
chaos
classical and quantum
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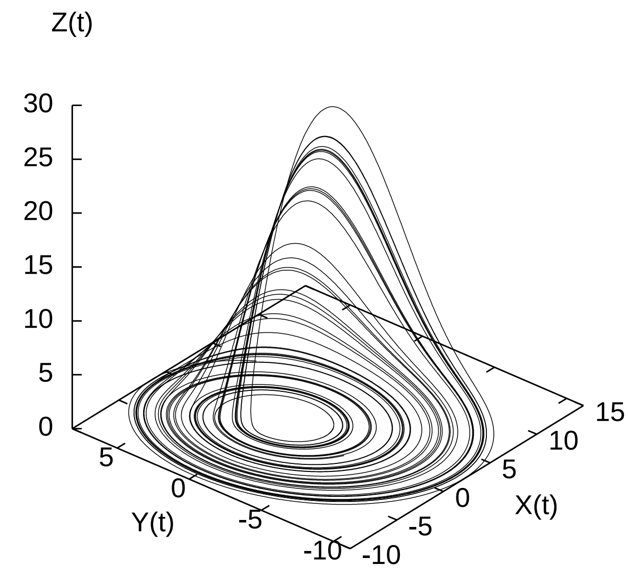
"Clouds over Croatia"

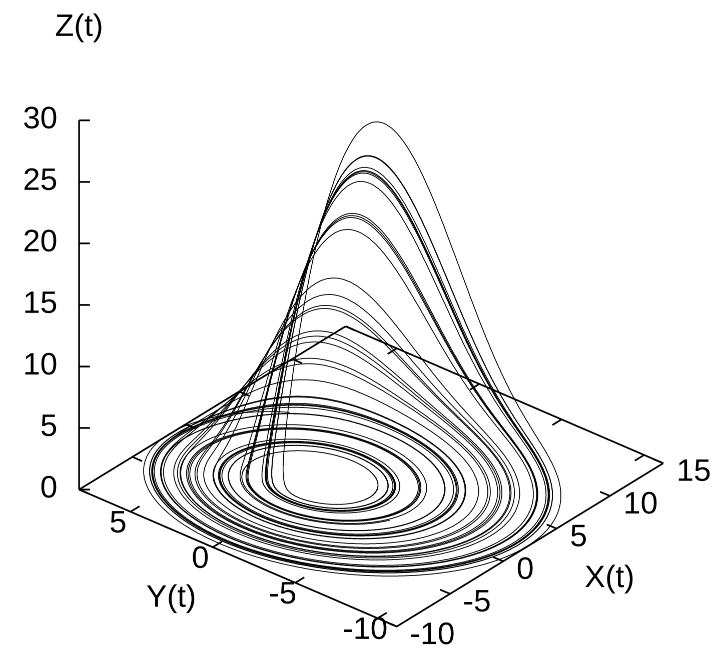
Rössler flow

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c), \qquad a = b = 0.2, \quad c = 5.7.$$



A typical numerically integrated long-time trajectory

A strange attractor?



A trajectory of the Rössler flow up to time t = 250.

Trajectories that start out sufficiently close to the origin seem to converge to a strange attractor.

Poincaré sections

Successive trajectory intersections with a Poincaré section, a d-dimensional set of hypersurfaces $\mathcal{P}$ embedded in the $(d+1)$-dimensional phase space $\mathcal{M}$,

define the Poincaré return map

$$x' = P(x) = f^{\tau(x)}(x) , \qquad x', x \in \mathcal{P} .$$

first return function $\tau(x)$ – time of flight to the next section

Hénon map

Multinomial approximations

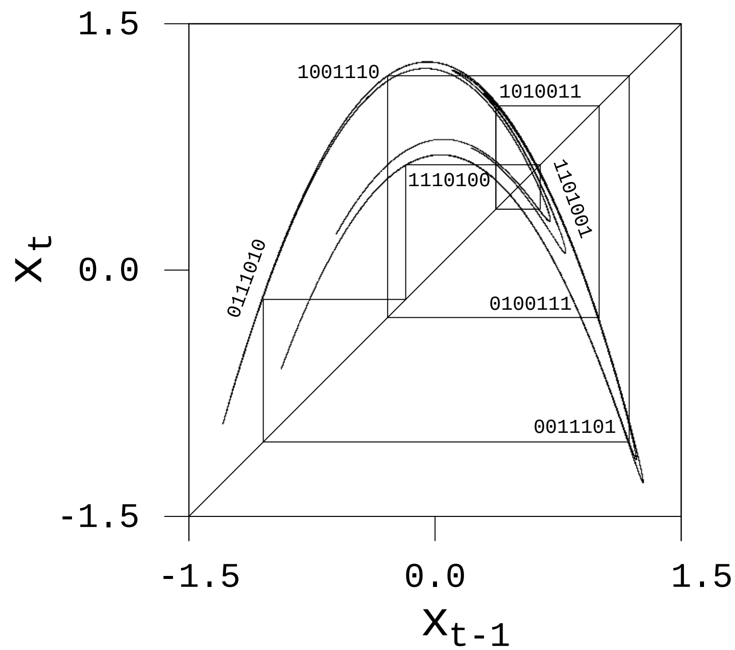
$$P_k(x) = a_k + \sum_{j=1}^{d+1} b_{kj} x_j + \sum_{i,j=1}^{d+1} c_{kij} x_i x_j + \ldots, \qquad x \in \mathcal{P} \tag{3}$$

to Poincaré return maps

$$\begin{pmatrix} x_{1,n+1} \\ x_{2,n+1} \\ \ldots \\ x_{d,n+1} \end{pmatrix} = \begin{pmatrix} P_1(x_n) \\ P_2(x_n) \\ \ldots \\ P_d(x_n) \end{pmatrix}, \qquad x_n, x_{n+1} \in \mathcal{P}$$

motivate model mappings such as the Hénon map

$$x_{n+1} = 1 - a x_n^2 + b y_n$$
$$y_{n+1} = x_n$$

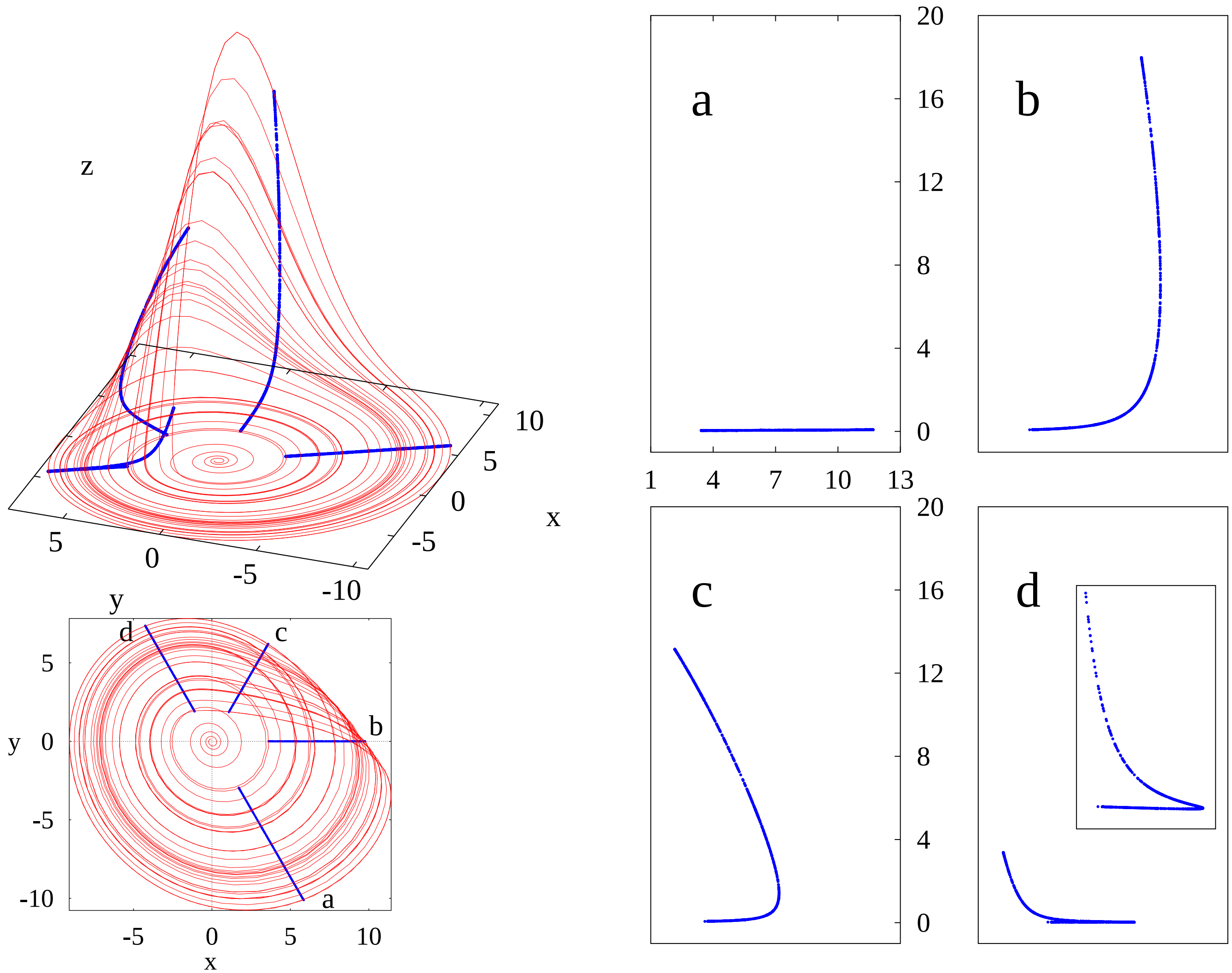


strange attractor and an unstable period 7 cycle of the Hénon map, $a = 1.4$, $b = 0.3$.

for vanishingly small b the Hénon map → parabola:

$$x_{n+1} = 1 - a x_n^2 \,. \qquad (4)$$

lose determinism : the inverse of map has two preimages $\{x^+_{n-1}, x^-_{n-1}\}$ for most x_n.



Poincaré sections, Rössler strange attractor:
planes at angles (a) -60^o (b) 0^o, (c) 60^o, (d) 120^o.

Rössler stretch and mix

A line segment [A, B] starts close to the x–y plane, stretching (a) → (b)

flow is expanding

followed by the folding (c) → (d):
the folded segment returns close to the x–y plane

C from the interior mapped into the outer edge
edge point B lands in the interior

flow is mixing

In one Poincaré return the [A, B] interval is stretched, folded and mapped onto itself

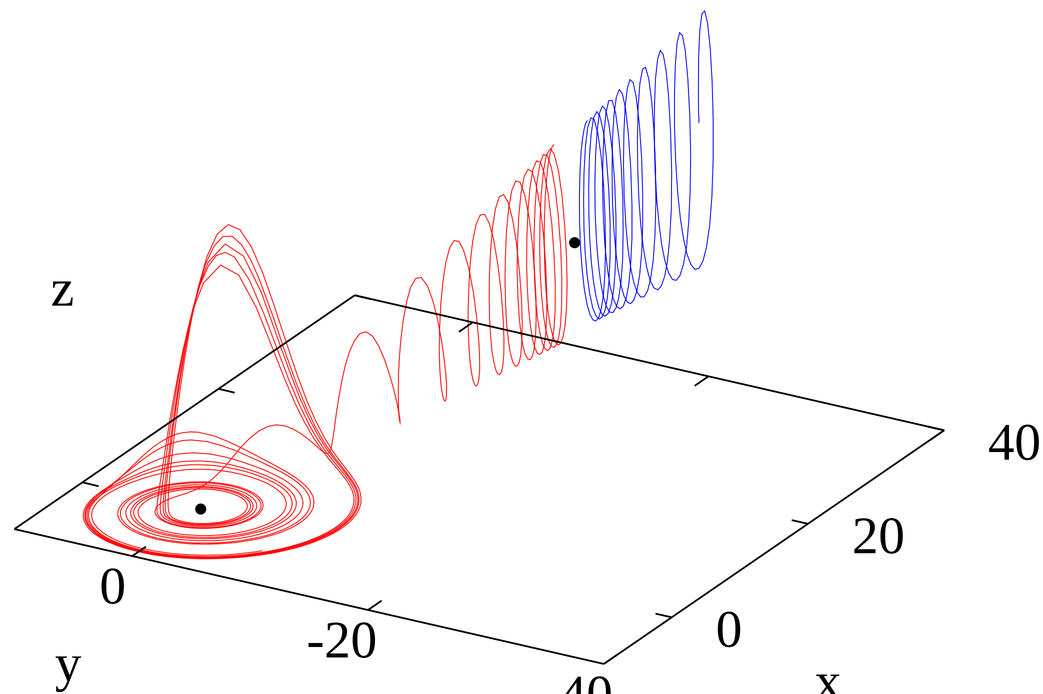
A strange attractor???

no proof that this – or any attractor of interest – is asymptotically aperiodic – it might well be that what we see is but a long transient on a way to an attractive periodic orbit.

pragmatist: I accept that is a "strange attractor"

Equilibria of Rössler flow

Two trajectories of the Rössler flow initiated in the neighborhood of the "+" equilibrium point

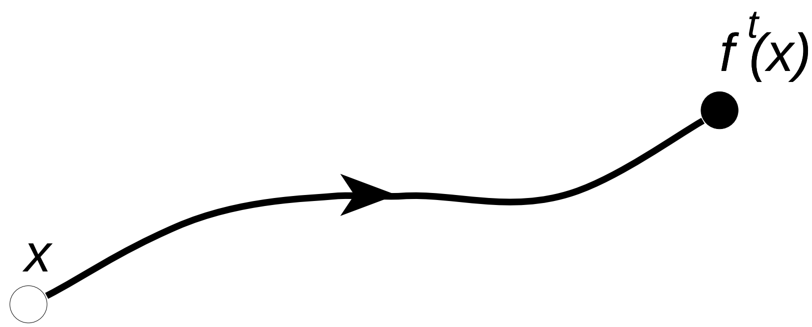


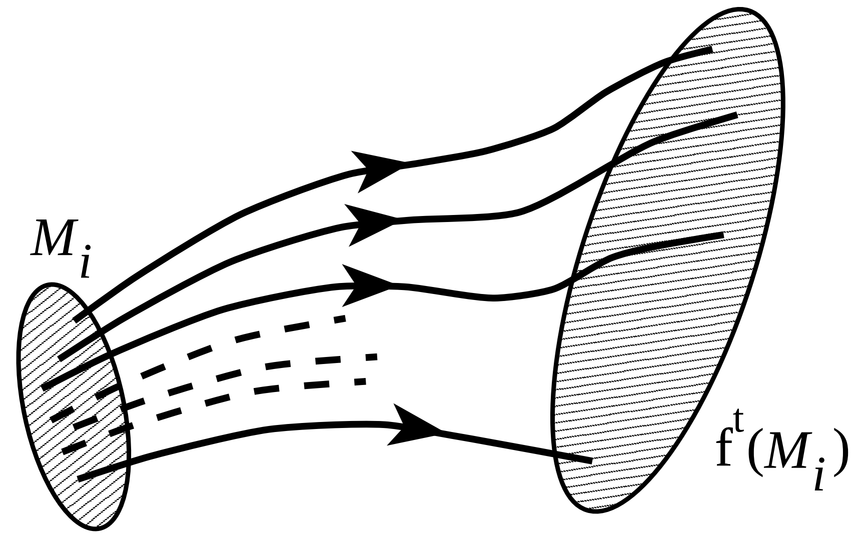
.

2 repelling equilibrium points (no dynamics there!)

$$(x^-, y^-, z^-) = (\,0.0070,\ -0.0351,\ 0.0351\,)$$
$$(x^+, y^+, z^+) = (\,5.6929,\ -28.464,\ 28.464\,)$$

Flows transport neighborhoods





so far: trajectory of a single initial point

next: transport the neighbor-hood of x(t)

Equations of variations

Flow transports displacement $x(t) + \delta x(t)$
along trajectory $x(t) = f^t(x_0)$.

equations of variations for infinitesimal neighborahood:

$$\dot{x}_i + \delta\dot{x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j}\delta x_j \,.$$

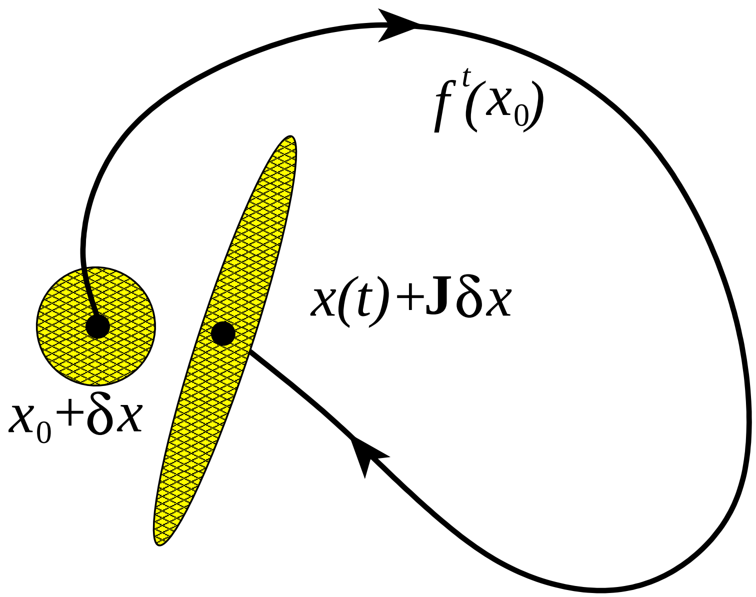
Together

$$\dot{x}_i = v_i(x)\,, \qquad \delta\dot{x}_i = \sum_j A_{ij}(x)\delta x_j$$

where matrix of variations

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j}$$

is the instantaneous rate of shearing of $x(t)$ neighborhood



Jacobian maps a spherical neighborhood of x_0 into an ellipsoidal neighborhood time t later

Neighbors separate along unstable directions,
approach each other along stable directions,
creep along the marginal directions

Stability of equilibria

Matrix of variations $\mathbf{A} = \mathbf{A}(x_q)$ evaluated at an equilibrium point x_q is constant

$$f^t(x) = x_q + e^{\mathbf{A}t}(x - x_q) + \cdots ,$$
$$\mathbf{J}^t(x_q) = e^{\mathbf{A}t} \qquad \mathbf{A} = \mathbf{A}(x_q) .$$

For a constant $\mathbf{A}$ the Jacobian matrix

$$x(t) = e^{t\mathbf{A}} x(0) .$$

is the solution of the linear equation

$$\dot{x} = \mathbf{A} x .$$

so study linear flows first:

Linear flows

Stability eigenvalues, diagonal case: If $\mathbf{A}$ diagonal matrix $\mathbf{A}_D$ with eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_d)$

$$\mathbf{J}^t = e^{t\mathbf{A}_D} = \begin{pmatrix} e^{t\lambda_1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & e^{t\lambda_d} \end{pmatrix} .$$

Λ_k = kth stability eigenvalue of the finite time Jacobian matrix $\mathbf{J}^t$
λ_k = kth stability exponent

$$|\Lambda_k| = e^{t\lambda_k} .$$

Complex stability eigenvalues

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

The eigenvalues λ_1, λ_2 of $\mathbf{A}$

$$\lambda_{1,2} = \frac{1}{2}\left(\mathrm{tr}\,\mathbf{A} \pm \sqrt{(\mathrm{tr}\,\mathbf{A})^2 - 4\det\mathbf{A}}\right)$$

can form a complex conjugate pair

$$\lambda_1 = \lambda + i\vartheta\,, \quad \lambda_2 = \lambda_1^* = \lambda - i\vartheta\,.$$

Complex stability eigenvalues, diagonal example:

The Jacobian matrix $\mathbf{J}$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{t\lambda} \begin{pmatrix} e^{it\vartheta} & 0 \\ 0 & e^{-it\vartheta} \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} .$$

exponent $\lambda > 0$: trajectory $x(t)$ spirals out

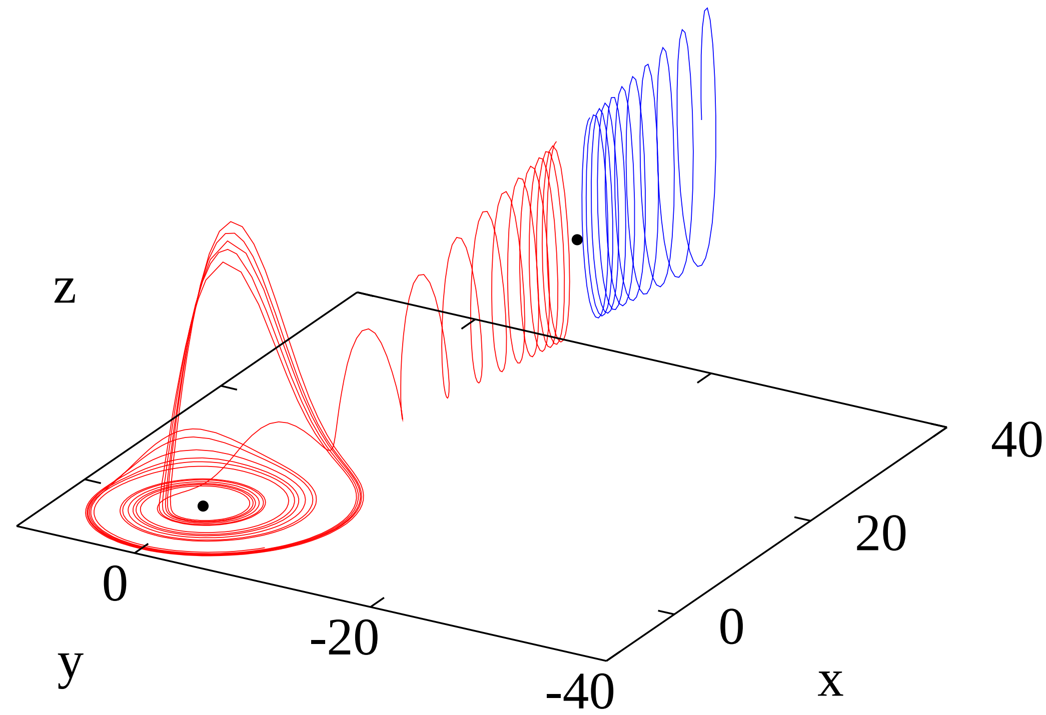
exponent $\lambda < 0$: it spirals in.

$$\vartheta \rightarrow \text{speed of rotation}$$

Stability of Rössler flow equilibria

two equilibrium points (x^-, y^-, z^-) (x^+, y^+, z^+)

stable manifold of "+" equilibrium point = attraction basin boundary:



right of the "+" equilibrium trajectories escape,

left of the "+" spiral toward the "−" equilibrium point

→ seem to wander chaoticaly for all times.

linearized stability exponents

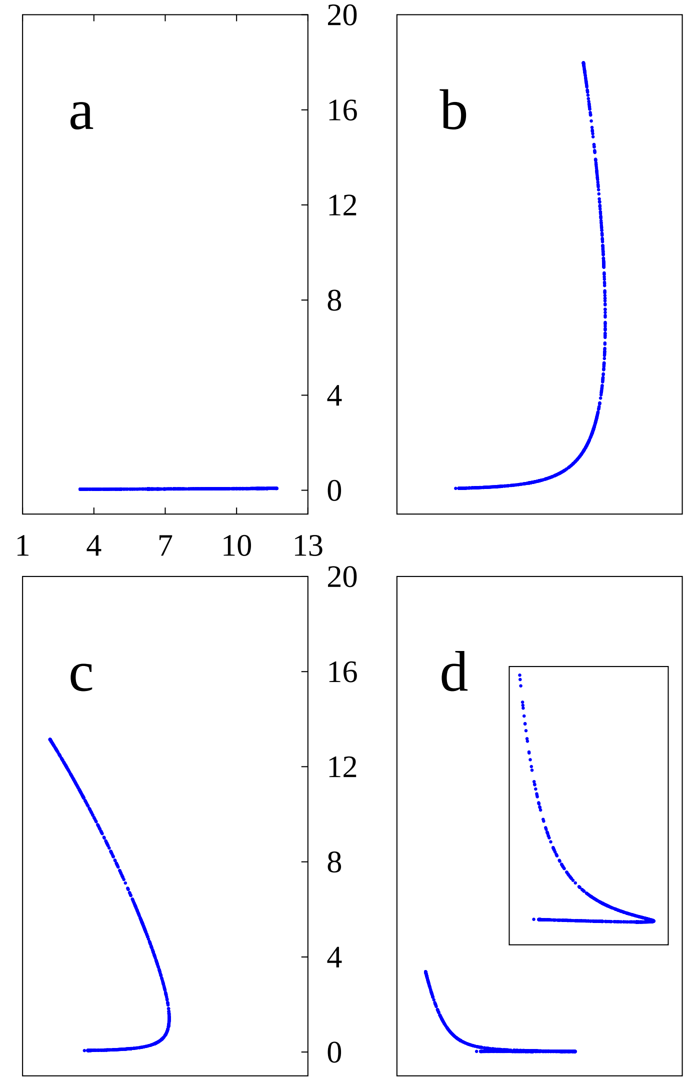
$$(\lambda_1^-, \lambda_2^- \pm i\vartheta_2^-) = (-5.686, \quad 0.0970 \pm i\, 0.9951)$$
$$(\lambda_1^+, \lambda_2^+ \pm i\vartheta_2^+) = (0.1929, \quad -4.596 \times 10^{-6} \pm i\, 5.428)$$

The $\lambda_2^- \pm i\vartheta_2^-$ eigenvectors span a plane

this plane rotates with angular period $T_- \approx \left|2\pi/\vartheta_2^-\right| = 6.313$

a trajectory that starts near the "−" equilibrium point spirals away per one rotation with multiplier $\Lambda_{radial} \approx \exp(\lambda_2^- T_-) = 1.84$

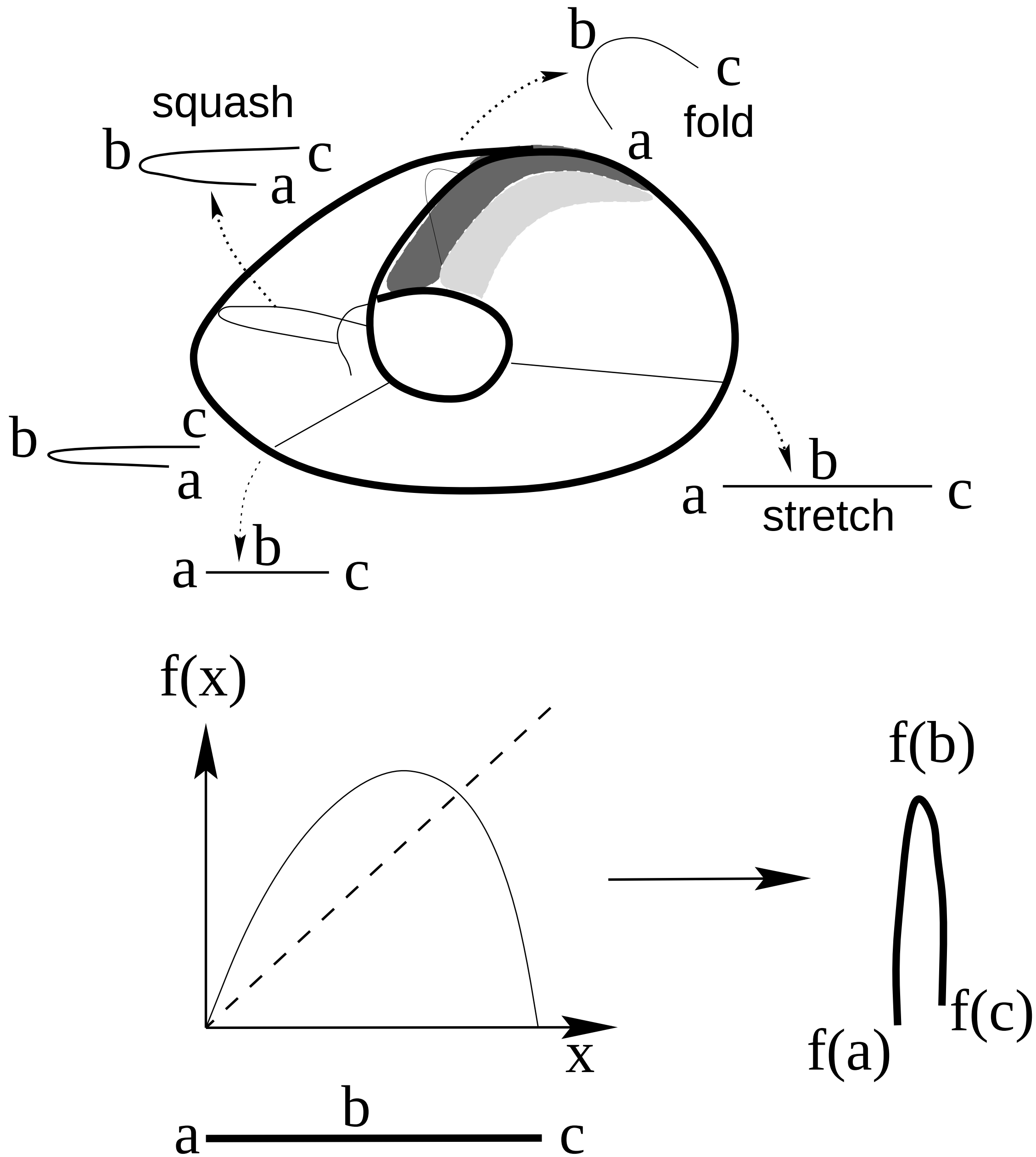
each Poincaré section return, contracted into the stable manifold by amazing factor of $\Lambda_1 \approx \exp(\lambda_1^- T_-) = 10^{-15.6}$ (!)



start with a 1 mm interval pointing in the contracting Λ_1 eigendirection.

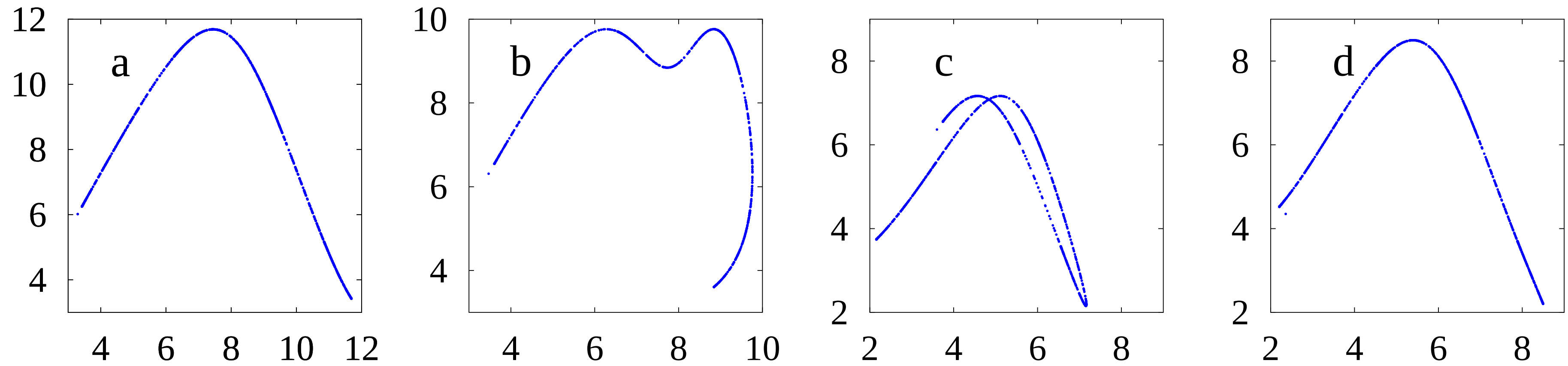
After one Poincaré return the interval is of order of 10^{-4} fermi

Rössler Poincaré return map is in practice 1 - dimensional



(a) A recurrent flow that stretches and folds.

Return maps: Poincaré sections projected onto radial distance $R_n \to R_{n+1}$



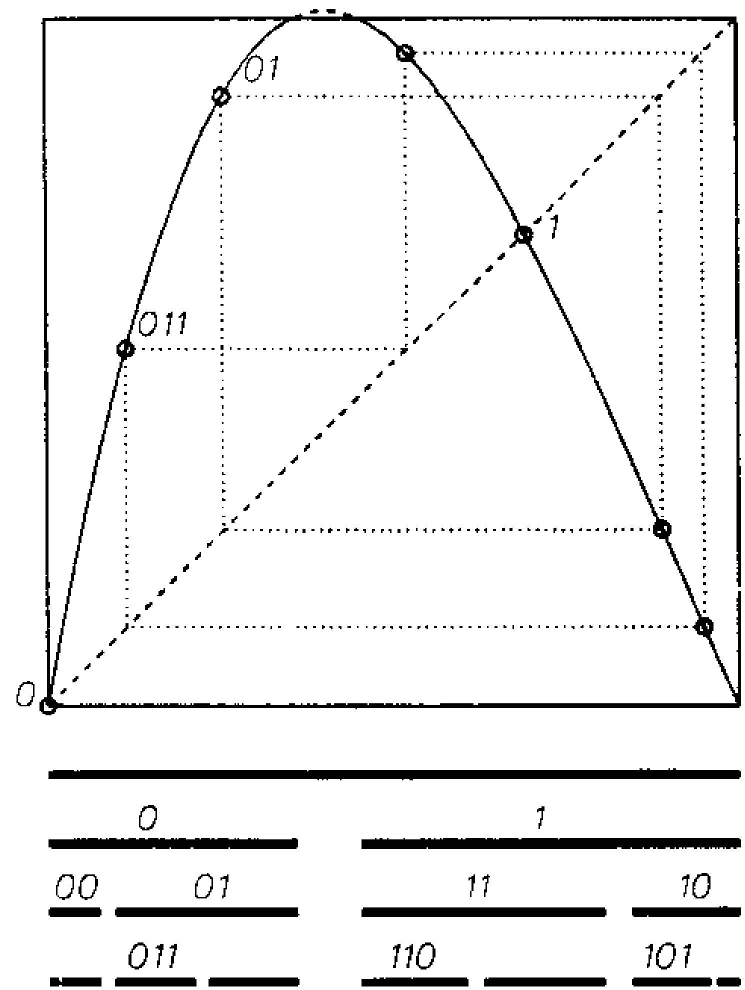
(a) and (d) nice 1-to-1 return maps

(b) and (c) appear multimodal and non-invertible artifacts of projections $(R_n, z_n) \to (R_{n+1}, z_{n+1})$ onto a 1-dimensional subspace $R_n \to R_{n+1}$

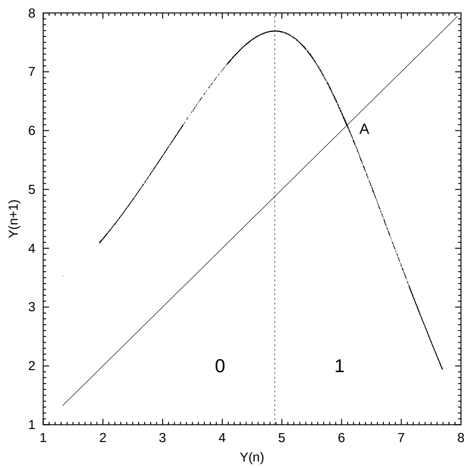
A repeller after 1, 2 and 3 iterations. Intervals marked $s_1 s_2 \cdots s_n$ are unions of all points that do not escape in n iterations, and follow the itinerary $S^+ = s_1 s_2 \cdots s_n$.

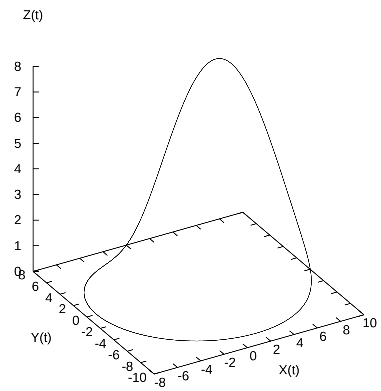
spatial ordering does not respect the binary ordering; for example $x_{00} < x_{01} < x_{11} < x_{10}$.

Also indicated: the fixed points x_0, x_1, the 2-cycle $\overline{01}$, and the 3-cycle $\overline{011}$.



Rössler short cycles

(a) 

(b) 

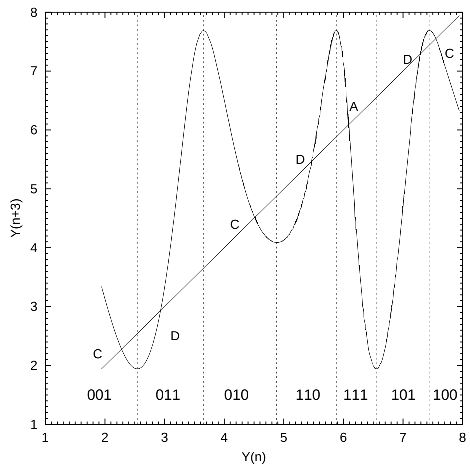
(a) $y \to P_1(y,z)$ return map for $x = 0$, $y > 0$ Poincaré section

(b) The $\overline{1}$-cycle found by taking the fixed point $y_{k+n} = y_k$ as initial guess $(0, y(0), 0)$ for the Newton-Raphson search

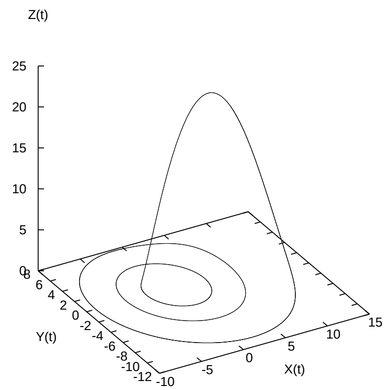
$$\overline{1}\text{-cycle:} \qquad T_1 = 5.88108845586$$

$$(\Lambda_{1,e}, \Lambda_{1,m}, \Lambda_{1,c}) = (-2.40395353, 1, -1.29 \times 10^{-14})$$

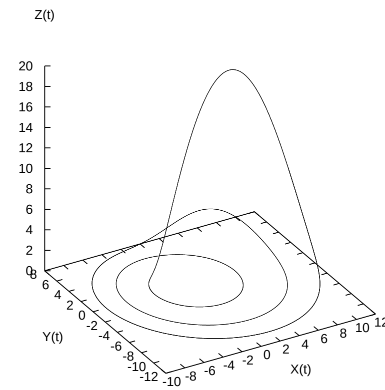
$$(\lambda_{1,e}, \lambda_{1,m}, \lambda_{1,c}) = (0.149141556, 0, -5.44)\,.$$



(c)



(d)



(e)

(c) $y_{k+3} = P_1^3(y_k, z_k)$, the third iterate of Poincaré return map is used to pick starting guesses for the Newton-Raphson searches for the two 3-cycles:

(d) the $\overline{001}$ cycle, and

(e) the $\overline{011}$ cycle.