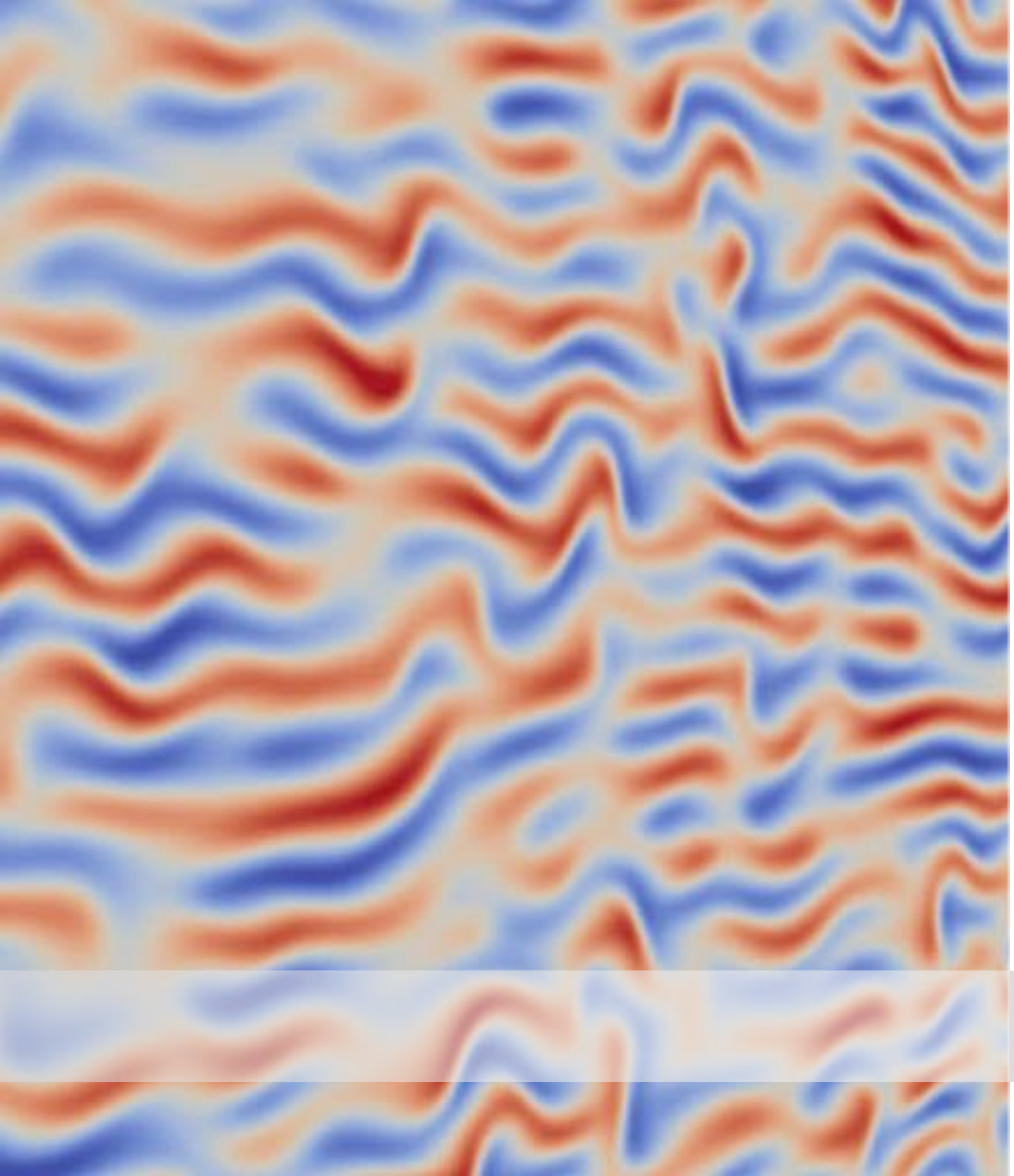

Nonlinear invariant solutions underlying spatio-temporal patterns in thermally driven shear flows

Florian Reetz[1], Priya Subramanian[2] & Tobias M. Schneider[1]

Emergent Complexity in Physical Systems

EPFL Lausanne, Switzerland[1]

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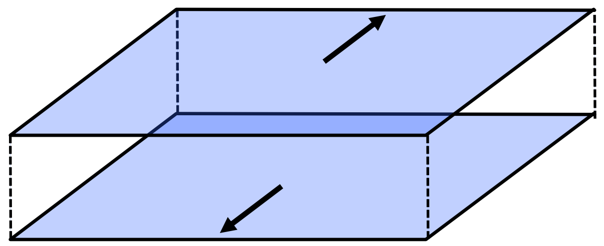
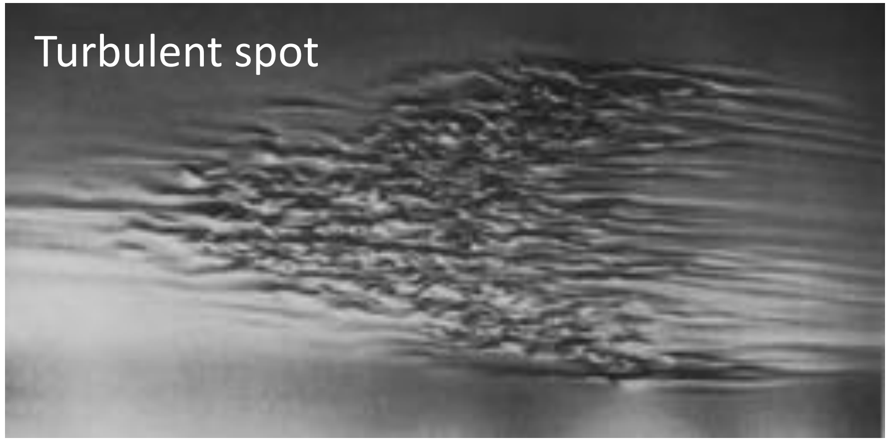
Supported by SNSF grant 200021-160088

Inclined layer convection
Interaction of buoyancy and shear

Shear flows (e.g. plane Couette)



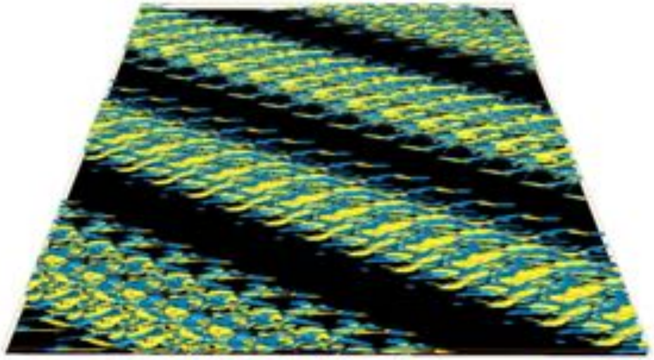
Shear flows (e.g. plane Couette)



Emmons, 1951

Inclined layer convection
Interaction of buoyancy and shear

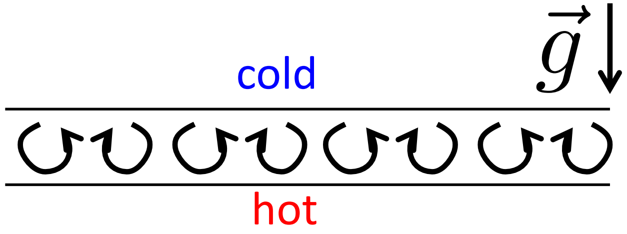
Shear flows (e.g. plane Couette)



Barkley & Tuckerman, 2005

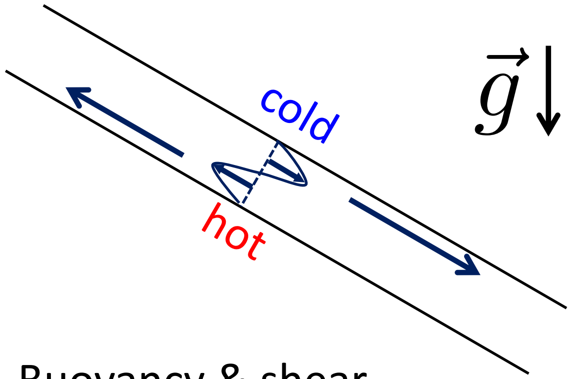
Self-organized turbulent-laminar patterns
Challenging to understand

Convection (Rayleigh-Benard)



Long studied pattern-forming system

Inclined layer convection



Buoyancy & shear

Three control parameters

Angle of incline γ

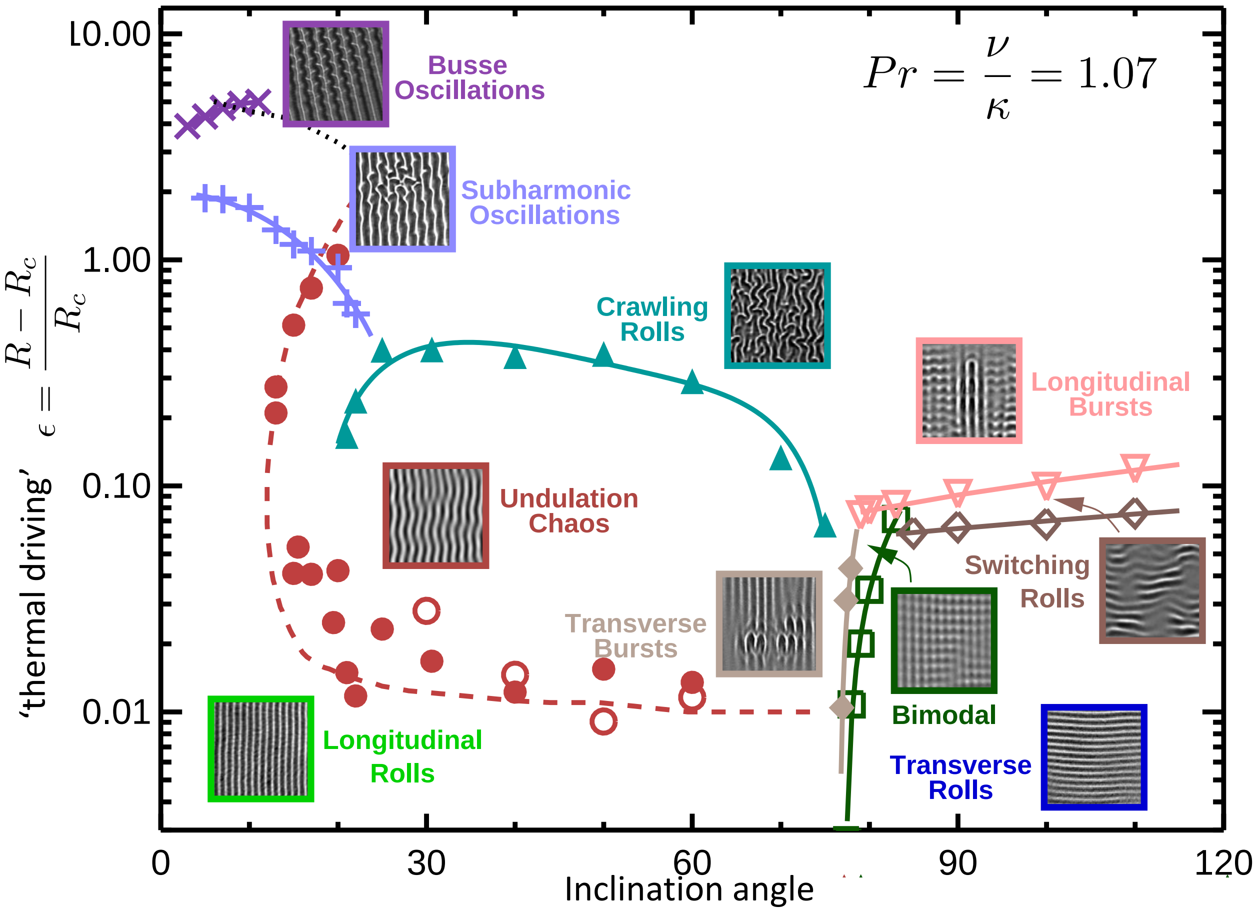
Driving $R = \frac{\alpha \rho g (T_1 - T_2) d^3}{\nu \kappa}$

Material properties $Pr = \frac{\nu}{\kappa}$

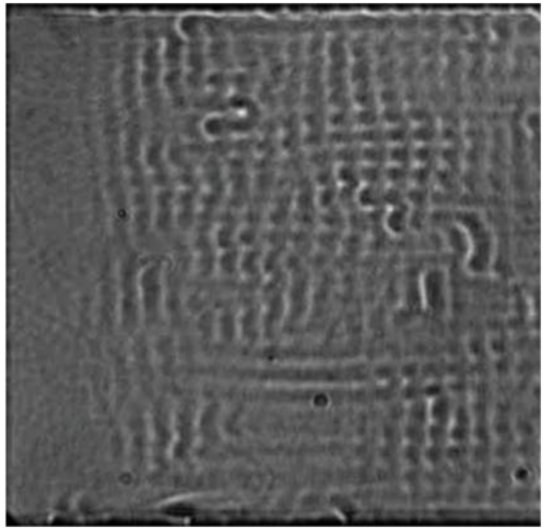
Questions: Patterns where buoyancy and shear compete?

Inclined layer convection
Rich dynamics

Phase diagram (Experiments: Daniels & Bodenschatz)



Bursts (experiment)



Weakly turbulent, localized

Questions: Mechanism underlying observed range of patterns?

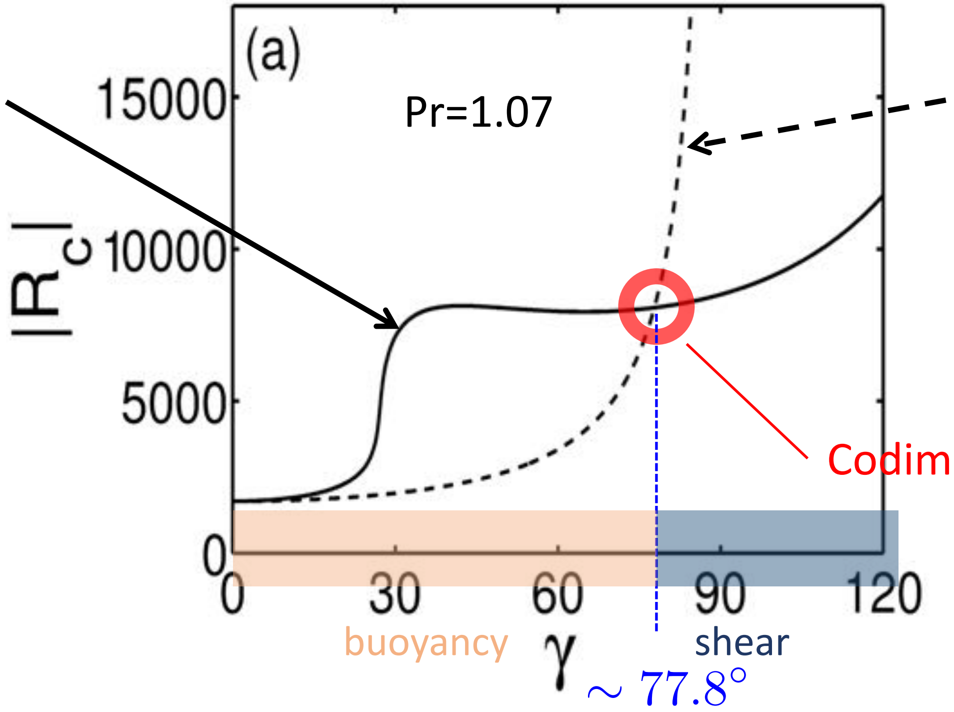
Primary instability of the base state

Periodic role patterns

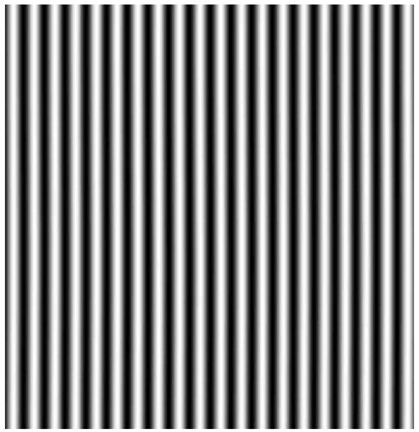
Longitudinal roles
(buoyancy)

Along the incline

Neutral curves



Transverse roles
(shear)



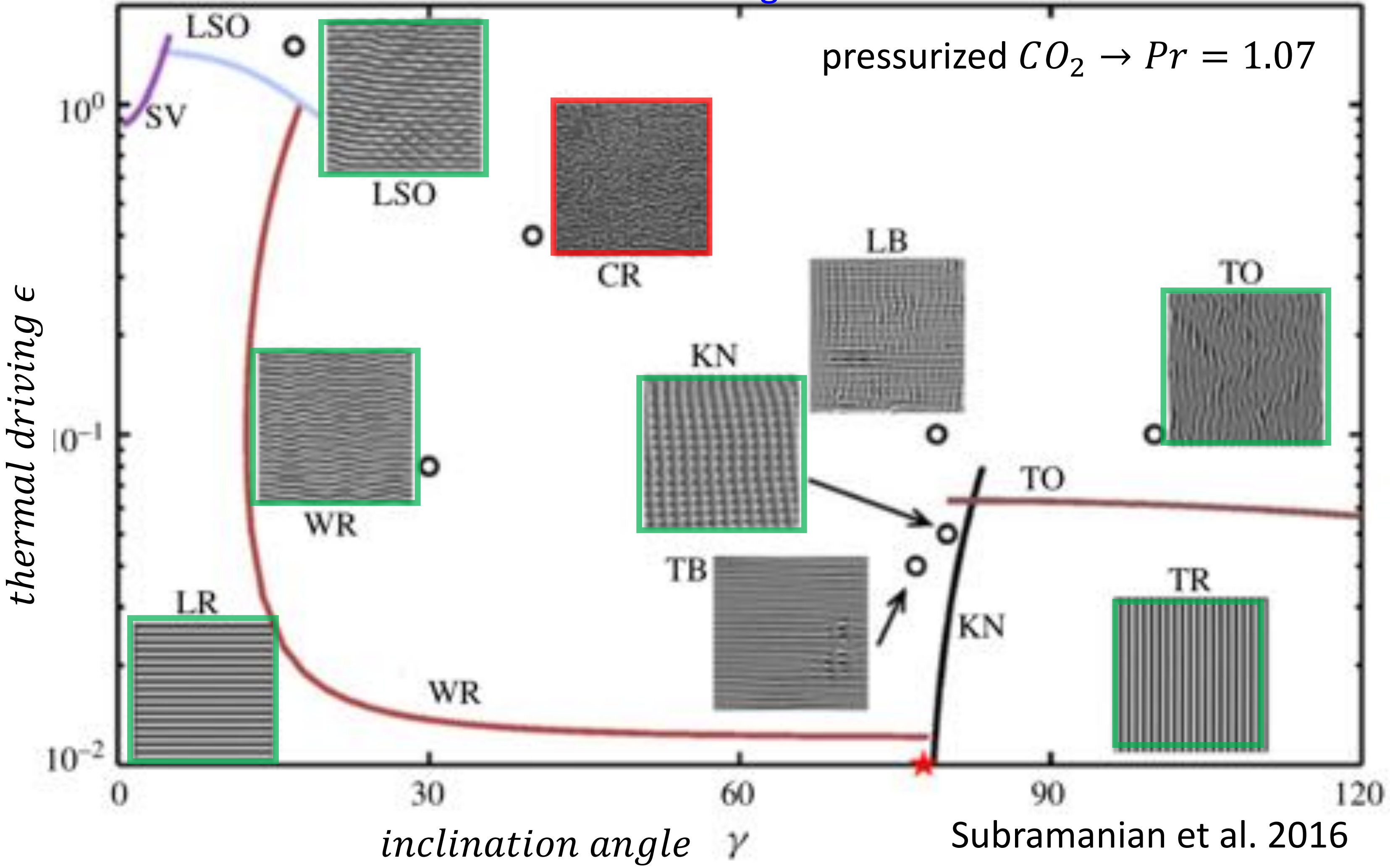
Question: Beyond primary instability of role patterns?

Rescaled Rayleigh: $$\epsilon = \frac{R}{R_c} - 1$$

Status: Linear theory and secondary instabilities

Convection patterns

Phase diagram



Subramanian et al. 2016

o experimental observation (Daniels & Bodenschatz)

Linear stability theory of base state:
Thresholds for onset of convection
2 types: longitudinal and transverse
straight convection rolls

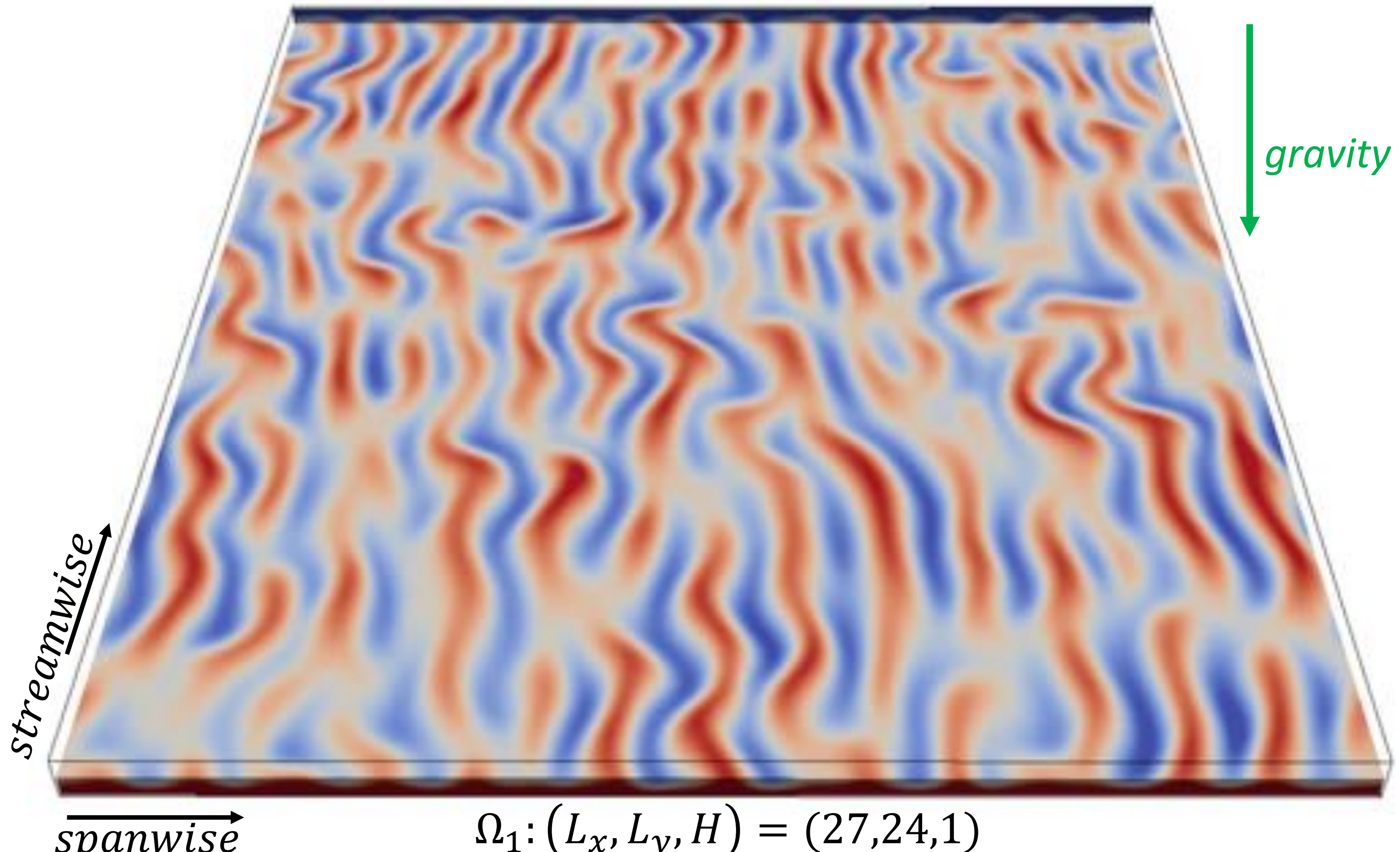
Secondary bifurcations of rols:
Thresholds for tertiary patterns
(Busse, Pesch & coworkers)

Question: Explain rich dynamics far beyond threshold?
Chaos, bursting, break up, spatial localization, defects,…

Example: crawling roles pattern

Crawling roles (CR) convection pattern
Direct numerical simulation

Temperature at the midplane



$\Omega_1: (L_x, L_y, H) = (27,24,1)$

Direct numerical simulations

$\gamma = 40°$

$Pr = 1.07$

$\epsilon = 0.5$

$(\rightarrow Ra = 3344)$

Grid: 384 x 384 x 25

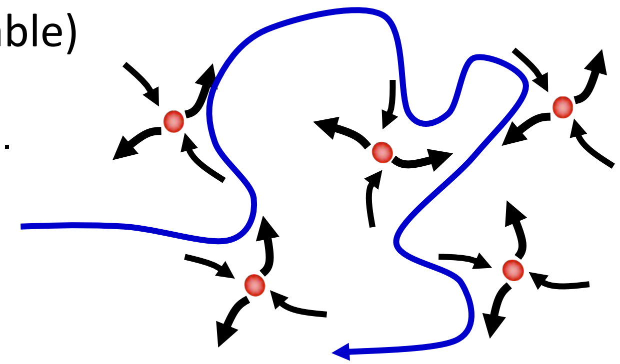
Dynamical systems approach
Role of exact invariant solutions

EPFL

Idea: Exact invariant solutions of Navier-Stokes support chaotic dynamics:

Random 'bouncing' between exact solutions (weakly unstable)

Cvitanovic, Eckhardt, Kerswell, Nagata, Waleffe,….



Model: pinball

Evolution equations: 3D *Oberbeck-Boussinesq equations*, inclined by angle γ against gravity

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}p + \sqrt{\frac{Pr}{Ra}}\nabla^2\vec{u} + (\sin(\gamma)\,\vec{e}_x + \cos(\gamma)\,\vec{e}_z)\theta$$

$$\frac{\partial \theta}{\partial t} + (\vec{u}\cdot\vec{\nabla})\theta = \frac{1}{\sqrt{\mathrm{Pr}\,Ra}}\nabla^2\theta$$

released: www.channelflow.ch

Exact invariant solutions:

Solution $\vec{x}^* = [\vec{u},\theta](x,y,z,t)$ of evolution equations

$$\vec{x}(t+T) = f^T(\vec{x}(t))$$

such that

$$\sigma\, f^T(\vec{x}^*) - \vec{x}^* = 0\,.$$

T: time period of integration

σ: symmetry operator

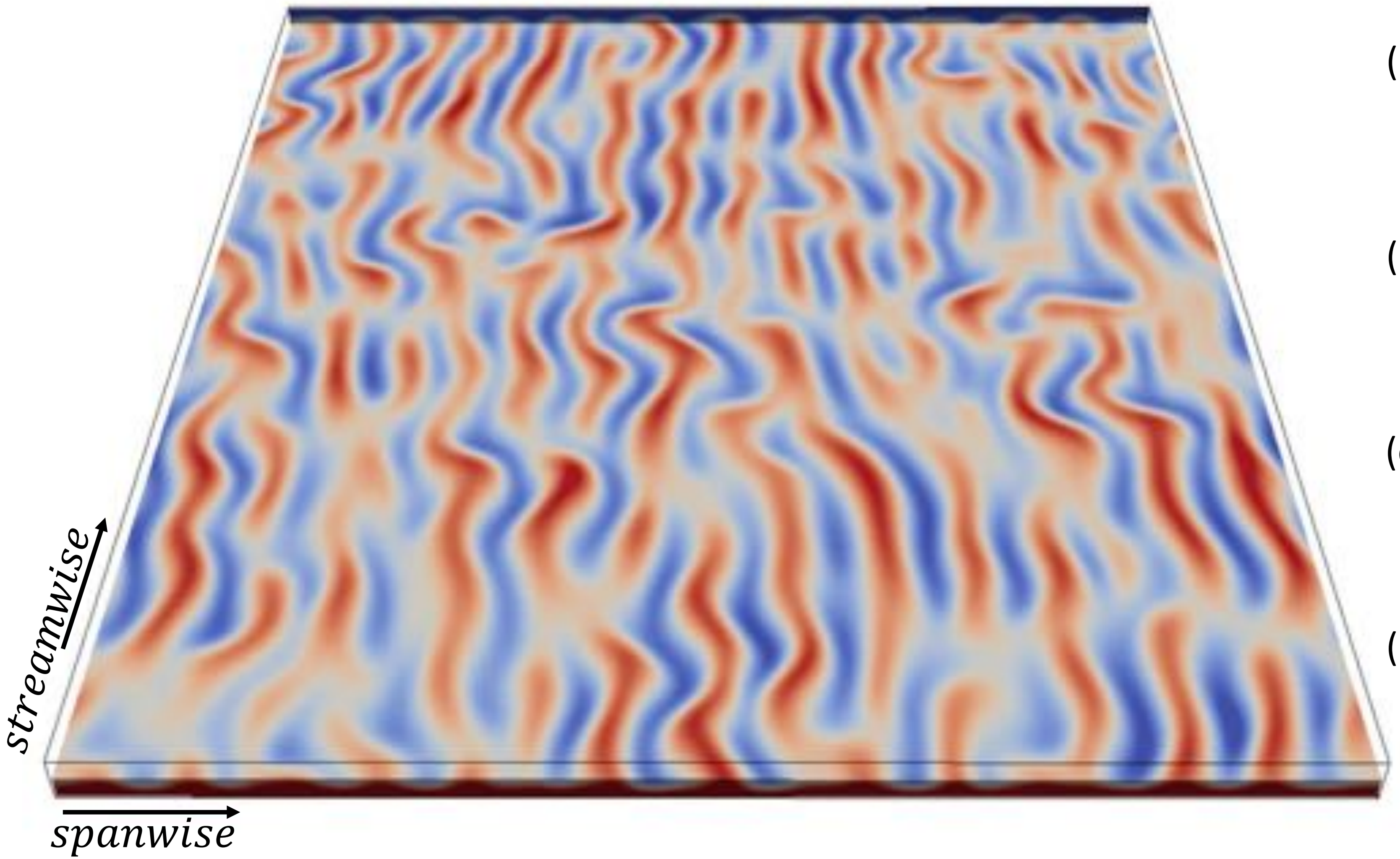
equilibria, traveling waves, (relative) periodic orbits

Numerical tool: Extension of CHANNELFLOW 2.0

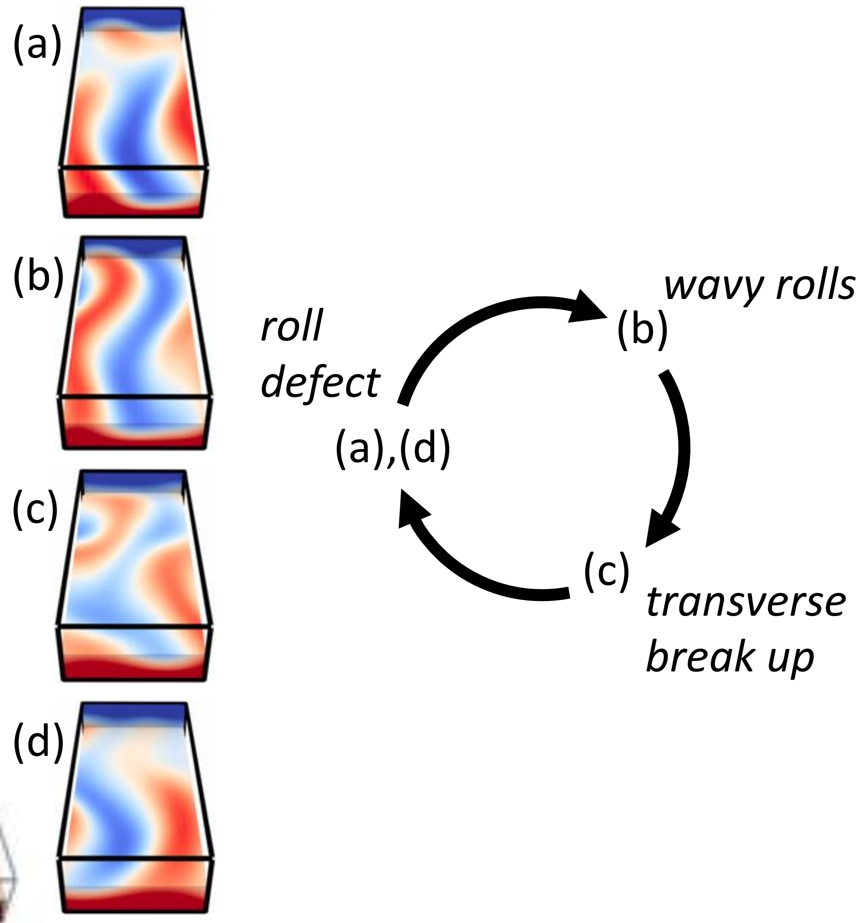
- Pseudo-spectral Direct Numerical Simulations (DNS) code (MPI-parallelized)
- Matrix-free Newton solver based on iterative Krylov subspace methods
- Computation of eigenvalue spectra and parametric continuation

Crawling roles (CR) convection pattern

Question: Exact invariant solutions underlying chaotic dynamics?



Observation: cyclic dynamics



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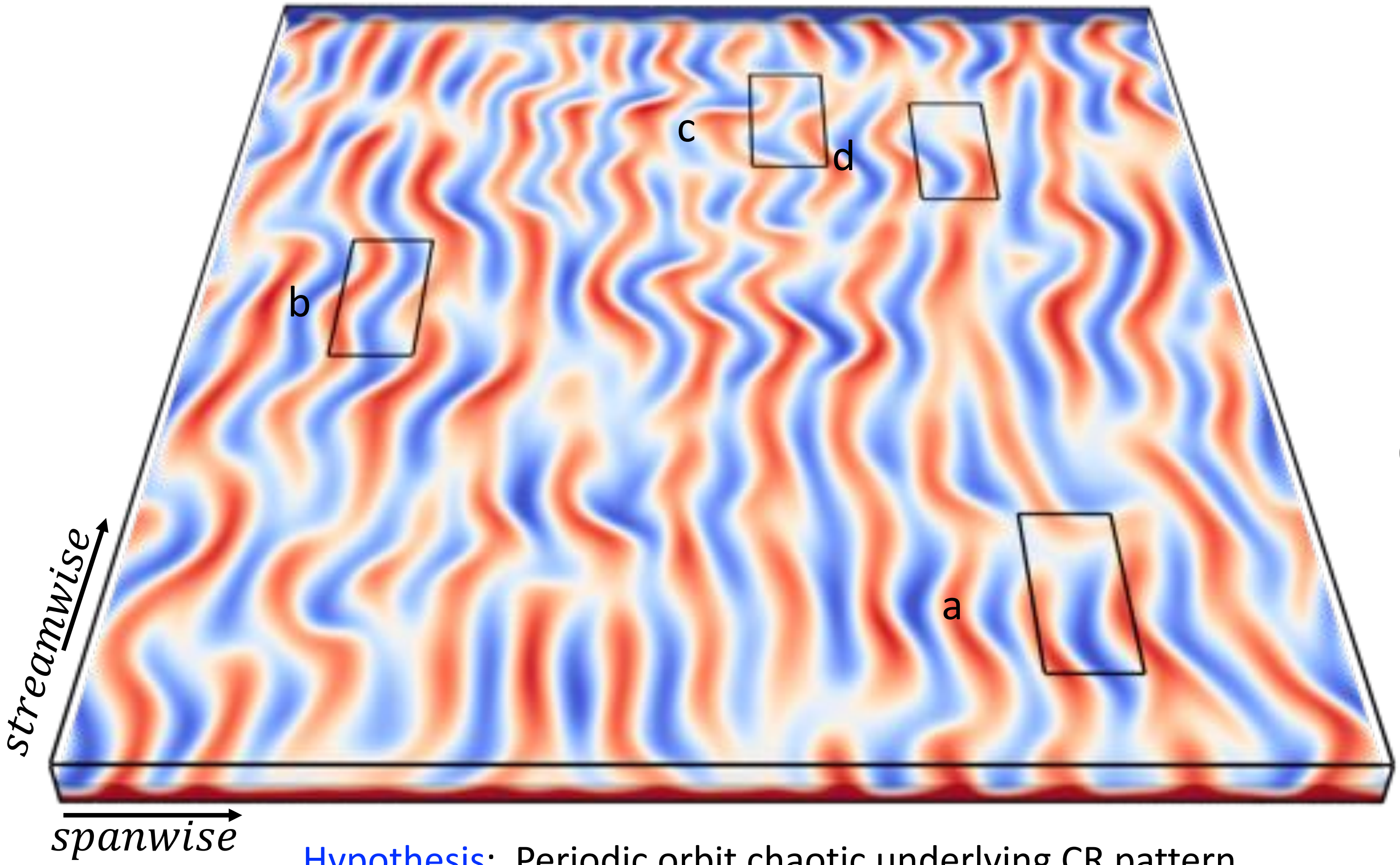

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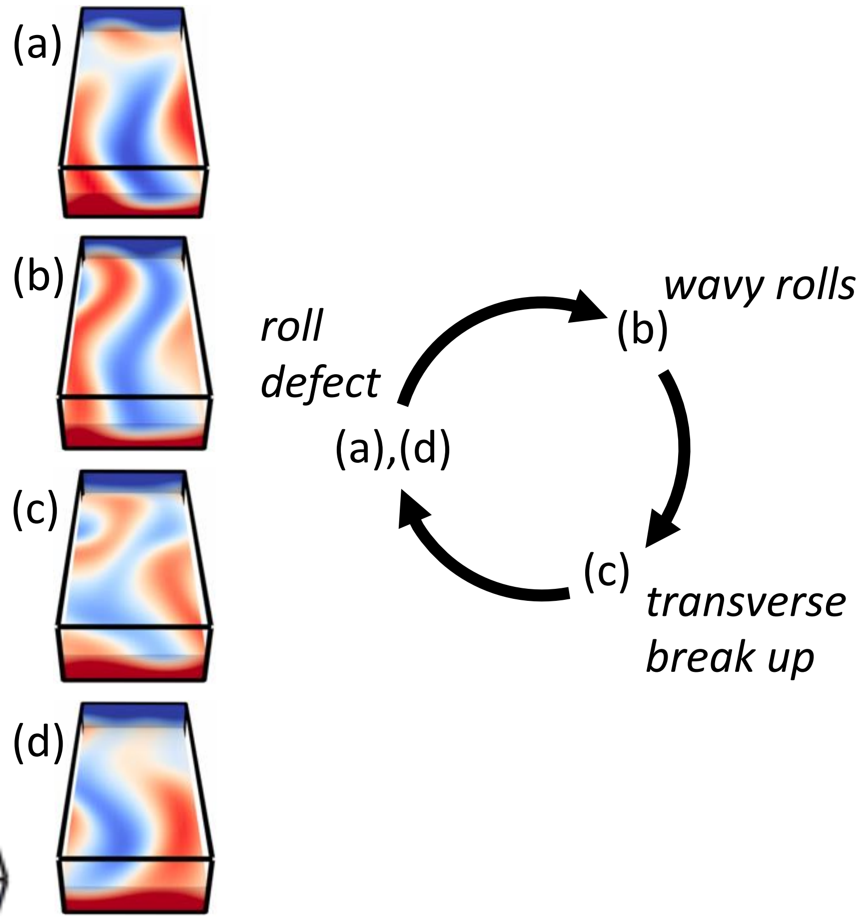


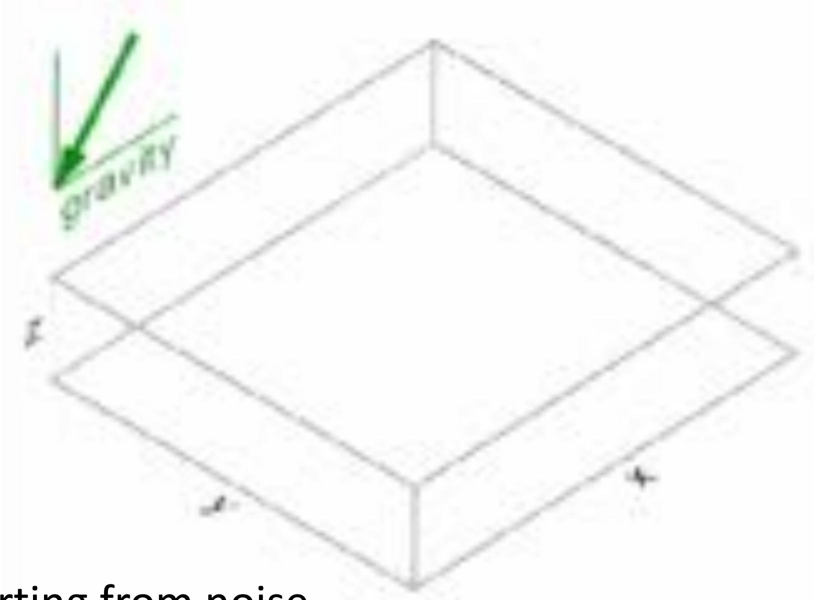
Hypothesis: Periodic orbit chaotic underlying CR pattern

Task: Identify exact invariant solutions underlying dynamics

Approach: Impose symmetry constraints to reduce complexity!

Imposing symmetries – step 1
DNS in small domain with periodic boundary conditions

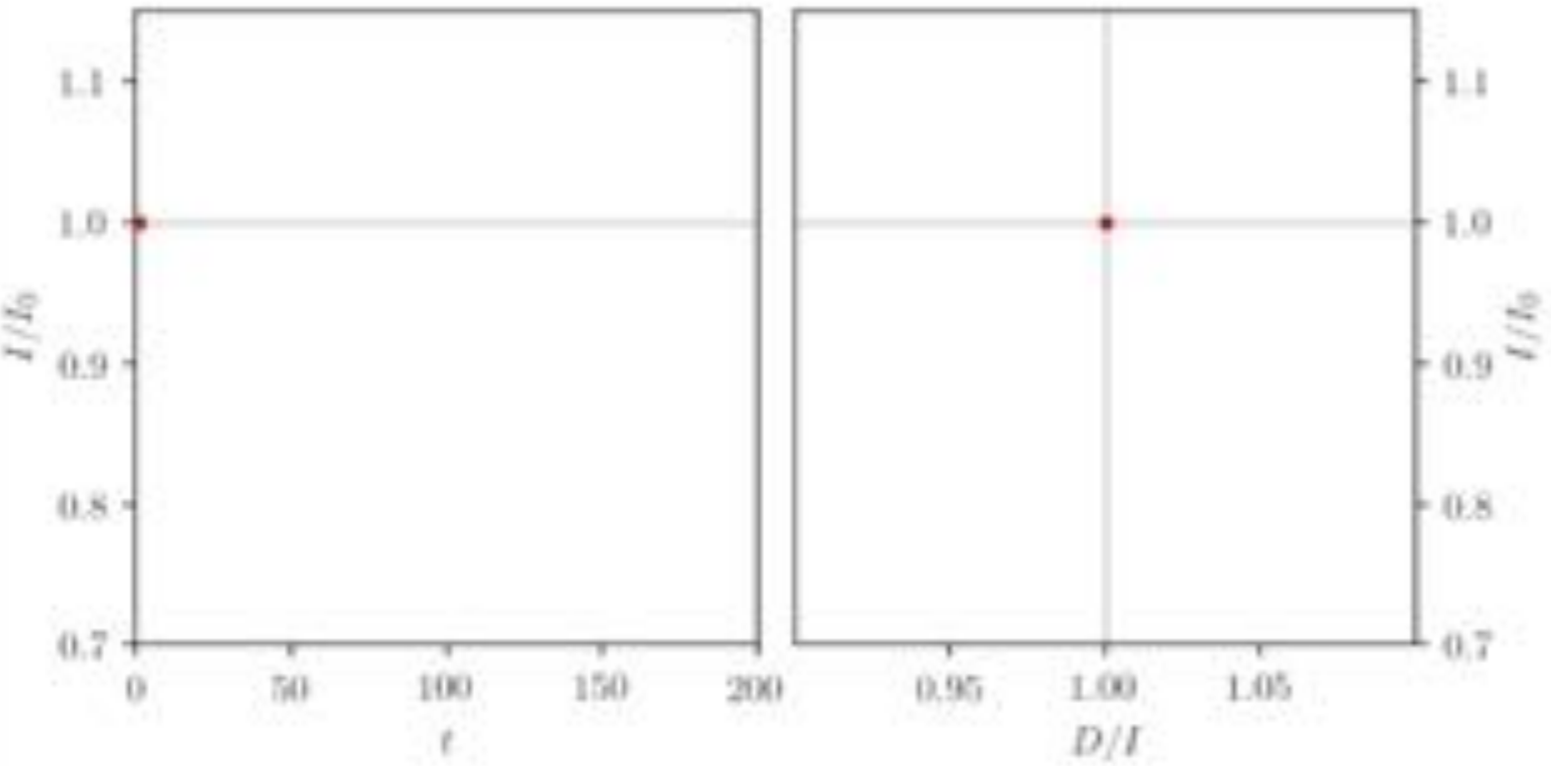
Flow visualization
(iso-contours of temperature fluctuations)



Starting from noise

Time series
(energy input I)

Phase portrait
Energy input vs dissipation/input



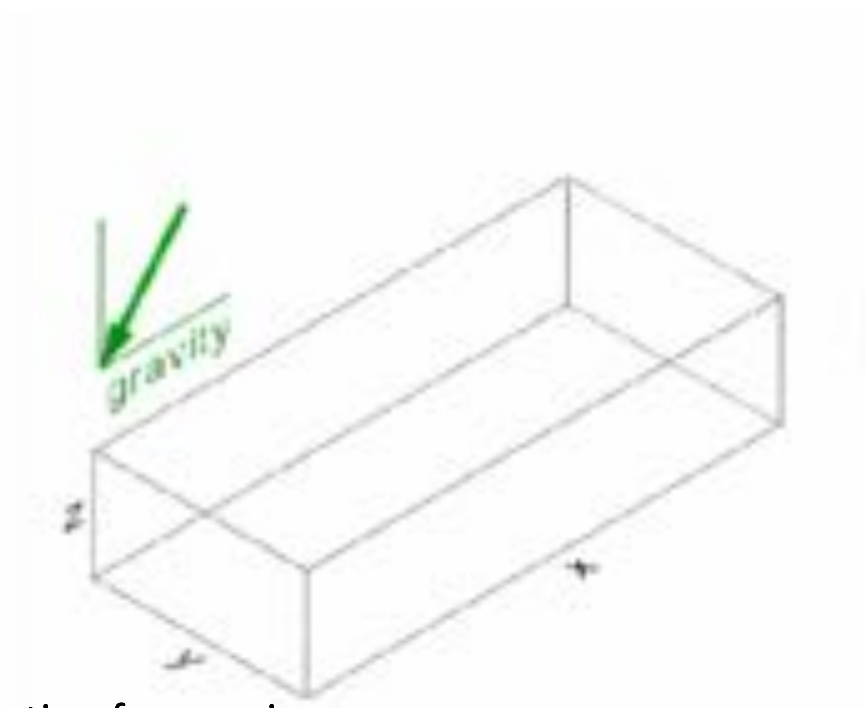
Observation: Chaotic dynamics in domain supporting two pair of roles
Evidence for transient visits to fixed point solutions

Imposing symmetries – step 2

DNS in half of the previous domain (only one pair or roles)

Flow visualization

(iso-contours of temperature fluctuations)



Starting from noise

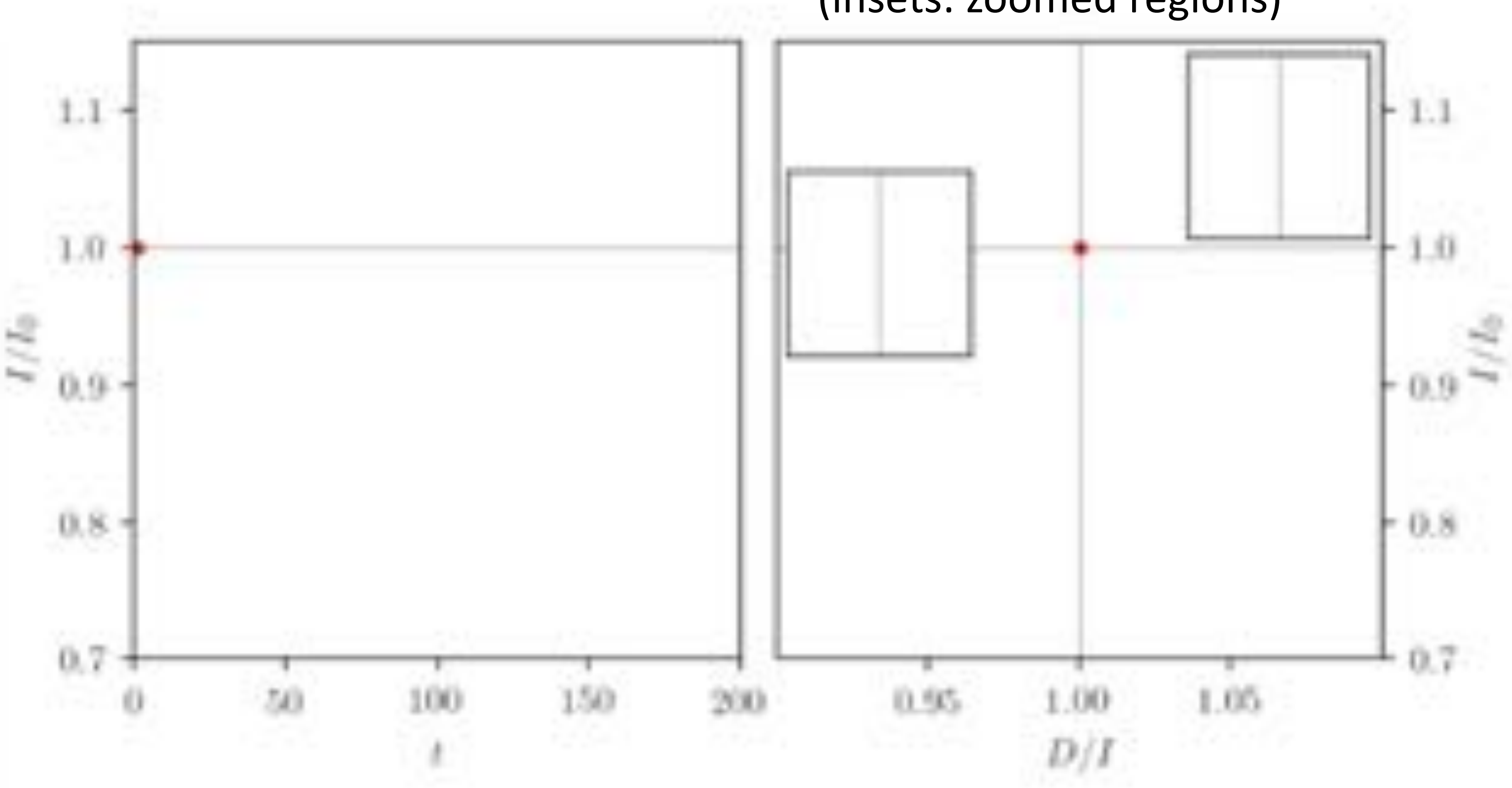
Time series

(energy input I)

Phase portrait

Energy input vs dissipation/input

(insets: zoomed regions)



Observation: Chaotic dynamics becomes transient

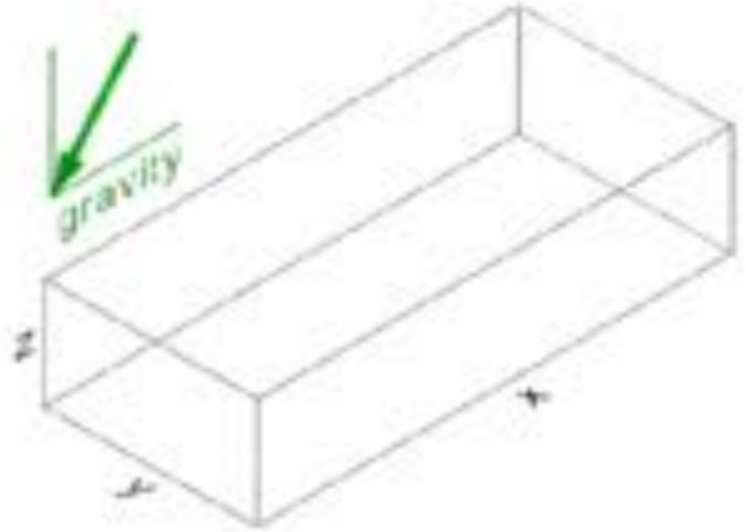
Transient visits to two non-trival fixed point solutions

Attractor: homoclinic orbit of a third fixed point

Imposing symmetries – step 3
Impose shift-and-rotate symmetry

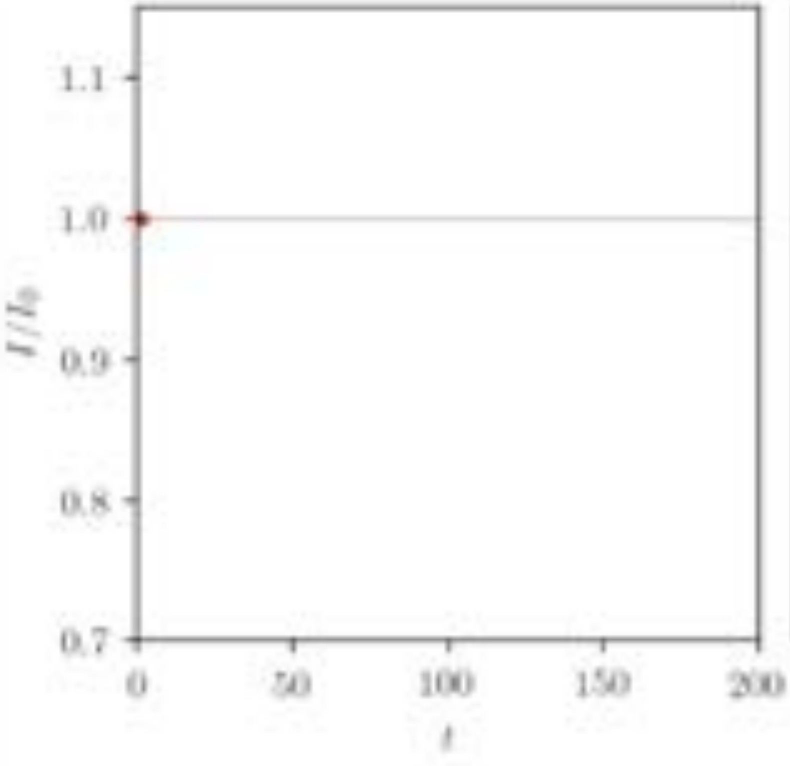
Flow visualization
(iso-contours of temperature fluctuations)

$$s[\vec{u},\theta] = [-u, v, -w, -\theta](-x + \frac{L_x}{2}, y + \frac{L_y}{2}, -z)$$

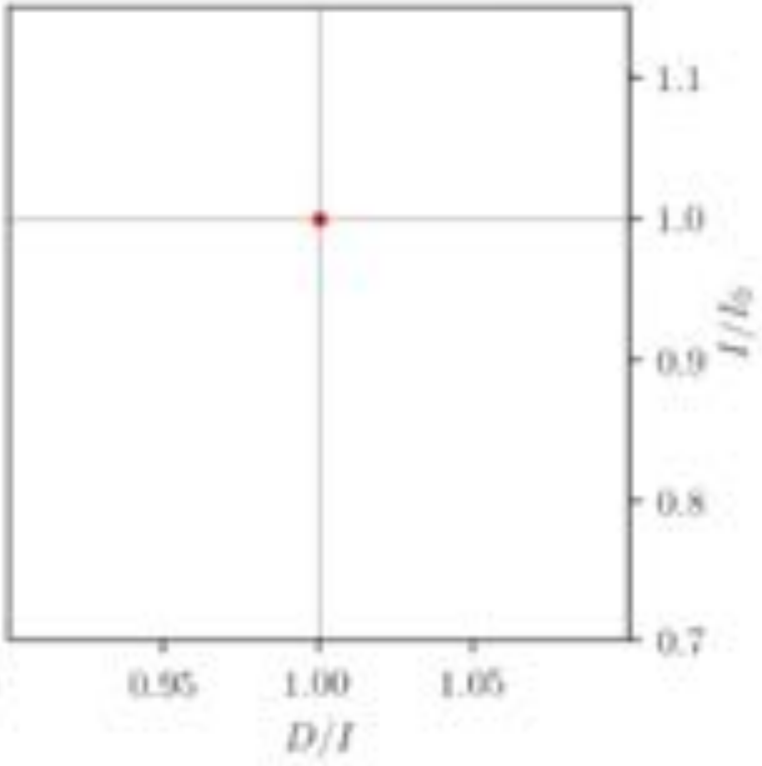


Starting from noise

Time series
(energy input I)



Phase portrait
Energy input vs dissipation/input

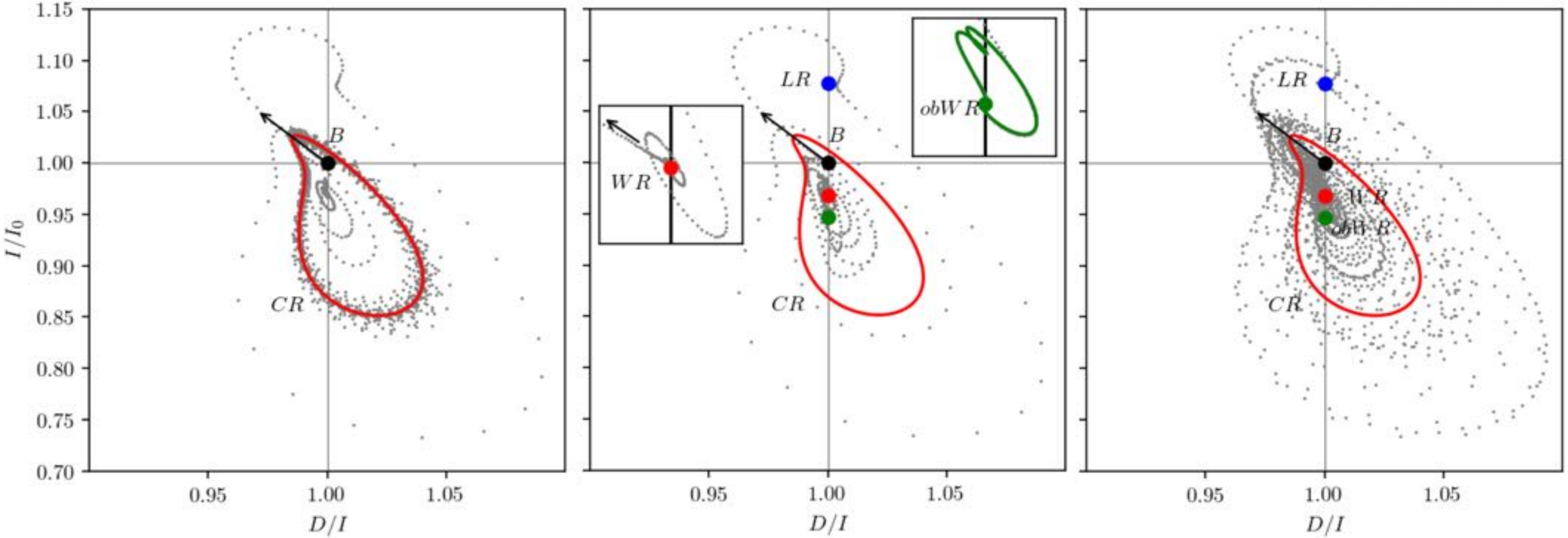


Observation: Transient visits to two nontrival fixed point solutions
Attractor: periodic orbit !!!

Exact invariant solutions underlying crawling roles

Symmetry reduced DNS gives access to unstable states

EPFL



periodic attractor | **transient near unstable states** | **chaos**

Impose symmetries

Results: Existence of 5 invariant solutions: base state (**B**), long. rolls (**LR**), wavy rolls (**WR**), oblique WR + homoclinic orbit, periodic orbit of crawling rolls (**CR**)

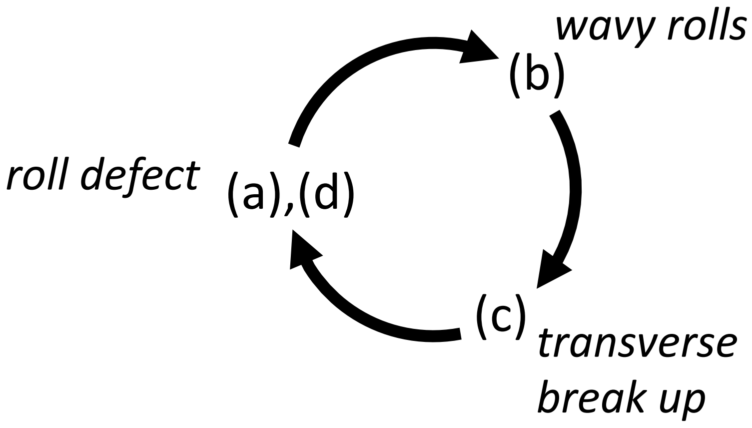
Invariant solutions are transiently visited by chaotic dynamics

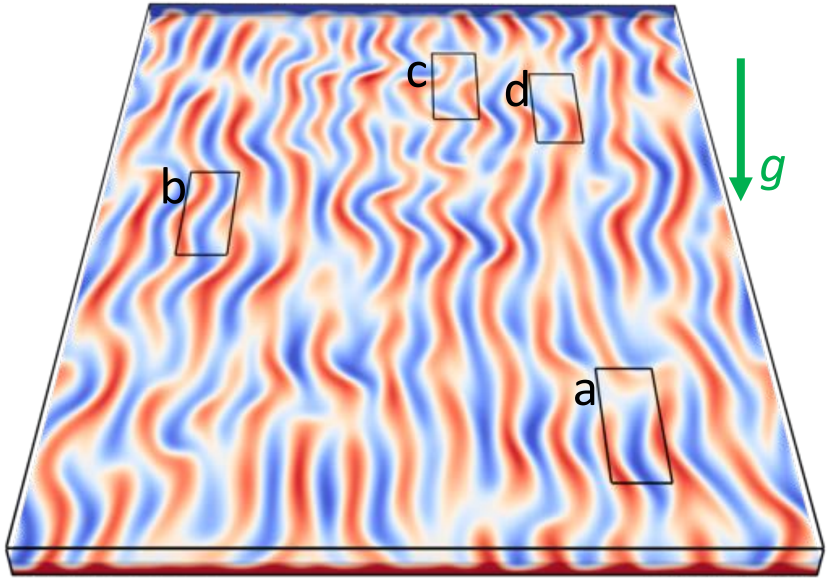
capture key features of the flow

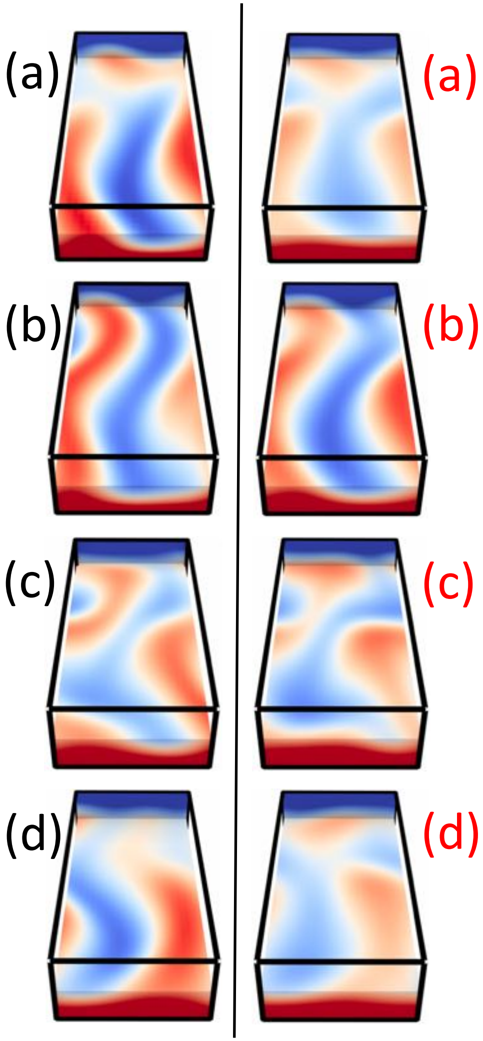
Exact invariant solutions underlying crawling roles

The relative periodic orbit underlying the pattern

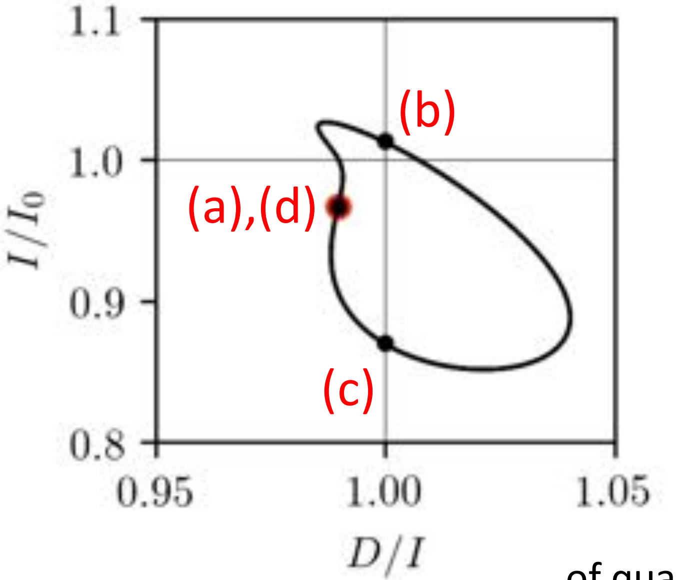
Observation: cyclic dynamics





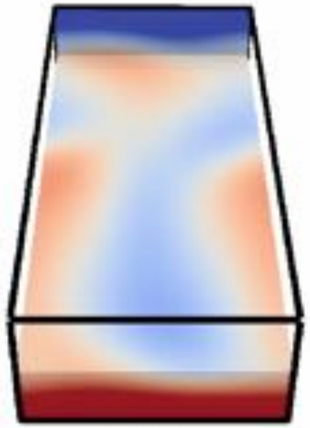


Exact periodic orbit captures cycle



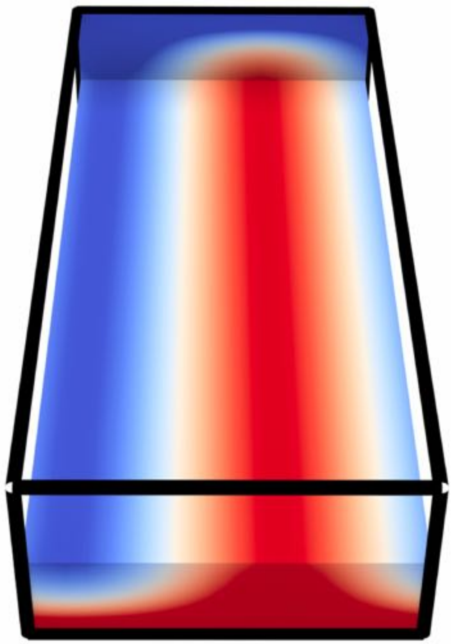
Relative periodic orbit of **period** $T = 39.9$ in free fall units with **symmetry operation** of quarter shifts + y-reflection

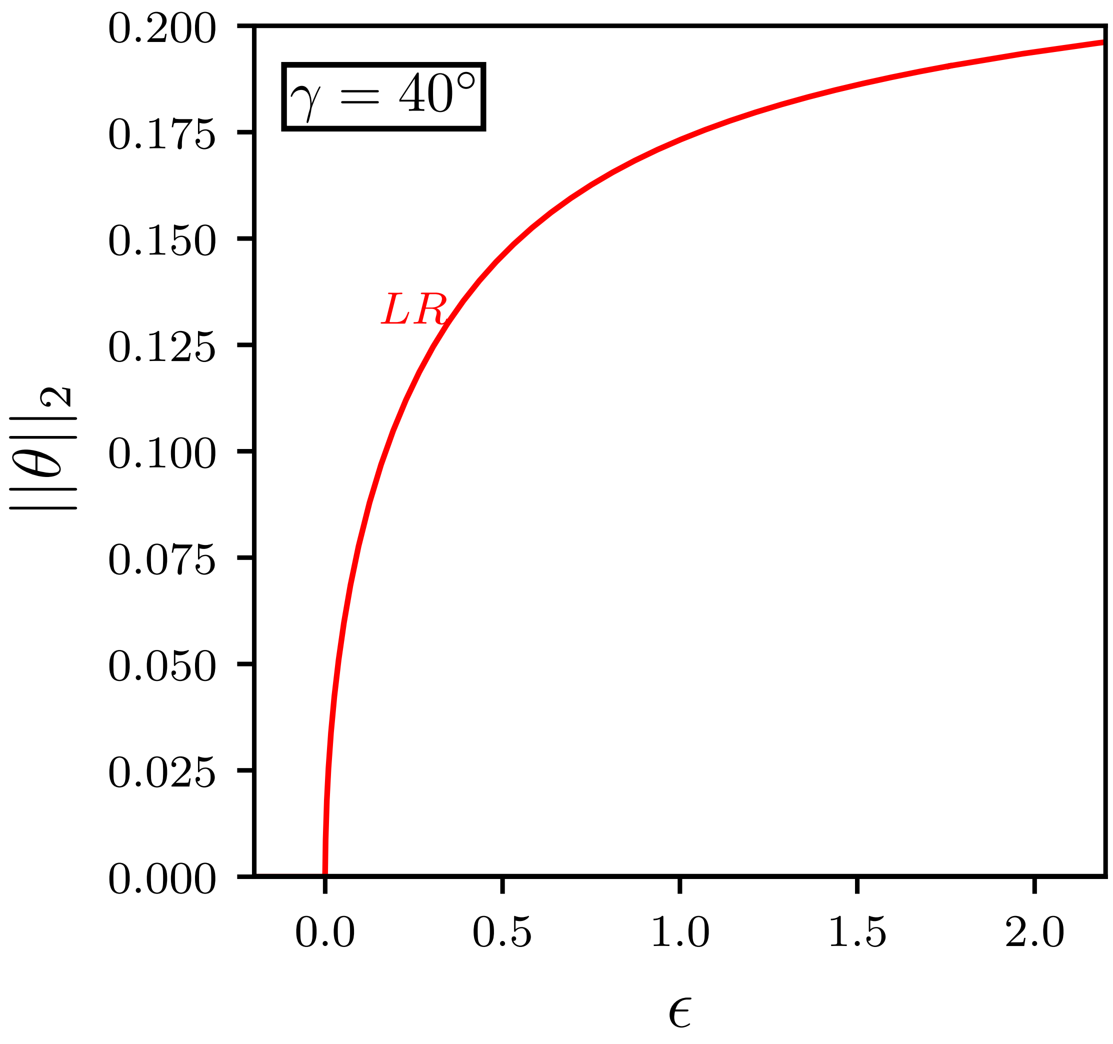
$$\sigma[\vec{u},\theta] = [u,-v,w,\theta](x,-y-\frac{L_y}{4},z)$$



Origin of exact invariant solutions – bifurcation analysis
Parametric continuation

Longitudinal Rolls (LR)

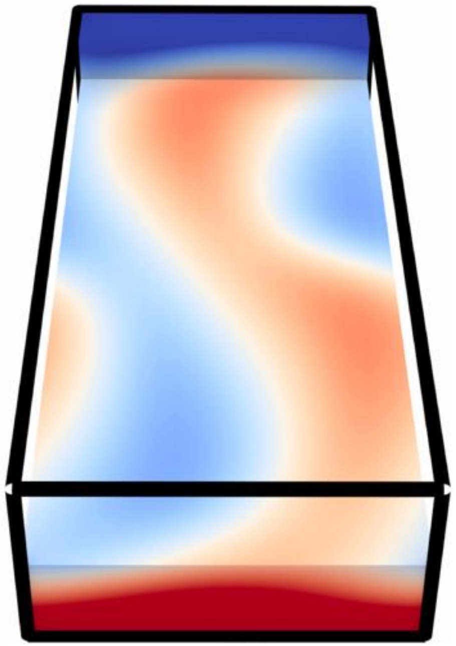


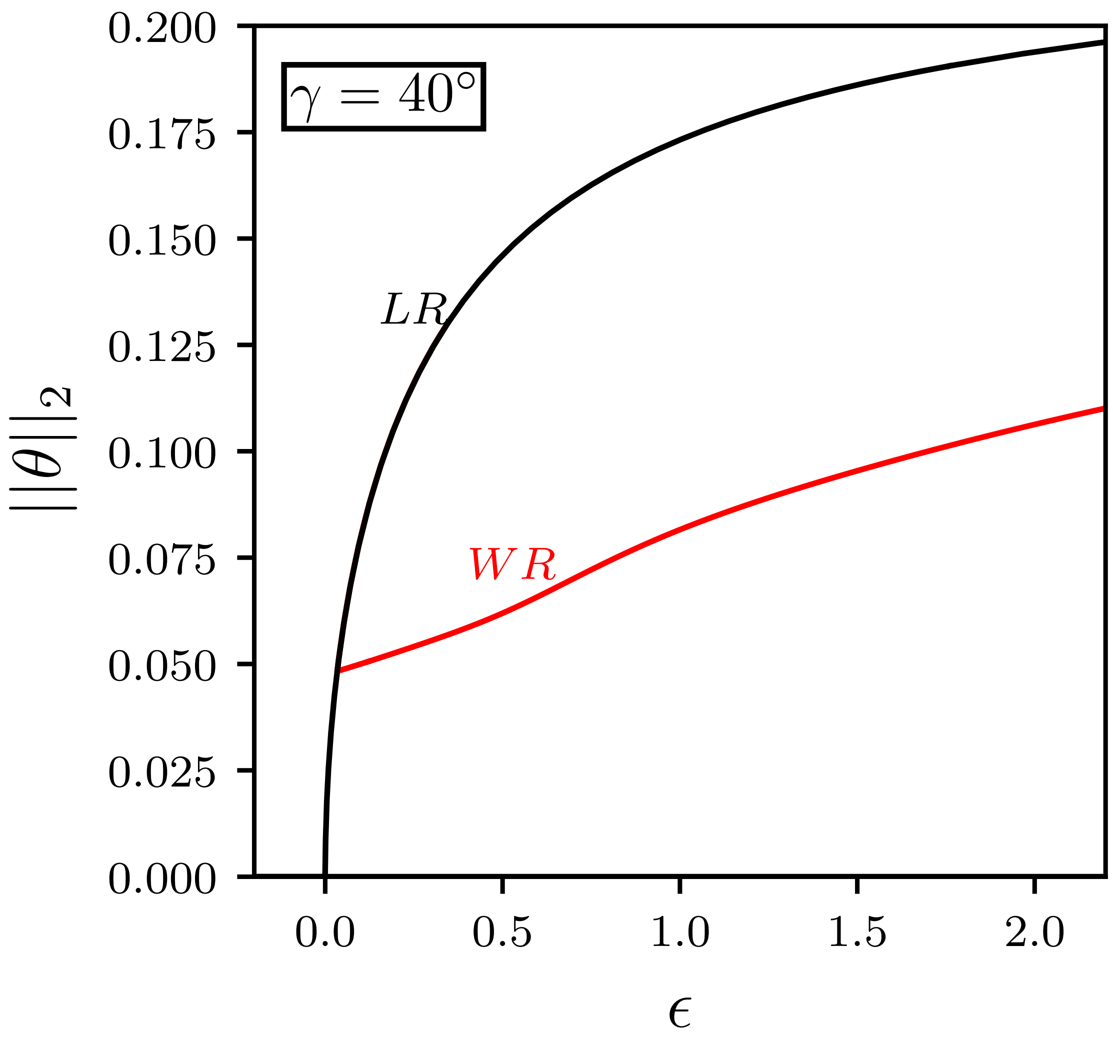


Origin of exact invariant solutions – bifurcation analysis

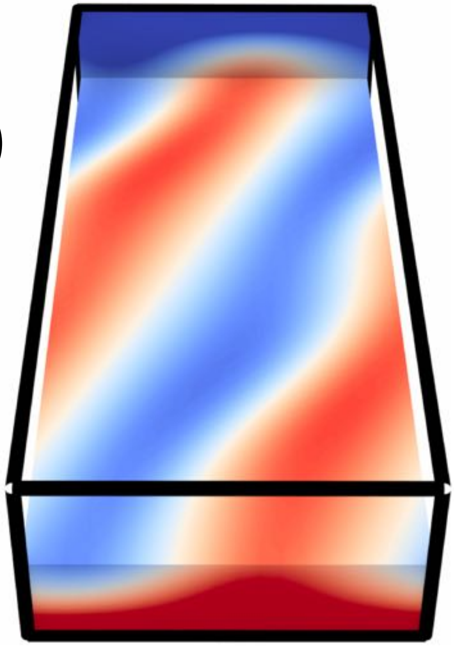
Parametric continuation

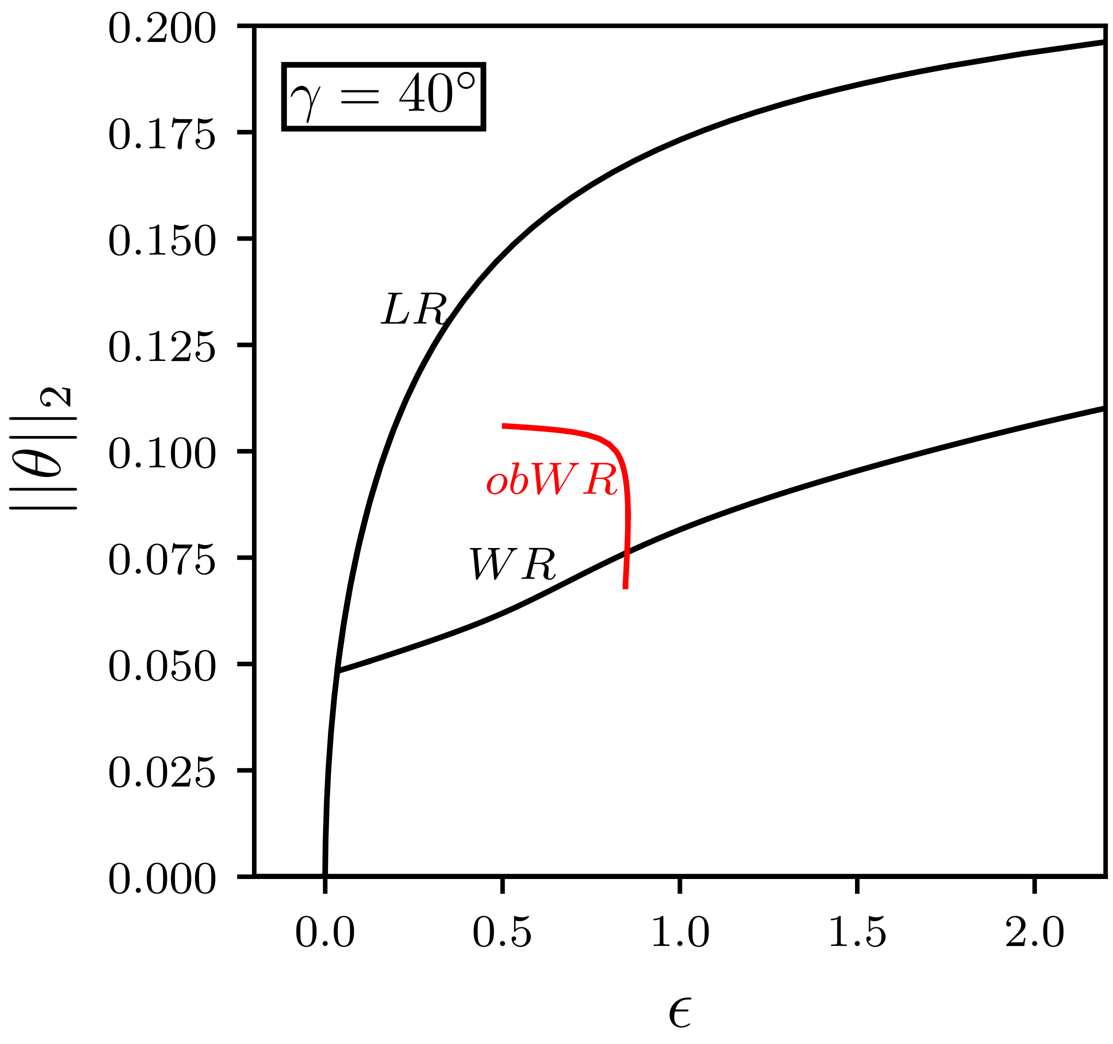
Wavy Rolls (WR)



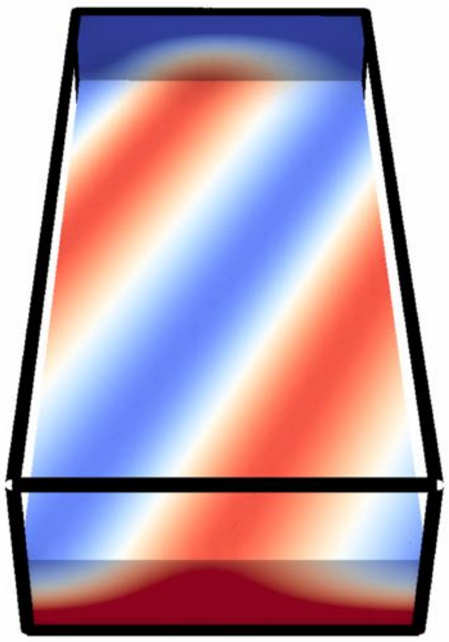


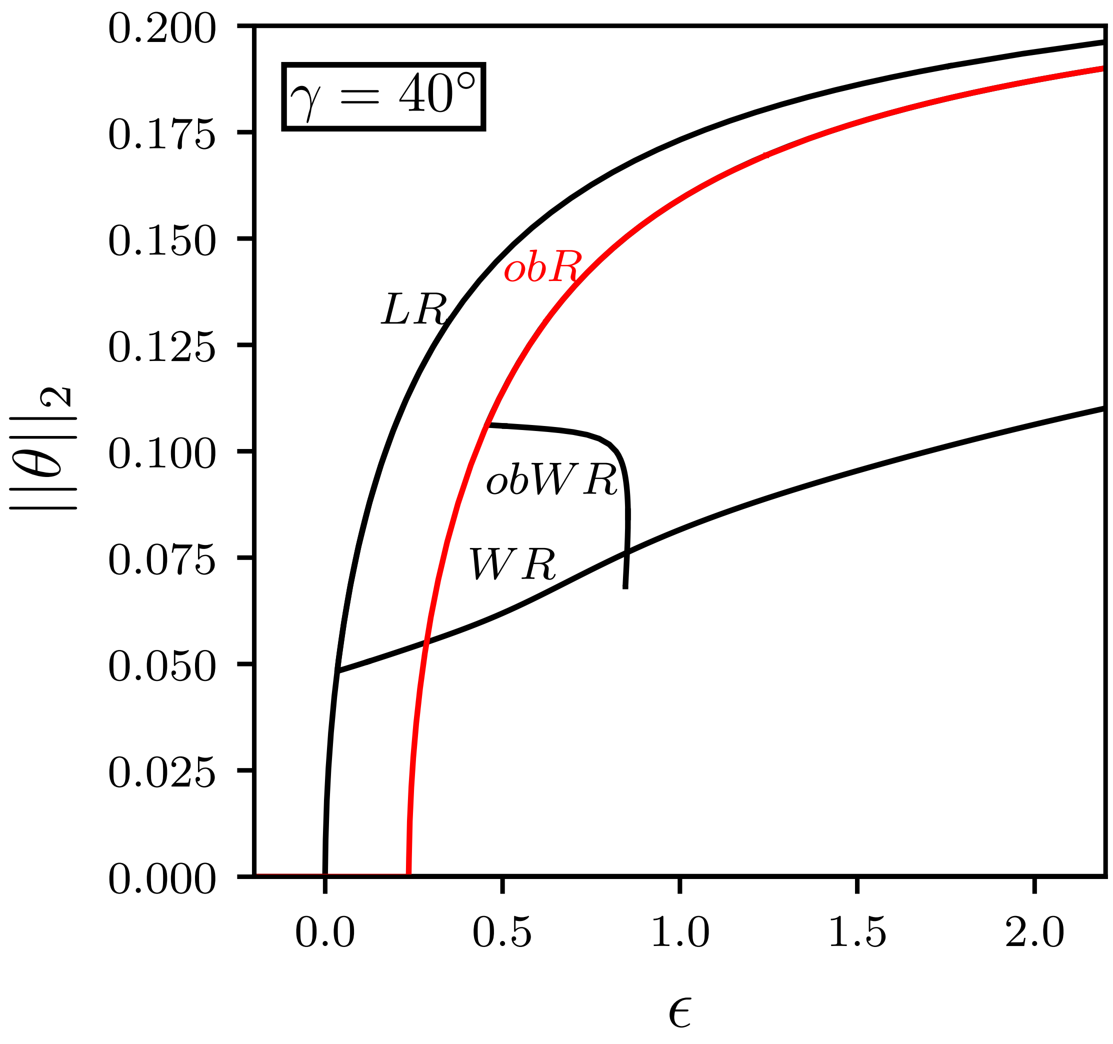
Oblique Wavy Rolls (obWR)



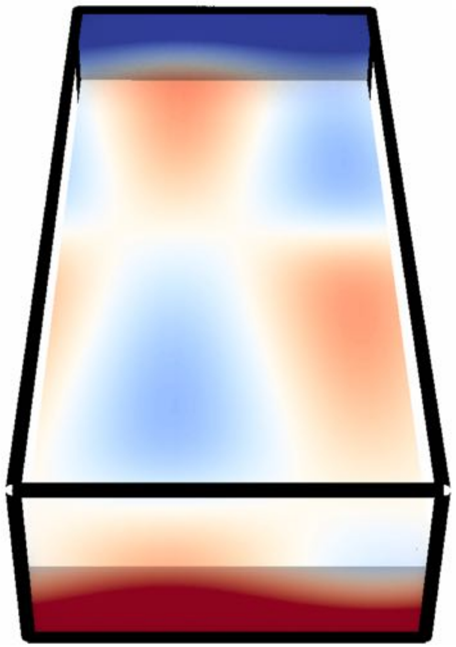


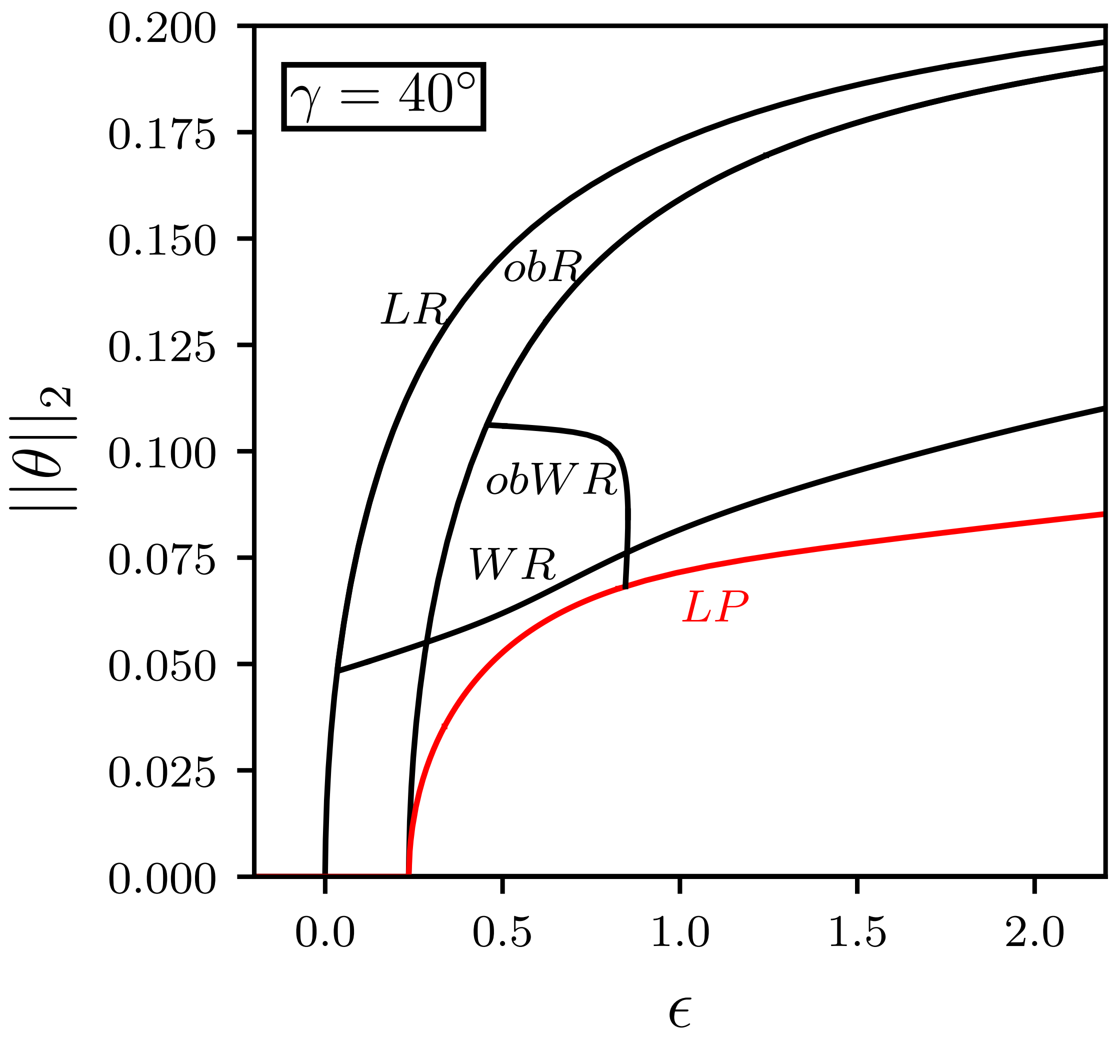
Oblique Rolls (obR)





Longitudinal Plumes (LP)

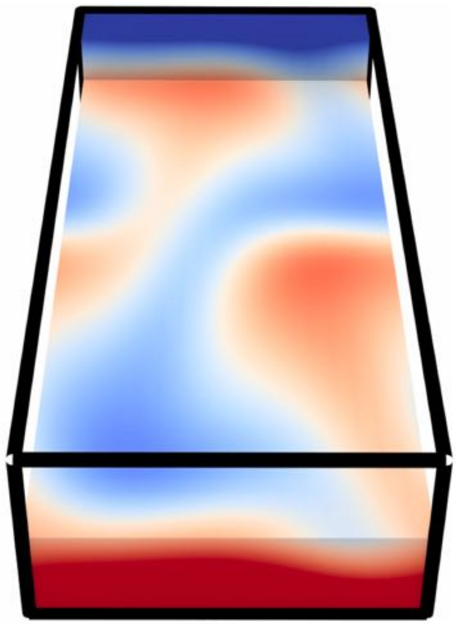


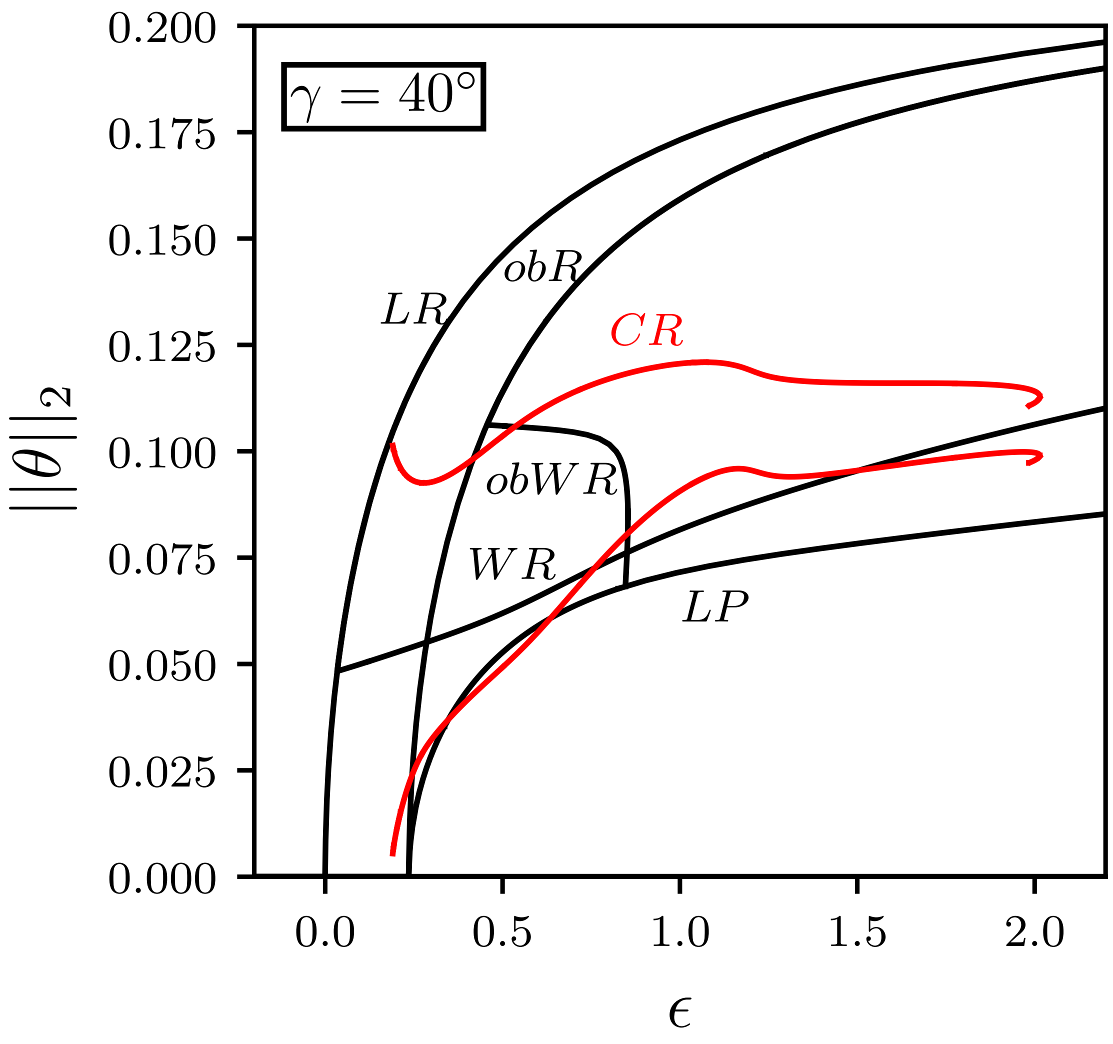


Origin of exact invariant solutions – bifurcation analysis

Parametric continuation

Crawling Rolls (CR)

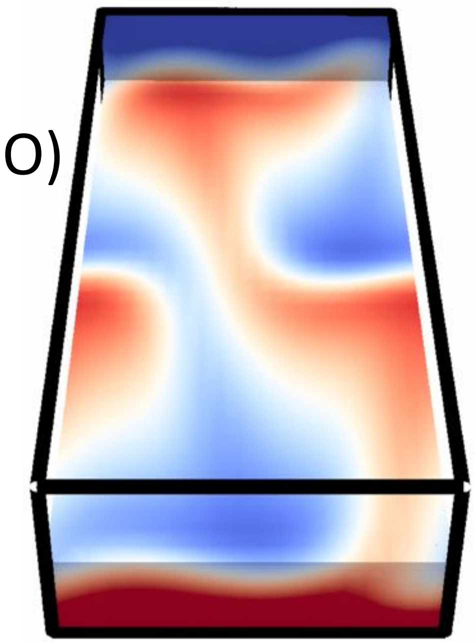


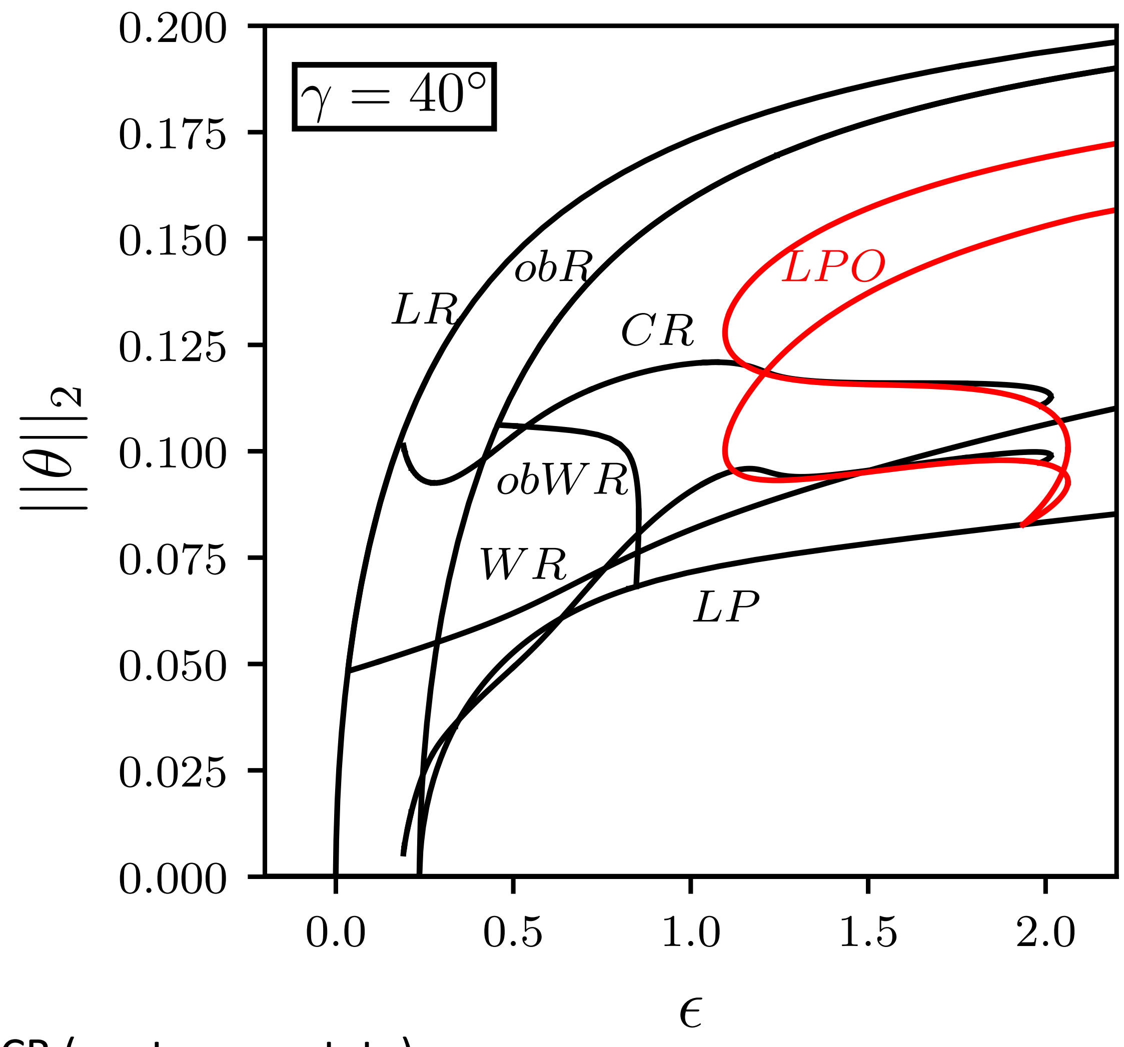


Origin of exact invariant solutions – bifurcation analysis

Parametric continuation

Long. Plumes Oscillation (LPO)

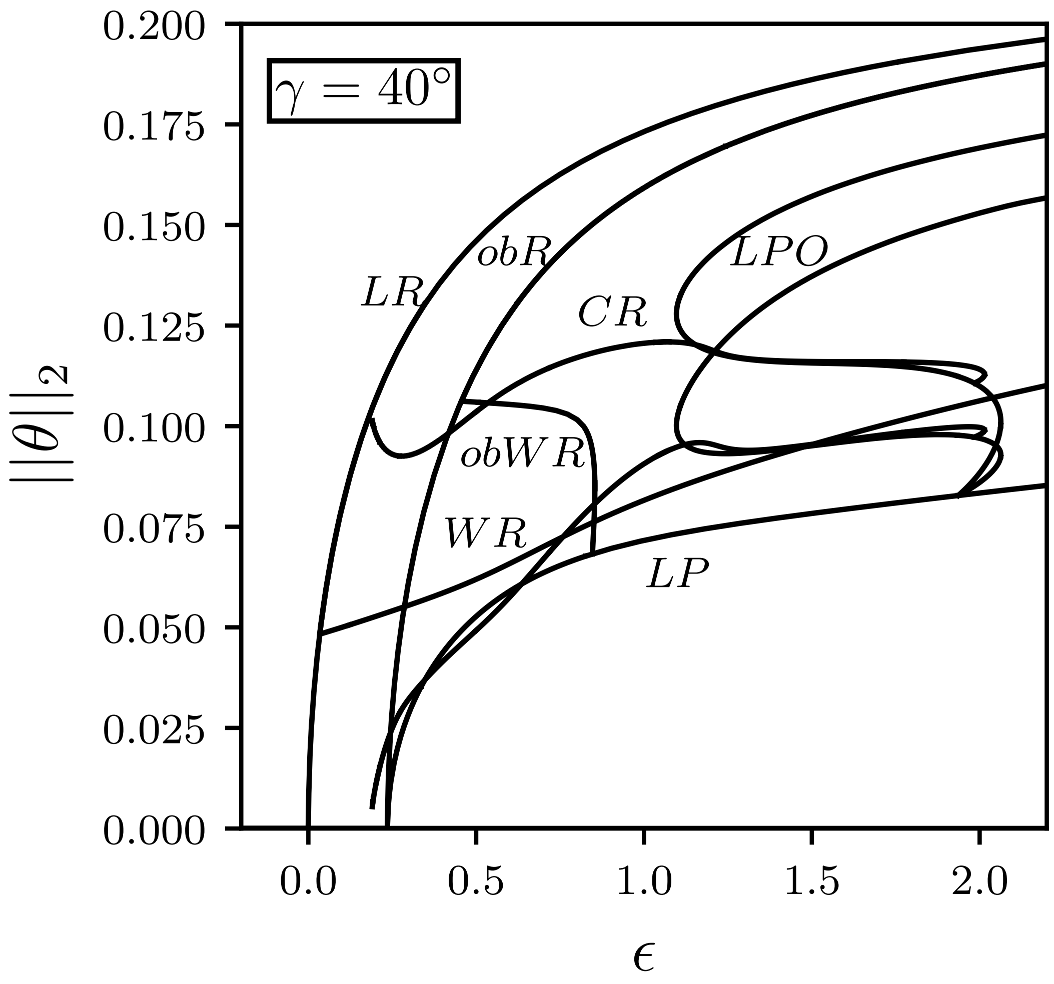




Result: Crawling roles connected to base flow: B -> LP -> LP0 -> CR (quaternary state)

Exact solutions for other inclination angles

Question: More invariant solutions underlying other convection patterns?



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EPFL

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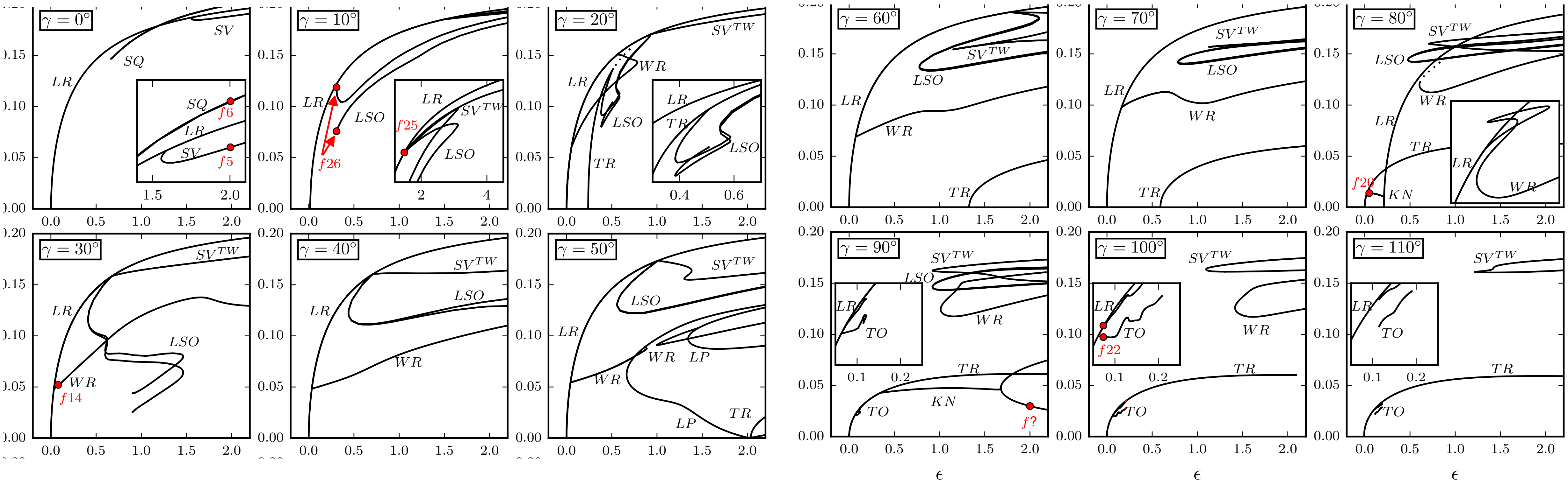


Observation: Multitude of many different invariant solutions

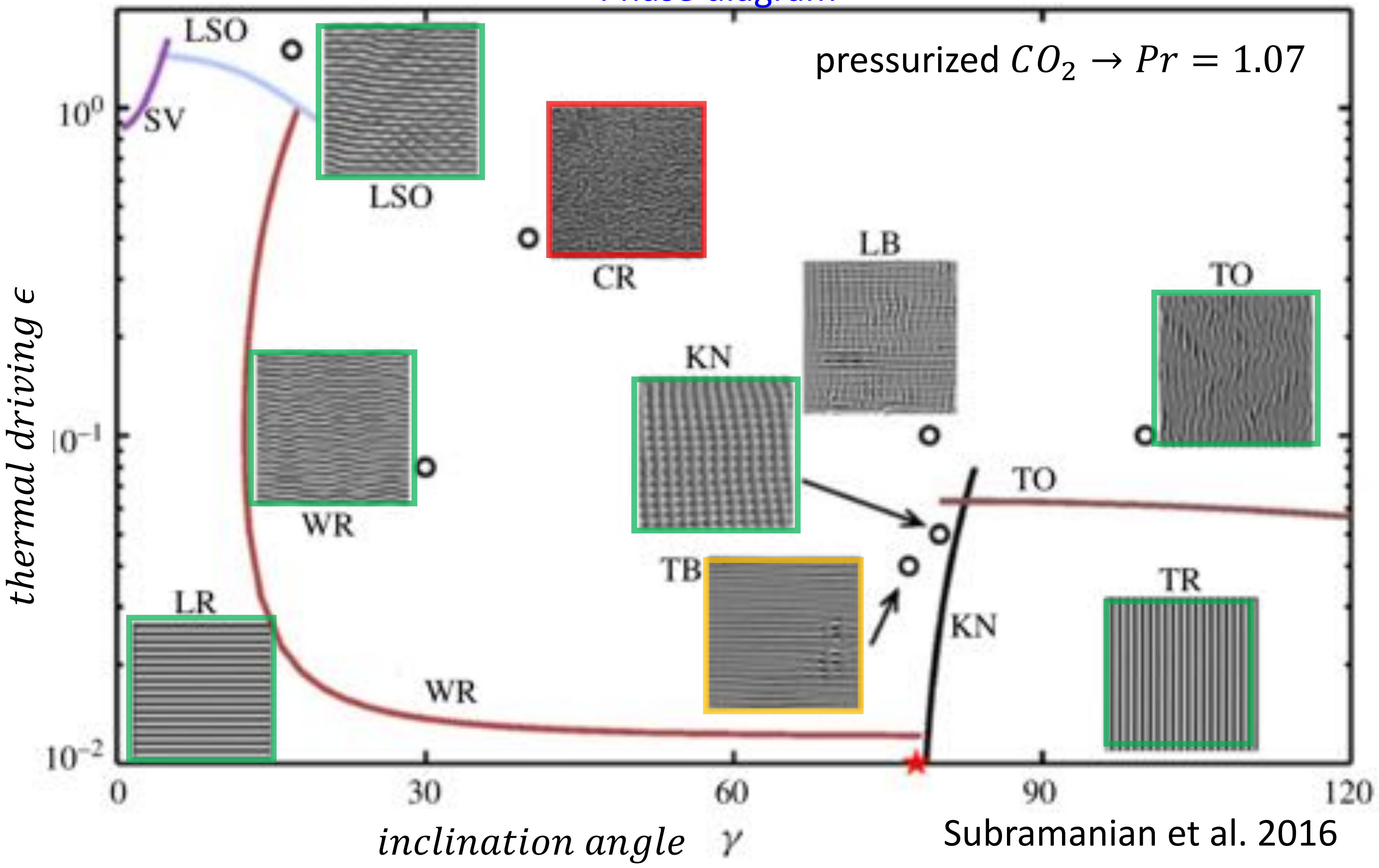
Interpretation: Underlie complex spatio-temporal convection patterns observed at different parameters

➡ Exact invariant solutions as 'building blocks' of complex fully nonlinear dynamics

Question: What about large-scale modulations, defects, spatial localization?

Transverse bursts
Localization in space

Phase diagram



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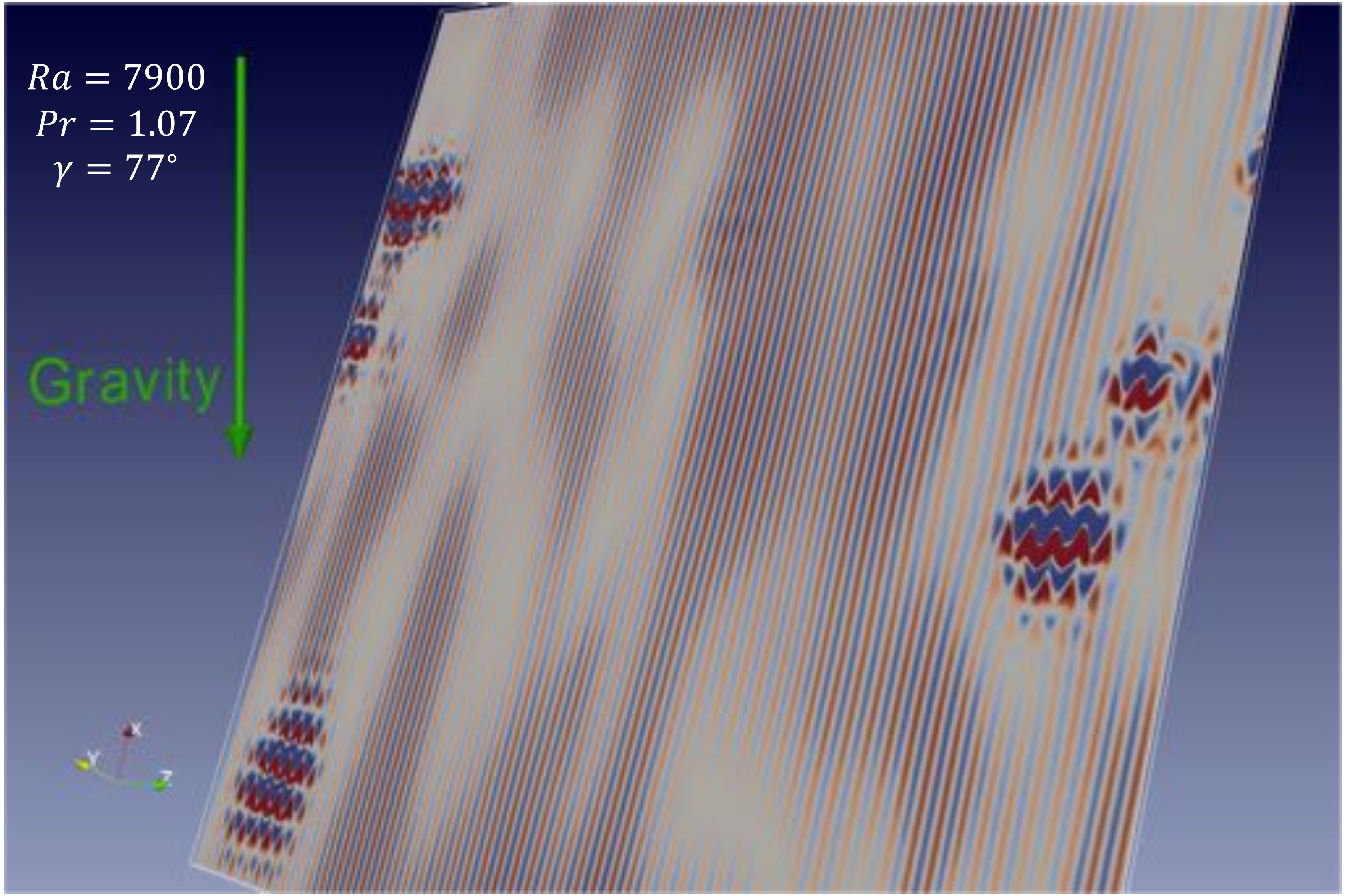
o experimental observation (Daniels & Bodenschatz)

Modulated stripes and bursting spots

DNS of transverse bursts



Features to explain:

(A): Bursting in time

(B): Localized in space

Question: Exact invariant solution(s) underlying bursting spots?

<u>system parameters</u>

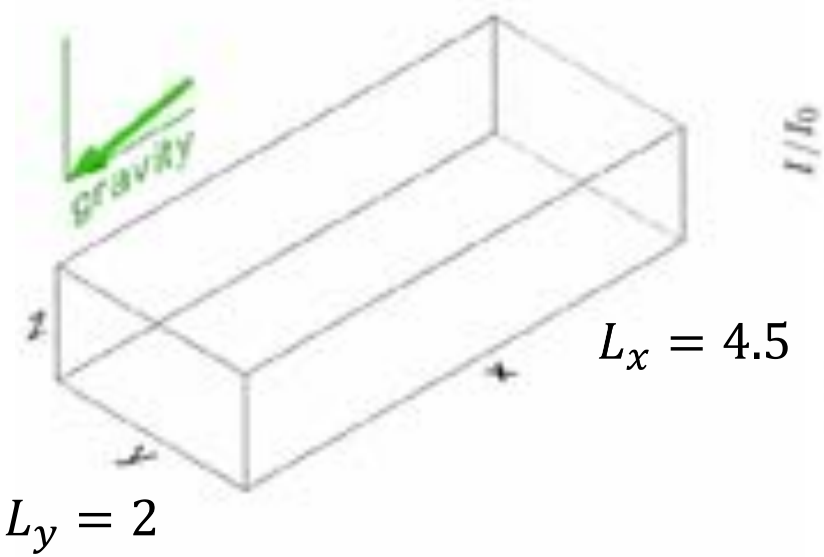
$Pr = 1.07$

$Ra = 8700$

$\gamma = 78^\circ$

DNS in a small domain:

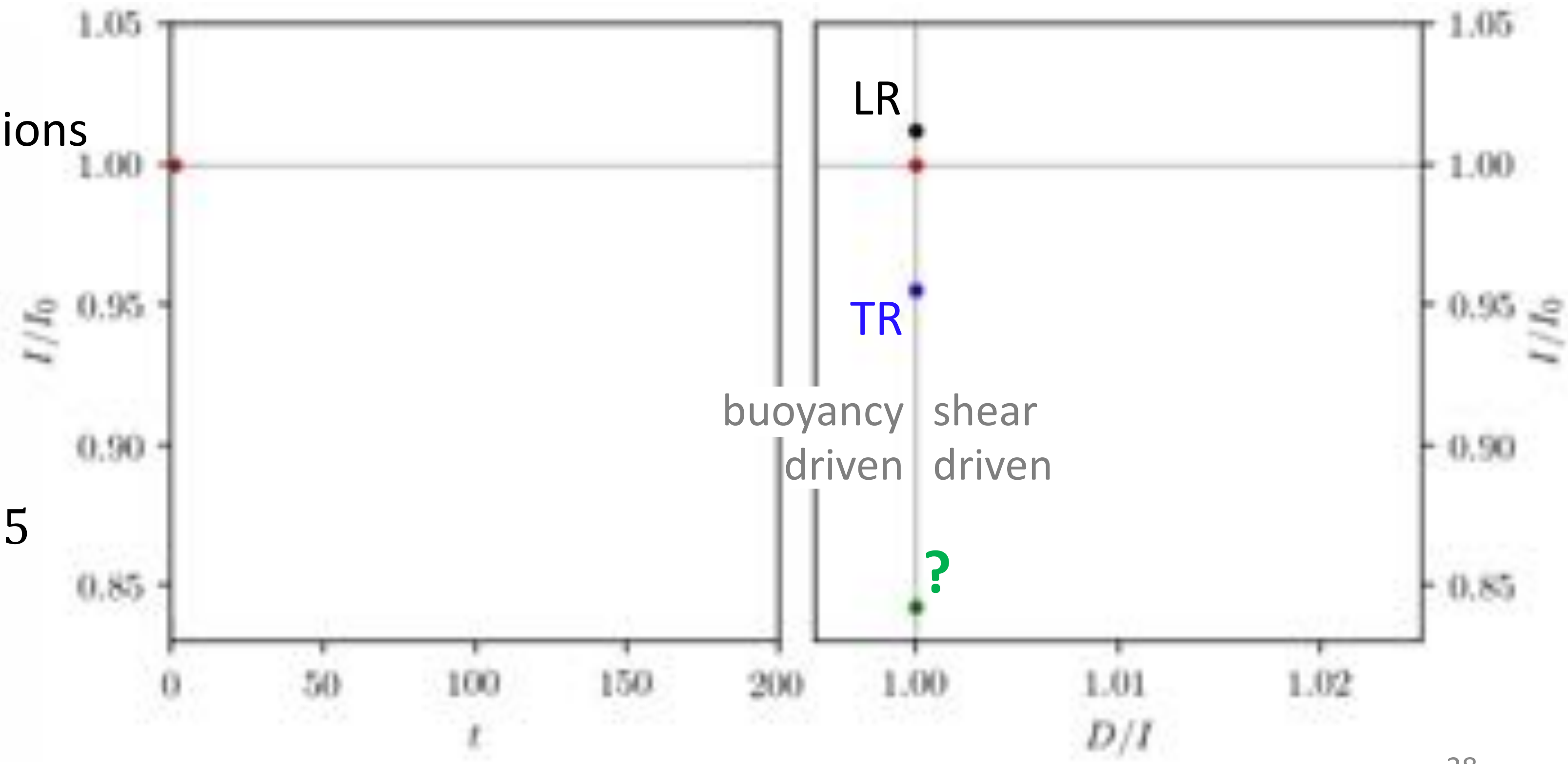
- Periodic boundary conditions
- Low amplitude noisy initial conditions



Phase portrait:

$$\frac{1}{2}\frac{\partial}{\partial t}\langle U^2\rangle_\Omega = I - D = \left\langle \hat{g}\vec{U}T \right\rangle_\Omega - \sqrt{\frac{Pr}{Ra}}\left\langle \left(\nabla\times\vec{U}\right)^2 \right\rangle_\Omega$$

Change of kinetic energy = Input by buoyancy − Dissipation by shear



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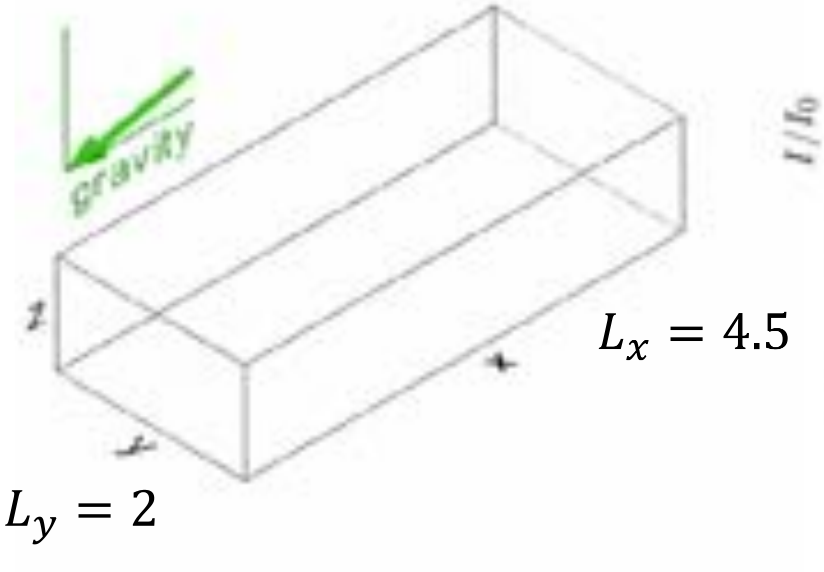

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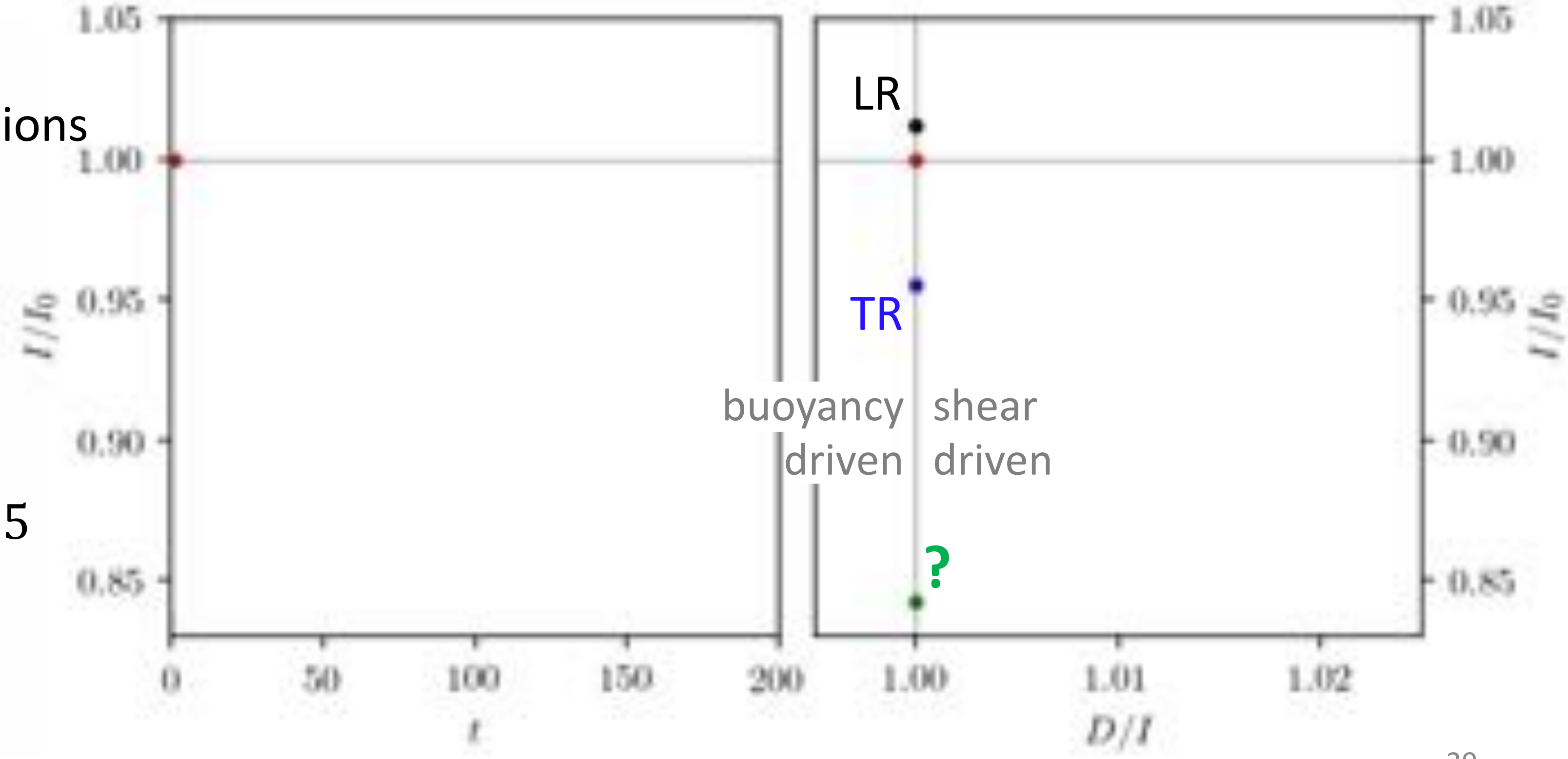

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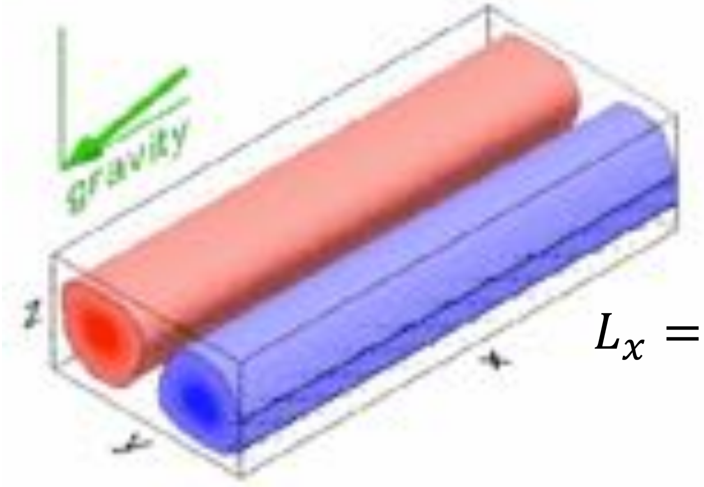


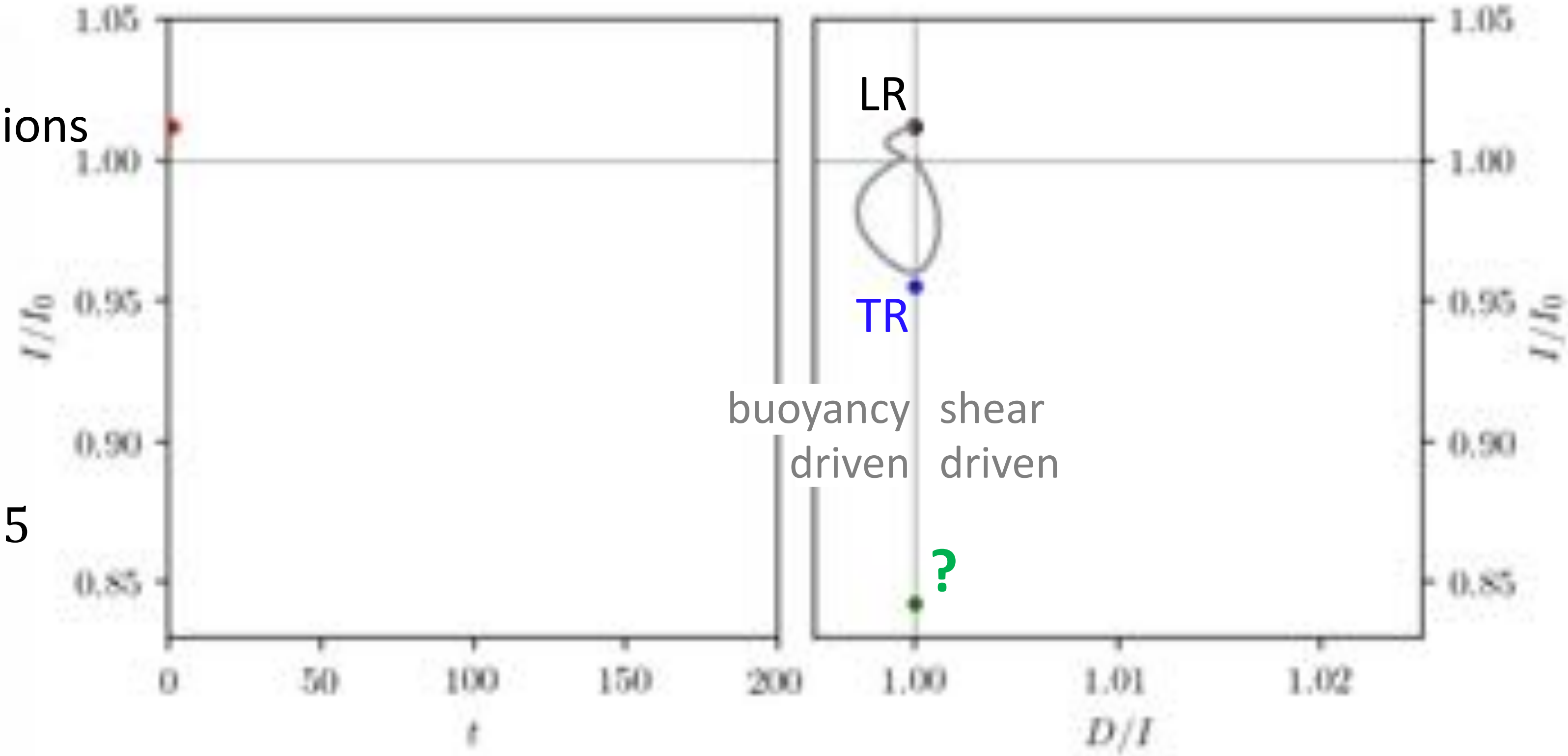
$L_x = 4.5$

$L_y = 2$

Phase portrait:

$$\frac{1}{2}\frac{\partial}{\partial t}\langle U^2\rangle_\Omega = I - D = \langle \hat{g}\vec{U}T\rangle_\Omega - \sqrt{\frac{Pr}{Ra}}\left\langle\left(\nabla\times\vec{U}\right)^2\right\rangle_\Omega$$

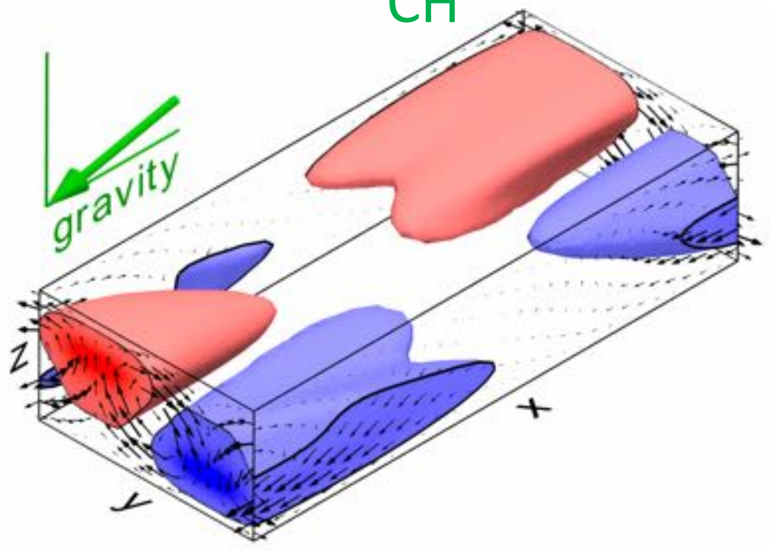
Change of kinetic energy = Input by buoyancy − Dissipation by shear



system parameters
$Pr = 1.07$
$Ra = 8700$
$\gamma = 78^\circ$

New invariant solution for chevron pattern (CH):

CH

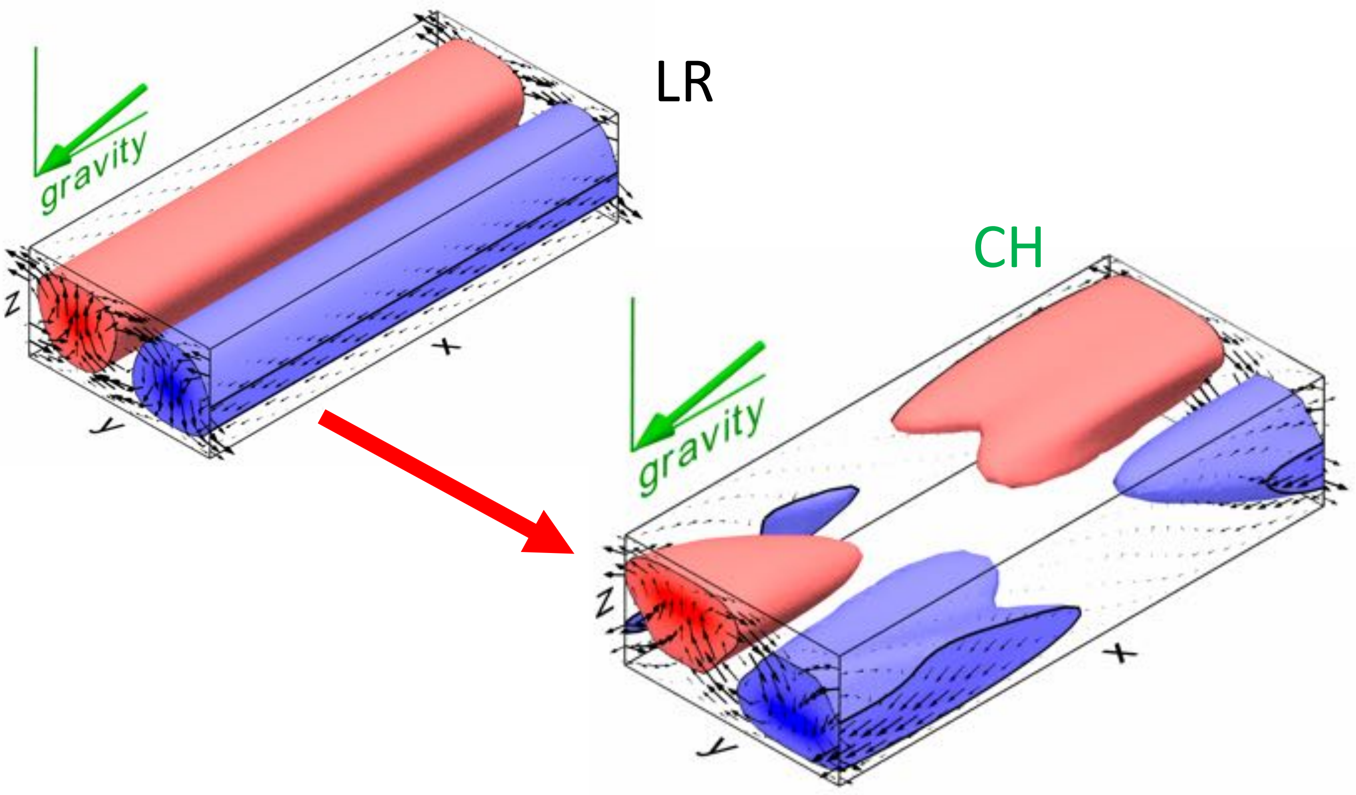


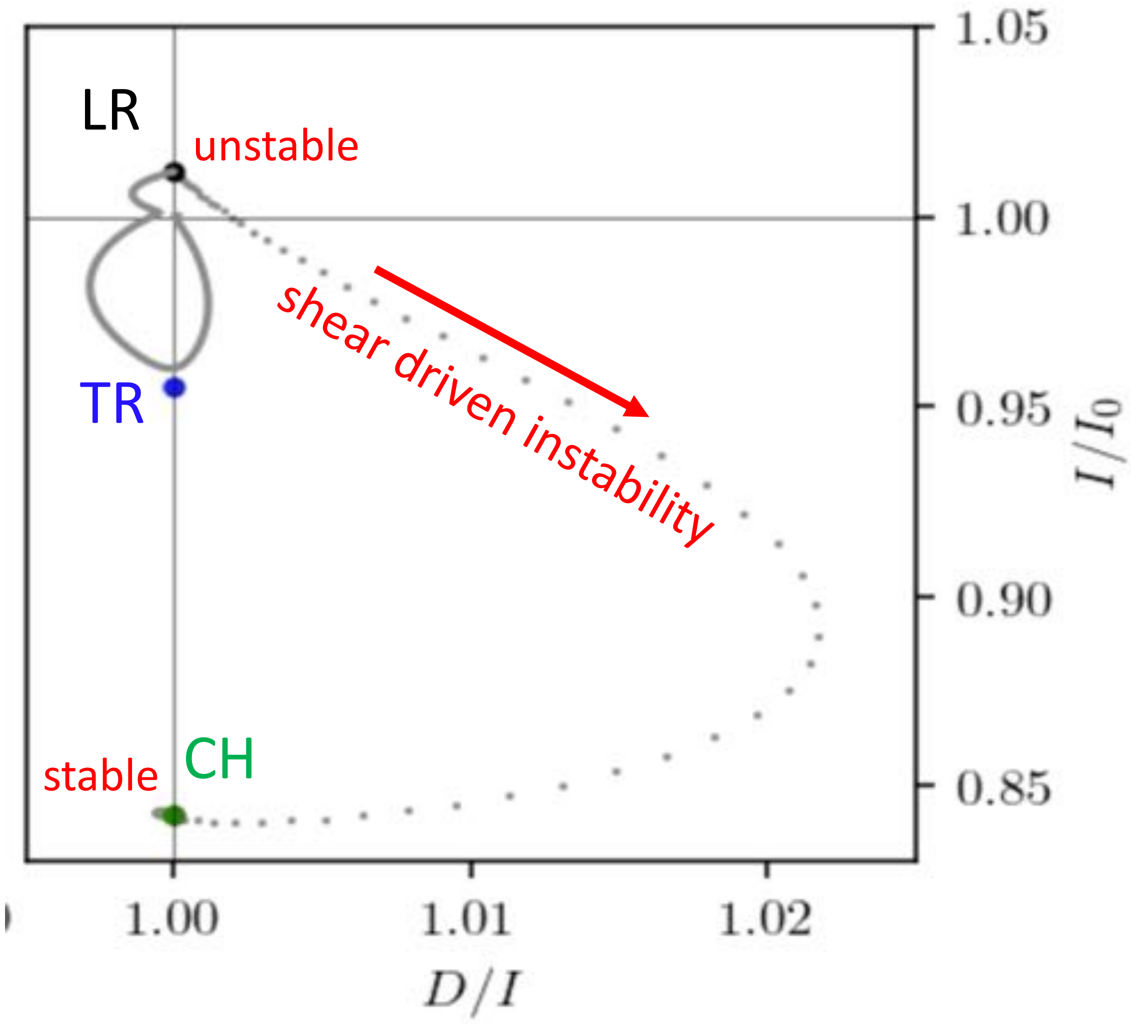
Captures the periodic pattern of the bursting core structure!

system parameters
$Pr = 1.07$
$Ra = 8700$
$\gamma = 78°$

Question
Feature (A): Bursting in time?

Result:
Heteroclinic orbit **LR** → **CH**





EPFL

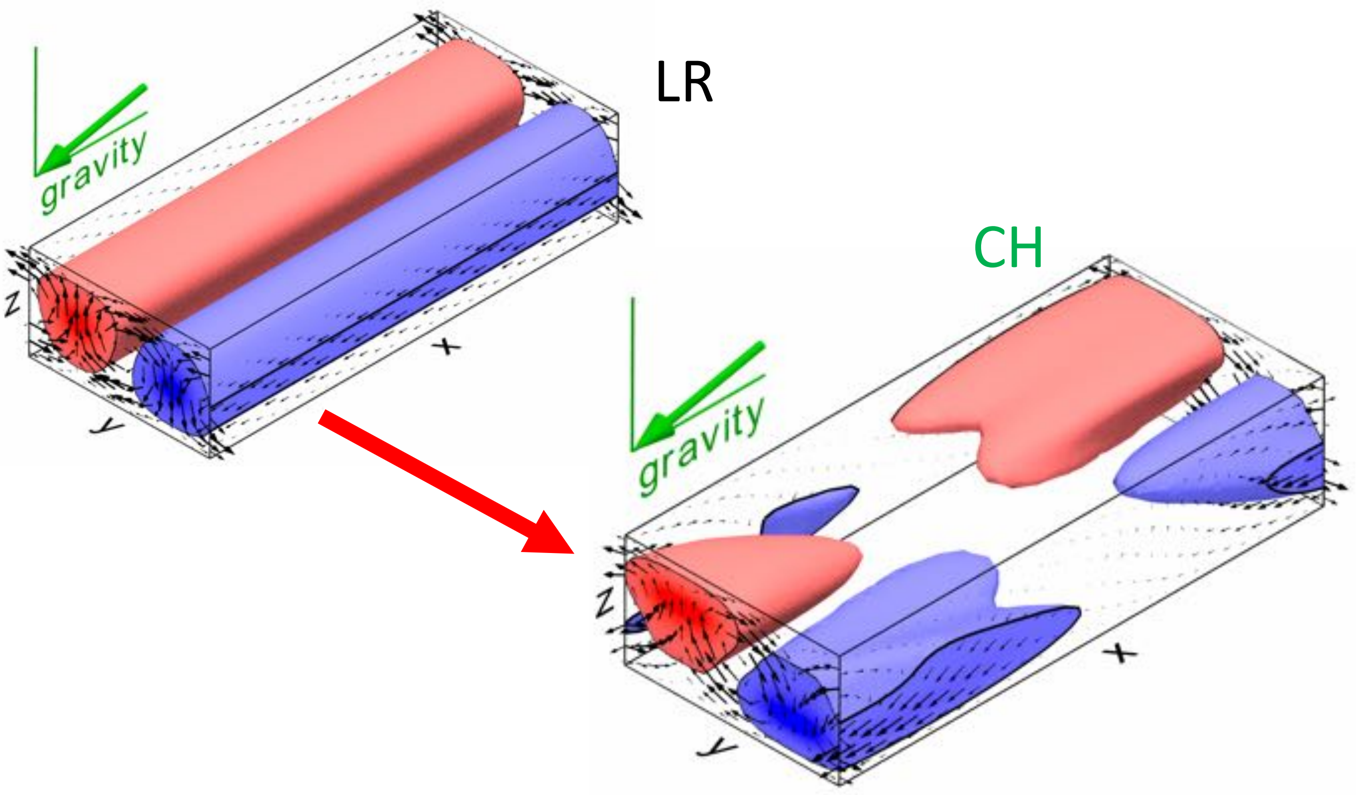
system parameters
$Pr = 1.07$
$Ra = 8700$
$\gamma = 78^\circ$

Question

Feature (A): Bursting in time?

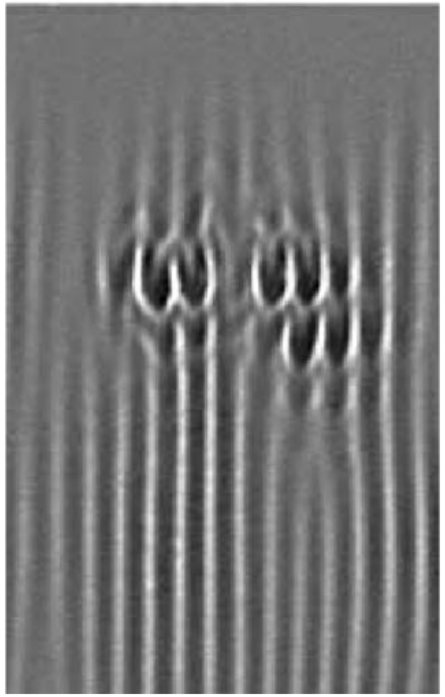
Result:

Heteroclinic orbit **LR** → **CH**



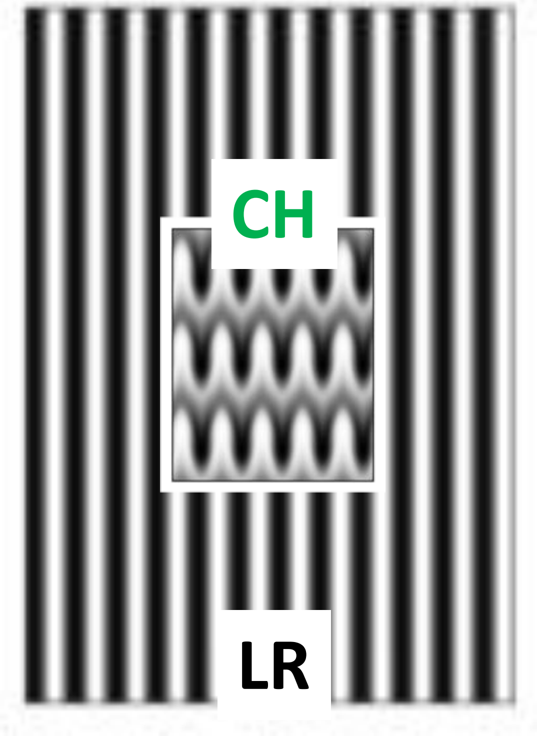
Feature (B): Localization in space?

Experiment



Daniels et al., 2000

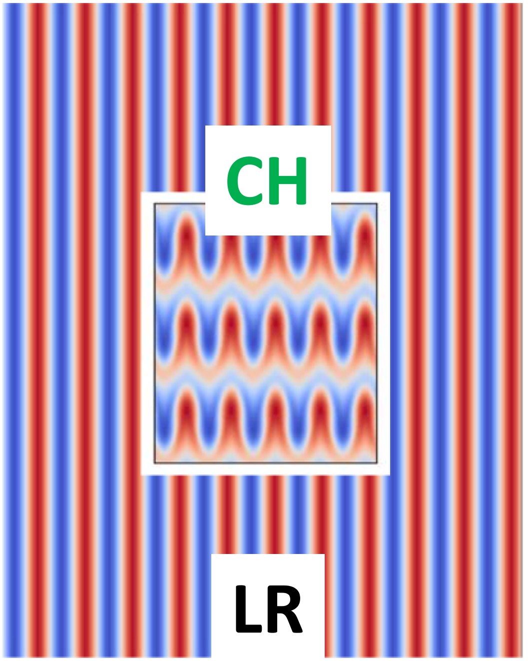
Localized invariant solution???



Approach:

Find pattern forming bifurcations

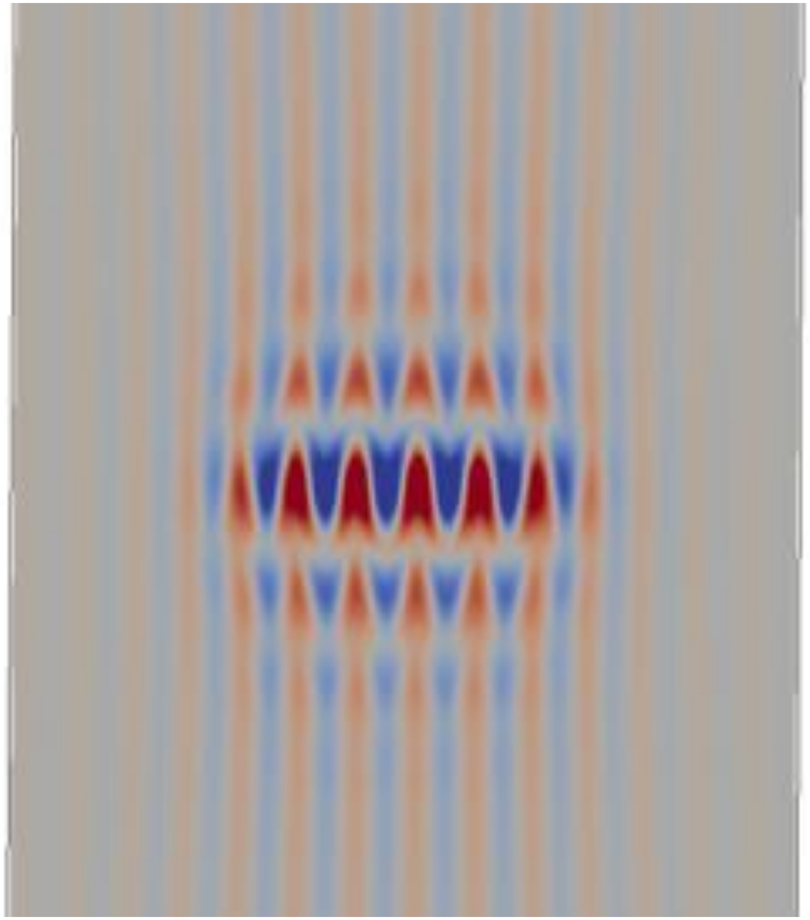
Bifurcation analysis: yields parameters where both LR **and** CH are stable



Hypothesis: There are invariant solutions as coexisting **LR** and **CH** in the **bistability range**.

Transverse bursts
Spatially localized invariant solution

doubly localized
invariant solution



Chevron (CH) in longitudinal role (LR) background

DNS (time evolution)

Interpretation: Unstable invariant solution clearly capture dynamics

Summary & conclusions

System: Convection in an inclined layer (ILC)

Question: Describe chaotic dynamics via ‘bouncing’ between unstable invariant solutions?

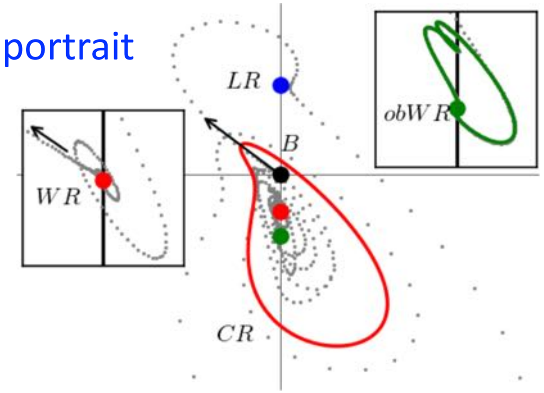
Tools: Symmetry-reduced DNS, fixed point search, parametric continuation

Identified: Exact invariant solutions of fully nonlinear 3D Navier-Stokes equations

Capture key features of the complex spatio-temporal patterns

Are transiently visited -> form *backbone* of the nonlinear dynamics

Phase portrait



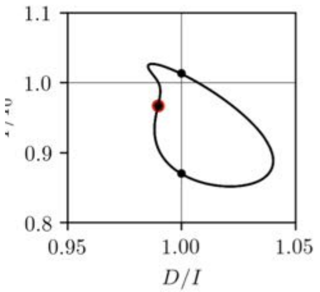
Origin: Sequence of bifurcations from laminar base flow

Relevance: (1) Explain convection patterns not captured by linear / weakly nonlinear theory

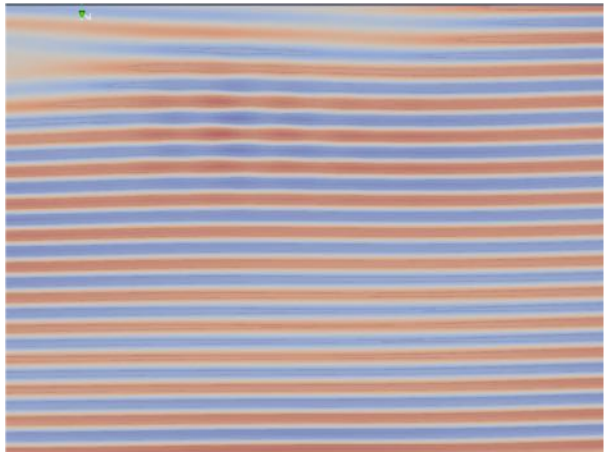
(2) Suggestion: ILC great system for dynamical systems approach to turbulence

(3) General method for nonlinear PDEs (incl. non-variational ones)

Periodic orbit

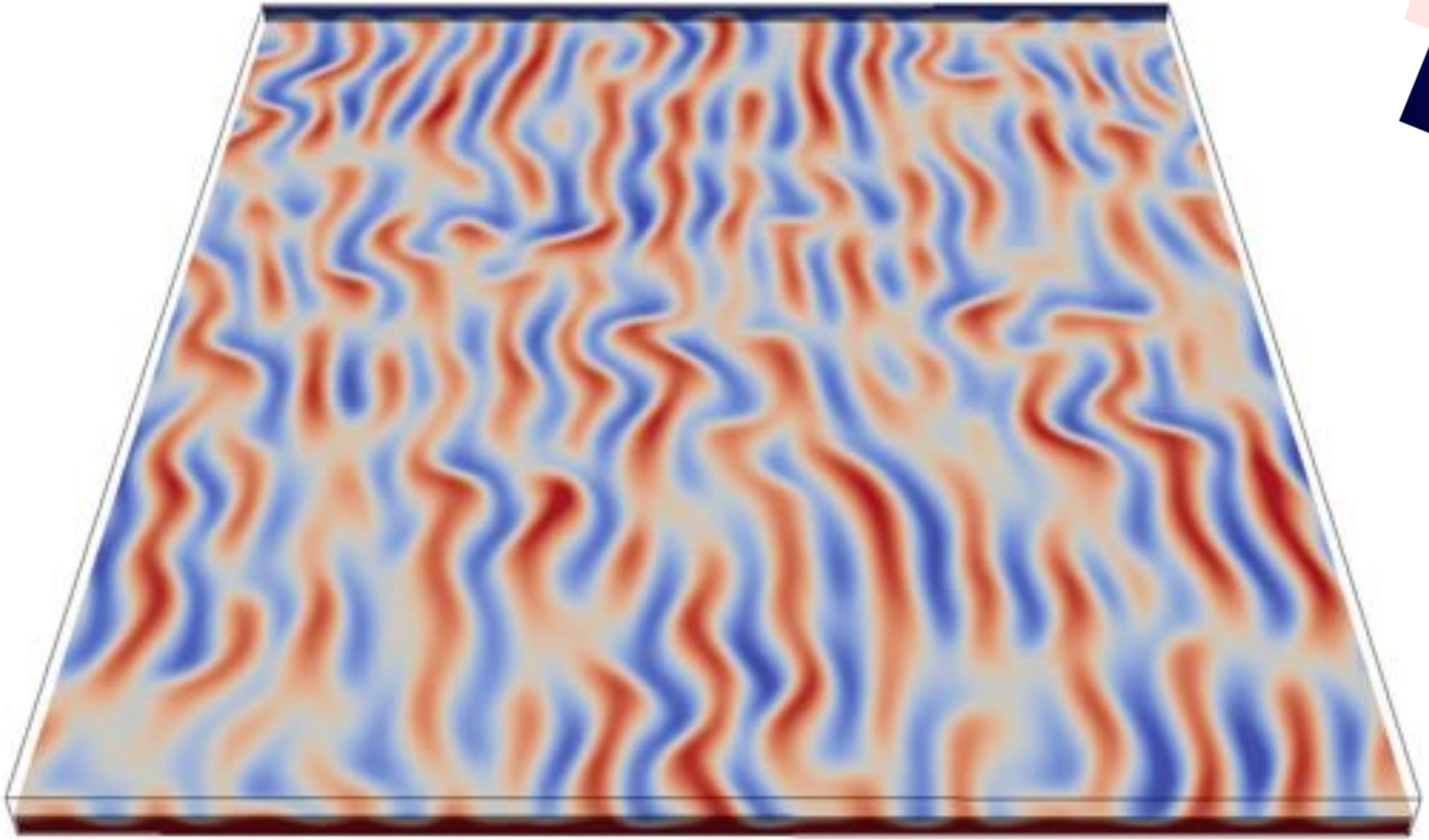


Bursting pattern



Thank you Bruno!!

Summary & conclusions



released: www.channelflow.ch

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F Reetz, TM Schneider - arXiv preprint arXiv:1911.02836, 2019
F Reetz, P Subramanian, TM Schneider - arXiv preprint arXiv:1911.02873, 2019