# Armbruster-Guckenheimer-Holmes flow

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## Contents

# 1 Go with the flow

## 1.1 Armbruster-Guckenheimer-Holmes flow

The dynamical system $(\mathbf{C}^2, f)$ proposed by Armbruster, Guckenheimer and Holmes [1] is given by the set of 2 complex differential equations

$$
\begin{aligned}
\dot{z}_1 &= \bar{z}_1 z_2 + z_1 \left(\mu_1 + e_{11}|z_1|^2 + e_{12}|z_2|^2\right) \\
\dot{z}_2 &= \pm z_1^2 + z_2 \left(\mu_2 + e_{21}|z_1|^2 + e_{22}|z_2|^2\right),
\end{aligned}
\tag{1}
$$

where $z_1, z_2 \in \mathbf{C}$, and $\mu_j, e_{jk} \in \mathbf{R}$. It can be reformulated with $z_j = x_j + y_j$ as a flow on the state space $\mathbf{R}^4$ created by 4 real differential equations

$$
\begin{aligned}
\dot{x}_1 &= x_1 x_2 + y_1 y_2 + x_1(\mu_1 + e_{11}(x_1^2 + y_1^2) + e_{12}(x_2^2 + y_2^2) \\
\dot{y}_1 &= x_1 y_2 - y_1 x_2 + y_1(\mu_1 + e_{11}(x_1^2 + y_1^2) + e_{12}(x_2^2 + y_2^2) \\
\dot{x}_2 &= \pm(x_1^2 - y_1^2) + x_2(\mu_2 + e_{21}(x_1^2 + y_1^2) + e_{22}(x_2^2 + y_2^2) \\
\dot{y}_2 &= \pm 2 x_1 y_1 + y_2(\mu_2 + e_{21}(x_1^2 + y_1^2) + e_{22}(x_2^2 + y_2^2))
\end{aligned}
\tag{2}
$$

or, letting $z_j = r_j e^{i\theta_j}$, as a flow on $\mathbf{R}^3$ through

$$
\begin{aligned}
\dot{r}_1 &= r_1 r_2 cos\phi + r_1(\mu + e_{11} r_1^2 + e_{12} r_2^2) \\
\dot{r}_2 &= \pm r_1^2 cos\phi + r_1(\mu_2 + e_{21} r_1^2 + e_{22} r_2^2) \\
\dot{\phi} &= -(2r_2 \pm r_1^2/r_2) sin\phi,
\end{aligned}
\tag{3}
$$

where $\phi = 2\theta_1 - \theta_2$.

## 1.2 Symmetry

The flow $(\mathbf{C}^2$,f) is invariant under the symmetry group $O(2)$, i.e.

$$
\forall g \in O(2)\ \forall (z_1, z_2) \in \mathbf{C}^2 : \qquad f(g(z_1, z_2)) = g(f(z_1, z_2)).
$$

This can easily be seen by doing the straightforward calculation using the representation

$$
\begin{aligned}
R(\theta)(z_1, z_2) &= (e^{i\theta} z_1, e^{i2\theta} z_2), \quad \theta \in [0, 2\pi), \\
\kappa(z_1, z_2) &= (\bar{z}_1, \bar{z}_2)
\end{aligned}
$$

of $O(2)$. What is special about it, is, that this yields a continuous symmetry, which is responsible for special phenomena in the further examination.

### 1.3 Fixed-point subspaces

Subgroups $H$ of the symmetry group $G$ operating on a set $M$ give naturally rise to fixed-point subspaces of the state space, where a fixed-point subspace $Fix(H)$ is defined as

$$Fix(H) = \{x \in M \mid \forall h \in H: \; hx = x\}.$$

Fixed-point subspaces are very important, as the action of the flow leaves them invariant, $f(Fix(H)) \subseteq Fix(H)$, and can be therefore used as a starting point for a further analysis of the flow.
In the case of the flow at hand, there exist, up to conjugation of the subgroups by other group elements of $O(2)$, three non-trivial fixed-point subspaces:

- the real subspace $Fix(\kappa) = \{(z_1, z_2) \in \mathbf{C}^2 \mid Im(z_1) = Im(z_2) = 0\}$
- the 2-subspace $Fix(R(\pi)) = \{(z_1, z_2) \in \mathbf{C}^2 \mid z_1 = 0\}$
- the $x_2$-axis $Fix(\kappa) \cap Fix(R(\pi))$

## 2 Equilibria

Equilibria of a flow are defined as the zeros of the velocity function, that means for the flow at hand the zeros of (1) or, equivalently, of (2). To determine their behaviour in terms of stable and unstable manifolds, the real stability matrix $\mathbf{A}$ is needed:

$$\mathbf{A} = \begin{bmatrix} x_2 + 2e_{11}x_1^2 + a_1 & y_2 + 2e_{11}x_1y_1 & x_1 + 2e_{12}x_1x_2 & y_1 + 2e_{12}x_1y_2 \\ y_2 + 2e_{12}x_1y_1 & -x_2 + 2e_{11}y_1^2 + a_1 & -y_1 + 2e_{12}x_2y_1 & x_1 + 2e_{12}y_1y_2 \\ \pm 2x_1 + 2e_{21}x_1x_2 & \mp 2y_1 + 2e_{21}x_2y_1 & 2e_{22}x_2^2 + a_2 & 2e_{22}x_2y_2 \\ \pm 2y_1 + 2e_{21}x_1y_2 & \pm 2x_1 + 2e_{21}y_1y_2 & 2e_{22}x_2y_2 & 2e_{22}y_2^2 + a_2 \end{bmatrix},$$

where

$$\begin{aligned} a_1 &= \mu_1 + e_{11}(x_1^2 + y_1^2) + e_{12}(x_2^2 + y_2^2) \\ a_2 &= \mu_2 + e_{21}(x_1^2 + y_1^2) + e_{22}(x_2^2 + y_2^2). \end{aligned}$$

### 2.1 Equilibria in the 2-subspace

In the 2-subspace (1) reduces because of $z_1 = 0$ to

$$\dot{z}_2 = z_2\left(\mu_2 + e_{22}|z_2|^2\right).$$

So one gets the trivial equilibrium $(z_1, z_2) = (0, 0)$, or equivalently, $(x_1, y_1, x_2, y_2) = (0, 0, 0, 0)$ with eigenvalues $\mu_1$ and $\mu_2$ with (un)stable manifolds along the four coordinate axis, and for $z_2 \neq 0$ in case $\mu_2/e_{22} < 0$ infinitely many fixed points for $|z_2| = \sqrt{-\mu_2/e_{22}}$. For those equilibria, the stability matrix $\mathbf{A}$ gets block-diagonal, so that one can determine the Floquet exponents $\mu_1 - \mu_2 e_{12}/e_{22} \pm \sqrt{\mu_2/e_{22}}$ with eigenvectors in the $z_2 = 0$-plane and 0 and $-2\mu_2$ in the orthogonal $z_1 = 0$-plane. The precise eigenvectors differ for the different equilibria and cannot be given in general.

### 2.2 Equilibria in the real subspace

For $y_1 = y_2 = 0$, (2) reduces to

$$\begin{aligned} \dot{x}_1 &= x_1x_2 + x_1\left(\mu_1 + e_{11}x_1^2 + e_{12}x_2^2\right) \\ \dot{x}_2 &= \pm x_1^2 + x_2\left(\mu_2 + e_{21}x_1^2 + e_{22}x_2^2\right). \end{aligned} \tag{4}$$

While the solutions with $x_1 = 0$ have already been looked at in the previous section, there of course remain solutions with $x_1 \neq 0$. After a short analytical intermezzo, they can best be evaluated numerically.
Generally, for a given setting of the parameters one can only expect a finite number of equilibria in the real subspace. Setting the first equation of (4)

to zero, yields, if $x_1 \neq 0$, an equation for $x_1^2$ as a polynomial in $x_2$ of degree 2, which can be used in the second equation. This in turn results in a polynomial of degree 3 in $x_2$, assuring, that all together there are at most six solutions of (4). This procedure results in the set of equations

$$0 = (e_{22} - e_{21}e_{12}/e_{11})\, x_2^3 + (\mp e_{12}/e_{11} - e_{21}/e_{11})x_2^2 + (\mu_2 \mp 1/e_{11} - \mu_1 e_{21}/e_{11})\, x_2 \mp \mu_1/e_{11}$$
$$x_1^2 = -(\mu_1 + x_2 + e_1 2x_2^2)/e_{11}$$

and can be solved for a given set of parameters easily numerically. For example, one can find for $\mu_1 = \mu_2 = -1, e_{11} = e_{12} = e_{21} = e_{22} = 1$ in the case of the upper sign the fixed points $(x_1, y_1, x_2, y_2) = (\pm 1, 0, -1, 0)$ and $(x_1, y_1, x_2, y_2) = (\pm 0.5, 0, 0.5, 0)$. A general analysis of Floquet exponents gets very hard; for a specific setting of the parameters, a numerical analysis is in spite easily possible (see chapter about periodic orbits in the real subspace).

## 3 Relative equilibria

The problem of finding equilibria and periodic orbits of the Armbruster-Guckenheimer-Holmes flow reveals some interesting consequences of the symmetry. When (3) was derived from (1), the two equations

$$\begin{aligned}\dot{\theta}_1 &= -r_2 sin\phi \\ \dot{\theta}_2 &= \pm(r_1^2/r_2) sin\phi\end{aligned}$$

were combined to get the equation for $\dot{\phi}$. The full set of 4 equations shows, that the zeros of (1) or (2) imply zeros of (3). But that means, that they belong to solutions of (3) with $\phi = 0$ or $\pi$; those are the *steady solutions.*
The other fixed points of (3) with $\phi \neq 0$, $\pi$ are so-called *travelling waves* or *relative equilibria* of (1) with a constant phase difference. In order to get a zero of the third equation of (3) for a fixed $\phi \neq 0$, $\pi$, it has to hold, that $2r_2^2 = \mp r_1^2$. Therefore, such equilibria only can occur for the lower sign in the flow. Setting (2) to 0 in this special case yields the solutions

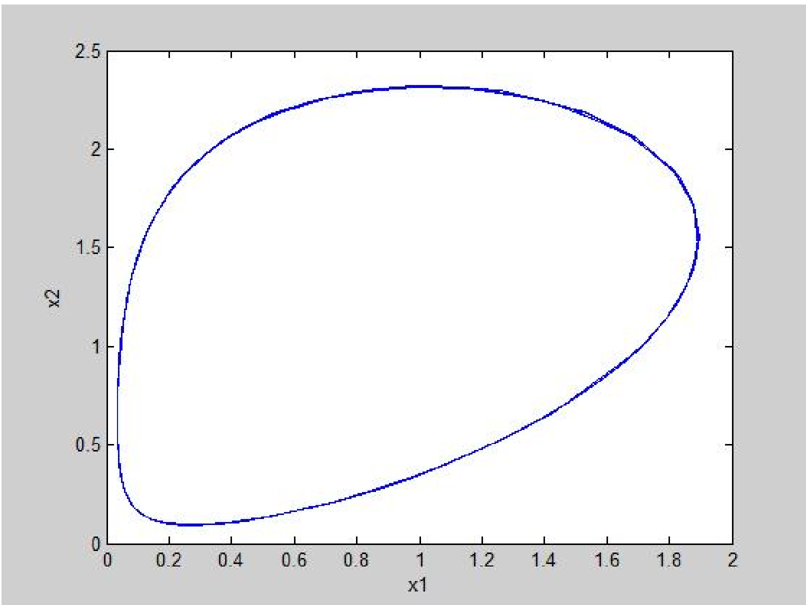
$$\begin{aligned}r_2^2 &= \frac{-(2\mu_1 + \mu_2)}{4e_{11} + 2e_{12} + 2e_{21} + e_{22}} \\ cos(\phi) &= \frac{\mu_2(2e_{11} + e_{12}) - \mu_1(2e_{21} + e_{22})}{[-(2\mu_1 + \mu_2)(4e_{11} + 2e_{12} + 2e_{21} + e_{22})]^{1/2}} \\ r_1^2 &= 2r_2^2\end{aligned}$$

Under consideration of the natural restriction $r_1$, $r_2 > 0$, for $\mu_1 = \mu_2 = -1, e_{11} = e_{12} = e_{21} = e_{22} = 1$, these equations have for example the solution $(r_1, r_2, \phi) = (0.8165, 0.5774, \pi/2)$. This relative equilibrium passes for example the point $(x_1, y_1, x_2, y_2) = (0, 0.8165, 0, 0.5774)$ and travels through the state space with the constant phase difference $\phi = \pi/2$.

# 4 Periodic orbits

As an example for finding periodic orbits, one in the real subspace is considered. For $\mu_2/e_{22} > 0$, there are just the fixed points of the real subspace and the origin. For example for the parameters $\mu_1 = 1; \mu_2 = -1; e_{11} = 2; e_{12} = -4; e_{21} = 3; e_{22} = -0.5$ and in the case of the upper sign one finds the equilibrium $(0, 0, 0, 0)$ with a repelling Floquet exponent along the $x_2$- and an attracting one along the $x_1$-axis. Also, there is the fixed point $(0.5493, 0, 0.7703)$ with a complex conjugate pair of eigenvalues $0.1112 \pm 3.0188i$ and eigenvectors in the real plane resulting in a spiraling out-movement. In addition, it exists a third equilibrium at $(0.8934, 0, -0.6903, 0)$ with a similar behavior: a complex pair of eigenvalues $1.9362 \pm 3.0938i$ leading to a spiraling out in the real plane. For the analysis in the real subspace in general the eigenvalues with eigenvectors in the complex directions are not important and therefore not given.

The combination of the origin with its repelling and attracting properties and one of the spiraling out equilibria looks promising in order to find a periodic orbit of the flow. A plot of the trajectory points in the same direction, and finally with the starting guess $(0.1, 0, 0.1, 0)$, period $T = 4.2$, a run of the Newton-routine yields the exact orbit point $(0.1274, 0, 0.1338, 0)$ with period $T = 4.0261$.



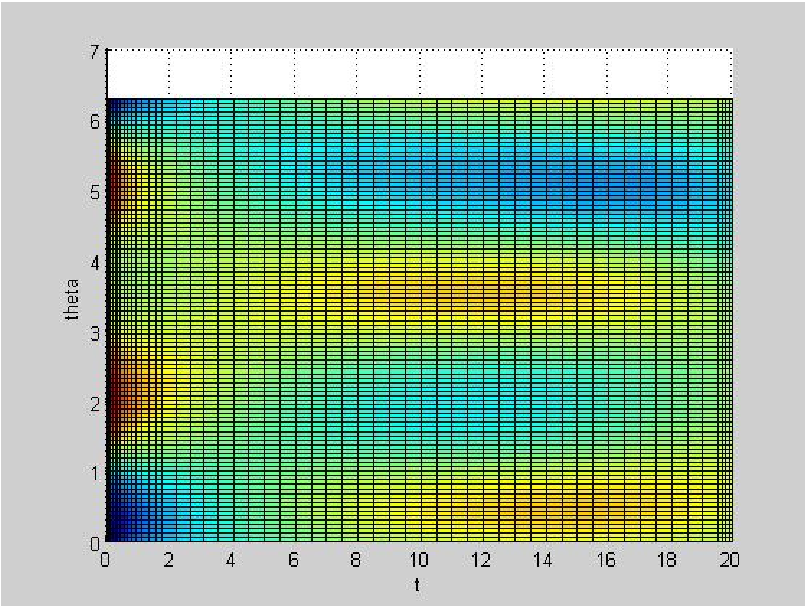

It is not surprising, that this periodic orbit is easy to find, as it is attracting for the given parameter set. The Floquet multipliers with eigenvectors in the real subspace are almost 1 and 0.4870.

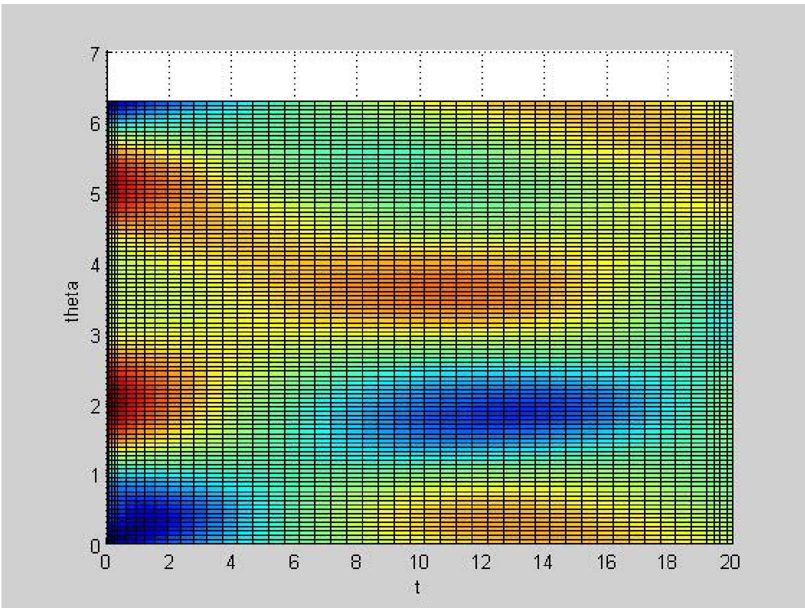
## 5 Relative periodic orbits

In [1] it is proven, that there cannot exist any relative periodic orbits with $\phi \neq 0$ or $\pi$ in the case of the upper sign. The proof is quite simple and is done by a look at the phase equation in (3). For the upper, the '+' sign, the term in the parenthesis is for $r_2 \neq 0$ strictly greater than 0. For $\phi \in (0, \pi)$, it holds that $\dot{\phi} < 0$ and for $\phi \in (\pi, 2\pi)$ clearly $\dot{\phi} > 0$. Thus $\phi(t) \to 0$, and therefore there cannot exist a stable solution with $\phi \neq 0$ or $\pi$. Also for $r_2 = 0$ these observation still holds. For $\phi \neq \pi$, $\phi$ just goes to inifinity, and for $\phi = \pi$, (3) tells, that the trajectory must have passed points near $\phi = 0, 2\pi$ in order for $\dot{r_2}$ to be $neq0$. But then, as $\dot{r_2} = r_1^2 cos(\phi)$ holds for $r_2 = 0$, the solution cannot reach $r_2 = 0$ again and is therefore no periodic solution.
In the case of the lower, the '-' sign, one might encounter all kinds of relative periodic orbits. To find them, one first has to come up with good initial guesses, before one can run a slightly modified Newton routine to really track them down.
So, what can be done to find initial guesses, is, to run the dynamics for an arbitrary starting point in space and then evaluate possible recurrences on a two-dimensional grid. Therefore, for discrete steps of time $t$, one evaluates for discrete steps of $\theta$

$$\parallel R(\theta) f^t(x_0) - x_0 \parallel$$

and color-codes this quantity in a graph. It is a measure, how close the flow comes back to its initial value with respect to a symmetry operation. In the following, two examples of these color-codes for the parameter set $\mu_1 = 0.135, \mu_2 = 0.2, e_{11} = -4, e_{12} = -1, e_{21} = e_{22} = -2$ also used in [1] are given, where blue stands for small and red for high values. The starting points in space were $(0.5, 0.2, 0.4, 0.5)$ and $(0.2, 0.2, 0.3, 0.2)$ respectively.




Out of these graphs one can get the initial guesses $T = 15, \theta = -5$ for the first and $T = 13, \theta = -2$ for the second point.
The Newton routine is slightly enlarged and is the result of the linearisation of the equation

$$R(\theta)f^T(x_0) = x_0.$$

It can be implemented, if one recalls, that

$$\frac{\partial R(\theta)}{\partial \theta} = \frac{\partial e^{g\theta}}{\partial \theta} = gR(\theta),$$

where

$$R(\theta) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & cos(2\theta) & -sin(2\theta) \\ 0 & 0 & sin(2\theta) & cos(2\theta) \end{bmatrix}, g = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

is the rotation operation $R(\theta)$ represented in four real dimensions and $g$ its generator. With the additional condition $gx_0\Delta x = 0$, it can be taken care of the symmetry by implementing it as a sixth row in the Newton matrix. Unfortunately, out of numerical problems it was not possible to implement the Newton routine properly, so I cannot provide points on periodic orbits other than the starting guesses. But as this is not a principal problem, I am sure, that in this way relative periodic orbits can be found.

## References


[1] D. Armbruster, J. Guckenheimer, and P. Holmes. Heteroclinic cycles and modulated travelling waves in systems with O(2) symmetry. *Physica D*, 29:257–282, 1988.