ChaosBook.org chapter

measure

June 26, 2014 version 14.5.6, ▸ you can't always get what you want

Outline

1. from trajectories to densities thereof

2. invariant measures

measure

measure

$d\mu(x) = \rho(x)dx\,, \qquad \int_{\mathcal{M}} dx\, \rho(x) = 1$

coarse-grained measure

assign “mass”

$$\Delta\mu_i = \int_{\mathcal{M}_i} d\mu(x) = \int_{\mathcal{M}_i} dx\, \rho(x)$$

fraction of trajectories $\in$ ith region, $\mathcal{M}_i \subset \mathcal{M}$ of the state space

normalized

$$\sum_i^{(n)} \Delta\mu_i = 1$$

sum over subregions i at the nth level of partitioning

transporting densities

conservation of representative points

$$\int_{f^t(\mathcal{M}_i)} dx\, \rho(x,t) = \int_{\mathcal{M}_i} dx_0\, \rho(x_0,0)\,.$$

transform the integration variable

initial points $x_0 = f^{-t}(x)$,

$$\int_{\mathcal{M}_i} dx_0\, \rho(f^t(x_0),t) \left|\det J^t(x_0)\right| = \int_{\mathcal{M}_i} dx_0\, \rho(x_0,0)\,.$$

density varies in time

$$\rho(x,t) = \frac{\rho(x_0,0)}{|\det J^t(x_0)|}\,, \qquad x = f^t(x_0)$$

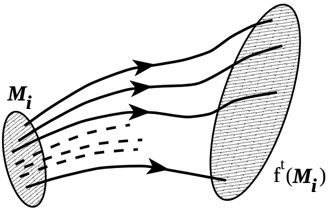
measure

density evolution

$$\rho(x,t) = \frac{\rho(x_0,0)}{|\det J^t(x_0)|}, \qquad x = f^t(x_0)$$

makes sense:

evolution rule



the density varies inversely with the volume occupied by the trajectories of the flow

Perron-Frobenius operator

transformation linear in ρ, so recast as an operator

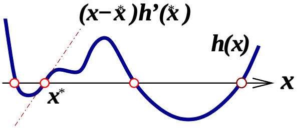
$$\rho(x,t) = \left(\mathcal{L}^t \circ \rho\right)(x) = \int_{\mathcal{M}} dx_0 \, \delta\big(x - f^t(x_0)\big) \, \rho(x_0, 0) \, .$$

check

integrating Dirac delta in d dimensions

$\int_{\mathcal{M}} dx\, \delta(x) = 1$ if $0 \in \mathcal{M}$, zero otherwise.

$$\int dx\, \delta(h(x)) \quad = $$



$$= \sum_j \int_{\mathcal{M}_j} dx\, \delta(h(x)) = \sum_j \frac{1}{\left|\det \frac{\partial h(x_j)}{\partial x}\right|}\,,$$

check

the semi-group property, and the measure evolution yourself.

Perron-Frobenius operator

we refer to

$$\mathcal{L}^t(y,x) = \delta\big(y - f^t(x)\big)$$

as the *Perron-Frobenius operator*

it assembles the density $\rho(y,t)$ at time t by going back in time to the density $\rho(x,0)$ at time $t = 0$.

family of Perron-Frobenius operators $\{\mathcal{L}^t\}_{t\in\mathbb{R}_+}$ forms

semigroup parameterized by time

(a) $\mathcal{L}^0 = I$

(b) $\mathcal{L}^t\mathcal{L}^{t'} = \mathcal{L}^{t+t'} \qquad t, t' \geq 0$

invariant measures

stationary or invariant density

is a density left unchanged by the flow

$$\rho(x,t) = \rho(x,0) = \rho(x)$$

invariant measures are fixed points

(in the infinite-dimensional function space of ρ densities) of the Perron-Frobenius operator, with the unit eigenvalue

$$\mathcal{L}^t \rho(x) = \int_{\mathcal{M}} dy\, \delta(x - f^t(y))\rho(y) = \rho(x).$$

“natural” measure

an invarinat measure

limit of transformation of an initial smooth distribution $\rho(x)$ under the action of f,

$$\rho_0(x) = \lim_{t\to\infty} \int_{\mathcal{M}} dy\, \delta(x - f^t(y))\, \rho(y,0)\,, \qquad \int_{\mathcal{M}} dy\, \rho(y,0) = 1\,.$$

Intuitively, the “natural” measure should be the least sensitive to the external noise

SRB or Sinai-Bowen-Ruelle measure

dynamics of measure

In computer experiments, as the Hénon example of figure **??**, the long time evolution of many “typical" initial conditions leads to the same asymptotic distribution. Hence
is defined as the limit

$$\overline{\rho}_{x_0}(y) \;=\; \begin{cases} \lim_{t\to\infty} \frac{1}{t}\int_0^t d\tau\, \delta(y - f^\tau(x_0)) & \text{flows} \\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \delta\big(y - f^k(x_0)\big) & \text{maps}\,, \end{cases}$$

x_0 is a generic initial point.

names

natural measure is also called equilibrium measure, physical measure, invariant density, natural density, or even “natural invariant”

computational natural measure

coarse-grained visitation of $\mathcal{M}_i$ region

$$\Delta\overline{\mu}_i = \lim_{t\to\infty} \frac{t_i}{t} \,,$$

t_i is the accumulated time that a trajectory of total duration t spends in the $\mathcal{M}_i$ region

observable

let $a = a(x)$ be a function that associates to each point in state space a number or a set of numbers

physical applications

observable $a(x)$ is necessarily a smooth function, which reports on some property of the dynamical system

averages

space average

of observable *a* with respect to a measure ρ is given by the *d*-dimensional integral over the state space $\mathcal{M}$:

$$\left\langle a \right\rangle_\rho = \frac{1}{|\rho_\mathcal{M}|} \int_\mathcal{M} dx\, \rho(x) a(x)$$

$$|\rho_\mathcal{M}| = \int_\mathcal{M} dx\, \rho(x) = \text{mass in } \mathcal{M}\,.$$

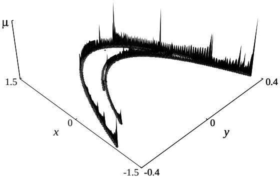
measure

time average of the observable *a* along a trajectory of the initial point x_0 is

$$\overline{a_{x_0}} = \lim_{t\to\infty} \frac{1}{t} \int_0^t d\tau\, a(f^\tau(x_0))\,.$$

natural measures are nasty

Hénon map attractor



natural measure for the Hénon strange attractor

fugget about it!

compute natural measure?

natural measure no place differentiable, everywhere singular, support on fractal sets

this is hopeless

the point of ChaosBook

is to teach you how to never compute them :)