Write down the complete expression for the energy, including the gravitational potential energy of the weight.

- Find the differential equation and boundary conditions at $z = 0, L$ that arise from minimizing the total energy. In doing this take care not to throw away any term arising from the integration by parts. You may find the following identity to be of use:

$$\frac{d}{dz}(f'g'' - fg''') = f''g'' - fg''''.$$

- Solve the equation. You should find that the displacement of the end of the beam is $y(L) = -\frac{1}{3}MgL^3/YI$.

Exercise 1.5: Suppose that an elastic body Ω of density ρ is slightly deformed so that the point that was at cartesian co-ordinate x_i is moved to $x_i + \eta_i(x)$. We define the resulting *strain tensor* e_{ij} by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial \eta_j}{\partial x_i} + \frac{\partial \eta_i}{\partial x_j}\right).$$

It is automatically symmetric in its indices. The Lagrangian for small-amplitude elastic motion of the body is

$$L[\boldsymbol{\eta}] = \int_\Omega \left\{\frac{1}{2}\rho\dot{\eta}_i^2 - \frac{1}{2}e_{ij}c_{ijkl}e_{kl}\right\} d^3x.$$

Here, c_{ijkl} is the tensor of *elastic constants*, which has the symmetries

$$c_{ijkl} = c_{klij} = c_{jikl} = c_{ijlk}.$$

By varying the η_i, show that the equation of motion for the body is

$$\rho\frac{\partial^2 \eta_i}{\partial t^2} - \frac{\partial}{\partial x_j}\sigma_{ji} = 0,$$

where

$$\sigma_{ij} = c_{ijkl}e_{kl}$$

is the *stress tensor*. Show that variations of η_i on the boundary $\partial\Omega$ give as boundary conditions

$$\sigma_{ij}n_j = 0,$$

where n_i are the components of the outward normal on $\partial\Omega$.

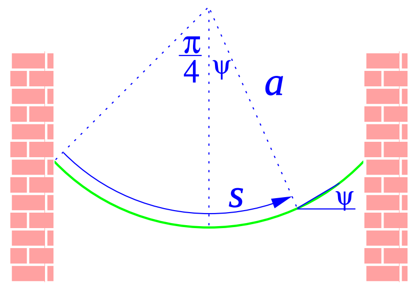

Figure 1.15: *Weighted line.*

Problem 1.6:The catenary revisited. We can describe a catenary curve in parametric form as $x(s)$, $y(s)$, where s is the arc-length. The potential energy is then simply $\int_0^L \rho g y(s) ds$ where ρ is the mass per unit length of the hanging chain. The x, y are not independent functions of s, however, because $\dot{x}^2 + \dot{y}^2 = 1$ at every point on the curve. Here a dot denotes a derivative with respect to s.

a) Introduce infinitely many Lagrange multipliers $\lambda(s)$ to enforce the $\dot{x}^2 + \dot{y}^2$ constraint, one for each point s on the curve. From the resulting functional derive two coupled equations describing the catenary, one for $x(s)$ and one for $y(s)$. By thinking about the forces acting on a small section of the cable, and perhaps by introducing the angle ψ where $\dot{x} = \cos\psi$ and $\dot{y} = \sin\psi$, so that s and ψ are *intrinsic coordinates* for the curve, interpret these equations and show that $\lambda(s)$ is proportional to the position-dependent tension $T(s)$ in the chain.
b) You are provided with a light-weight line of length $\pi a/2$ and some lead shot of total mass M. By using equations from the previous part (suitably modified to take into account the position dependent $\rho(s)$) or otherwise, determine how the lead should be distributed along the line if the loaded line is to hang in an arc of a circle of radius a (see figure 1.15) when its ends are attached to two points at the same height.

Problem 1.7: Another model for Lobachevski geometry (see exercise 1.3) is the *Poincaré disc*. This space consists of the interior of the unit disc $D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ equipped with Riemann metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2}.$$

The geodesic paths are found by minimizing the arc-length functional

$$s[\mathbf{r}] \equiv \int ds = \int \left\{ \frac{1}{1-x^2-y^2} \sqrt{\dot{x}^2+\dot{y}^2} \right\} dt,$$

where $\mathbf{r}(t) = (x(t), y(t))$ and a dot indicates a derivative with respect to the parameter t.

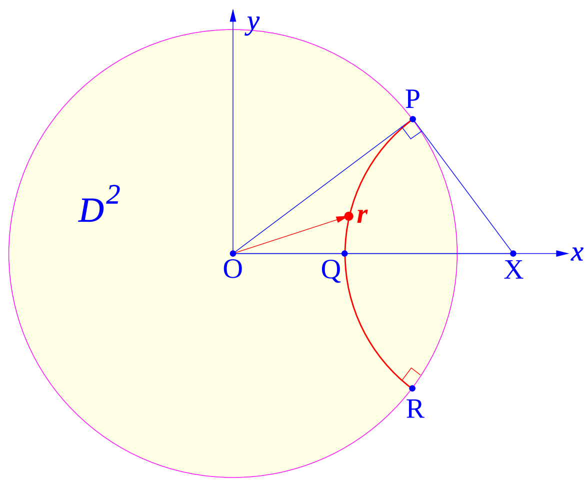

Figure 1.16: *The Poincaré disc of exercise 1.7. The radius* OP *of the Poincare disc is unity, while the radius of the geodesic arc* PQR *is* PX = QX = RX = *R. The distance between the centres of the disc and arc is* OX = x_0. *Your task in part c) is to show that* ∠OPX = ∠ORX = 90°.

a) Either by manipulating the two Euler-Lagrange equations that give the conditions for $s[\mathbf{r}]$ to be stationary under variations in $\mathbf{r}(t)$, or, more efficiently, by observing that $s[\mathbf{r}]$ is invariant under the infinitesimal rotation

$$\begin{aligned} \delta x &= \varepsilon y \\ \delta y &= -\varepsilon x \end{aligned}$$

and applying Noether's theorem, show that the parameterised geodesics obey

$$\frac{d}{dt}\left(\frac{1}{1-x^2-y^2} \frac{x\dot{y}-y\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0.$$

b) Given a point (a, b) within D^2, and a direction through it, show that the equation you derived in part a) determines a unique geodesic curve passing through (a, b) in the given direction, but does not determine the parametrization of the curve.

c) Show that there exists a solution to the equation in part a) in the form

$$\begin{aligned} x(t) &= R\cos t + x_0 \\ y(t) &= R\sin t. \end{aligned}$$

Find a relation between x_0 and R, and from it deduce that the geodesics are circular arcs that cut the bounding unit circle (which plays the role of the line at infinity in the Lobachevski plane) at right angles.

Exercise 1.8: The Lagrangian for a particle of charge q is

$$L[\mathbf{x}, \dot{\mathbf{x}}] = \frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi(\mathbf{x}) + q\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}).$$

Show that Lagrange's equation leads to

$$m\ddot{\mathbf{x}} = q(\mathbf{E} + \dot{\mathbf{x}}\times\mathbf{B}),$$

where

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \operatorname{curl}\mathbf{A}.$$

Exercise 1.9: Consider the action functional

$$S[\boldsymbol{\omega}, \mathbf{p}, \mathbf{r}] = \int \left(\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 + \mathbf{p}\cdot(\dot{\mathbf{r}} + \boldsymbol{\omega}\times\mathbf{r})\right\} dt,$$

where $\mathbf{r}$ and $\mathbf{p}$ are time-dependent three-vectors, as is $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, Apply the action principle to obtain the equations of motion for $\mathbf{r}, \mathbf{p}, \boldsymbol{\omega}$ and show that they lead to Euler's equations

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= 0, \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= 0, \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= 0. \end{aligned}$$

governing the angular velocity of a freely-rotating rigid body.

Problem 1.10: *Piano String*. A elastic piano string can vibrate both transversely and longitudinally, and the two vibrations influence one another. A Lagrangian that takes into account the lowest-order effect of stretching on the local string tension, and can therefore model this coupled motion, is

$$L[\xi, \eta] = \int dx \left\{\frac{1}{2}\rho_0\left[\left(\frac{\partial\xi}{\partial t}\right)^2 + \left(\frac{\partial\eta}{\partial t}\right)^2\right] - \frac{\lambda}{2}\left[\frac{\tau_0}{\lambda} + \frac{\partial\xi}{\partial x} + \frac{1}{2}\left(\frac{\partial\eta}{\partial x}\right)^2\right]^2\right\}.$$

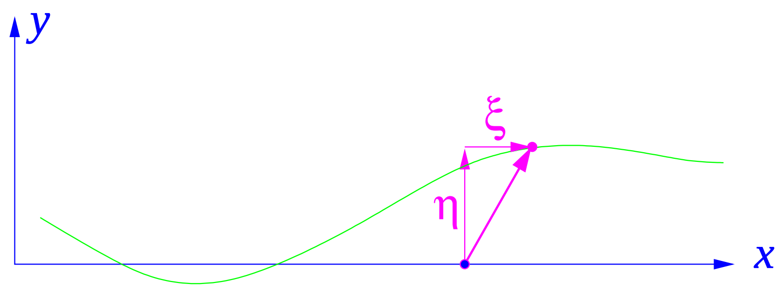

Figure 1.17: *Vibrating piano string.*

Here $\xi(x,t)$ is the longitudinal displacement and $\eta(x,t)$ the transverse displacement of the string. Thus, the point that in the undisturbed string had co-ordinates $[x,0]$ is moved to the point with co-ordinates $[x+\xi(x,t),\eta(x,t)]$. The parameter τ_0 represents the tension in the undisturbed string, λ is the product of Young's modulus and the cross-sectional area of the string, and ρ_0 is the mass per unit length.

a) Use the action principle to derive the two coupled equations of motion, one involving $\dfrac{\partial^2\xi}{\partial t^2}$ and one involving $\dfrac{\partial^2\eta}{\partial t^2}$.
b) Show that when we linearize these two equations of motion, the longitudinal and transverse motions decouple. Find expressions for the longitudinal (c_L) and transverse (c_T) wave velocities in terms of τ_0, ρ_0 and λ.
c) Assume that a given transverse pulse $\eta(x,t) = \eta_0(x-c_Tt)$ propagates along the string. Show that this induces a concurrent longitudinal pulse of the form $\xi(x-c_Tt)$. Show further that the longitudinal Newtonian momentum density in this concurrent pulse is given by

$$\rho_o\frac{\partial\xi}{\partial t} = \frac{1}{2}\frac{c_L^2}{c_L^2-c_T^2}T^0{}_1$$

where

$$T^0{}_1 \equiv -\rho_0\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial t}$$

is the associated pseudo-momentum density.

The forces that created the transverse pulse will also have created other longitudinal waves that travel at c_L. Consequently the Newtonian x-momentum moving at c_T is not the only x-momentum on the string, and the total "true" longitudinal momentum density is not simply proportional to the pseudo-momentum density.

Exercise 1.11: Obtain the canonical energy-momentum tensor ${T^\nu}_\mu$ for the barotropic fluid described by (1.119). Show that its conservation leads to both the momentum conservation equation (1.128), and to the energy conservation equation

$$\partial_t \mathcal{E} + \partial_i \{v_i(\mathcal{E} + P)\},$$

where the energy density is

$$\mathcal{E} = \frac{1}{2}\rho(\nabla\phi)^2 + u(\rho).$$

Interpret the energy flux as being the sum of the convective transport of energy together with the rate of working by an element of fluid on its neighbours.

Problem 1.12: Consider the action functional[3]

$$S[\mathbf{v}, \rho, \phi, \beta, \gamma] = \int d^4x \left\{ -\frac{1}{2}\rho\mathbf{v}^2 - \phi\left(\frac{\partial\rho}{\partial t} + \mathrm{div}\,(\rho\mathbf{v})\right) + \rho\beta\left(\frac{\partial\gamma}{\partial t} + (\mathbf{v}\cdot\nabla)\gamma\right) + u(\rho)\right\},$$

which is a generalization of (1.177) to include two new scalar fields β and γ. Show that varying $\mathbf{v}$ leads to

$$\mathbf{v} = \nabla\phi + \beta\nabla\gamma.$$

This is the *Clebsch representation* of the velocity field. It allows for flows with non-zero vorticity

$$\boldsymbol{\omega} \equiv \mathrm{curl}\,\mathbf{v} = \nabla\beta \times \nabla\gamma.$$

Show that the equations that arise from varying the remaining fields ρ, ϕ, β, γ, together imply the mass conservation equation

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\,(\rho\mathbf{v}) = 0,$$

and Bernoulli's equation in the form

$$\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\omega}\times\mathbf{v} = -\nabla\left(\frac{1}{2}\mathbf{v}^2 + h\right).$$

(Recall that $h = du/d\rho$.) Show that this form of Bernoulli's equation is equivalent to Euler's equation

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla h.$$

Consequently S provides an action principle for a general inviscid barotropic flow.

[3]H. Bateman, *Proc. Roy. Soc. Lond.* A **125** (1929) 598-618; C. C. Lin, *Liquid Helium* in *Proc. Int. Sch. Phys. "Enrico Fermi", Course XXI* (Academic Press 1965).

Exercise 1.13: *Drums and Membranes.* The shape of a distorted drumskin is described by the function $h(x,y)$, which gives the height to which the point (x,y) of the flat undistorted drumskin is displaced.

a) Show that the area of the distorted drumskin is equal to

$$\text{Area}[h] = \int dx\, dy \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2},$$

where the integral is taken over the area of the flat drumskin.

b) Show that for small distortions, the area reduces to

$$\mathcal{A}[h] = \text{const.} + \frac{1}{2}\int dx\, dy\, |\nabla h|^2.$$

c) Show that if h satisfies the two-dimensional Laplace equation then $\mathcal{A}$ is stationary with respect to variations that vanish at the boundary.

d) Suppose the drumskin has mass ρ_0 per unit area, and surface tension T. Write down the Lagrangian controlling the motion of the drumskin, and derive the equation of motion that follows from it.

Problem 1.14: *The Wulff construction.* The surface-area functional of the previous exercise can be generalized so as to find the equilibrium shape of a crystal. We describe the crystal surface by giving its height $z(x,y)$ above the x-y plane, and introduce the direction-dependent surface tension (the surface free-energy per unit area) $\alpha(p,q)$, where

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}. \quad (\star)$$

We seek to minimize the total surface free energy

$$F[z] = \int dxdy \left\{\alpha(p,q)\sqrt{1+p^2+q^2}\right\},$$

subject to the constraint that the volume of the crystal

$$V[z] = \int z\, dxdy$$

remains constant.

a) Enforce the volume constraint by introducing a Lagrange multiplier $2\lambda^{-1}$, and so obtain the Euler-Lagrange equation

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial p}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial q}\right) = 2\lambda^{-1}.$$

Here

$$f(p, q) = \alpha(p.q)\sqrt{1 + p^2 + q^2}.$$

b) Show in the isotropic case, where α is constant, that

$$z(x, y) = \sqrt{(\alpha\lambda)^2 - (x - a)^2 - (y - b)^2} + \text{const.}$$

is a solution of the Euler-Lagrange equation. In this case, therefore, the equilibrium shape is a sphere.

An obvious way to satisfy the Euler-Lagrange equation in the general anisotropic case would be to arrange things so that

$$x = \lambda\frac{\partial f}{\partial p}, \quad y = \lambda\frac{\partial f}{\partial q}. \quad (\star\star)$$

c) Show that $(\star\star)$ is exactly the relationship we would have if $z(x, y)$ and $\lambda f(p, q)$ were Legendre transforms of each other—*i.e.* if

$$\lambda f(p, q) = px + qy - z(x, y),$$

where the x and y on the right-hand side are functions of p q obtained by solving $(\star)$. Do this by showing that the inverse relation is

$$z(x, y) = px + qy - \lambda f(p, q)$$

where now the p, q on the right-hand side become functions of x and y, and are obtained by solving $(\star\star)$.

For real crystals, $\alpha(p, q)$ can have the property of a being a continuous-but-nowhere-differentiable function, and so the differential calculus used in deriving the Euler-Lagrange equation is inapplicable. The Legendre transformation, however, has a geometric interpretation that is more robust than its calculus-based derivation.

Recall that if we have a two-parameter family of surfaces in $\mathbb{R}^3$ given by $F(x, y, z; p, q) = 0$, then the equation of the *envelope* of the surfaces is found by solving the equations

$$0 = F = \frac{\partial F}{\partial p} = \frac{\partial F}{\partial q}$$

so as to eliminate the parameters p, q.

d) Show that the equation

$$F(x, y, z; p, q) \equiv px + qy - z - \lambda\alpha(p, q)\sqrt{1 + p^2 + q^2} = 0$$

describes a family of planes perpendicular to the unit vectors

$$\mathbf{n} = \frac{(p, q, -1)}{\sqrt{1 + p^2 + q^2}}$$

and at a distance $\lambda\alpha(p, q)$ away from the origin.

e) Show that the equations to be solved for the envelope of this family of planes are exactly those that determine $z(x, y)$. Deduce that, for smooth $\alpha(p, q)$, the profile $z(x, y)$ is this envelope.

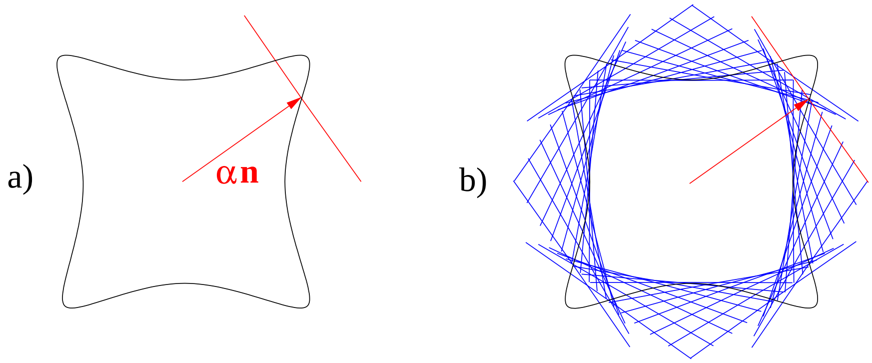

Figure 1.18: *Two-dimensional Wulff crystal. a) Polar plot of surface tension α as a function of the normal $\mathbf{n}$ to a crystal face, together with a line perpendicular to $\mathbf{n}$ at distance α from the origin. b) Wulff's construction of the corresponding crystal surface as the envelope of the family of perpendicular lines. In this case, the minimum-energy crystal has curved faces, but sharp corners. The envelope continues beyond the corners, but these parts are unphysical.*

Wulff conjectured[4] that, even for non-smooth $\alpha(p, q)$, the minimum-energy shape is given by an equivalent geometric construction: erect the planes from part d) and, for each plane, discard the half-space of $\mathbb{R}^3$ that lies on the far side of the plane from the origin. The convex region consisting of the intersection of the retained half-spaces is the crystal. When $\alpha(p, q)$ is smooth this "Wulff

[4] G. Wulff, "Zur frage der geschwindigkeit des wachsturms under auflosung der kristallflachen," *Zeitschrift für Kristallografie,* **34** (1901) 449-530.

body" is bounded by part of the envelope of the planes. (The parts of the envelope not bounding the convex body—the "swallowtails" visible in figure 1.18—are unphysical.) When $\alpha(p.q)$ has cusps, these singularities can give rise to flat facets which are often joined by rounded edges. A proof of Wulff's claim had to wait until 43 years until 1944, when it was established by use of the Brunn-Minkowski inequality.[5]

[5] A. Dinghas, "Uber einen geometrischen Satz von Wulff für die Gleichgewichtsform von Kristallen, *Zeitshrift für Kristallografie,* **105** (1944) 304-314. For a readable modern account see: R. Gardner, "The Brunn-Minkowski inequality," *Bulletin Amer. Math. Soc.* **39** (2002) 355-405.