kicked rotor
cat in 1 spacetime dimension

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ChaosBook.org/overheads/spatiotemporal
$\to$ Chaotic field theory slides

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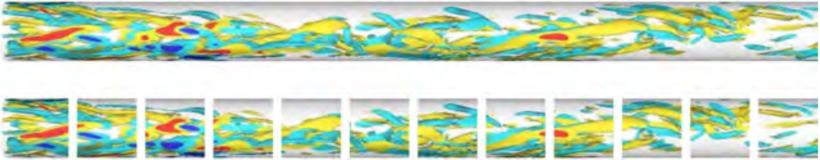
building blocks od turbulence



have : a detailed theory of small turbulent cells

construct : the infinite state by coupling turbulent cells[1]

what would that theory look like ?

[1] M. N. Gudorf et al., *Spatiotemporal tiling of the Kuramoto-Sivashinsky flow*, In preparation, 2021.

coin toss ? that's not physics !

Field Theory should be Hamiltonian and energy conserving
Quantum Mechanics requires it

because that is physics !

need a system as simple as the Bernoulli, but mechanical

so, we move on from running in circles,

to a mechanical rotor to kick.

next : a kicked rotor

Du mußt es dreimal sagen!
— Mephistopheles

1. what this is about
2. coin toss
3. **kicked rotor**
4. spatiotemporal cat
5. bye bye, dynamics

field theory in 1 spacetime dimension

we now define

the cat map in 1 spacetime dimension

then we generalize to

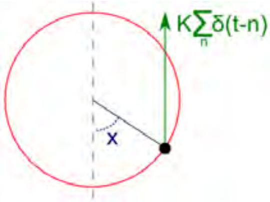
d-dimensional spatiotemporal cat

- cat map in Hamiltonian formulation
- cat map in Lagrangian formulation
  (so much more elegant!)

time-evolution formulation

example of a "small domain" dynamics : a single kicked rotor

an electron circling an atom, subject to
a discrete time sequence of angle-dependent kicks $F(x_t)$



Taylor, Chirikov and Greene standard map

$$
\begin{aligned}
x_{t+1} - x_t &= p_{t+1} \qquad \text{mod } 1 \\
p_{t+1} - p_t &= F(x_t)
\end{aligned}
$$

→ chaos in Hamiltonian systems

the simplest example : a cat map evolving in time

force $F(x) = Kx$ linear in the displacement x , $K \in \mathbb{Z}$

$$\begin{aligned} x_{t+1} &= x_t + p_{t+1} & \mod 1 \\ p_{t+1} &= p_t + Kx_t & \mod 1 \end{aligned}$$

Continuous Automorphism of the Torus, or

time-evolution cat map

a linear, area preserving map of a 2-torus onto itself

$$\begin{bmatrix} \phi_t \\ \phi_{t+1} \end{bmatrix} = J \begin{bmatrix} \phi_{t-1} \\ \phi_t \end{bmatrix} - \begin{bmatrix} 0 \\ m_t \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & 1 \\ -1 & s \end{bmatrix}$$

for integer 'stretching' $s = \mathrm{tr}\, J > 2$ the map is
beloved by ergodicists :
hyperbolic $\Rightarrow$ perfect chaotic Hamiltonian dynamical system

a cat is literally Hooke's wild, 'anti-harmonic' sister

for $s < 2$ Hooke rules

local restoring oscillations
around the sleepy z-z-z-zzz resting state

for $s > 2$ cats rule

exponential runaway
wrapped global around a phase space torus

cat is to chaos what harmonic oscillator is to order

there is no more fundamental example of chaos in mechanics

lattice formulation

cat map in lattice formulation

replace momentum by velocity

$$p_{t+1} = (\phi_{t+1} - \phi_t)/\Delta t$$

obtain

$$\begin{bmatrix} \phi_t \\ \phi_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} \phi_{t-1} \\ \phi_t \end{bmatrix} - \begin{bmatrix} 0 \\ m_t \end{bmatrix}$$

temporal lattice formulation is pretty[2] :

2-step difference equation

$$-\phi_{t+1} + s\,\phi_t - \phi_{t-1} = m_t$$

integer m_t ensures that

ϕ_t lands in the unit interval

$$m_t \in \mathcal{A}\,, \quad \mathcal{A} = \{\text{finite alphabet}\}$$

[2] I. Percival and F. Vivaldi, Physica D **27**, 373–386 (1987).

think globally, act locally

spatiotemporal cat at every instant t, local in time

$$-\phi_{t+1} + s\,\phi_t - \phi_{t-1} = m_t$$

is enforced by the global equation

$$\mathcal{J}\,\Phi = \mathsf{M}\,,$$

where

orbit Jacobian matrix

$$\mathcal{J}\,\Phi - \mathsf{M} = 0$$

with

$$\Phi = (\phi_{t+1}, \cdots, \phi_{t+n})\,, \quad \mathsf{M} = (m_{t+1}, \cdots, m_{t+n})$$

a lattice state, and a symbol block

and $[n\times n]$ orbit Jacobian matrix $\mathcal{J}$ is

$$-r + s\,\mathbb{1} - r^{-1} = \begin{pmatrix} s & -1 & & & -1 \\ -1 & s & -1 & & \\ & -1 & & \ddots & \\ & & & s & -1 \\ -1 & & & -1 & s \end{pmatrix}$$

think globally, act locally

solving the spatiotemporal cat equation

$$\mathcal{J}\Phi = \mathsf{M}\,,$$

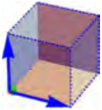
with the $[n\times n]$ matrix $\qquad \mathcal{J} = -r + s\,\mathbb{1} - r^{-1}$

can be viewed as a search for zeros of the function

$$F[\Phi] = \mathcal{J}\Phi - \mathsf{M} = 0$$

where the entire global lattice state Φ_{M} is

a single fixed point $\Phi_{\mathsf{M}} = (\phi_1, \phi_2, \cdots, \phi_n)$

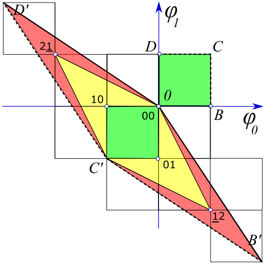


in the n-dimensional unit hyper-cube $\qquad \Phi \in [0,1)^n$

fundamental fact in action

temporal cat fundamental parallelepiped for period 2

square $[0BCD] \Rightarrow \mathcal{J} = $ fundamental parallelepiped $[0B'C'D']$



$N_2 = |\mathrm{Det}\,\mathcal{J}| = 5$

fundamental parallelepiped
$=$ 5 unit area quadrilaterals

a periodic point per each unit volume

spatiotemporal cat zeta function

is the generating function that counts orbits

substituting the Hill determinant count of periodic lattice states

$$N_n = \mathrm{Det}\,\mathcal{J}$$

into the topological zeta function

$$1/\zeta_{\mathrm{top}}(z) = \exp\left(-\sum_{n=1} \frac{z^n}{n} N_n\right)$$

leads to the elegant explicit formula[3]

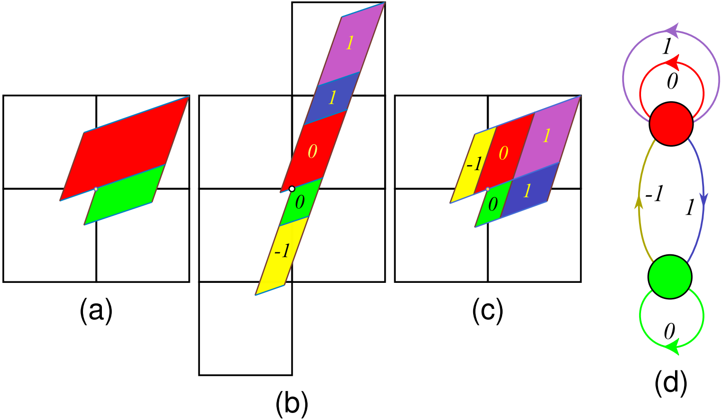
$$1/\zeta_{\mathrm{top}}(z) = \frac{1 - sz + z^2}{1 - 2z + z^2}$$

solved!

[3] S. Isola, Europhys. Lett. **11**, 517–522 (1990).

a side remark to experts

slicing cats



is not the way

Adler-Weiss generating partition of the unit torus
is a distraction. Klein-Gordon is a deeper insight

what continuum theory is temporal cat discretization of?

have

2-step difference equation

$$-\phi_{t+1} + s\,\phi_t - \phi_{t-1} = m_t$$

discrete lattice

Laplacian in 1 dimension

$$\phi_{t+1} - 2\phi_t + \phi_{t-1} = \Box\,\phi_t$$

so temporal cat is an (anti)oscillator chain, known as

$d = 1$ Klein-Gordon (or damped Poisson) equation (!)

$$(-\Box + \mu^2)\,\phi_t = m_t\,, \qquad \mu^2 = s - 2$$

did you know that a cat map can be so cool?

a reminder slide, to skip : Helmholtz equation in continuum

inhomogeneous Helmoltz equation

is an elliptical equation of form

$$(\Box + k^2)\,\phi(x) = -m(x)\,, \qquad x \in \mathbb{R}^d$$

where $\phi(x)$ is a C^2 function, and $m(x)$ is a function with compact support

for the $\mu^2 = -k^2 > 0$ (imaginary k), the equation is known as the Klein-Gordon, Yukawa, or screened Poisson equation[4] equation

[4] A. L. Fetter and J. D. Walecka, *Theoretical Mechanics of Particles and Continua*, (Dover, New York, 2003).

that's it! for spacetime of 1 dimension

lattice Klein-Gordon equation

$$(-\Box + \mu^2)\,\phi_t = m_t$$

solved completely and analytically!

think globally, act locally - summary

the problem of determining all global solutions stripped to its bare essentials :

1. each solution a zero of the global fixed point condition

$$F[\Phi] = 0$$

2. compute the orbit Jacobian matrix

$$\mathcal{J}_{ij} = \frac{\delta F[\Phi]_i}{\delta \phi_j}$$

3. fundamental fact $\quad N_n = |\mathrm{Det}\, \mathcal{J}| =$ period-n states

4. $\Rightarrow$ zeta function $1/\zeta_{\mathrm{top}}(z)$