Chapter 18

Counting

I'm gonna close my eyes
And count to ten
I'm gonna close my eyes
And when I open them again
Everything will make sense to me then

—Tina Dico, 'Count To Ten'

WE ARE NOW in a position to apply the periodic orbit theory to the first and the easiest problem in theory of chaotic systems: cycle counting. This is the simplest illustration of the raison d'etre of periodic orbit theory; we derive a duality transformation that relates *local* information - in this case the next admissible symbol in a symbol sequence - to *global* averages, in this case the mean rate of growth of the number of cycles with increasing cycle period. In chapter 17 we have transformed, by means of the transition matrices / graphs, the topological dynamics of chapter 14 into a multiplicative operation. Here we show that the nth power of a transition matrix counts all itineraries of length n. The asymptotic growth rate of the number of admissible itineraries is therefore given by the leading eigenvalue of the transition matrix; the leading eigenvalue is in turn given by the leading zero of the characteristic determinant of the transition matrix, which is - in this context - called the *topological zeta function*.

For flows with finite transition graphs this determinant is a finite *topological polynomial* which can be read off the graph. However, (a) even something as humble as the quadratic map generically requires an infinite partition (sect. 18.5), but (b) the finite partition approximants converge exponentially fast.

The method goes well beyond the problem at hand, and forms the core of the entire treatise, making tangible the abstract notion of "spectral determinants" yet to come.

18.1 How many ways to get there from here?

In the 3-disk system of example 14.2 the number of admissible trajectories doubles with every iterate: there are $K_n = 3 \cdot 2^n$ distinct itineraries of length n. If disks are too close and a subset of trajectories is pruned, this is only an upper bound and explicit formulas might be hard to discover, but we still might be able to establish a lower exponential bound of the form $K_n \geq Ce^{n\hat{h}}$. Bounded exponentially by $3e^{n \ln 2} \geq K_n \geq Ce^{n\hat{h}}$, the number of trajectories must grow exponentially as a function of the itinerary length, with rate given by the *topological entropy*:

$$h = \lim_{n\to\infty} \frac{1}{n} \ln K_n \,. \tag{18.1}$$

We shall now relate this quantity to the spectrum of the transition matrix, with the growth rate of the number of topologically distinct trajectories given by the leading eigenvalue of the transition matrix.

The transition matrix element $T_{ij} \in \{0, 1\}$ in (17.1) indicates whether the transition from the starting partition j into partition i in one step is allowed or not, and the (i, j) element of the transition matrix iterated n times

exercise 18.1

$$(T^n)_{ij} = \sum_{k_1,k_2,\ldots,k_{n-1}} T_{ik_1} T_{k_1k_2} \ldots T_{k_{n-1}j} \tag{18.2}$$

receives a contribution 1 from every admissible sequence of transitions, so $(T^n)_{ij}$ is the number of admissible n symbol itineraries starting with j and ending with i.

example 18.1
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The total number of admissible itineraries of n symbols is

$$K_n = \sum_{ij} (T^n)_{ij} = \begin{bmatrix}1, 1, \ldots, 1\end{bmatrix} T^n \begin{bmatrix}1\\1\\ \vdots \\1\end{bmatrix} . \tag{18.3}$$

A finite $[N \times N]$ matrix T has eigenvalues $\{\lambda_0, \lambda_1, \cdots, \lambda_{m-1}\}$ and (right) eigenvectors $\{\varphi_0, \varphi_1, \cdots, \varphi_{m-1}\}$ satisfying $T\varphi_\alpha = \lambda_\alpha \varphi_\alpha$. Expressing the initial vector in (18.3) in this basis,

$$T^n \begin{bmatrix}1\\1\\ \vdots \\1\end{bmatrix} = T^n \sum_{\alpha=0}^{m-1} b_\alpha \varphi_\alpha = \sum_{\alpha=0}^{m-1} b_\alpha \lambda_\alpha^n \varphi_\alpha \,,$$

and contracting with $\left[1, 1, \ldots, 1\right]$, we obtain

$$K_n = \sum_{\alpha=0}^{m-1} c_\alpha \lambda_\alpha^n \,.$$

The constants c_α depend on the choice of initial and final partitions: In this example we are sandwiching T^n between the vector $\left[1, 1, \ldots, 1\right]$ and its transpose, but any other pair of vectors would do, as long as they are not orthogonal to the leading eigenvector φ_0. In an experiment the vector $\left[1, 1, \ldots, 1\right]$ would be replaced by a description of the initial state, and the right vector would describe the measurement time n later.

exercise 18.3

As n increases, the sum

is dominated by the leading eigenvalue of the transition matrix, $\lambda_0 > |\mathrm{Re}\,\lambda_\alpha|$, $\alpha = 1, 2, \cdots, m-1$, and the topological entropy (18.1) is given by

$$\begin{aligned} h &= \lim_{n\to\infty} \frac{1}{n} \ln c_0 \lambda_0^n \left[1 + \frac{c_1}{c_0} \left(\frac{\lambda_1}{\lambda_0}\right)^n + \cdots \right] \\ &= \ln \lambda_0 + \lim_{n\to\infty} \left[\frac{\ln c_0}{n} + \frac{1}{n} \frac{c_1}{c_0} \left(\frac{\lambda_1}{\lambda_0}\right)^n + \cdots \right] \\ &= \ln \lambda_0 \,. \end{aligned} \tag{18.4}$$

What have we learned? The transition matrix T is a one-step, *short time* operator, advancing the trajectory from one partition to the next admissible partition. Its eigenvalues describe the rate of growth of the total number of trajectories at the *asymptotic times*. Instead of painstakingly counting $K_1, K_2, K_3, \ldots$ and estimating (18.1) from a slope of a log-linear plot, we have the *exact* topological entropy if we can compute the leading eigenvalue of the transition matrix T.

18.2 Topological trace formula

There are two standard ways of computing eigenvalues of a matrix - by evaluating the trace $\mathrm{tr}\, T^n = \sum \lambda_\alpha^n$, or by evaluating the determinant $\det(1 - zT)$. We start by evaluating the trace of transition matrices. The main lesson will be that the trace receives contributions only from itineraries that return to the initial partition, i.e., periodic orbits.

Consider an M-step memory transition matrix, like the 1-step memory example (17.11). The trace of the transition matrix counts the number of partitions that

Table 18.1: Prime cycles for the binary symbolic dynamics up to length 9. The numbers of prime cycles are given in table 18.3.

n_p	p	n_p	p	n_p	p	n_p	p	n_p	p
1	0	7	0001001	8	00001111	9	000001101	9	001001111
	1		0000111		00010111		000010011		001010111
2	01		0001011		00011011		000010101		001011011
3	001		0001101		00011101		000011001		001011101
	011		0010011		00100111		000100011		001100111
4	0001		0010101		00101011		000100101		001101011
	0011		0001111		00101101		000101001		001101101
	0111		0010111		00110101		000001111		001110101
5	00001		0011011		00011111		000010111		010101011
	00011		0011101		00101111		000011011		000111111
	00101		0101011		00110111		000011101		001011111
	00111		0011111		00111011		000100111		001101111
	01011		0101111		00111101		000101011		001110111
	01111		0110111		01010111		000101101		001111011
6	000001		0111111		01011011		000110011		001111101
	000011	8	00000001		00111111		000110101		010101111
	000101		00000011		01011111		000111001		010110111
	000111		00000101		01101111		001001011		010111011
	001011		00001001		01111111		001001101		001111111
	001101		00000111	9	000000001		001010011		010111111
	001111		00001011		000000011		001010101		011011111
	010111		00001101		000000101		000011111		011101111
	011111		00010011		000001001		000101111		011111111
7	0000001		00010101		000010001		000110111		
	0000011		00011001		000000111		000111011		
	0000101		00100101		000001011		000111101		

map into themselves. More generally, each closed walk through n concatenated entries of T contributes to $\mathrm{tr}\, T^n$ the product (18.2) of the matrix entries along the walk. Each step in such a walk shifts the symbolic string by one symbol; the trace ensures that the walk closes on a periodic string c. Define t_c to be the *local trace*, the product of matrix elements along a cycle c, each term being multiplied by a book keeping variable z.

The quantity $z^n \mathrm{tr}\, T^n$ is then the sum of t_c for all cycles of period n. The t_c = (product of matrix elements along cycle c is manifestly cyclically invariant, $t_{100} = t_{010} = t_{001}$, so a prime cycle p of period n_p contributes n_p times, once for each periodic point along its orbit. For the purposes of periodic orbit counting, the local trace takes values

$$t_p = \begin{cases} z^{n_p} & \text{if } p \text{ is an admissible cycle} \\ 0 & \text{otherwise,} \end{cases} \tag{18.5}$$

i.e., (setting $z = 1$) the local trace is $t_p = 1$ if the cycle is admissible, and $t_p = 0$ otherwise.


example 18.2
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Table 18.2: The total numbers N_n of periodic points of period n for binary symbolic dynamics. The numbers of contributing prime cycles illustrates the preponderance of long prime cycles of period n over the repeats of shorter cycles of periods n_p, where $n = rn_p$. Further enumerations of binary prime cycles are given in tables 18.1 and 18.3. (L. Rondoni)

n	N_n	# of prime cycles of period n_p									
		1	2	3	4	5	6	7	8	9	10
1	2	2									
2	4	2	1								
3	8	2		2							
4	16	2	1		3						
5	32	2				6					
6	64	2	1	2			9				
7	128	2						18			
8	256	2	1		3				30		
9	512	2		2						56	
10	1024	2	1			6					99

Hence $\mathrm{tr}\, T^n = N_n$ counts the number of *admissible periodic points* of period n. The nth order trace (18.28) picks up contributions from all repeats of prime cycles, with each cycle contributing n_p periodic points, so N_n, the total number of periodic points of period n is given by

$$z^n N_n = z^n \mathrm{tr}\, T^n = \sum_{n_p|n} n_p t_p^{p:n/n_p} = \sum_p n_p \sum_{r=1}^{\infty} \delta_{n,n_p r} t_p^r \,. \tag{18.6}$$

Here $m|n$ means that m is a divisor of n. An example is the periodic orbit counting in table 18.2.

In order to get rid of the awkward divisibility constraint $n = n_p r$ in the above sum, we introduce the generating function for numbers of periodic points

$$\sum_{n=1}^{\infty} z^n N_n = \mathrm{tr} \frac{zT}{1 - zT} \,. \tag{18.7}$$

The right hand side is the geometric series sum of $N_n = \mathrm{tr}\, T^n$. Substituting (18.6) into the left hand side, and replacing the right hand side by the eigenvalue sum $\mathrm{tr}\, T^n = \sum \lambda_\alpha^n$, we obtain our first example of a trace formula, the *topological trace formula*

$$\sum_{\alpha=0} \frac{z\lambda_\alpha}{1 - z\lambda_\alpha} = \sum_p \frac{n_p t_p}{1 - t_p} \,. \tag{18.8}$$

A trace formula relates the spectrum of eigenvalues of an operator - here the transition matrix - to the spectrum of periodic orbits of a dynamical system. It is a statement of duality between the short-time, local information - in this case the next admissible symbol in a symbol sequence - to long-time, global averages, in this case the mean rate of growth of the number of cycles with increasing cycle period.

18.3 Determinant of a graph

Our next task is to determine the zeros of the *spectral determinant* of an $[m \times m]$ transition matrix

$$\det(1 - zT) = \prod_{\alpha=0}^{m-1} (1 - z\lambda_\alpha) \,. \tag{18.9}$$

We could now proceed to diagonalize T on a computer, and get this over with. It pays, however, to dissect $\det(1 - zT)$ with some care; understanding this computation in detail will be the key to understanding the cycle expansion computations of chapter 23 for arbitrary dynamical averages. For T a finite matrix, (18.9) is just the characteristic polynomial for T. However, we shall be able to compute this object even when the dimension of T and other such operators becomes infinite, and for that reason we prefer to refer to (18.9) loosely as the "spectral determinant."

There are various definitions of the determinant of a matrix; we will view the determinant as a sum over all possible permutation cycles composed of the traces $\mathrm{tr}\, T^k$, in the spirit of the determinant–trace relation (1.16):

exercise 4.1

$$\begin{aligned}
\det(1 - zT) &= \exp(\mathrm{tr}\, \ln(1 - zT)) = \exp\left(-\sum_{n=1} \frac{z^n}{n} \mathrm{tr}\, T^n\right) \\
&= 1 - z\, \mathrm{tr}\, T - \frac{z^2}{2}\left((\mathrm{tr}\, T)^2 - \mathrm{tr}\, T^2\right) - \ldots
\end{aligned} \tag{18.10}$$

Formally, the right hand is a Taylor series in z about $z = 0$. If T is an $[m \times m]$ finite matrix, then the characteristic polynomial is at most of order m. In that case the coefficients of z^n must vanish *exactly* for $n > m$.

We now proceed to relate the determinant in (18.10) to the corresponding transition graph of chapter 17: toward this end, we start with the usual textbook expression for a determinant as the sum of products of all permutations

$$\det M = \sum_{\{\pi\}} (-1)^\pi M_{1,\pi_1} M_{2,\pi_2} \cdots M_{m,\pi_m} \tag{18.11}$$

where $M = 1 - zT$ is a $[m \times m]$ matrix, $\{\pi\}$ denotes the set of permutations of m symbols, π_k is the permutation π applied to k, and $(-1)^\pi = \pm 1$ is the parity of permutation π. The right hand side of (18.11) yields a polynomial in T of order m

in z: a contribution of order n in z picks up $m-n$ unit factors along the diagonal, the remaining matrix elements yielding

$$(-z)^n(-1)^{\pi} T_{s_1 \pi s_1} \cdots T_{s_n \pi s_n} \tag{18.12}$$

where π is the permutation of the subset of n distinct symbols $s_1 \cdots s_n$ indexing T matrix elements. As in (18.28), we refer to any combination $t_c = T_{s_1 s_k} T_{s_3 s_2} \cdots T_{s_2 s_1}$, for a given itinerary $c = s_1 s_2 \cdots s_k$, as the *local trace* associated with a closed loop c on the transition graph. Each term of the form (18.12) may be factored in terms of local traces $t_{c_1} t_{c_2} \cdots t_{c_k}$, i.e., loops on the transition graph. These loops are non-intersecting, as each node may only be reached by *one* link, and they are indeed loops, as if a node is reached by a link, it has to be the starting point of another *single* link, as each s_j must appear exactly *once* as a row and column index.

So the general structure is clear, a little more thinking is only required to get the sign of a generic contribution. We consider only the case of loops of length 1 and 2, and leave to the reader the task of generalizing the result by induction. Consider first a term in which only loops of unit length appear in (18.12), i.e., only the diagonal elements of T are picked up. We have $k = m$ loops and an even permutation π so the sign is given by $(-1)^k$, where k is the number of loops. Now take the case in which we have i single loops and j loops of length $n = 2j + i$. The parity of the permutation gives $(-1)^j$ and the first factor in (18.12) gives $(-1)^n = (-1)^{2j+i}$. So once again these terms combine to $(-1)^k$, where $k = i + j$ is the number of loops. Let f be the maximal number of non-intersecting loops. We may summarize our findings as follows: exercise 18.4

> The characteristic polynomial of a transition matrix is given by the sum of all possible partitions π of the corresponding transition graph into products of k non-intersecting loops, with each loop trace t_p carrying a minus sign:
>
> $$\det(1 - zT) = \sum_{k=0}^{f} \sum_{\pi}{}' (-1)^k t_{p_1} \cdots t_{p_k} \tag{18.13}$$

Any self-intersecting loop is *shadowed* by a product of two loops that share the intersection point. As both the long loop t_{ab} and its shadow $t_a t_b$ in the case at hand carry the same weight $z^{n_a+n_b}$, the cancelation is exact, and the loop expansion (18.13) is finite. In the case that the local traces count prime cycles (18.5), $t_p = 0$ or z^n, we refer to $\det(1 - zT)$ as the *topological polynomial*.

We refer to the set of all non-self-intersecting loops $\{t_{p_1}, t_{p_2}, \cdots t_{p_f}\}$ as the *fundamental cycles* (for an explicit example, see the loop expansion of example 18.6). If the graph has m nodes, no fundamental cycle is of period longer than m, as any longer cycle is of necessity self-intersecting.

The above loop expansion of a determinant in terms of traces is most easily grasped by working through a few examples. The complete binary dynamics transition graph of figure 17.5 is a little bit too simple, but let us start humbly and consider it anyway.

example 18.3
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Similarly, for the complete symbolic dynamics of N symbols the transition graph has one node and N links, yielding

$$\det(1 - zT) = 1 - Nz\,, \tag{18.14}$$

which gives the topological entropy $h = \ln N$.

example 18.4
p. 337

18.4 Topological zeta function

What happens if there is no finite-memory transition matrix, if the transition graph is infinite? If we are never sure that looking further into the future will reveal no further forbidden blocks? There is still a way to define the determinant, and this idea is central to the whole treatise: the determinant is then defined by (18.10)

exercise 4.1

$$\det(1 - zT) = 1 - \sum_{n=1}^{\infty} \hat{c}_n z^n\,. \tag{18.15}$$

For finite dimensional matrices the expansion is a finite polynomial, and (18.15) is an identity; however, for infinite dimensional operators the cumulant expansion coefficients $\hat{c}_n$ *define* the determinant.

Let us now evaluate the determinant in terms of traces for an arbitrary transition matrix. In order to obtain an expression for the spectral determinant (18.9) in terms of cycles, substitute (18.6) into (18.15) and sum over the repeats of prime cycles using $\ln(1-x) = -\sum_r x^r/r$,

$$\begin{aligned} \det(1-zT) &= \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{t_p^r}{r}\right) = \exp\left(\sum_p \ln(1-t_p)\right) \\ \prod_\alpha (1-z\lambda_\alpha) &= \prod_p (1-t_p)\,, \end{aligned} \tag{18.16}$$

where for the topological entropy the weight assigned to a prime cycle p of period n_p is $t_p = z^{n_p}$ if the cycle is admissible, or $t_p = 0$ if it is pruned. This determinant is called the *topological* or the *Artin-Mazur* zeta function, conventionally denoted by

$$1/\zeta_{\text{top}}(z) = \prod_p (1-z^{n_p}) = 1 - \sum_{n=1} \hat{c}_n z^n\ . \tag{18.17}$$

Counting cycles amounts to giving each admissible prime cycle p weight $t_p = z^{n_p}$ and expanding the Euler product (18.17) as a power series in z. The number of prime cycles p is infinite, but if T is an $[m\times m]$ finite matrix, then the number of roots λ_α is at most m, the characteristic polynomial is at most of order m, and the coefficients of z^n vanish for $n > m$.

The topological entropy h can now be determined from the leading zero $z = e^{-h}$ of the topological zeta function. For a finite $[m\times m]$ transition matrix, the number of terms in the characteristic equation (18.13) is finite, and we refer to this expansion as the *topological polynomial* of order $\leq m$.

18.6 Shadowing

The topological zeta function is a pretty function, but the infinite product (18.16) should make you pause. For finite transition matrices the left hand side is a determinant of a finite matrix, therefore a finite polynomial; so why is the right hand side an infinite product over the infinitely many prime periodic orbits of all periods?

The way in which this infinite product rearranges itself into a finite polynomial is instructive, and crucial for all that follows. You can already take a peek at the full cycle expansion (23.8) of chapter 23; all cycles beyond the fundamental t_0 and t_1 appear in the shadowing combinations such as

$$t_{s_1 s_2 \cdots s_n} - t_{s_1 s_2 \cdots s_m} t_{s_{m+1} \cdots s_n} .$$

For subshifts of finite type such shadowing combinations cancel *exactly*, if we are counting cycles as we do in (18.29) and (18.36), or if the dynamics is piecewise linear, as in exercise 22.2. As we argue in sect. 1.5.4, for nice hyperbolic flows whose symbolic dynamics is a subshift of finite type, the shadowing combinations *almost* cancel, and the spectral determinant is dominated by the fundamental cycles from (18.13), with longer cycles contributing only small "curvature" corrections.

These exact or nearly exact cancelations depend on the flow being smooth and the symbolic dynamics being a subshift of finite type. If the dynamics requires an infinite state space partition, with pruning rules for blocks of increasing length, most of the shadowing combinations still cancel, but the few corresponding to new forbidden blocks do not, leading to a finite radius of convergence for the spectral determinant, as depicted in figure 18.2.

18.7 Counting cycles

Chaos is merely order waiting to be deciphered
— José Saramago, *The Double*

In what follows, we shall occasionally need to compute all cycles up to topological period n, so it is important to know their exact number. The formulas are fun to derive, but a bit technical for plumber on the street, and probably best skipped on the first reading.

18.7.1 Counting periodic points

The number of periodic points of period n is denoted N_n. It can be computed from (18.15) and (18.7) as a logarithmic derivative of the topological zeta function

$$
\begin{aligned}
\sum_{n=1} N_n z^n &= \mathrm{tr}\left(-z\frac{d}{dz}\ln(1-zT)\right) = -z\frac{d}{dz}\ln\det\,(1-zT) \\
&= \frac{-z\frac{d}{dz}(1/\zeta_{\mathrm{top}})}{1/\zeta_{\mathrm{top}}}\,.
\end{aligned}
\tag{18.24}
$$

Observe that the trace formula (18.8) diverges at $z \to e^{-h}$, because the denominator has a simple zero there.

18.7.2 Counting prime cycles

Table 18.3: Number of prime cycles for various alphabets and grammars up to period 10. The first column gives the cycle period, the second gives the formula (18.26) for the number of prime cycles for complete N-symbol dynamics, and columns three through five give the numbers of prime cycles for $N = 2, 3$ and 4.

n	$M_n(N)$	$M_n(2)$	$M_n(3)$	$M_n(4)$
1	N	2	3	4
2	$N(N-1)/2$	1	3	6
3	$N(N^2-1)/3$	2	8	20
4	$N^2(N^2-1)/4$	3	18	60
5	$(N^5-N)/5$	6	48	204
6	$(N^6-N^3-N^2+N)/6$	9	116	670
7	$(N^7-N)/7$	18	312	2340
8	$N^4(N^4-1)/8$	30	810	8160
9	$N^3(N^6-1)/9$	56	2184	29120
10	$(N^{10}-N^5-N^2+N)/10$	99	5880	104754

We list the number of prime cycles up to period 10 for 2-, 3- and 4-letter complete symbolic dynamics in table 18.3, obtained by Möbius inversion (18.26).

example 18.10 p. 340 example 18.11 p. 340 example 18.12 p. 341

Résumé

The main result of this chapter is the cycle expansion (18.17) of the topological zeta function (i.e., the spectral determinant of the transition matrix):

$$1/\zeta_{\text{top}}(z) = 1 - \sum_{k=1} \hat{c}_k z^k \,.$$

Table 18.4: List of 3-disk prime cycles up to period 10. Here n is the cycle period, M_n is the number of prime cycles, N_n is the number of periodic points, and S_n the number of distinct prime cycles under D_3 symmetry (see chapter 25 for further details). Column 3 also indicates the splitting of N_n into contributions from orbits of periods that divide n. The prefactors in the fifth column indicate the degeneracy m_p of the cycle; for example, 3·12 stands for the three prime cycles $\overline{12}$, $\overline{13}$ and $\overline{23}$ related by $2\pi/3$ rotations. Among symmetry-related cycles, a representative $\hat{p}$ which is lexically lowest is listed. The cycles of period 9 grouped with parentheses are related by time reversal symmetry, but not by any D_3 transformation.

n	M_n	N_n	S_n	$m_p \cdot \hat{p}$
1	0	0	0	
2	3	6=3·2	1	3·12
3	2	6=2·3	1	2·123
4	3	18=3·2+3·4	1	3·1213
5	6	30=6·5	1	6·12123
6	9	66=3·2+2·3+9·6	2	6·121213 + 3·121323
7	18	126=18·7	3	6·1212123 + 6·1212313 + 6·1213123
8	30	258=3·2+3·4+30·8	6	6·12121213 + 3·12121313 + 6·12121323 + 6·12123123 + 6·12123213 + 3·12132123
9	56	510=2·3+56·9	10	6·121212123 + 6·(121212313 + 121212323) + 6·(121213123 + 121213213) + 6·121231323 + 6·(121231213 + 121232123) + 2·121232313 + 6·121321323
10	99	1022	18	

Table 18.5: The 4-disk prime cycles up to period 8. The symbols is the same as shown in table 18.4. Orbits related by time reversal symmetry (but no C_{4v} symmetry) already appear at cycle period 5. Cycles of period 7 and 8 have been omitted.

n	M_n	N_n	S_n	$m_p \cdot \hat{p}$
1	0	0	0	
2	6	12=6·2	2	4·12 + 2·13
3	8	24=8·3	1	8·123
4	18	84=6·2+18·4	4	8·1213 + 4·1214 + 2·1234 + 4·1243
5	48	240=48·5	6	8·(12123 + 12124) + 8·12313 + 8·(12134 + 12143) + 8·12413
6	116	732=6·2+8·3+116·6	17	8·121213 + 8·121214 + 8·121234 + 8·121243 + 8·121313 + 8·121314 + 4·121323 + 8·(121324 + 121423) + 4·121343 + 8·121424 + 4·121434 + 8·123124 + 8·123134 + 4·123143 + 4·124213 + 8·124243
7	312	2184	39	
8	810	6564	108	

For subshifts of finite type, the transition matrix is finite, and the topological zeta function is a finite polynomial evaluated by the loop expansion (18.13) of $\det(1 - zT)$. For infinite grammars the topological zeta function is defined by its cycle expansion. The topological entropy h is given by the leading zero $z = e^{-h}$. This expression for the entropy is *exact*; in contrast to the initial definition (18.1), no $n \to \infty$ extrapolations of $\ln K_n/n$ are required.

What have we accomplished? We have related the number of topologically distinct paths from one state space region to another region to the leading eigenvalue of the transition matrix T. The spectrum of T is given by topological zeta function, a certain sum over traces $\mathrm{tr}\, T^n$, and in this way the periodic orbit theory has entered the arena through the trace formula (18.8), already at the level of the topological dynamics.

Contrary to claims one all too often encounters in the literature, "exponential proliferation of trajectories" is <u>not</u> the problem; what limits the convergence of cycle expansions is the proliferation of the grammar rules, or the "algorithmic complexity," as illustrated by sect. 18.5, and figure 18.2 in particular.

18.8 Examples

Example 18.1 ***3-disk itinerary counting.*** *The* $(T^2)_{13} = T_{12}T_{23} = 1$ *element of* T^2 *for the 3-disk transition matrix (17.9)*

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}. \tag{18.27}$$

corresponds to path $3 \to 2 \to 1$*, the only 2-step path from 3 to 1, while* $(T^2)_{33} = T_{31}T_{13} + T_{32}T_{23} = 2$ *counts the two returning, periodic paths* $\overline{31}$ *and* $\overline{32}$*. Note that the trace* $\mathrm{tr}\, T^2 = (T^2)_{11} + (T^2)_{22} + (T^2)_{33} = 2T_{13}T_{31} + 2T_{21}T_{12} + 2T_{32}T_{23}$ *has a contribution from each 2-cycle* $\overline{12}$, $\overline{13}$, $\overline{23}$ *twice, one contribution from each periodic point.*

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Example 18.2 ***Traces for binary symbolic dynamics.*** *For example, for the [8×8] transition matrix $T_{s_1s_2s_3,s_0s_1s_2}$ version of (17.11), or any refined partition $[2^n{\times}2^n]$ transition matrix, n arbitrarily large, the periodic point $\overline{100}$ contributes $t_{100} = z^3 T_{\overline{100},\overline{010}} T_{\overline{010},\overline{001}} T_{\overline{001},\overline{100}}$ to $z^3 \mathrm{tr}\, T^3$. This product is manifestly cyclically invariant, $t_{100} = t_{010} = t_{001}$, so a prime cycle $p = \overline{001}$ of period 3 contributes 3 times, once for each periodic point along its orbit.*

exercise 14.7

For the binary labeled non–wandering set the first few traces are given by (consult tables 18.1 and 18.2)

$$
\begin{aligned}
z\,\mathrm{tr}\, T &= t_0 + t_1, \\
z^2 \mathrm{tr}\, T^2 &= t_0^2 + t_1^2 + 2t_{10}, \\
z^3 \mathrm{tr}\, T^3 &= t_0^3 + t_1^3 + 3t_{100} + 3t_{101}, \\
z^4 \mathrm{tr}\, T^4 &= t_0^4 + t_1^4 + 2t_{10}^2 + 4t_{1000} + 4t_{1001} + 4t_{1011}.
\end{aligned} \tag{18.28}
$$

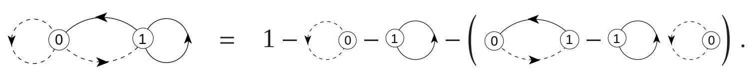
In the binary case the trace picks up only two contributions on the diagonal, $T_{0\cdots 0,0\cdots 0} + T_{1\cdots 1,1\cdots 1}$, no matter how much memory we assume. We can even take infinite memory $M \to \infty$, in which case the contributing partitions are shrunk to the fixed points, $\mathrm{tr}\, T = T_{\overline{0},\overline{0}} + T_{\overline{1},\overline{1}}$.

If there are no restrictions on symbols, the symbolic dynamics is complete, and all *binary sequences are* admissible *(or* allowable*) itineraries. As this type of symbolic dynamics pops up frequently, we list the shortest binary prime cycles in table 18.1.*

exercise 14.2
click to return: p. 323

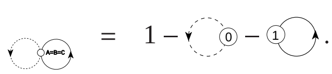
Example 18.3 ***Topological polynomial for complete binary dynamics:*** *(continuation of example 17.1) There are only two non-intersecting loops, yielding*

$$
\det(1 - zT) = 1 - t_0 - t_1 - (t_{01} - t_0 t_1) = 1 - 2z \tag{18.29}
$$



Due to the symmetry under $0 \leftrightarrow 1$ interchange, this is a redundant graph (the 2-cycle t_{01} is exactly shadowed by the 1-cycles). Another way to see is that itineraries are labeled by the $\{0, 1\}$ links, node labels can be omitted. As both nodes have 2 in-links and 2 out-links, they can be identified, and a more economical presentation is in terms of the $[1{\times}1]$ adjacency matrix (17.12)

$$
\det(1 - zA) = 1 - t_0 - t_1 = 1 - 2z \tag{18.30}
$$



The leading (and only) zero of this characteristic polynomial yields the topological entropy $e^h = 2$. As there are $K_n = 2^n$ binary strings of length N, this comes as no surprise.

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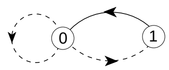
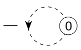
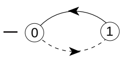
Example 18.4 ***Golden mean pruning:*** *The "golden mean" pruning of example 17.5 has one grammar rule: the substring $\_11\_$ is forbidden. The corresponding transition*

exercise 18.5

graph non-intersecting loops are of length 1 and 2, so the topological polynomial is given by

$$\det(1 - zT) = 1 - t_0 - t_{01} = 1 - z - z^2 \tag{18.31}$$

 $= 1 -$  $-$  .

The leading root of this polynomial is the golden mean, so the entropy ([18.4]) is the logarithm of the golden mean, $h = \ln \frac{1+\sqrt{5}}{2}$.

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Exercises

18.1. **A transition matrix for 3-disk pinball.**

a) Draw the transition graph corresponding to the 3-disk ternary symbolic dynamics, and write down the corresponding transition matrix corresponding to the graph. Show that iteration of the transition matrix results in two coupled linear difference equations, - one for the diagonal and one for the off diagonal elements. (Hint: relate $\mathrm{tr}\, T^n$ to $\mathrm{tr}\, T^{n-1} + \ldots$.)

b) Solve the above difference equation and obtain the number of periodic orbits of length n. Compare your result with table 18.4.

c) Find the eigenvalues of the transition matrix $\mathbf{T}$ for the 3-disk system with ternary symbolic dynamics and calculate the topological entropy. Compare this to the topological entropy obtained from the binary symbolic dynamics $\{0, 1\}$.

18.2. **3-disk prime cycle counting.** A *prime* cycle p of length n_p is a single traversal of the orbit; its label is a non-repeating symbol string of n_p symbols. For example, $\overline{12}$ is prime, but $\overline{2121}$ is not, since it is $\overline{21} = \overline{12}$ repeated.

Verify that a 3-disk pinball has $3, 2, 3, 6, 9, \cdots$ prime cycles of length $2, 3, 4, 5, 6, \cdots$.

18.3. **Sum of A_{ij} is like a trace.** Let A be a matrix with eigenvalues λ_k. Show that

$$\Gamma_n := \sum_{i,j} [A^n]_{ij} = \sum_k c_k \lambda_k^n .$$

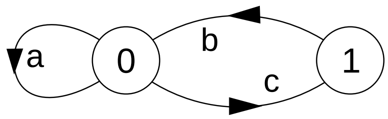
(a) Under what conditions do $\ln |\mathrm{tr}\, A^n|$ and $\ln |\Gamma_n|$ have the same asymptotic behavior as $n \to \infty$, i.e., their ratio converges to one?

(b) Do eigenvalues λ_k need to be distinct, $\lambda_k \neq \lambda_l$ for $k \neq l$? How would a degeneracy $\lambda_k = \lambda_l$ affect your argument for (a)?

18.4. **Loop expansions.** Prove by induction the sign rule in the determinant expansion (18.13):

$$\det(1 - z\mathbf{T}) = \sum_{k \geq 0} \sum_{p_1 + \cdots + p_k} (-1)^k t_{p_1} t_{p_2} \cdots t_{p_k} .$$

18.5. **Transition matrix and cycle counting.** Suppose you are given the transition graph



This diagram can be encoded by a matrix T, where the entry T_{ij} means that there is a link connecting node i to node j. The value of the entry is the weight of the link.

a) Walks on the graph are given a weight that is the product of the weights of all links crossed by the walk. Convince yourself that the transition matrix for this graph is:

$$T = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}.$$

b) Enumerate all the walks of length three on the transition graph. Now compute T^3 and look at the entries. Is there any relation between the terms in T^3 and all the walks?

c) Show that T^n_{ij} is the number of walks from point i to point j in n steps. (Hint: one might use the method of induction.)

d) Estimate the number K_n of walks of length n for this simple transition graph.

e) The topological entropy h measures the rate of exponential growth of the total number of walks K_n as a function of n. What is the topological entropy for this transition graph?

18.6. **Alphabet {0,1}, prune _00_.** The transition graph example 17.8 implements this pruning rule which implies that "0" must always be bracketed by "1"s; in terms of a new symbol 2 := 10, the dynamics becomes unrestricted symbolic dynamics with with binary alphabet {1,2}. The cycle expansion (18.13) becomes

$$\begin{aligned} 1/\zeta &= (1-t_1)(1-t_2)(1-t_{12})(1-t_{112})\ldots \\ &= 1-t_1-t_2-(t_{12}-t_1t_2) \qquad (18.41) \\ &\quad -(t_{112}-t_{12}t_1)-(t_{122}-t_{12}t_2)\ldots \end{aligned}$$

In the original binary alphabet this corresponds to:

$$\begin{aligned} 1/\zeta &= 1-t_1-t_{10}-(t_{110}-t_1t_{10}) \qquad (18.42) \\ &\quad -(t_{1110}-t_{110}t_1)-(t_{11010}-t_{110}t_{10})\ldots \end{aligned}$$

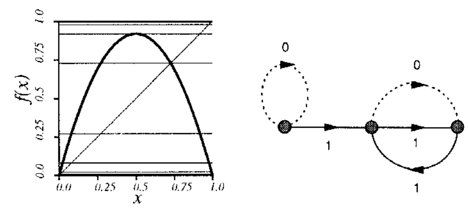
This symbolic dynamics describes, for example, circle maps with the golden mean winding number. For unimodal maps this symbolic dynamics is realized by the tent map of exercise 14.6.

18.7. **"Golden mean" pruned map.** (continuation of exercise 14.6) Show that the total number of periodic orbits of length n for the "golden mean" tent map is

$$\frac{(1+\sqrt{5})^n+(1-\sqrt{5})^n}{2^n}.$$

Continued in exercise 22.1. See also exercise 18.8.

18.8. **A unimodal map with golden mean pruning.** Consider the unimodal map



for which the critical point maps into the right hand fixed point in three iterations, $S^+ = 100\overline{1}$. Show that the admissible itineraries are generated by the above transition graph, with transient neighborhood of $\overline{0}$ fixed point, and _00_ pruned from the recurrent set. (K.T. Hansen)

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