Problem 3.2.6 (Eliminating the cubic term)

Consider the system

$$\dot{X} = RX - X^2 + aX^3 + O(X^4),$$

where $R \neq 0$. We want to find a new variable x such that the system transforms into

$$\dot{x} = Rx - x^2 + O(x^4).$$

This would be a big improvement, since the cubic term has been eliminated and the error term has been bumped up to fourth order.

Let $x = X + bX^3 + O(X^4)$, where b will be chosen later to eliminate the cubic term in the differential equation for x. This is called a *near-identity transformation*, since x and X are practically equal; they differ by a tiny cubic term. (We have skipped the quadratic term X^2, because it is not needed–you should check this later.) Now we need to rewrite the system in terms of x; this calculation requires a few steps.

(a)

Show that the near-identity transformation can be inverted to yield $X = x + cx^3 + O(x^4)$, and solve for c.

(b)

Write $\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)$, and substitute for X and $\dot{X}$ on the right-hand side, so that everything depends only on x. Multiply the resulting series expansions and collect terms, to obtain $\dot{x} = Rx - x^2 + kx^3 + O(x^4)$, where k depends on a, b, and R.

(c)

Now the moment of triumph: choose b so that $k = 0$.

(d)

Is it really necessary to make the assumption $R \neq 0$? Explain.