Exercise 2.6 **Runge-Kutta integration.** *Implement the fourth-order Runge-Kutta integration formula (see, for example, ref. [2.7]) for $\dot{x} = v(x)$:*

$$\begin{aligned} x_{n+1} &= x_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(\delta\tau^5) \\ k_1 &= \delta\tau v(x_n)\,, \qquad k_2 = \delta\tau v(x_n + k_1/2) \\ k_3 &= \delta\tau v(x_n + k_2/2)\,, \quad k_4 = \delta\tau v(x_n + k_3)\,. \end{aligned} \tag{2.17}$$

If you already know your Runge-Kutta, program what you believe to be a better numerical integration routine, and explain what is better about it.

Exercise 2.7 **Rössler system.** *Use the result of exercise 2.6 or some other integration routine to integrate numerically the Rössler system (2.14). Does the result look like a "strange attractor"? If you happen to already know what fractal dimensions are, argue (possibly on basis of numerical integration) that this attractor is of dimension smaller than $\mathbb{R}^3$.*

Exercise 2.8 **Equilibria of the Rössler system.**

(a) *Find all equilibrium points (x^q, y^q, z^q) of the Rössler system (2.14). How many are there?*

(b) *Assume that $b = a$. Define parameters*

$$\begin{aligned} &\epsilon = a/c \\ &D = 1 - 4\epsilon^2 \\ &p^\pm = (1 \pm \sqrt{D})/2 \end{aligned} \tag{2.18}$$

Express all the equilibria in terms of $(c, \epsilon, D, p^\pm)$. Expand equilibria to the first order in ϵ. Note that it makes sense because for $a = b = 0.2$, $c = 5.7$ in (2.14), $\epsilon \approx 0.035$.

(continued as exercise 3.1)

(Rytis Paškauskas)

Exercise 2.9 **Can you integrate me?** *Integrating equations numerically is not for the faint of heart. It is not always possible to establish that a set of nonlinear ordinary differential equations has a solution for all times and there are many cases were the solution only exists for a limited time interval, as, for example, for the equation* $\dot{x} = x^2\,, \qquad x(0) = 1\,.$

(a) *For what times do solutions of*

$$\dot{x} = x(x(t))$$

exist? Do you need a numerical routine to answer this question?

(b) *Let's test the integrator you wrote in exercise 2.6. The equation $\ddot{x} = -x$ with initial conditions $x(0) = 2$ and $\dot{x} = 0$ has as solution $x(t) = e^{-t}(1 + e^{2\,t})$. Can your integrator reproduce this solution for the interval $t \in [0, 10]$? Check you solution by plotting the error as compared to the exact result.*

Exercises

Exercise 3.1 **Poincaré sections of the Rössler flow.** *(continuation of exercise 2.8) Calculate numerically a Poincaré section (or several Poincaré sections) of the Rössler flow. As the Rössler flow phase space is 3-dimensional, the flow maps onto a 2-dimensional Poincaré section. Do you see that in your numerical results? How good an approximation would a replacement of the return map for this section by a 1-dimensional map be? More precisely, estimate the thickness of the strange attractor. (continued as exercise 4.3)*

(Rytis Paškauskas)

Exercise 3.2 **Arbitrary Poincaré sections.** *We will generalize the construction of Poincaré sections so that they can have any shape, as specified by the equation* $U(x) = 0$.

(a) *Start by modifying your integrator so that you can change the coordinates once you get near the Poincaré section. You can do this easily by writing the equations as*

$$\frac{dx_k}{ds} = \kappa f_k \,, \tag{3.16}$$

with $dt/ds = \kappa$, *and choosing* κ *to be* 1 *or* $1/f_1$. *This allows one to switch between* t *and* x_1 *as the integration "time."*

(b) *Introduce an extra dimension* x_{n+1} *into your system and set*

$$x_{n+1} = U(x) \,. \tag{3.17}$$

How can this be used to find a Poincaré section?

Exercise 3.3 **Classical collinear helium dynamics.** *(continuation of exercise 2.10)*

Make a Poincaré surface of section by plotting (r_1, p_1) *whenever* $r_2 = 0$: *Note that for* $r_2 = 0$, p_2 *is already determined by (5.6). Compare your results with figure 34.3(b).*

(Gregor Tanner, Per Rosenqvist)

Exercise 3.4 **Hénon map fixed points.** *Show that the two fixed points* (x_0, x_0), (x_1, x_1) *of the Hénon map (3.12) are given by*

$$\begin{aligned} x_0 &= \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2a} \,, \\ x_1 &= \frac{-(1-b) + \sqrt{(1-b)^2 + 4a}}{2a} \,. \end{aligned} \tag{3.18}$$

Exercise 3.5 **How strange is the Hénon attractor?**

Exercises

Exercise 4.1 **Trace-log of a matrix.** *Prove that*

$$\det M = e^{\mathrm{tr} \ln M} .$$

for an arbitrary finite dimensional matrix M.

Exercise 4.2 **Stability, diagonal case.** *Verify the relation (4.16)*

$$\mathbf{J}^t = e^{t\mathbf{A}} = \mathbf{U}^{-1} e^{t\mathbf{A}_D} \mathbf{U} , \quad \textit{where} \quad \mathbf{A}_D = \mathbf{U}\mathbf{A}\mathbf{U}^{-1} .$$

Exercise 4.3 **Topology of the Rössler flow.** *(continuation of exercise 3.1)*

(a) *Show that equation $|\det(A - \lambda \mathbf{1})| = 0$ for Rössler system in the notation of exercise 2.18 can be written as*

$$\lambda^3 + \lambda^2 c\,(p^{\mp} - \epsilon) + \lambda(p^{\pm}/\epsilon + 1 - c^2 \epsilon p^{\mp}) \mp c\sqrt{D} = 0 \tag{4.41}$$

(b) *Solve (4.41) for eigenvalues $\lambda^{\pm}$ for each equilibrium point as an expansion in powers of ϵ. Derive*

$$\begin{array}{ll} \lambda_1^- &= -c + \epsilon c/(c^2+1) + o(\epsilon) \\ \lambda_2^- &= \epsilon c^3/[2(c^2+1)] + o(\epsilon^2) \\ \theta_2^- &= 1 + \epsilon/[2(c^2+1)] + o(\epsilon) \\ \lambda_1^+ &= c\epsilon(1-\epsilon) + o(\epsilon^3) \\ \lambda_2^+ &= -\epsilon^5 c^2/2 + o(\epsilon^6) \\ \theta_2^+ &= \sqrt{1+1/\epsilon}\,(1 + o(\epsilon)) \end{array} \tag{4.42}$$

Compare with exact eigenvalues. What are dynamical implications of the extravagant value of λ_1^-?

(continued as exercise 4.3) *(Rytis Paškauskas)*

Exercise 4.4 **A contracting baker's map.** *Consider a contracting (or "dissipative") baker's map, acting on a unit square $[0,1]^2 = [0,1] \times [0,1]$, defined by*

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/3 \\ 2y_n \end{pmatrix} \quad y_n \le 1/2$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/3 + 1/2 \\ 2y_n - 1 \end{pmatrix} \quad y_n > 1/2$$

This map shrinks strips by a factor of 1/3 in the x-direction, and then stretches (and folds) them by a factor of 2 in the y-direction.

By how much does the phase space volume contract fo one iteration of the map?