4.10 Solving for the pressure we find

$$P = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2} \ . \tag{4-A2}$$

a) Differentiating we get

$$K_T = -V \left(\frac{\partial p}{\partial V}\right)_T = \frac{nRTV}{(V - nb)^2} - \frac{2an^2}{V^2} \tag{4-A3}$$

b) It can become negative for

$$nRT < \frac{2an^2 (V - nb)^2}{V^3} \tag{4-A4}$$

When $K < 0$ the gas must condense.

4.11 a) The differential of a function is

$$dQ = \frac{\partial Q}{\partial T}\, dT + \frac{\partial Q}{\partial V}\, dV \ . \tag{4-A5}$$

so that $A = \partial Q/\partial T$ and $B = \partial Q/\partial B$. Then $\partial A/\partial V = \partial B/\partial T = \partial^2 Q/\partial V \partial T$.

b) We have $A = C_V$ and $B = nRT/V$, and thus $\partial A/\partial V = 0$ and $\partial B/\partial T = nR/V \neq 0$.

5 Buoyancy

5.1 Under a shift of the origin of the coordinate system $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{a}$ the moment of force transforms to

$$\boldsymbol{\mathcal{M}} = \int_V \boldsymbol{x} \times \boldsymbol{f}\, dv \to \int_V (\boldsymbol{x} + \boldsymbol{a}) \times \boldsymbol{f}\, dv = \boldsymbol{\mathcal{M}} + \boldsymbol{a} \times \boldsymbol{\mathcal{F}}$$

If $\boldsymbol{\mathcal{F}} = \boldsymbol{0}$, the moment of force is unchanged.

5.2 0.04 m^3. 2500 $\mathsf{kg/m}^3$.

5.3 Use $M = \rho_0 V_0 = \rho_1 V_1$ and $\pi a^2 h = V_1 - V_0 = M(1/\rho_1 - 1/\rho_0)$ to get $h = 20$ mm.

5.4 a) Displacement $M_1 + M_2 = \rho_0((1 - f)V_1 + V_2)$ with $M_1 = \rho_1 V_1$ and $M_2 = \rho_2 V_2$. Then

$$\frac{M_1}{M_2} = \frac{1 - \dfrac{\rho_0}{\rho_2}}{(1 - f)\dfrac{\rho_0}{\rho_1} - 1} = 2.36$$

b) $f \leq 1 - \rho_1/\rho_0 = 0.35$.

5.5

a) $d_0 = \dfrac{M}{2aL\rho_0}$.

b) $d = h + d_0$.

c) $z_{G_0} = b - \dfrac{h(2b-h)}{2d}$.

d) The movement of the real water inside the hull shifts the center of mass of the ferry horizontally by exactly the same amount as the center of buoyancy was shifted by the displaced water (provided $h \gg \alpha a$ and provided the water is free to move and can adjust itself fast enough). This means that stability condition reverts to that of absolute stability $z_{G_0} < z_{B_0}$. Stability can only be obtained if there is so much water on the car deck that the center of gravity becomes lower than the center of buoyancy. This happens for $h > b - d_0/2$, *i.e.* for sufficiently much water on the deck (at this point $d > b + d_0/2$ so the ferry is well into sinking when it finally becomes stable). The conclusion is that any small, but not too small, amount of water inside the hull tends to destabilize the ferry.

5.6 Using Gauss theorem, the moment of buoyancy is

$$(\boldsymbol{\mathcal{M}}_B)_i = -\int_S \sum_{jk} \epsilon_{ijk} x_j p \, dS_k = -\int_V \sum_{jk} \epsilon_{ijk} \nabla_k (x_j p) \, dV$$
$$= -\int_V \sum_{jk} \epsilon_{ijk} x_j \nabla_k p \, dV = -\int_V (\boldsymbol{x} \times \boldsymbol{\nabla} p)_i \; dV$$

Using local hydrostatic equilibrium (4-18) we get,

$$\boldsymbol{\mathcal{M}}_B = -\int_V \boldsymbol{x} \times \rho_{\text{fluid}} \boldsymbol{g} \; dV \tag{5-A1}$$

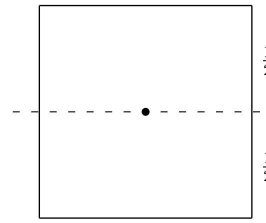
5.10 Under a rotation of the coordinate system by ϕ, the coordinates transform according to (**??**). The components of the area moment tensor transform as

$$I_{x'x'} = I_{xx} \cos^2\phi + I_{yy} \sin^2\phi + 2I_{xy} \sin\phi \cos\phi \tag{5-A2}$$
$$I_{y'y'} = I_{xx} \sin^2\phi + I_{yy} \cos^2\phi - 2I_{xy} \sin\phi \cos\phi \tag{5-A3}$$
$$I_{x'y'} = (I_{yy} - I_{xx}) \sin\phi \cos\phi + I_{xy}(\cos^2\phi - \sin^2\phi) \tag{5-A4}$$

If the area has a discrete symmetry such there will be at least one angle $\phi \neq 0, \pi$ such that $I_{x'x'} = I_{xx}$, $I_{y'y'} = I_{yy}$ and $I_{x'y'} = I_{xy}$. For this angle we thus have

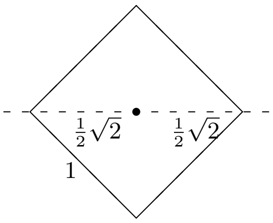
$$(I_{yy} - I_{xx}) \sin^2\phi + 2I_{xy} \sin\phi \cos\phi = 0 \tag{5-A5}$$
$$(I_{xx} - I_{yy}) \sin^2\phi - 2I_{xy} \sin\phi \cos\phi = 0 \tag{5-A6}$$
$$(I_{yy} - I_{xx}) \sin\phi \cos\phi - 2I_{xy} \sin^2\phi = 0 \tag{5-A7}$$

The two first equations are the same, and the last two can only be satisfied if $I_{xx} = I_{yy}$ and $I_{xy} = 0$ when $\phi \neq 0, \pi$.



Here the cube floats with faces horizontal and vertical. This configura is unstable. The x-axis into the paper.

5.11 The center of gravity is at the center of the cube $z_G = 0$ and the submerged volume is $V = \frac{1}{2}$ in all orientations. For symmetry reasons the center of buoyancy is vertically below the center of gravity in all three cases:



Here the cube floats with one edge horizontal and two edges in the waterline. This configuration is marginally stable (or unstable). The x-axis goes into the paper.

a) The block is floating with two faces horizontal and the other faces vertical. Here we use (5-32) with $a = b = c = 2d = 1$ and find the metacentric height $z_M = -\frac{1}{12}$ which is is below the center of gravity. This configuration is manifestly unstable.

b) The block is floating with one horizontal edge below the water, one above the water, and two in the waterline. In this configuration the waterline area is a rectangle with sides 1 and $\sqrt{2}$. Taking the x-axis along the horizontal edges, we find the moments

$$I_{xx} = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{-\frac{1}{2}\sqrt{2}}^{\frac{1}{2}\sqrt{2}} dy\, y^2 = \frac{1}{6}\sqrt{2} \tag{5-A8}$$

$$I_{yy} = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{-\frac{1}{2}\sqrt{2}}^{\frac{1}{2}\sqrt{2}} dy\, x^2 = \frac{1}{12}\sqrt{2} \tag{5-A9}$$

$$I_{xy} = 0 \tag{5-A10}$$

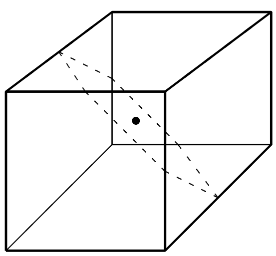
The center of buoyancy is given by (5-29) with $A(z) = \sqrt{2} + 2z$ at depth z,

$$z_B = 2\int_{-\frac{1}{2}\sqrt{2}}^{0} z(\sqrt{2} + 2z)\, dz = -\frac{1}{6}\sqrt{2} \tag{5-A11}$$

Thus the metacenter (for the smallest moment) is at

$$z_M = z_B + \frac{I_{yy}}{V} = 0 \tag{5-A12}$$

This floating configuration is hus marginally stable for rotations around the y-axis which lower a corner and raise another. That brings us to the last configuration.

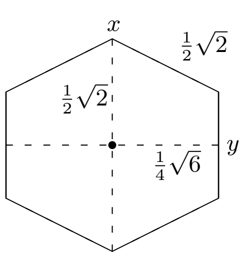


Here the cube floats with the lower left corner vertically below the center of gravity (gravity points downwards to the left in this picture). The waterline area is a regular hexagon (dashed).

c) The block is floating with one corner vertically below the center of the cube. In this case the waterline area is a hexagon with sides of length $\frac{1}{2}\sqrt{2}$. Because of the symmetry we may calculate the area moment around any axis we choose, for example one that connects two opposite corners of the hexagon, with length $\sqrt{2}$. Integrating over the first quadrant we have we have from the geometry of the hexagon

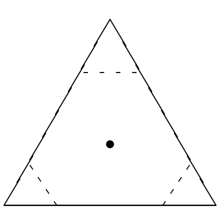
$$I = 4\int_0^{\frac{1}{4}\sqrt{6}} dy \int_0^{\frac{1}{2}\sqrt{2} - y\frac{1}{3}\sqrt{3}} dx\, y^2 \tag{5-A13}$$

$$= 4\int_0^{\frac{1}{4}\sqrt{6}} y^2 \left(\frac{1}{2}\sqrt{2} - y\frac{1}{3}\sqrt{3}\right) dy = \frac{5\sqrt{3}}{64} \tag{5-A14}$$



Hexagonal waterline area.

The center of buoyancy is of the same form as before given by (5-29), but in this case it is a bit harder to determine the area $A(z)$ at depth z because the shape of the area changes from a triangle to a hexagon at $z = -\frac{1}{6}\sqrt{3}$. For $-\frac{1}{2}\sqrt{3} < z < -\frac{1}{6}\sqrt{3}$ the shape is an equilateral triangle. Since its side length must vary linearly with z from $s = 0$ at $z = -\frac{1}{2}\sqrt{3}$ to $s = \sqrt{2}$ for $z = -\frac{1}{6}\sqrt{3}$, we have $s = \frac{3}{2}\sqrt{2} + z\sqrt{6}$ and area $A(z) = \frac{1}{4}\sqrt{3}\ s^2$. For $-\frac{1}{6}\sqrt{3} < z < 0$ the (irregular) hexagon can be obtained from this triangle by removing smaller equilateral triangles from



The general hexagon is an equilateral triangle with