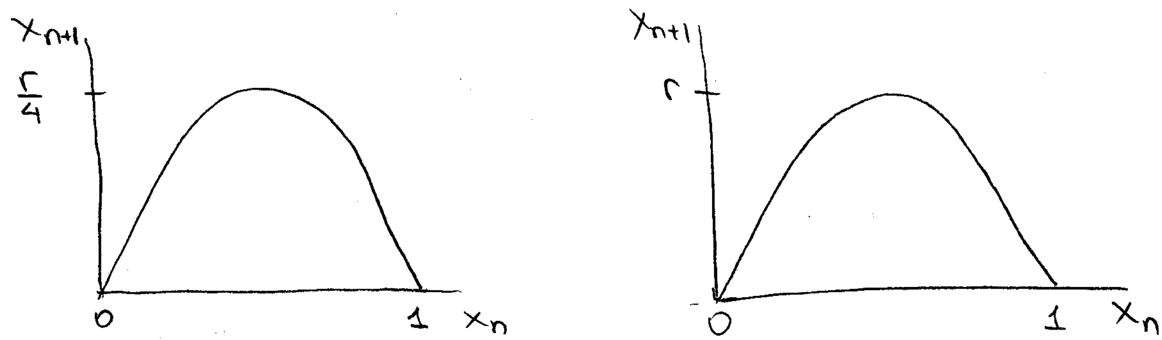
Lecture 21

Universality

Compare two maps:

Logistic map: $x_{n+1} = r x_n (1 - x_n)$

Sine map: $x_{n+1} = r \sin \pi x_n$



Both are unimodal:
- smooth
- convex
- have one maximum

Surprise: qualitative dynamics are IDENTICAL!

→ figure: orbit diagrams

Same sequence of bifurcations, called:

U-sequence

Theorem (Metropolis 1973):
All unimodal maps $(f(0) = f(1) = 0)$ have a universal bifurcation sequence: 1, 2, 2×2, 6, 5, 3, 3×2, 5, 6, 4, 6, 5, 6
(up to period-6)

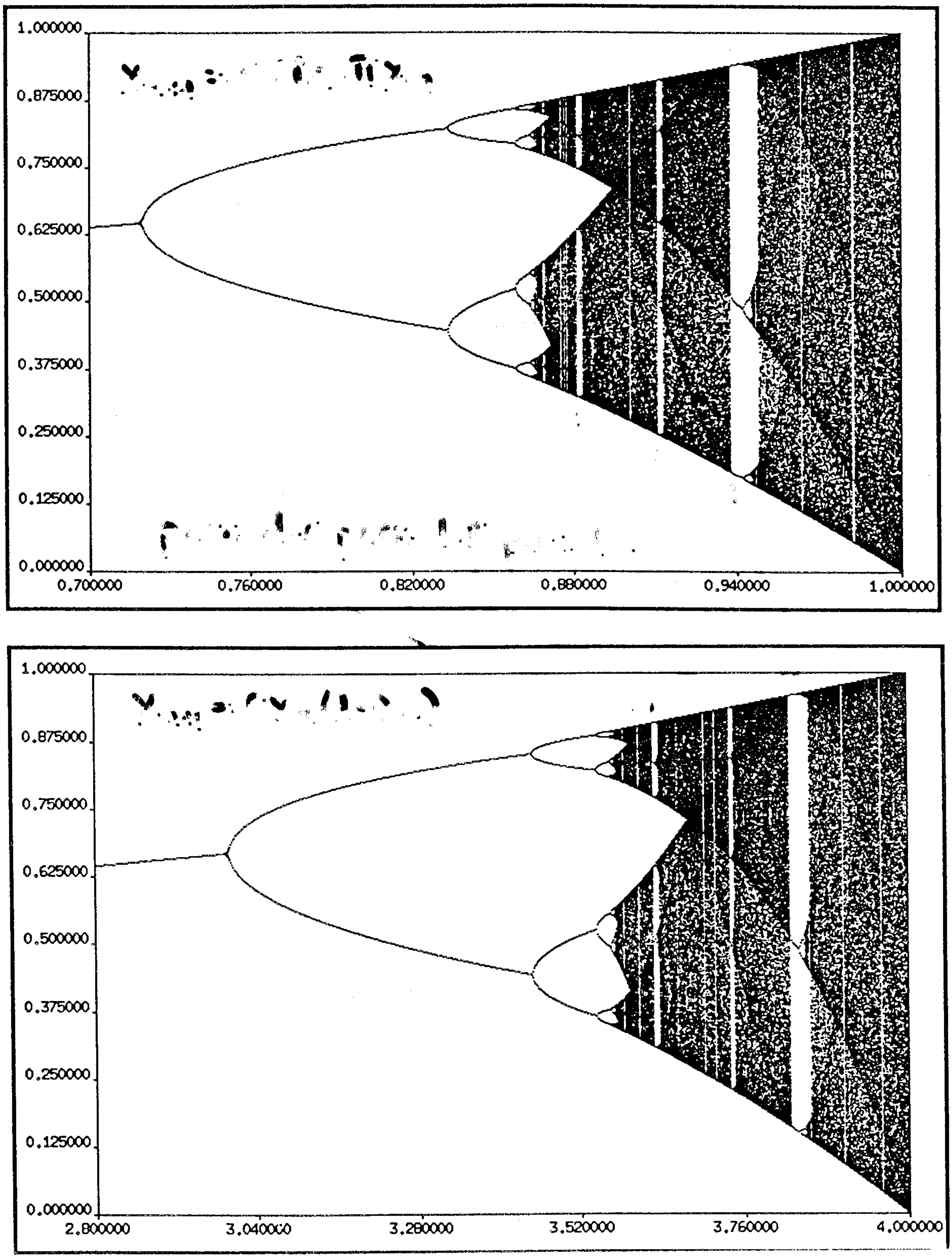


Figure 10.6.2 Courtesy of Andy Christian

The beginning of this sequence is familiar: periods 1, 2, and 2×2 are the first stages in the period-doubling scenario. (The later period-doublings give periods greater than 6, so they are omitted here.) Next, periods 6, 5, 3 correspond to the large windows mentioned in the discussion of Figure 10.6.2. Period 2×3 is the

More important:

same sequence in BZ reaction (Simoyi et al. 1982)

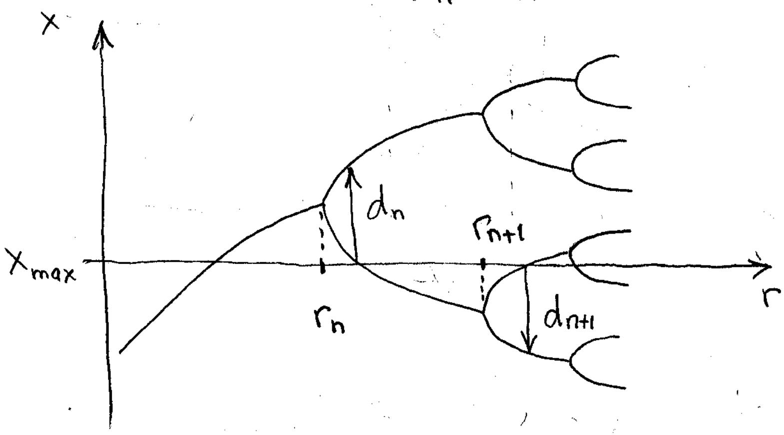
— continuous-time experimental system

⇒ Important result that transcends discrete systems!

Quantitative Universality

- r_n – n-th period doubling (arbitrary unimodal map)

$$\frac{r_n - r_{n-1}}{r_{n+1} - r_n} \to \delta = 4.669 \ , \ n \to \infty \ \text{(Feigenbaum, 1975)}$$

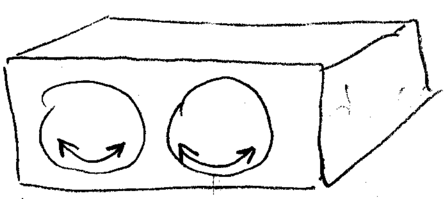


Another universal constant:

$$\frac{d_n}{d_{n+1}} \to \alpha = -2.503 \ , \ n \to \infty$$

Experimental Results

- RB convection in mercury (Libchaber 1982)



→ figure: convection

Other experiments: $\delta = 4.3 - 4.7$ (from ≈ 4 period doubl.)
(water, mercury) convection, (diode, transistor, Josephson junction) circuits. (1981–1982)

…nce of period doublings as the Rayleigh number is increased.

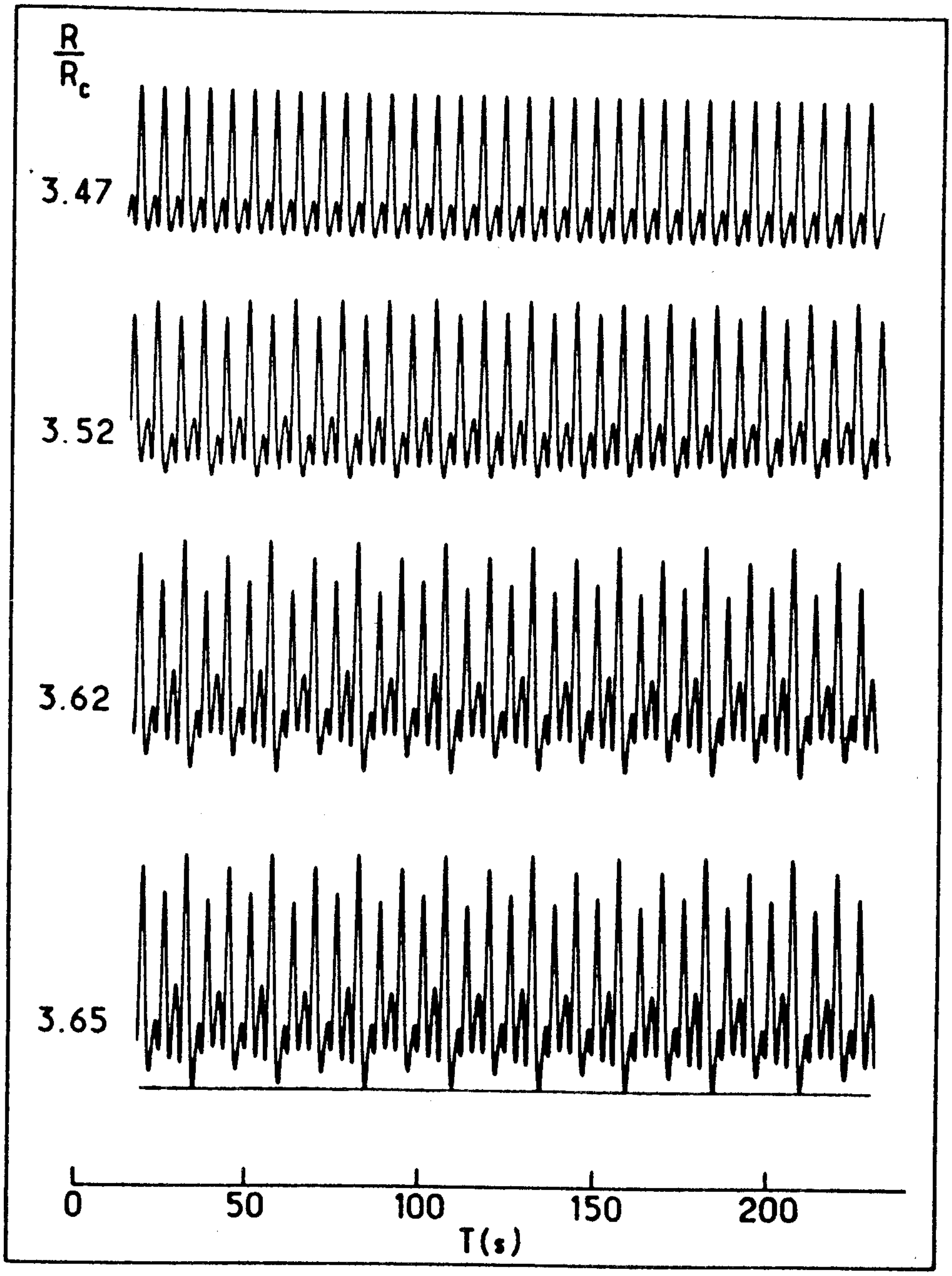


Figure 10.6.5 Libchaber et al. (1982), p. 213

…ries shows the temperature variations at one point in the fluid. For … the temperature varies periodically. This may be regarded as the basic

Question: How come 1-d maps predict physics of continuous-time systems so well?

Example: (Rossler System)

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x - c) \end{cases}$$

→ figure: orbits for $a = b = 0.2$

"Lorenz" map: successive maxima of $x(t)$

⇒ unimodal!

→ figure: map for $c = 5$

Orbit diagram: all maxima of $x(t)$ vs. c

⇒ U-sequence!

→ figure: orbit diagram

! If "Lorenz" map is nearly 1-d and unimodal ⇒ general theory applies

Consequences:
- attractor is nearly 2-d (flat)
- highly dissipative (2÷3 active DOF)

Doesn't work for strongly chaotic systems:

- turbulence
- fibrillating hearts

where a, b, and c are parameters. This system contains only one nonlinear term, zx, and is even simpler than the Lorenz system (Chapter 9), which has two nonlinearities.

Figure 10.6.6 shows two-dimensional projections of the system's attractor for different values of c (with $a = b = 0.2$ held fixed).

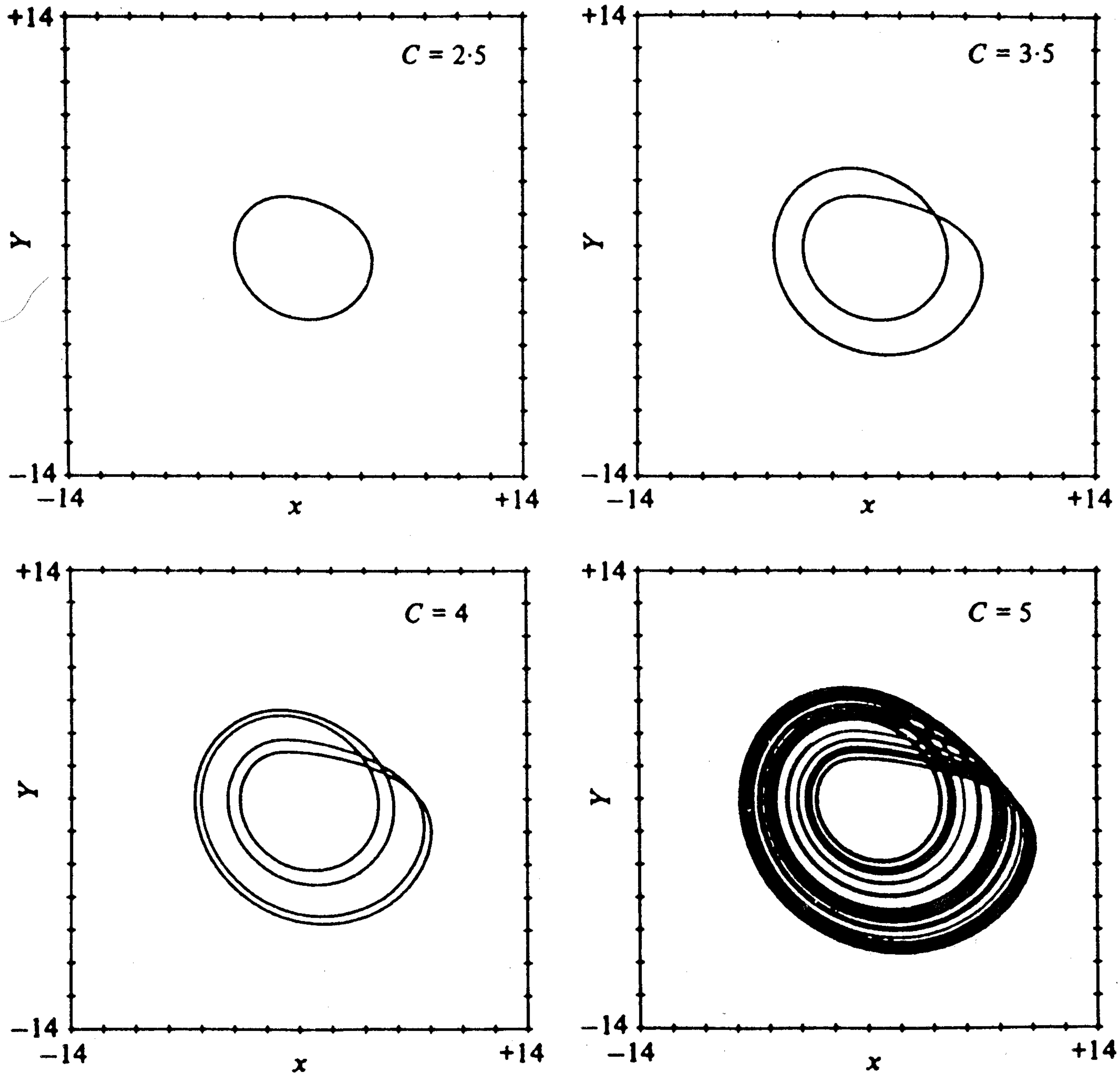


Figure 10.6.6 Olsen and Degn (1985), p. 185

At $c = 2.5$ the attractor is a simple limit cycle. As c is increased to 3.5, the limit cycle goes around twice before closing, and its period is approximately twice that of the original cycle. This is what period-doubling looks like in a continuous-time system! In fact, somewhere between $c = 2.5$ and 3.5, a ***period-doubling bifurcation of cycles*** must have occurred. (As Figure 10.6.6 suggests, such a bifurcation

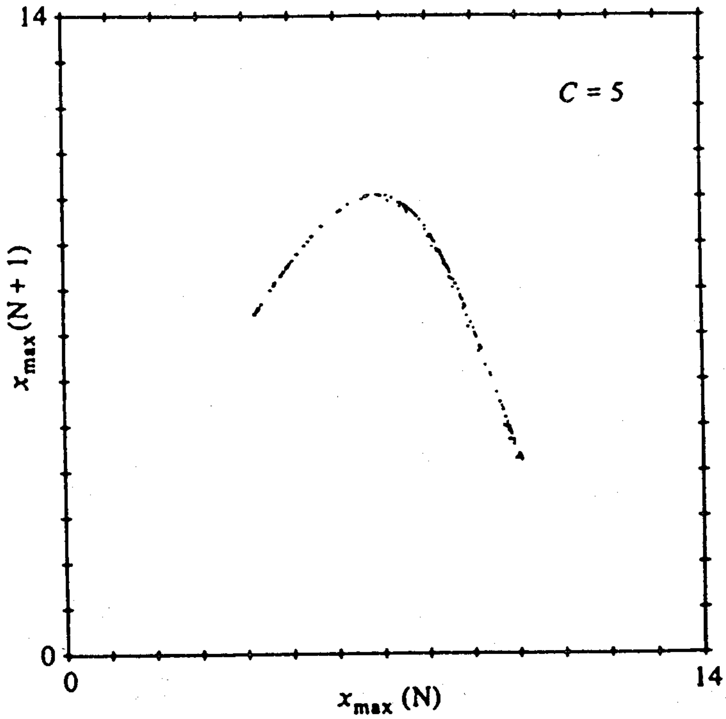


Figure 10.6.7 Olsen and Degn (1985), p. 186

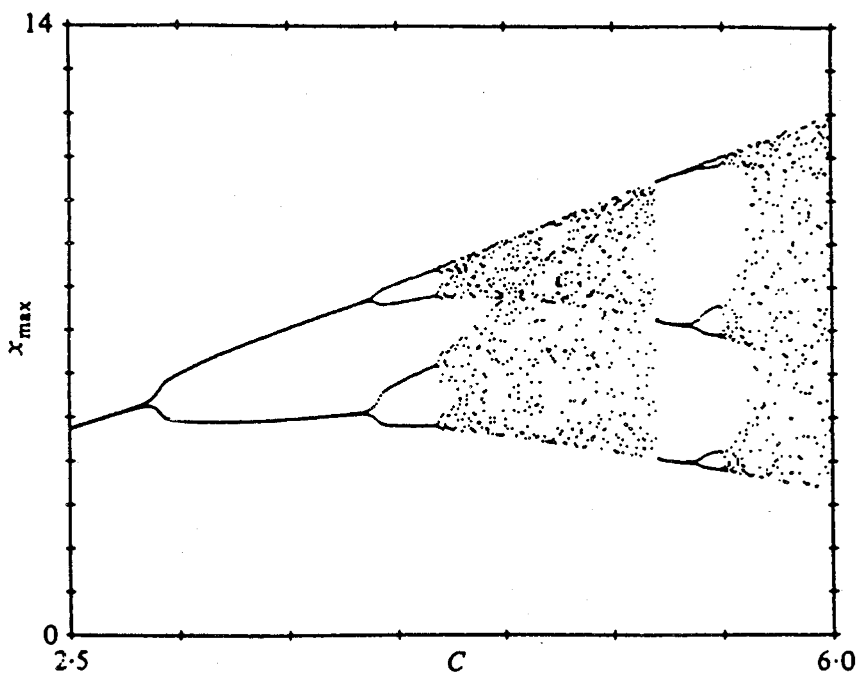


Figure 10.6.8 Olsen and Degn (1985), p. 186