Introduction to nonlinear dynamics and chaos

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Physics D60-0, Spring quarter 1999

Final exam

due 2:30 PM tuesday, June 8, 1999, Predrag's office, Tech F332

1 The prelude

The only way to developed intuition about chaotic dynamics is by computing, and you are urged to try to work through the essential steps in this take-home exam, combining the techniques learned in the course with whatever other explorations that seem of of interest to you. The exam is a real-life example of how one uses general tools of nonlinear dynamics to explore a real-life research problem. Here is Yueheng and he would like to understand turbulence. How does he get started? The steps are:

1. a problem is posed and formulated - here as the approximate equation (1) describing the dynamics of a flame front.
2. ways of reformulating the dynamics suited to numerical explorations are explored. Here the Fourier representation (5) seems a good starting point.
3. determine the fixed point(s)
4. compute the fixed point stabilities
5. explore the fixed point stabilities and the associated bifurcations as system parameter(s) are varied

So far all work has been analytic, and Yueheng has some fingertips feeling for the topology of solutions and bifurcation sequences. This is all one can extract from the problem by analysis not assisted by numerical experimentation. Next,

1. implement a numerical simulator for your problem
2. plot a variety of long orbits to get some sense for the attractors for different values of system parameters
3. find numerically stable cycles, if any
4. compute stable cycle stabilities (this might be too hard)
5. determine values of parameters for stable cycles $\rightarrow$ unstable cycles bifurcations (at least estimate by trial and error)
6. diagnose parameter values at which onset of chaos seems to be taking place
7. firm up your hunch by estimating by numerical simulation some chaos diagnostic, like existence of a positive Lyapunov exponent (though this might take too much time as part of the exam)
8. try to study such bifurcations in a Poincaré section
9. perhaps compute the unstable cycle eigenvectors (this might be too hard for this exam)
10. if a cycle is unstable, and you have succeeded in computing its unstable eigenvector(s), attempt to trace out its unstable manifold by starting with a set of points close to its Poincaré section fixed point, sprinkled along the unstable eigenvector
11. have a Carlsberg, perhaps the best beer in some parts of Copenhagen

In the exam tour guide that follows, I have indicated which steps are exam questions, and which are optional. I do not expect you to get through the whole length of the exam in the time allotted; do as much as feels right.

The list of references is appended only to amuse you, and none are needed in order to get through the exam.

2 Hopf's last hope, or: Turbulence, and what to do about it?

Hopf[1] and Spiegel[2, 3, 4] have proposed that the turbulence in spatially extended systems be described in terms of recurrent spatiotemporal patterns.

Pictorially, dynamics drives a given spatially extended system through a repertoire of unstable patterns; as we watch a turbulent system evolve, every so often we catch a glimpse of a familiar pattern. For any finite spatial resolution, for a finite time the system follows approximately a pattern belonging to a finite alphabet of admissible patterns, and the long term dynamics can be thought of as a walk through the space of such patterns, just as chaotic dynamics with a low dimensional attractor can be thought of as a succession of nearly periodic (but unstable) motions.

In this exam we explore such ideas in a spatially extended system claimed to describe the flutter of the flame front of gas burning in a cylindrically symmetric burner on your kitchen stove. Carrying out Hopf's program in a systematic manner is an open research problem, far too difficult as an exam question. Here we are happy if in a few days of analysis we succeed in simulating the system numerically, and develop some intuition about such systems on the level of Strogatz's textbook: determine the fixed points, stabilities, stability eigenvectors, bifurcations, onset of chaos.

3 Fluttering flame front

The Kuramoto-Sivashinsky equation[5, 6] is one of the simplest partial differential equations that exhibits chaos. It is a dynamical system extended in one spatial dimension, defined by

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx} \,. \tag{1}$$

In this equation $t \geq 0$ is the time and $x \in [0, 2\pi]$ is the space coordinate. The subscripts x and t denote the partial derivatives with respect to x and t; $u_t = du/dt$, u_{xxxx} stands for 4th spatial derivative of the "height of the flame front" $u = u(x,t)$ at position x and time t. ν is a "viscosity" damping parameter; its role is to suppress solutions with fast spatial variations. The term $(u^2)_x$ makes this a *nonlinear system.*

How good description of a flame front this is need not concern us here; suffice it to say that such model amplitude equations for interfacial instabilities arise in a variety of contexts - see e.g. reference [7] - and this one is perhaps the simplest physically interesting spatially extended nonlinear system. The salient feature of such partial differential equations is that for any finite value of the phase-space contraction parameter ν a theorem says that the asymptotic dynamics is describable by a *finite* set of "inertial manifold" ordinary differential equations[8]. We shall verify by numerical experimentation that even though $u(x,t)$ is in principle infinite dimensional (it has a component for each spatial point x), the attractors are indeed of finite dimension.

The "flame front" $u(x,t) = u(x+2\pi,t)$ is periodic on the $x \in [0,2\pi]$ interval, so the standard strategy is to expand it in a discrete spatial Fourier series:

$$u(x,t) = \sum_{k=-\infty}^{+\infty} b_k(t) e^{ikx} \,. \tag{2}$$

Since $u(x,t)$ is real,

$$b_k = b^*_{-k} \,. \tag{3}$$

Show that substituting (2) into (1) yields the infinite ladder of evolution equations for the Fourier coefficients b_k: Exam question

$$\dot{b}_k = (k^2 - \nu k^4) b_k + ik \sum_{m=-\infty}^{\infty} b_m b_{k-m} \,. \tag{4}$$

As $\dot{b}_0 = 0$, the solution integrated over space is constant in time. In what follows we shall assume that this average is zero, $\int dx\, u(x,t) = 0$.

The coefficients b_k are in general complex functions of time. We can simplify the system (4) further by considering the case of b_k pure imaginary, $b_k = i a_k$, where a_k are real, with the evolution equations

$$\dot{a}_k = (k^2 - \nu k^4) a_k - k \sum_{m=-\infty}^{\infty} a_m a_{k-m} \,. \tag{5}$$

Argue that this picks out the subspace of odd solutions $u(x,t) = -u(-x,t)$ Exam question (the optional reading in sect. 7 discusses this in more detail). Use (3) to further simplify the tower of evolution equations.

How you solve the equations numerically is up to you. Here are some of the options:

1. You can divide the x interval into a sufficiently fine discrete grid of N points, replace space derivatives (1) by approximate discrete derivatives, and integrate a finite set of first order differential equations for the discretized spatial components $u_j(t) = u(2\pi j/N, t)$, by any integration routine you feel comfortable with.

2. You can integrate numerically the Fourier modes (5), truncating the ladder of equations to a finite length N, i.e., set $a_k = 0$ for $k > N$. In my experience, for this exploration $N \leq 16$ truncations were sufficiently accurate.

3. If you happen to have such things handy, you can use more sophisticated numerical methods, such as pseudo-spectral methods or implicit methods. But that is much more sophisticated than what is expected here.

If your integration takes days and gobbles up terabits of memory, you are using some brain-damaged "high level" software. You should have written a few lines of the Runge-Kuta code in some mundane everyday language.

In my own simulations, I have have determined the solutions in the space of Fourier coefficients, and then reconstituted from them the spatiotemporal solutions of (1).

The trivial solution $u(x,t) = 0$ is a fixed point of (1).

Show that from (5) it follows that the $|k| < 1/\sqrt{\nu}$ long wavelength modes of this fixed point are linearly unstable, and the $|k| > 1/\sqrt{\nu}$ short wavelength modes are stable. For $\nu > 1$, $u(x,t) = 0$ is the globally attractive stable fixed point, i.e., the dissipation is so strong that any flame front burns out. Exam question

Starting with $\nu = 1$ the solutions go through a rich sequence of bifurcations, studied e.g. in reference [7].

What kind of bifurcation takes place as $\nu > 1 \to \nu < 1$? As ν decreases, are there any further bifurcations from the $u(x,t) = 0$ fixed point, and if so, of what type? Exam question

Try to determine some further fixed points of (5); if any, discuss bifurcations that lead to them, etc. (I have not checked myself in any significant detail what interesting fixed points are there beyond $u(x,t) = 0$, but there is surely a whole zoo, so do not spend too much time on this.) Optional

Detailed investigation of the parameter dependence of bifurcations sequences is too laborious for the time alloted; from here on we turn to numerical experimentation. We shall take $\sqrt{\nu}$ sufficiently small so that the dynamics can be spatiotemporally chaotic, but not so small that we would be overwhelmed by too many short wavelength modes needed in order to accurately represent the dynamics.

My advice how to do such exploration: start on *terra firma*, at $\nu = 1$, low N, and decrease ν a little bit, integrate until the trajectory has settled down; then decrease ν a little bit again, integrate until the trajectory has settled down. Repeat. Stop incrementing ν for a bit, increment N instead and check how sensitive is your attarctor to truncation size. You will sail thorugh a sequence of bifurcations and enter chaos, most likely via period-doubling route. This "adiabatic" approach has advantage of (almost) always starting you close to the attractor and avoids potentially long transients of arbitrary starting conditions.

4 Fourier modes truncations

The growth of the unstable long wavelengths (low $|k|$) excites the short wavelengths through the nonlinear term in (5). The excitations thus transferred are dissipated by the strongly damped short wavelengths, and a sort of "chaotic equilibrium" can emerge. The very short wavelengths $|k| \gg 1/\sqrt{\nu}$ will remain small for all times, but the intermediate wavelengths of order $|k| \sim 1/\sqrt{\nu}$ will play an important role in maintaining the dynamical equilibrium. Hence, while one may truncate the high modes in the expansion (5), care has to be exercised to ensure that no modes essential to the dynamics are chopped away. In practice one does this by repeating the same calculation at different truncation cutoffs N, and making sure that inclusion of additional modes has no effect within the accuracy desired.

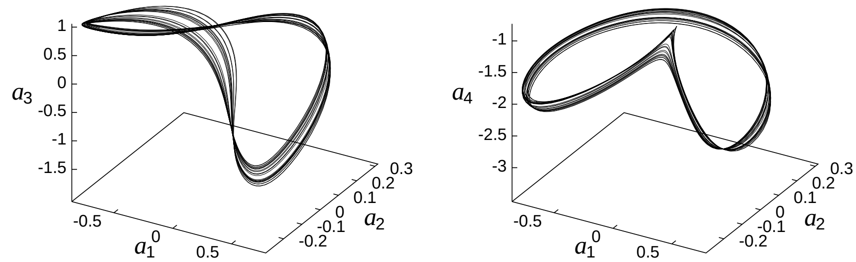

Figure 1: Projections of a typical 16-dimensional trajectory onto different 3-dimensional subspaces, coordinates (a) $\{a_1, a_2, a_3\}$, (b) $\{a_1, a_2, a_4\}$. $N = 16$ Fourier modes truncation with $\nu = 0.029910$.

When we simulate the equation (5) on a computer, we have to truncate the ladder of equations to a finite length N, i.e., set $a_k = 0$ for $k > N$. N has to be sufficiently large that no harmonics a_k important for the dynamics with $k > N$ are truncated. On the other hand, computation time increases with the increase of N.

For reasons that will be explained below, I have performed my numerical calculations taking $N = 16$.

The problem with such high dimensional truncations of (5) is that the dynamics is difficult to visualize. Best we can do without much programming is to examine trajectory's projections onto any three axes a_i, a_j, a_k.

Plot your trajectory for the same ν, the same two or three axes as in figure 1; is your dynamics qualitatively the same as in my plots? Exam question

5 Poincaré section

Optional

The question is how to look at such flow? Usually one of the first steps in analysis of such flows is to restrict the dynamics to a Poincaré section. I fix (arbitrarily) the Poincaré section to be the hyperplane $a_1 = 0$, and integrate (5) with the initial conditions $a_1 = 0$, and arbitrary values of the coordinates $a_2, \ldots, a_N$, where N is the truncation order. When a_1 becomes 0 the next time, the coordinates $a_2, \ldots, a_N$ are mapped into $(a'_2, \ldots a'_N) =$

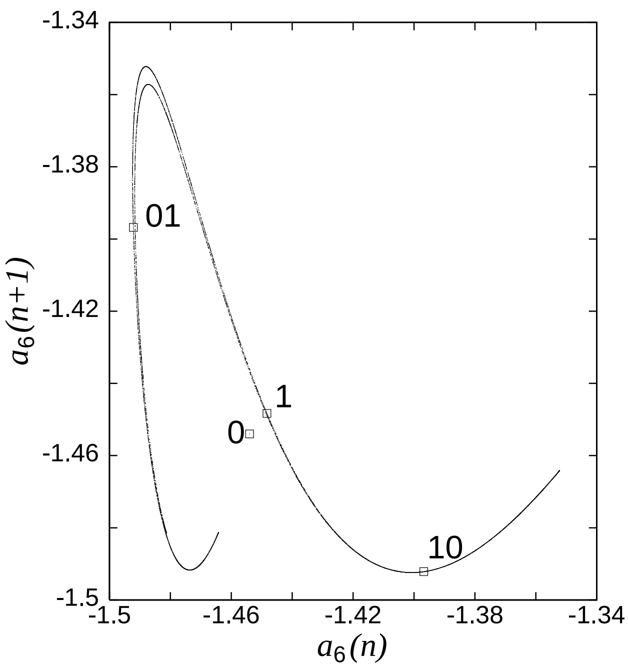

Figure 2: The attractor of the system (5), plotted as the a_6 component of the $a_1 = 0$ Poincaré section return map, 10,000 Poincaré section returns of a typical trajectory. Indicated are the periodic points $\overline{0}$, $\overline{1}$ and $\overline{01}$. $N = 16$ Fourier modes truncation with $\nu = 0.029910$.

$P(a_2, \ldots, a_N)$, where P is the Poincaré mapping of a $N-1$ dimensional hyperplane into itself. Figure 2 is an example of a results that one gets. While the topology of the attractor is still obscure, one thing is clear - as claimed in the introduction, the attractor is finite and thin, barely thicker than a line.

6 Bifurcation trees

Optional

Provided we have figured out how to generate numerically a Poincaré section, we can let computer run and do some numerical fishing on our behalf. Figure 3 is a representative bifurcation diagram for the system at hand. To obtain this figure, we took a random initial point, iterated it for a some time to let it settle on the attractor and then plotted the a_6 coordinate of the next 1000 intersections with the Poincaré section. Repeating this for different values of the damping parameter ν, one can obtain a picture of the attractor as a function of ν. For an intermediate range of values of ν, the dynamics exhibits a rich variety of behaviors, such as period-doubling, strange attractors, stable limit cycles, and so on.

I have found that the minimum value of N to get any chaotic behavior at all was $N = 9$.

Do you get any chaos for $N < 9$?

Exam question

The dynamics for the $N = 9$ truncated system is rather different from the full system dynamics, and therefore I have performed all calculations reported here for $N = 16$, which seemed a reasonable cutoff. Having been there, done that, I recommend examining in particular two values of the damping parameter: $\nu = 0.029910$, for which the system is chaotic, and $\nu = 0.029924$, for which the system has a stable period-3 cycle.

Do you get a stable cycle for $\nu = 0.029924$?

Exam question

7 Symmetry decomposition

Optional

Before proceeding with the calculations, we take into account the symmetries of the solutions. Consider the spatial flip and shift symmetry op-

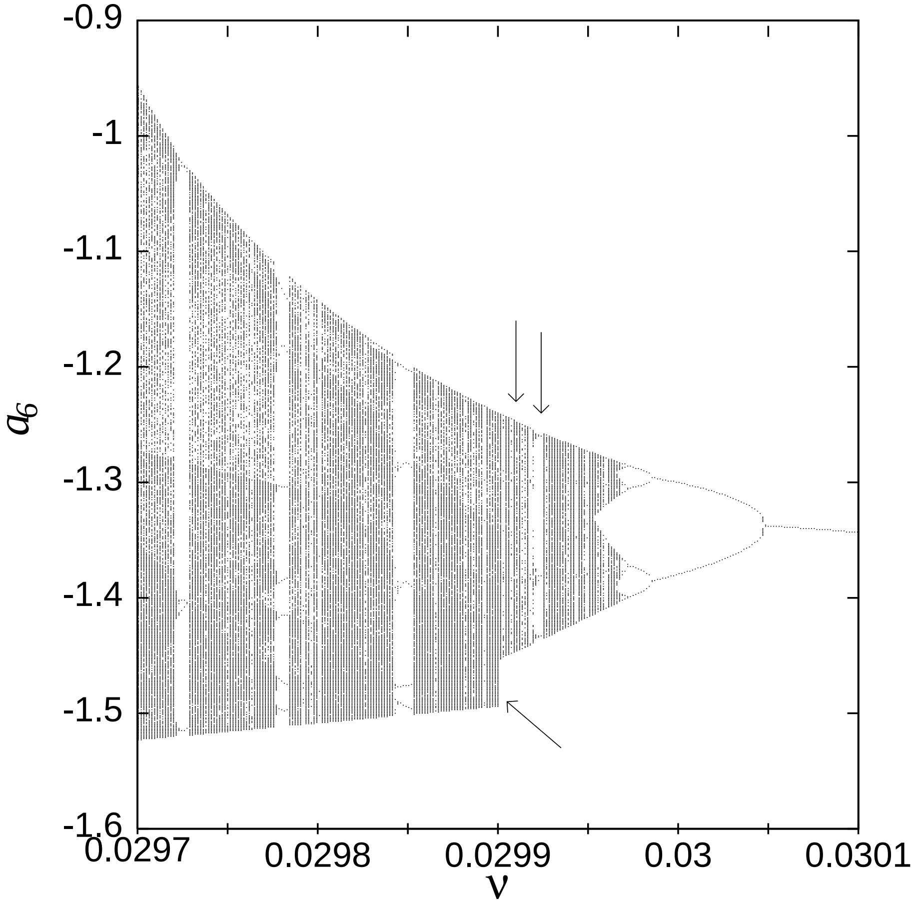

Figure 3: Period-doubling tree for coordinate a_6, $N = 16$ Fourier modes truncation of (5). The two upper arrows indicate the values of damping parameter that we use in our numerical investigations; $\nu = 0.029910$ (chaotic) and $\nu = 0.029924$ (period-3 window). Truncation to $N = 17$ modes yields a similar figure, with values for specific bifurcation points shifted by $\sim 10^{-5}$ with respect to the $N = 16$ values. The choice of the coordinate a_6 is arbitrary; projected down to any coordinate, the tree is qualitatively the same.

erations $Ru(x) = u(-x)$, $Su(x) = u(x+\pi)$. The latter symmetry reflects the invariance under the shift $u(x,t) \to u(x+\pi,t)$, and is a particular case of the translational invariance of the Kuramoto-Sivashinsky equation (1). In the Fourier modes decomposition (5) this symmetry acts as $S: a_{2k} \to a_{2k}, a_{2k+1} \to -a_{2k+1}$. Relations $R^2 = S^2 = 1$ induce decomposition of the space of solutions into 4 invariant subspaces[7]; the restriction to $b_k = ia_k$ that lead to simplified set of equations (5) amounts to specializing to a subspace of odd solutions $u(x,t) = -u(-x,t)$.

Now, with the help of the symmetry S the whole attractor $\mathcal{A}_{tot}$ can be decomposed into two pieces: $\mathcal{A}_{tot} = \mathcal{A}_0 \cup S\mathcal{A}_0$ for some set $\mathcal{A}_0$. It can happen that the set $\mathcal{A}_0$ (the symmetrically decomposed attractor) can be decomposed even further into four disjoint sets: $\mathcal{A}_{tot} = \mathcal{A} \cup S\mathcal{A} \cup \Theta\mathcal{A} \cup \Theta S\mathcal{A}$.

8 Strange interlude

You might have wondered why am I giving you values of the viscosity parameter ν accurate to 5 significant figures, if all we want is to get a qualitative feeling for the flame front flutter?

The problem is that it is extremely hard to prove that an attractor is chaotic. Adding an extra dimension to a truncation of the system (5) introduces a small perturbation, and this can (and often will) throw the system into a totally different asymptotic state. A chaotic attractor for $N = 15$ can become a period three window for $N = 16$, and so on.

Let us switch gears for a moment, and perform a numerical experiment that will enable you to do a part of this exam even if all your integration programs are in shambles.

8.1 How strange is the Hénon attractor?

Numerical studies indicate that for $a = 1.4$, $b = 0.3$ the attractor of the Hénon map (see pictures in the Strogatz's book)

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + by_n \\ y_{n+1} &= x_n \,. \end{aligned}$$

is "strange". Reproduce the Hénon picture of his "strange attractor" by numerical iteration of the map. Next, repeat the numerical experiment for the map with parameter variation as minute as changing a to $a = 1.39945219$. If you wait long enough (100,000's of iterations), the attractor should undergo a dramatic change. What do you get?

The moral of this experiment is that "strange attractors" are not structurally stable. If we compute, for example, the Lyapunov exponent $\lambda(\nu, N)$ for the strange attractor of the system (5), there is no reason to expect $\lambda(\nu, N)$ to smoothly converge to the limit value $\lambda(\nu, \infty)$ as $N \to \infty$.

9 Tour of a few numerical results

If we are integrating an unstable, chaotic solution in the Fourier space, we can go back to the configuration space using (2) and plot the corresponding spatiotemporal solution $u(x, t)$.

Plot a spatiotemporal solution $u(x, t)$ for the chaotic, $\nu = 0.029910$ attractor. Exam question

Staring at the solution as it evolves in time we should start getting a glimpse of the repertoire of the spatiotemporal patterns that Hopf wanted us to see in turbulent dynamics.

More precisely, he wanted us to see *recurrent* patterns, that is to say, the unstable spatiotemporally periodic solutions of our equations. This can be done, but is hard work - I list a few computed by Freddy Christiansen in table 1, and plot the shortest one in figure 4, just to give you a feeling for the form and stability of such solutions. Other solutions exhibit the same overall gross structure - a few wiggles here and there, continuously in flux and yet so alike.

One of the objectives of a theory of turbulence is to predict measurable global averages over turbulent flows, such as velocity-velocity correlations and transport coefficients. With the present parameter values we are far from any strongly turbulent regime, and in fact we are lucky if in the time alloted we manage to implement even the simplest test of chaotic dynamics: evaluation of the Lyapunov exponents.

Table 1: A few unstable cycles for the $N = 16$ Fourier modes truncation of the Kuramoto-Sivashinsky equation (5), damping parameter $\nu = 0.029910$ (chaotic attractor) and $\nu = 0.029924$ (period-3 window), periods, the first four stability eigenvalues. The deviation from unity of Λ_2, the eigenvalue along the flow, is an indication of the accuracy of the numerical integration.

p	T_p	Λ_1	$\Lambda_2 - 1$	Λ_3	Λ_4
Chaotic, $\nu = 0.029910$					
0	0.897653	3.298183	$5{\cdot}10^{-12}$	$2.793085{\cdot}10^{-3}$	$2.793085{\cdot}10^{-3}$
1	0.870729	-2.014326	$5{\cdot}10^{-12}$	$6.579608{\cdot}10^{-3}$	$3.653655{\cdot}10^{-4}$
10	1.751810	-3.801854	$8{\cdot}10^{-12}$	$3.892045{\cdot}10^{-5}$	$2.576621{\cdot}10^{-7}$
Period-3 window, $\nu = 0.029924$					
0	0.897809	3.185997	$7{\cdot}10^{-13}$	$2.772435{\cdot}10^{-3}$	$-2.772435{\cdot}10^{-3}$
1	0.871737	-1.914257	$5{\cdot}10^{-13}$	$6.913449{\cdot}10^{-3}$	$-3.676167{\cdot}10^{-4}$
10	1.752821	-3.250080	$1{\cdot}10^{-12}$	$4.563478{\cdot}10^{-5}$	$2.468647{\cdot}10^{-7}$

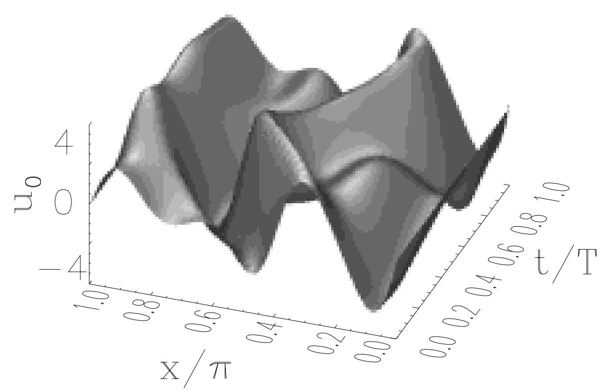

Figure 4: Spatiotemporally periodic solution $u_0(x,t)$. We have divided x by π and plotted only the $x > 0$ part, since we work in the subspace of the odd solutions, $u(x,t) = -u(-x,t)$. $N = 16$ Fourier modes truncation with $\nu = 0.029910$.

For the strange attractor at $\nu = 0.029910$ our numerical simulation estimate for the **Lyapunov exponent** is 0.629. What do you get?

Exam question

10 Wrapping up this guided tour

Hopf's proposal for a theory of turbulence was to think of turbulence as a sequence of near recurrences of a repertoire of unstable spatiotemporal patterns. This exam falls short of implementing the proposal, but it sheds some light on how such ideas are developed - numerical solutions that you have studied are both "turbulent" and recognizable to the eye.

Hopf's proposal is in its spirit very different from most ideas that animate current turbulence research. It is distinct from the Landau quasiperiodic picture of turbulence as a sum of infinite number of incommensurate frequencies, with dynamics taking place on a large-dimensional torus. It is not the Kolmogorov's 1941 homogeneous turbulence with no coherent structures fixing the length scale, here all the action is in specific coherent structures. And it is *not* probabilistic; everything is fixed by the deterministic dynamics with no probabilistic assumptions on the velocity distributions or external stochastic forcing.

The parameter ν values that we have played with correspond to the weakest nontrivial "turbulence", and it is an open question to what extent the approach remains implementable as the system goes more turbulent.

Have a Carlsberg, and a good summer.

References

[1] Hopf E 1942 *Abzweigung einer periodischen Lösung Bereich. Sächs. Acad. Wiss. Leipzig, Math. Phys. Kl.* **94** 19. This is presumably not the reference. The story so far goes like this: in 1960 E.A. Spiegel was P. Kraichnan's research associate. Kraichnan told him: "Flow follows a regular solution for a while, then another one, then switches to another one; that's turbulence". It was not too clear, but Kraichnan's vision of turbulence moved Spiegel. In 1962 E.A. Spiegel and D. Moore investigated a 3rd order convection equations which seemed to follow

one periodic solution, then another, and continued going from periodic solution to periodic solution. Ed told Derek: "This is turbulence!" and Derek said "This is wonderful!" and was moved. He went to give a lecture at Caltech sometime in 1964 and came back angry as hell. They pilloried him there: "Why is this turbulence?" they kept asking and he could not answer, so he expunged the word "turbulence" from their 1966 article[2] on periodic solutions. In 1970 E.A. Spiegel met P. Kraichnan and told him: "This vision of turbulence of yours has been very useful to me." Kraichnan said: "That wasn't my vision, that was Hopf's vision". What Hopf *actually* said and where he said it remains deeply obscure to this very day. There are papers that lump him together with Landau, as the "Landau-Hopf's incorrect theory of turbulence", but he did not seem to propose incommensurate frequencies as building blocks of turbulence, which is what Landau's guess was.

[2] Moore D W and Spiegel E A 1966 A thermally excited nonlinear oscillator *Astrophys. J.* **143** 871

[3] Baker N H, Moore D W and Spiegel E A 1971 *Quatr. J. Mech. and Appl. Math.* **24** 391

[4] Spiegel E A 1987 Chaos: a mixed metaphor for turbulence *Proc. Roy. Soc.* **A413** 87

[5] Kuramoto Y and Tsuzuki T 1976 Persistent propagation of concentration waves in dissipative media far from thermal equilibrium *Progr. Theor. Physics* **55** 365

[6] Sivashinsky G I 1977 Nonlinear analysis of hydrodynamical instability in laminar flames - I. Derivation of basic equations *Acta Astr.* **4** 1177

[7] Kevrekidis I G, Nicolaenko B and Scovel J C 1990 Back in the saddle again: a computer assisted study of the Kuramoto-Sivashinsky equation *SIAM J. Applied Math.* **50** 760

[8] See e.g. Foias C, Nicolaenko B, Sell G R and Témam R 1988 Kuramoto-Sivashinsky equation *J. Math. Pures et Appl.* **67** 197