

# Noise is your friend

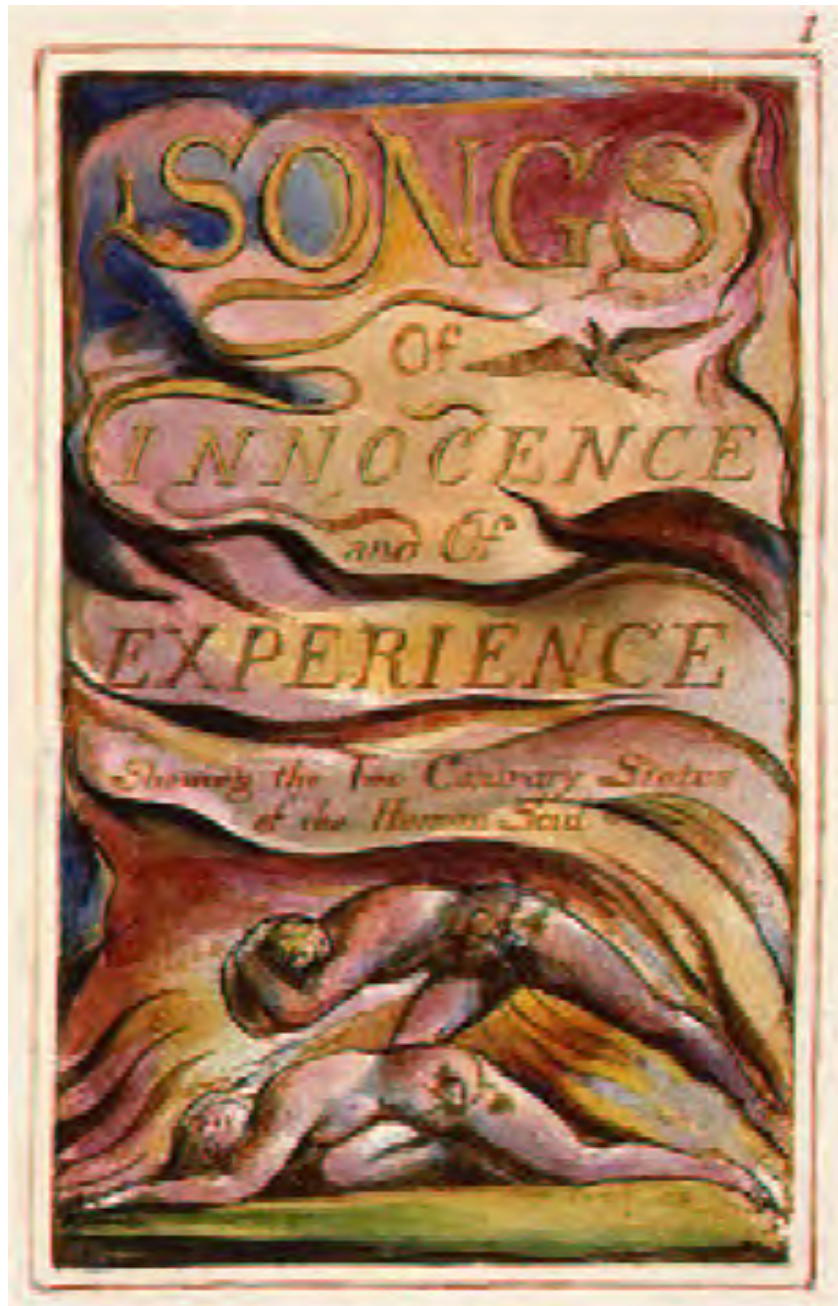
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Euromech Colloquium EC565 on  
Subcritical Transition to Turbulence,  
Cargese, Corsica

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## Outline

- 1 **what this talk is about**
  - knowing when to stop
- 2 **deterministic partitions**
  - idea #1: partition by periodic points
- 3 **dynamicist's view of noise**
  - idea #2: evolve densities, not noisy trajectories
  - idea #3: for unstable directions, look back
- 4 **optimal partition hypothesis**



# Turbulence? and what to do about it

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# Turbulence: A walk through a repertoire of unstable recurrent patterns?

As a turbulent flow evolves, every so often we catch a glimpse of a familiar pattern:



$\Rightarrow$  other swirls  $\Rightarrow$



For any finite spatial resolution, the system follows approximately for a finite time a pattern belonging to a **finite alphabet** of admissible patterns. The long term dynamics = a **walk through the space of such unstable patterns**.

# Conclusion: Hof is Hope renewed for Hopf's Last Hope for a Theory of Turbulence

Kuramoto-Sivashinsky periodic solutions embody Hopf's vision: They form the repertoire of the recurrent spatio-temporal patterns that is being explored by the turbulent dynamics.

detailed dynamics horrible, but much less so than feared: pieced together from 1-d return maps (!)

“To do” list:

Q: plane Couette-Taylor shear flow?

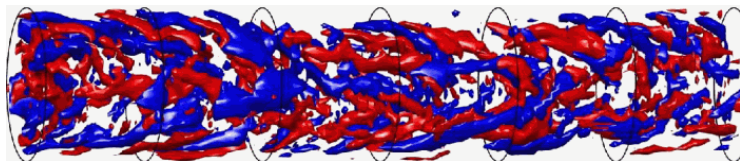
Waleffe; Kawahara & Kida: it can be done!

Q: Osborne Reynolds' turbulent pipe flow?

need to predict turbulent drag for a given  $Re \dots$

## knowing when to stop

computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes,



is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed.

when are we to stop calculating these solutions?

## knowing when to stop

need the 3D velocity field at **every**  $(x, y, z)$ !

**motions of fluids : require  $\infty$  bits?**

numerical simulations track  $10^2 - 10^6$  of computational degrees of freedom; terabytes of data, but how much information is there in all of this?

**knowing when to stop**

**motions of fluids : require  $\infty$  bits??**

**that cannot be right...**



## knowing when to stop

Science originates from curiosity and bad eyesight.

— Bernard de Fontenelle,  
*Entretiens sur la Pluralité des Mondes Habités*

### in practice

every physical problem is coarse partitioned and finite

## noise rules the state space

- any physical system experiences (some kind of) noise
- any numerical computation is 'noisy'
- any prediction only needs a desired finite accuracy

### mathematician's idealized state space

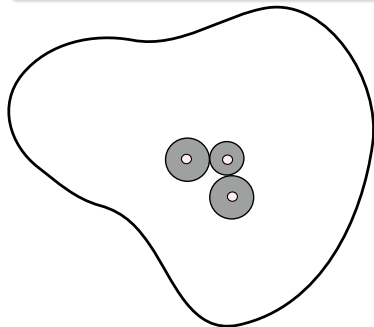
a manifold  $\mathcal{M} \in \mathbb{R}^d$  :  $d$  real numbers determine the state of the system  $x \in \mathcal{M}$

### noise-limited state space

a 'grid'  $\mathcal{M}'$  :  $N$  discrete states of the system  $a \in \mathcal{M}'$ , one for each noise covariance ellipsoid  $\Delta_a$

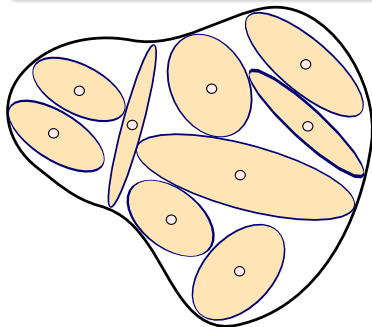
## noise limited state space partitions

noise limited cell



a resolvable neighborhood is no smaller than a ball whose radius is the noise amplitude

noise limited partition grid



state space noise-partitioned into neighborhoods indicated by their centers

## dynamics + noise: unique coarse-grained partition

**reasonable to assume that the noise**

limits the resolution

that can be attained in partitioning the state space

## dynamics + noise: unique coarse-grained partition

**reasonable to assume that the noise**

is uniform,

leading to a uniform grid partition of the state space

## dynamics + noise: unique coarse-grained partition

**reasonable to assume that the noise**

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leading to a uniform grid partition of the state space

**in dynamics, this is wrong!**

noise has memory

## dynamics + noise: unique coarse-grained partition

### noise memory

accumulated noise along dynamical trajectories

**always** coarsens the partition nonuniformly



## dynamics + noise: unique coarse-grained partition

### noise memory

accumulated noise along dynamical trajectories

**always** coarsens the partition nonuniformly

that is good, because

**dynamics + noise determine**

the **finest attainable** partition

## the challenge

turbulence.zip

**or 'equation assisted' data compression:**

replace the  $\infty$  of turbulent videos by the best possible

**small finite set**

of **videos** encoding all physically distinct motions of the turbulent fluid

## devil is in the details

### fluid dynamics

have equations: can compute the optimal partition

### Navier-Stokes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{R} \nabla^2 \mathbf{v} - \nabla p + \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0,$$

velocity field  $\mathbf{v} \in \mathbb{R}^3$  ; pressure field  $p$  ; driving force  $\mathbf{f}$

# dynamical system

## state space

a manifold  $\mathcal{M} \in \mathbb{R}^d$  :  $d$  numbers determine the state of the system

## representative point

$$x(t) \in \mathcal{M}$$

a state of physical system at instant in time

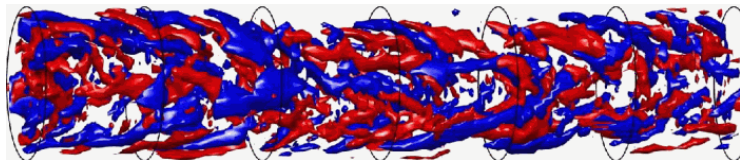
## today's experiments

### example of a representative point

$$x(t) \in \mathcal{M}, d = \infty$$

a state of turbulent pipe flow at instant in time

Stereoscopic Particle Image Velocimetry  $\rightarrow$  3- $d$  velocity field over the entire pipe<sup>1</sup>

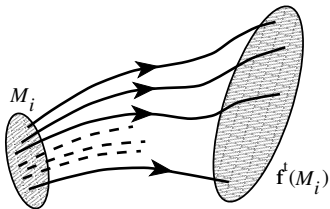


<sup>1</sup>Casimir W.H. van Doorne (PhD thesis, Delft 2004)

## dynamics

map  $f^t(x_0)$  = representative point time  $t$  later

## evolution in time



$f^t$  maps a region  $\mathcal{M}_i$  of the state space into the region  $f^t(\mathcal{M}_i)$

## dynamical description of turbulence

### dynamical system

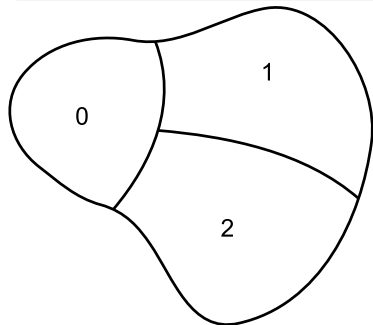
the pair  $(\mathcal{M}, f)$

### the problem

enumerate, classify all solutions of  $(\mathcal{M}, f)$

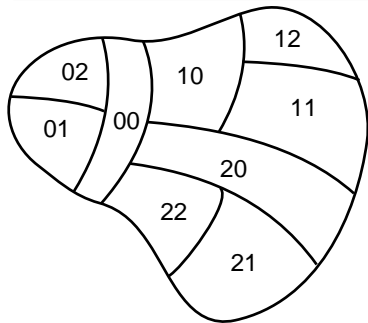
## deterministic partition into regions of similar states

1-step memory partition



$\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2$   
ternary alphabet  
 $\mathcal{A} = \{1, 2, 3\}$ .

2-step memory refinement



$\mathcal{M}_i = \mathcal{M}_{i0} \cup \mathcal{M}_{i1} \cup \mathcal{M}_{i2}$   
labeled by nine 'words'  
 $\{00, 01, 02, \dots, 21, 22\}$ .



**deterministic partitions are no good**

**deterministic dynamics: partitioning can be arbitrarily fine**  
requires exponential # of exponentially small regions

## deterministic partitions are no good

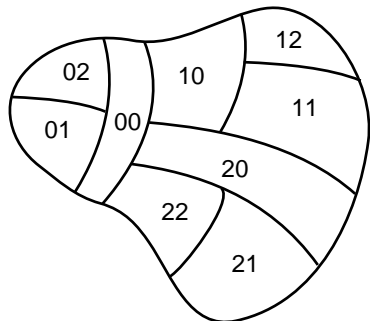
**deterministic dynamics: partitioning can be arbitrarily fine**  
requires exponential # of exponentially small regions

yet

**in practice**

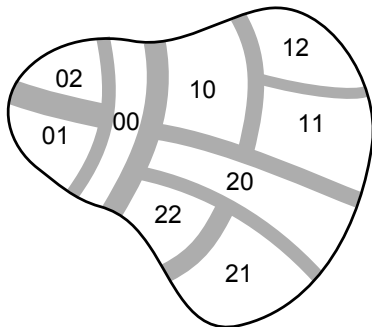
every physical problem must be coarse partitioned

## deterministic vs. noisy partitions



deterministic partition

can be refined  
*ad infinitum*



noise blurs the boundaries

when overlapping, no further  
refinement of partition

## periodic points instead of boundaries

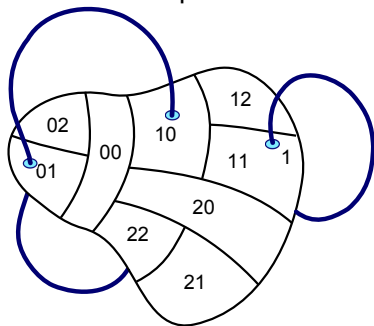
- mhm, do not know how to compute boundaries...

## periodic points instead of boundaries

- mhm, do not know how to compute boundaries...
- however, each partition contains a short periodic point

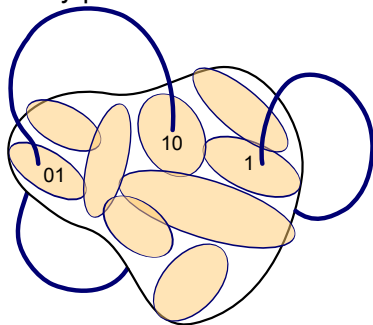
## periodic orbit partition

deterministic partition



some short periodic points:  
fixed point  $\bar{1} = \{x_1\}$   
two-cycle  $\overline{01} = \{x_{01}, x_{10}\}$

noisy partition



periodic points blurred by noise  
into cigar-shaped densities

## periodic points and their cigars

- each partition contains a short periodic point smeared into a 'cigar' by noise

## periodic points and their cigars

- each partition contains a short periodic point smeared into a 'cigar' by noise
- compute the size of a noisy periodic point neighborhood!



## how big is the neighborhood blurred by the accumulated noise?

**the** (well known) **key formula** that we now derive:

$$Q_{n+1} = M_n Q_n M_n^T + \Delta_n$$

density covariance matrix at time  $n$ :  $Q_n$

noise covariance matrix:  $\Delta_n$

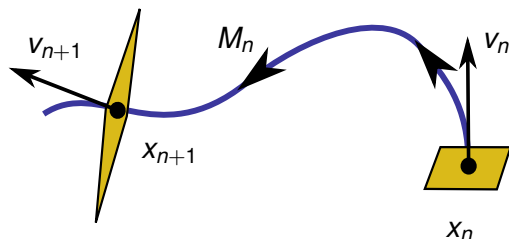
Jacobian matrix of linearized flow:  $M_n$

Lyapunov equation, doctoral dissertation 1892

Ornstein-Uhlenbeck 1930

Kalman filter 'prediction' 1960

## linearized deterministic flow



$$x_{n+1} + z_{n+1} = f(x_n) + M_n z_n, \quad M_{ij} = \partial f_i / \partial x_j$$

in one time step a linearized neighborhood of  $x_n$  is

- (1) advected by the flow
- (2) transported by the Jacobian matrix  $M_n$  into a neighborhood given by the  $M$  eigenvalues and eigenvectors

## covariance advection

let the initial density of deviations  $z$  from the deterministic center be a Gaussian whose covariance matrix is

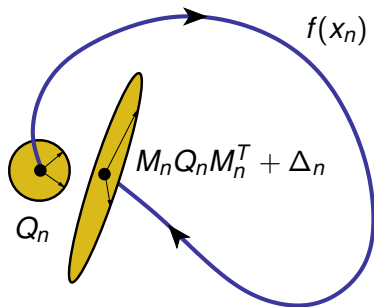
$$Q_{jk} = \langle z_j z_k^T \rangle$$

a step later the Gaussian is advected to

$$\begin{aligned} \langle z_j z_k^T \rangle &\rightarrow \langle (M z)_j (M z)_k^T \rangle \\ Q &\rightarrow M Q M^T \end{aligned}$$

next: add noise

## roll your own cigar



in one time step

a Gaussian density distribution with covariance matrix  $Q_n$  is

- (1) advected by the flow
- (2) smeared with additive noise

into a Gaussian 'cigar' whose widths and orientation are given by the singular values and vectors of  $Q_{n+1}$

## covariance evolution

$$Q_{n+1} = M_n Q_n M_n^T + \Delta_n$$

- (1) advect deterministically  
local density covariance matrix  $Q \rightarrow MQM^T$
- (2) add noise covariance matrix  $\Delta$

covariances add up as sums of squares

## cumulative noise along a trajectory

iterate  $Q_{n+1} = M_n Q_n M_n^T + \Delta_n$  along a trajectory

if  $M$  is contracting,  $|\Lambda_j| < 1$ ,

the memory of the covariance  $Q_0$  of the starting density is lost,  
with iteration leading to the limit distribution

$$Q_n = \Delta_n + M_{n-1} \Delta_{n-1} M_{n-1}^T + M_{n-2}^2 \Delta_{n-2} (M_{n-2}^2)^T + \dots$$

## example : noise and a single attractive fixed point

if all eigenvalues of  $M$  are strictly contracting, all  $|\lambda_j| < 1$

any initial compact measure converges to the unique invariant Gaussian measure  $\rho_0(z)$  whose covariance matrix satisfies

**Lyapunov equation: time-invariant measure condition**

$$Q = MQM^T + \Delta$$

[A. M. Lyapunov doctoral dissertation 1892]

## example : Ornstein-Uhlenbeck process

width of the natural measure concentrated at the attractive deterministic fixed point  $z = 0$

$$\rho_0(z) = \frac{1}{\sqrt{2\pi Q}} \exp\left(-\frac{z^2}{2Q}\right), \quad Q = \frac{\Delta}{1 - |\Lambda|^2},$$

- is balance between contraction by  $\Lambda$  and noisy smearing by  $\Delta$  at each time step
- for strongly contracting  $\Lambda$ , the width is due to the noise only
- As  $|\Lambda| \rightarrow 1$  the width diverges: the trajectories are no longer confined, but diffuse by Brownian motion



# example : 2D Brusselator limit cycle

214105-7 Nakanishi, Sakaue, and Wakou

J. Chem. Phys. **139**, 214105 (2013)

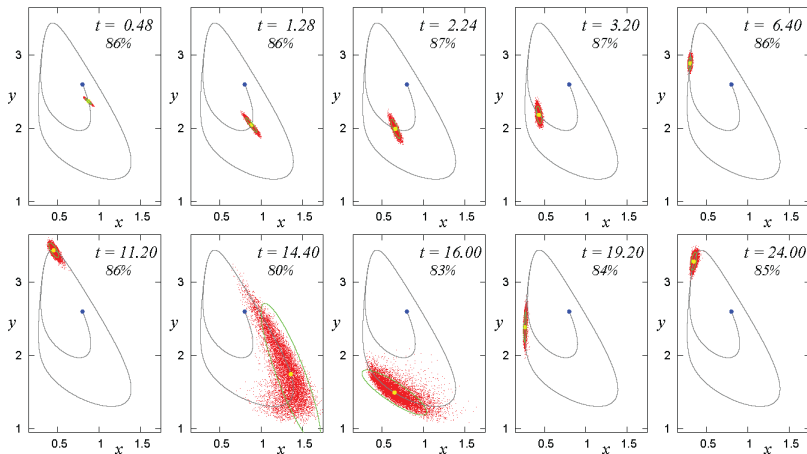


FIG. 2. Time development of distribution for Brusselator. 10 000 samples of Monte Carlo simulations are plotted by the red dots along with the covariance matrix  $\hat{M}$  estimated by Eq. (E7);  $\hat{M}$ 's are represented by the green ellipses given by  $\delta \mathbf{x}^T \hat{M}^{-1} \delta \mathbf{x} = 4/\Omega$ , where  $\delta \mathbf{x}^T \equiv (x - x^*(t), y - y^*(t))$ . The percentages of the samples that fall within the ellipses are shown in each panel. The gray curves represent the trajectory by the rate equation starting from the initial point marked by the blue circles. The system parameters are  $k_1 = 0.5$ ,  $k_2 = 1.5$ ,  $k_3 = 1.0$ ,  $k_4 = 1.0$ , and  $\Omega = 10^4$ . The initial point is  $(x_0^*, y_0^*) = (0.8, 2.6)$ .

## remembrance of things past

noisy dynamics of a nonlinear system is fundamentally different from Brownian motion, as the flow **ALWAYS** induces a local, history dependent effective noise

## things fall apart, centre cannot hold

but what if  $M$  has *expanding* eigenvalues?

both deterministic dynamics and noise tend to smear densities away from the fixed point: no peaked Gaussian in your future

## things fall apart, centre cannot hold

but what if  $M$  has *expanding* eigenvalues?

look into the past, for initial peaked distribution that spreads to the present state

## for unstable directions, look back

if  $M$  has only *expanding* eigenvalues,

balance between the two is attained by iteration from the past, and the evolution of the covariance matrix  $\tilde{Q}$  is now given by

$$\tilde{Q}_{n+1} + \Delta_n = M_n \tilde{Q}_n M_n^T,$$

[aside to control theorists: reachability and observability Gramians]

## solving the Lyapunov equation

iterate  $Q_{n+1} = M_n Q_n M_n^T + \Delta_n$

attractive fixed point,  $Q = Q_\infty$ ,  $M = M_n$ ,  $Q = Q_n$ :

$$Q = \Delta + M\Delta M^T + M^2\Delta(M^T)^2 + \dots = \sum_{m,n=0}^{\infty} \delta_{mn} M^n \Delta (M^T)^m$$

bring to resolvent form,  $\delta_{mn} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(m-n)}$

**for  $M$  contracting, expanding, or hyperbolic (!)**

$$Q = \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{\mathbf{1} - e^{-i\theta} M} \Delta \frac{1}{\mathbf{1} - e^{i\theta} M^T}$$

## Cauchy magic

a similarity transformation  $S$  separates the expanding and contracting subspaces

$$\Lambda \equiv S^{-1}MS = \begin{pmatrix} \Lambda_e & 0 \\ 0 & \Lambda_c \end{pmatrix}$$

transformed noise covariance matrix

$$\hat{\Delta} \equiv S^{-1}\Delta(S^{-1})^T = \begin{pmatrix} \Delta_{ee} & \Delta_{ec} \\ \Delta_{ce} & \Delta_{cc} \end{pmatrix}$$

## Cauchy magic

### contour integral representation

$$Q = \oint_{\Gamma} \frac{ds}{2\pi} (\mathbf{1} - s^{-1}M)^{-1} \Delta (\mathbf{1} - sM)^{-1}$$

separates  $Q$  into expanding and contracting covariances:

$$\tilde{Q}_e \equiv S \begin{pmatrix} Q_e & 0 \\ 0 & 0 \end{pmatrix} S^{\top}, \quad Q_c \equiv S \begin{pmatrix} 0 & 0 \\ 0 & Q_c \end{pmatrix} S^{\top}$$

two stationary 'cigars', one in the expanding manifold and the other in the contracting manifold (not orthogonal to each other!)



**local problem solved: can compute every cigar**

a periodic point of period  $n$  is a fixed point of  $n$ th iterate of dynamics

**global problem solved: can compute all cigars**

more algebra: can compute the noisy neighborhoods of all periodic points

## optimal partition challenge

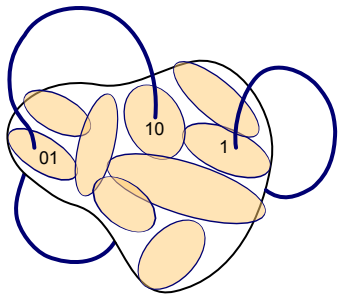
**now can address our challenge:**

*determine the finest possible partition for a given noise*

## noisy dynamics partitions: strategy

- use periodic orbits to partition state space
- compute local covariances at periodic points to determine their neighborhoods
- done once neighborhoods overlap

## optimal partition hypothesis



### optimal partition:

the maximal set of resolvable  
periodic point neighborhoods

## how noise frees us from determinism

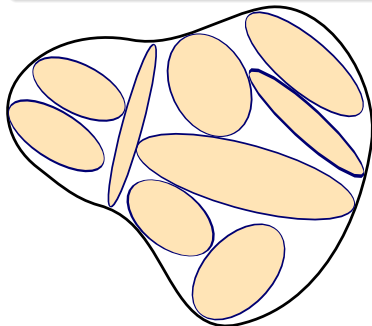
### noise memory

accumulated noise along a dynamical trajectory always coarsens the partition

### this partition is

- intrinsic to dynamics
- computable

turbulence.zip



## example: representative solutions of fluid dynamics

- Professor Zweistein, from the back of the auditorium:
  - (1) she has already done all this in 1969
  - (2) you must be kidding, it cannot be done for turbulence

## example: representative solutions of fluid dynamics

- Professor Zweistein, from the back of the auditorium:
  - (1) she has already done all this in 1969
  - (2) you must be kidding, it cannot be done for turbulence
- OK, OK, today we typically have about 10-100 exact invariant solutions populating the state space  
and we have their Jacobians  $M$  (that was hell to get)

## disclosure

- we have not yet tested the method on fluid dynamics



## disclosure

- we have not yet tested the method on fluid dynamics
- the brave candidates: step up after the talk

## take home message

computation of unstable periodic orbits in high-dimensional state spaces, such as Navier-Stokes,

is at the border of what is feasible numerically, and criteria to identify finite sets of the most important solutions are very much needed

we are to stop calculating these solutions when we attain

take home message

optimal partition hypothesis

---

## optimal partition hypothesis

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- the best of all possible state space partitions
- optimal for the given dynamical system, the given noise

Is this a cloud?



## references

- D. Lippolis and P. Cvitanović, *How well can one resolve the state space of a chaotic map?*, Phys. Rev. Lett. 104, 014101 (2010); [arXiv.org:0902.4269](https://arxiv.org/abs/0902.4269)
- P. Cvitanović and D. Lippolis, *Knowing when to stop: How noise frees us from determinism*, in M. Robnik and V.G. Romanovski, eds., *Let's Face Chaos through Nonlinear Dynamics* (Am. Inst. of Phys., Melville, New York, 2012); [arXiv.org:1206.5506](https://arxiv.org/abs/1206.5506)