stepsize $\Delta t = 0.1$. The solutions have the shape expected from Section 2.3. ■

Computers are indispensable for studying dynamical systems. We will use them liberally throughout this book, and you should do likewise.

EXERCISES FOR CHAPTER 2

2.1 A Geometric Way of Thinking

In the next three exercises, interpret $\dot{x} = \sin x$ as a flow on the line.

2.1.1 Find all the fixed points of the flow.

2.1.2 At which points x does the flow have greatest velocity to the right?

2.1.3

a) Find the flow's acceleration $\ddot{x}$ as a function of x.

b) Find the points where the flow has maximum positive acceleration.

2.1.4 (Exact solution of $\dot{x} = \sin x$) As shown in the text, $\dot{x} = \sin x$ has the solution $t = \ln\left|(\csc x_0 + \cot x_0)/(\csc x + \cot x)\right|$, where $x_0 = x(0)$ is the initial value of x.

a) Given the specific initial condition $x_0 = \pi/4$, show that the solution above can be inverted to obtain

$$x(t) = 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right).$$

Conclude that $x(t) \to \pi$ as $t \to \infty$, as claimed in Section 2.1. (You need to be good with trigonometric identities to solve this problem.)

b) Try to find the analytical solution for $x(t)$, given an *arbitrary* initial condition x_0.

2.1.5 (A mechanical analog)

a) Find a mechanical system that is approximately governed by $\dot{x} = \sin x$.

b) Using your physical intuition, explain why it now becomes obvious that $x^* = 0$ is an unstable fixed point and $x^* = \pi$ is stable.

2.2 Fixed Points and Stability

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions. Then try for a few minutes to obtain the analytical solution for $x(t)$; if you get stuck, don't try for too long since in several cases it's impossible to solve the equation in closed form!

2.2.1 $\dot{x} = 4x^2 - 16$

2.2.2 $\dot{x} = 1 - x^{14}$

2.2.3 $\dot{x} = x - x^3$

2.2.4 $\dot{x} = e^{-x} \sin x$

2.2.5 $\dot{x} = 1 + \frac{1}{2}\cos x$

2.2.6 $\dot{x} = 1 - 2\cos x$

2.2.7 $\dot{x} = e^x - \cos x$ (Hint: Sketch the graphs of e^x and $\cos x$ on the same axes, and look for intersections. You won't be able to find the fixed points explicitly, but you can still find the qualitative behavior.)

2.2.8 (Working backwards, from flows to equations) Given an equation $\dot{x} = f(x)$, we know how to sketch the corresponding flow on the real line. Here you are asked to solve the opposite problem: For the phase portrait shown in Figure 1, find an equation that is consistent with it. (There are an infinite number of correct answers—and wrong ones too.)

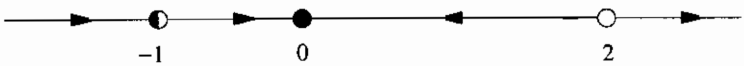

Figure 1

2.2.9 (Backwards again, now from solutions to equations) Find an equation $\dot{x} = f(x)$ whose solutions $x(t)$ are consistent with those shown in Figure 2.

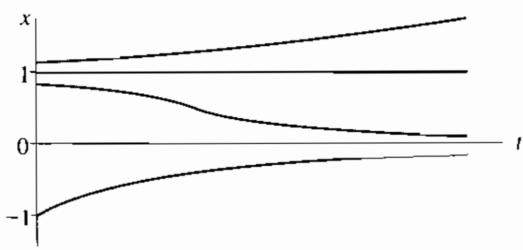

Figure 2

2.2.10 (Fixed points) For each of (a)–(e), find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not. (In all cases, assume that $f(x)$ is a smooth function.)

a) Every real number is a fixed point.

b) Every integer is a fixed point, and there are no others.

c) There are precisely three fixed points, and all of them are stable.

d) There are no fixed points.

e) There are precisely 100 fixed points.

2.2.11 (Analytical solution for charging capacitor) Obtain the analytical solution of the initial value problem $\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}$, with $Q(0) = 0$, which arose in Example 2.2.2.

2.2.12 (A nonlinear resistor) Suppose the resistor in Example 2.2.2 is replaced by a nonlinear resistor. In other words, this resistor does not have a linear

a) Separate variables and integrate, using partial fractions.

b) Make the change of variables $x = 1/N$. Then derive and solve the resulting differential equation for x.

2.3.2 (Autocatalysis) Consider the model chemical reaction

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X$$

in which one molecule of X combines with one molecule of A to form two molecules of X. This means that the chemical X stimulates its own production, a process called *autocatalysis*. This positive feedback process leads to a chain reaction, which eventually is limited by a "back reaction" in which $2X$ returns to $A + X$.

According to the *law of mass action* of chemical kinetics, the rate of an elementary reaction is proportional to the product of the concentrations of the reactants. We denote the concentrations by lowercase letters $x = [X]$ and $a = [A]$. Assume that there's an enormous surplus of chemical A, so that its concentration a can be regarded as constant. Then the equation for the kinetics of x is

$$\dot{x} = k_1 a x - k_{-1} x^2$$

where k_1 and k_{-1} are positive parameters called *rate constants*.

a) Find all the fixed points of this equation and classify their stability.

b) Sketch the graph of $x(t)$ for various initial values x_0.

2.3.3 (Tumor growth) The growth of cancerous tumors can be modeled by the Gompertz law $\dot{N} = -aN \ln(bN)$, where $N(t)$ is proportional to the number of cells in the tumor, and $a, b > 0$ are parameters.

a) Interpret a and b biologically.

b) Sketch the vector field and then graph $N(t)$ for various initial values.

The predictions of this simple model agree surprisingly well with data on tumor growth, as long as N is not too small; see Aroesty et al. (1973) and Newton (1980) for examples.

2.3.4 (The Allee effect) For certain species of organisms, the effective growth rate $\dot{N}/N$ is highest at intermediate N. This is the called the Allee effect (Edelstein–Keshet 1988). For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.

a) Show that $\dot{N}/N = r - a(N-b)^2$ provides an example of Allee effect, if r, a, and b satisfy certain constraints, to be determined.

b) Find all the fixed points of the system and classify their stability.

c) Sketch the solutions $N(t)$ for different initial conditions.

d) Compare the solutions $N(t)$ to those found for the logistic equation. What are the qualitative differences, if any?

2.4 Linear Stability Analysis

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, use a graphical argument to decide the stability.

2.4.1 $\dot{x} = x(1 - x)$

2.4.2 $\dot{x} = x(1 - x)(2 - x)$

2.4.3 $\dot{x} = \tan x$

2.4:4 $\dot{x} = x^2(6 - x)$

2.4.5 $\dot{x} = 1 - e^{-x^2}$

2.4.6 $\dot{x} = \ln x$

2.4.7 $\dot{x} = ax - x^3$, where a can be positive, negative, or zero. Discuss all three cases.

2.4.8 Using linear stability analysis, classify the fixed points of the Gompertz model of tumor growth $\dot{N} = -aN\ln(bN)$. (As in Exercise 2.3.3, $N(t)$ is proportional to the number of cells in the tumor and $a, b > 0$ are parameters.)

2.4.9 (Critical slowing down) In statistical mechanics, the phenomenon of "critical slowing down" is a signature of a second-order phase transition. At the transition, the system relaxes to equilibrium much more slowly than usual. Here's a mathematical version of the effect:

a) Obtain the analytical solution to $\dot{x} = -x^3$ for an arbitrary initial condition. Show that $x(t) \to 0$ as $t \to \infty$, but that the decay is not exponential. (You should find that the decay is a much slower algebraic function of t.)
b) To get some intuition about the slowness of the decay, make a numerically accurate plot of the solution for the initial condition $x_0 = 10$, for $0 \le t \le 10$. Then, on the same graph, plot the solution to $\dot{x} = -x$ for the same initial condition.

2.5 Existence and Uniqueness

2.5.1 (Reaching a fixed point in a finite time) A particle travels on the half-line $x \ge 0$ with a velocity given by $\dot{x} = -x^c$, where c is real and constant.

a) Find all values of c such that the origin $x = 0$ is a stable fixed point.
b) Now assume that c is chosen such that $x = 0$ is stable. Can the particle ever reach the origin in a *finite* time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of c?

2.5.2 ("Blow-up": Reaching infinity in a finite time) Show that the solution to $\dot{x} = 1 + x^{10}$ escapes to $+\infty$ in a finite time, starting from any initial condition. (Hint: Don't try to find an exact solution; instead, compare the solutions to those of $\dot{x} = 1 + x^2$.)

2.5.3 Consider the equation $\dot{x} = rx + x^3$, where $r > 0$ is fixed. Show that $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x_0 \ne 0$.

2.5.4 (Infinitely many solutions with the same initial condition) Show that the initial value problem $\dot{x} = x^{1/3}$, $x(0) = 0$, has an infinite number of solutions. (Hint: