ee

"This book is perhaps the most user-friendly introductory text to the essentials of quantum field theory and its many modern applications. With his physically intuitive approach, Professor Zee makes a serious topic more reachable for beginners, reducing the conceptual barrier while preserving enough mathematical details necessary for a firm grasp of the subject."
—Bei-Lok Hu, *University of Maryland*

"Like the famous Feynman Lectures on Physics, this book has the flavor of a good blackboard lecture. Zee presents technical details, but only insofar as they serve the larger purpose of giving insight into quantum field theory and bringing out its beauty."
—Stephen M. Barr, *University of Delaware*

"This is a fantastic book—exciting, amusing, unique, and very valuable."
—Clifford V. Johnson, *University of Durham*

"Tony Zee explains quantum field theory with a clear and engaging style. For budding or seasoned condensed matter physicists alike, he shows us that field theory is a nourishing nut to be cracked and savored."
—Matthew P. A. Fisher, *Kavli Institute for Theoretical Physics*

ERSITY PRESS

Quantum Field Theory
IN A NUTSHELL

Quantum Field Theory
IN A NUTSHELL

A. Zee

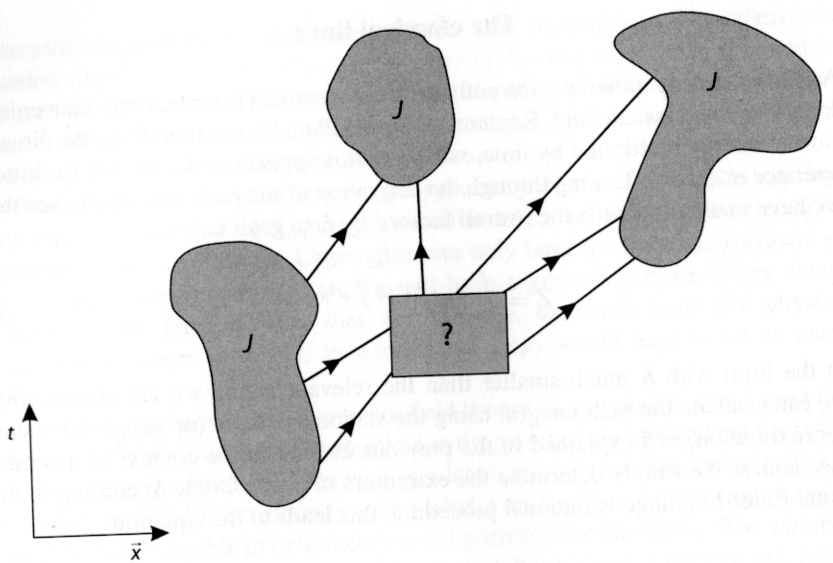

Figure I.3.1

Disturbing the vacuum

We would like to do something more exciting than watching a boiling sea of quantum fluctuations. We would like to disturb the vacuum. Somewhere in space, at some instant in time, we would like to create a particle, watch it propagate for a while, and then annihilate it somewhere else in space, at some later instant in time. In other words, we want to create a source and a sink (sometimes referred to collectively as sources) at which particles can be created and annihilated.

To see how to do this, let us go back to the mattress. Bounce up and down on it to create some excitations. Obviously, pushing on the mass labeled by a in the mattress corresponds to adding a term such as $J_a(t)q_a$ to the potential $V(q_1, q_2, \cdots, q_N)$. More generally, we can add $\sum_a J_a(t)q_a$. When we go to field theory this added term gets promoted to $J(x)\varphi(x)$ in the field theory Lagrangian, according to the promotion table (6).

This so-called source function $J(t, \vec{x})$ describes how the mattress is being disturbed. We can choose whatever function we like, corresponding to our freedom to push on the mattress wherever and whenever we like. In particular, $J(x)$ can vanish everywhere in spacetime except in some localized regions.

By bouncing up and down on the mattress we can get wave packets going off here and there (Fig. I.3.1). This corresponds precisely to sources (and sinks) for particles. Thus, we really want the path integral

$$Z = \int D\varphi e^{i\int d^4x[\frac{1}{2}(\partial\varphi)^2 - V(\varphi) + J(x)\varphi(x)]} \tag{11}$$

Free field theory

The functional integral in (11) is impossible to do except when

$$\mathcal{L}(\varphi) = \frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] \tag{12}$$

The corresponding theory is called the free or Gaussian theory. The equation of motion (9) works out to be $(\partial^2 + m^2)\varphi = 0$, known as the Klein-Gordon equation.[2] Being linear, it can be solved immediately to give $\varphi(\vec{x}, t) = e^{i(\omega t - \vec{k}\cdot\vec{x})}$ with

$$\omega^2 = \vec{k}^2 + m^2 \tag{13}$$

In the natural units we are using, $\hbar = 1$ and so frequency ω is the same as energy $\hbar\omega$ and wave vector $\vec{k}$ is the same as momentum $\hbar\vec{k}$. Thus, we recognize (13) as the energy-momentum relation for a particle of mass m, namely the sophisticate's version of the layperson's $E = mc^2$. We expect this field theory to describe a relativistic particle of mass m.

Let us now evaluate (11) in this special case:

$$Z = \int D\varphi e^{i\int d^4x\{\frac{1}{2}[(\partial\varphi)^2 - m^2\varphi^2] + J\varphi\}} \tag{14}$$

Integrating by parts under the $\int d^4x$ and not worrying about the possible contribution of boundary terms at infinity (we implicitly assume that the fields we are integrating over fall off sufficiently rapidly), we write

$$Z = \int D\varphi e^{i\int d^4x[-\frac{1}{2}\varphi(\partial^2 + m^2)\varphi + J\varphi]} \tag{15}$$

You will encounter functional integrals like this again and again in your study of field theory. The trick is to imagine discretizing spacetime. You don't actually have to do it: Just imagine doing it. Let me sketch how this goes. Replace the function $\varphi(x)$ by the vector $\varphi_i = \varphi(ia)$ with i an integer and a the lattice spacing. (For simplicity, I am writing things as if we were in 1-dimensional spacetime. More generally, just let the index i enumerate the lattice points in some way.) Then differential operators become matrices. For example, $\partial\varphi(ia) \to (1/a)(\varphi_{i+1} - \varphi_i) \equiv \sum_j M_{ij}\varphi_j$, with some appropriate matrix M. Integrals become sums. For example, $\int d^4x J(x)\varphi(x) \to a^4 \sum_i J_i\varphi_i$.

Now, lo and behold, the integral (15) is just the integral we did in (I.2.15)

---

[2] The Klein-Gordon equation was actually discovered by Schrödinger before he found the equation that now bears his name. Later, in 1926, it was written down independently by Klein, Gordon, Fock, Kudar, de Donder, and Van Dungen.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dq_1 dq_2\cdots dq_N\, e^{(i/2)q\cdot A\cdot q+iJ\cdot q}$$

$$=\left(\frac{(2\pi i)^N}{\det[A]}\right)^{\frac{1}{2}} e^{-(i/2)J\cdot A^{-1}\cdot J} \tag{16}$$

The role of A in (16) is played in (15) by the differential operator $-(\partial^2+m^2)$. The defining equation for the inverse $A\cdot A^{-1}=I$ or $A_{ij}A^{-1}_{jk}=\delta_{ik}$ becomes in the continuum limit

$$-(\partial^2+m^2)D(x-y)=\delta^{(4)}(x-y) \tag{17}$$

We denote the continuum limit of A^{-1}_{jk} by $D(x-y)$ (which we know must be a function of $x-y$, and not of x and y separately, since no point in spacetime is special). Note that in going from the lattice to the continuum Kronecker is replaced by Dirac. It is very useful to be able to go back and forth mentally between the lattice and the continuum.

Our final result is

$$Z(J)=\mathcal{C}e^{-(i/2)\iint d^4x d^4y J(x)D(x-y)J(y)}\equiv \mathcal{C}e^{iW(J)} \tag{18}$$

with $D(x)$ determined by solving (17). The overall factor $\mathcal{C}$, which corresponds to the overall factor with the determinant in (16), does not depend on J and, as will become clear in the discussion to follow, is often of no interest to us. As a rule I will omit writing $\mathcal{C}$ altogether. Clearly, $\mathcal{C}=Z(J=0)$ so that $W(J)$ is defined by

$$Z(J)\equiv Z(J=0)e^{iW(J)} \tag{19}$$

Observe that

$$W(J)=-\frac{1}{2}\iint d^4x d^4y J(x)D(x-y)J(y) \tag{20}$$

is a simple quadratic functional of J. In contrast, $Z(J)$ depends on arbitrarily high powers of J. This fact will be of importance in Chapter I.7.

Free propagator

The function $D(x)$, known as the propagator, plays an essential role in quantum field theory. As the inverse of a differential operator it is clearly closely related to the Green's function you encountered in a course on electromagnetism.

Physicists are sloppy about mathematical rigor, but even so, they have to be careful once in a while to make sure that what they are doing actually makes sense. For the integral in (15) to converge for large φ we replace $m^2\to m^2-i\varepsilon$ so that

the integrand contains a factor $e^{-\varepsilon\int d^4x\varphi^2}$, where ε is a positive infinitesimal[3] we will let tend to zero later.

We can solve (17) easily by going to momentum space and recalling the representation of the Dirac delta function

$$\delta^{(4)}(x-y)=\int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)} \tag{21}$$

The solution is

$$D(x-y)=\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2-m^2+i\varepsilon} \tag{22}$$

which you can check by plugging into (17). Note that the so-called $i\varepsilon$ prescription we just mentioned is essential; otherwise the k integral would hit a pole.

To evaluate $D(x)$ we first integrate over k^0 by the method of contours. Define $\omega_k=+\sqrt{\vec{k}^2+m^2}$. The integrand has two poles in the complex k^0 plane, at $\pm\sqrt{\omega_k^2-i\varepsilon}$, which in the $\varepsilon\to 0$ limit is equal to $+\omega_k-i\varepsilon$ and $-\omega_k+i\varepsilon$. For x^0 positive we can extend the integration contour that goes from $-\infty$ to $+\infty$ on the real axis to include the infinite semicircle in the upper half-plane, thus enclosing the pole at $-\omega_k+i\varepsilon$ and giving $-i\int[d^3k/(2\pi)^3 2\omega_k]e^{i(\omega_k t-\vec{k}\cdot\vec{x})}$. For x^0 negative we close the contour in the lower half-plane. Thus

$$D(x)=-i\int\frac{d^3k}{(2\pi)^3 2\omega_k}[e^{-i(\omega_k t-\vec{k}\cdot\vec{x})}\theta(x^0)+e^{i(\omega_k t-\vec{k}\cdot\vec{x})}\theta(-x^0)] \tag{23}$$

Physically, $D(x)$ describes the amplitude for a disturbance in the field to propagate from the origin to x. We expect drastically different behavior depending on whether x is inside or outside the lightcone. Without evaluating the integral we can see roughly how things go. For $x=(t,0)$ with, say, $t>0$, $D(x)=-i\int[d^3k/(2\pi)^3 2\omega_k]e^{-i\omega_k t}$ is a superposition of plane waves and thus we should have oscillatory behavior. In contrast, for $x^0=0$, we have, upon interpreting $\theta(0)=\frac{1}{2}$, $D(x)=-i\int\left[d^3k/(2\pi)^3 2\sqrt{\vec{k}^2+m^2}\right]e^{-i\vec{k}\cdot\vec{x}}$ and the square root cut starting at $\pm im$ leads to an exponential decay $\sim e^{-m|\vec{x}|}$, as we would expect. Classically, a particle cannot get outside the lightcone, but a quantum field can "leak" out over a distance of the order m^{-1}.

Exercises

I.3.1. Verify that $D(x)$ decays exponentially for spacelike separation.

I.3.2. Work out the propagator $D(x)$ for a free-field theory in $(1+1)$-dimensional spacetime and study the large x^1 behavior for $x^0=0$.

[3] As is customary, ε is treated as generic, so that ε multiplied by any positive number is still ε.

Chapter I.4

From Field to Particle to Force

From field to particle

In the previous chapter we obtained for the free theory

$$W(J) = -\frac{1}{2} \iint d^4x d^4y J(x) D(x-y) J(y) \quad (1)$$

which we now write in terms of the Fourier transform $J(k) \equiv \int d^4x e^{-ikx} J(x)$:

$$W(J) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} J(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J(k) \quad (2)$$

[Note that $J(k)^* = J(-k)$ for $J(x)$ real.]

We can jump up and down on the mattress any way we like. In other words, we can choose any $J(x)$ we want, and by exploiting this freedom of choice, we can extract a remarkable amount of physics.

Consider $J(x) = J_1(x) + J_2(x)$, where $J_1(x)$ and $J_2(x)$ are concentrated in two local regions 1 and 2 in spacetime (Fig. I.4.1). Then $W(J)$ contains four terms, of the form $J_1^* J_1$, $J_2^* J_2$, $J_1^* J_2$, and $J_2^* J_1$. Let us focus on the last two of these terms, one of which reads

$$W(J) = -\frac{1}{2} \int \frac{d^4k}{(2\pi)^4} J_2(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J_1(k) \quad (3)$$

We see that $W(J)$ is large only if $J_1(x)$ and $J_2(x)$ overlap significantly in their Fourier transform and if in the region of overlap in momentum space $k^2 - m^2$ almost vanishes. There is a "resonance type" spike at $k^2 = m^2$, that is, if the energy-momentum relation of a particle of mass m is satisfied. (We will use the language of the relativistic physicist, writing "momentum space" for energy-momentum space, and lapse into nonrelativistic language only when the context demands it, such as in "energy-momentum relation.")

We thus interpret the physics contained in our simple field theory as follows: In region 1 in spacetime there exists a source that sends out a "disturbance in the field," which is later absorbed by a sink in region 2 in spacetime. Experimentalists choose to call this disturbance in the field a particle of mass m. Our expectation

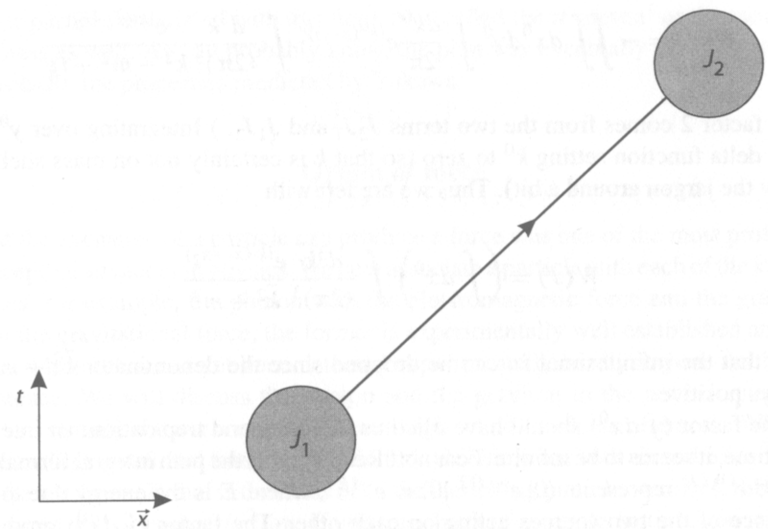

Figure I.4.1

based on the equation of motion that the theory contains a particle of mass m is fulfilled.

A bit of jargon: When $k^2 = m^2$, k is said to be on mass shell. Note, however, that in (3) we integrate over all k, including values of k far from the mass shell. For arbitrary k, it is a linguistic convenience to say that a "virtual particle" of momentum k propagates from the source to the sink.

From particle to force

We can now go on to consider other possibilities for $J(x)$ (which we will refer to generically as sources), for example, $J(x) = J_1(x) + J_2(x)$, where $J_a(x) = \delta^{(3)}(\vec{x} - \vec{x}_a)$. In other words, $J(x)$ is a sum of sources that are time-independent infinitely sharp spikes located at $\vec{x}_1$ and $\vec{x}_2$ in space. (If you like more mathematical rigor than is offered here, you are welcome to replace the delta function by lumpy functions peaking at $\vec{x}_a$. You would simply clutter up the formulas without gaining much.) More picturesquely, we are describing two massive lumps sitting at $\vec{x}_1$ and $\vec{x}_2$ on the mattress and not moving at all [no time dependence in $J(x)$].

What do the quantum fluctuations in the field φ, that is, the vibrations in the mattress, do to the two lumps sitting on the mattress? If you expect an attraction between the two lumps, you are quite right.

As before, $W(J)$ contains four terms. We neglect the "self-interaction" term $J_1 J_1$ since this contribution would be present in W regardless of whether J_2 is present or not. We want to study the interaction between the two "massive lumps" represented by J_1 and J_2. Similarly we neglect $J_2 J_2$.

Plugging into (1) and doing the integral over d^3x and d^3y we immediately obtain

$$W(J) = -\iint dx^0 dy^0 \int \frac{dk^0}{2\pi} e^{ik^0(x-y)^0} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{k^2 - m^2 + i\varepsilon} \tag{4}$$

(The factor 2 comes from the two terms $J_2 J_1$ and $J_1 J_2$.) Integrating over y^0 we get a delta function setting k^0 to zero (so that k is certainly not on mass shell, to throw the jargon around a bit). Thus we are left with

$$W(J) = \left(\int dx^0\right) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2 + m^2} \tag{5}$$

Note that the infinitesimal $i\varepsilon$ can be dropped since the denominator $\vec{k}^2 + m^2$ is always positive.

The factor $(\int dx^0)$ should have filled us with fear and trepidation: an integral over time, it seems to be infinite. Fear not! Recall that in the path integral formalism $Z = \mathcal{C}\, e^{iW(J)}$ represents $\langle 0| e^{-iHT} |0\rangle = e^{-iET}$, where E is the energy due to the presence of the two sources acting on each other. The factor $(\int dx^0)$ produces precisely the time interval T. All is well. Setting $iW = iET$ we obtain from (5)

$$E = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2 + m^2} \tag{6}$$

This energy is negative! The presence of two delta function sources, at $\vec{x}_1$ and $\vec{x}_2$, has lowered the energy. In other words, the two sources attract each other by virtue of their coupling to the field φ. We have derived our first physical result in quantum field theory!

We identify E as the potential energy between two static sources. Even without doing the integral we see that as the separation $\vec{x}_1 - \vec{x}_2$ between the two sources becomes large, the oscillatory exponential cuts off the integral. The characteristic distance is the inverse of the characteristic value of k, which is m. Thus, we expect the attraction between the two sources to decrease rapidly to zero over the distance $1/m$.

The range of the attractive force generated by the field φ is determined inversely by the mass m of the particle described by the field. Got that?

The integral is done in the appendix to this chapter and gives

$$E = -\frac{1}{4\pi r} e^{-mr} \tag{7}$$

The result is as we expected: The potential drops off exponentially over the distance scale $1/m$. Obviously, $dE/dr > 0$: The two massive lumps sitting on the mattress can lower the energy by getting closer to each other.

What we have derived was one of the most celebrated results in twentieth-century physics. Yukawa proposed that the attraction between nucleons in the atomic nucleus is due to their coupling to a field like the φ field described here. The known range of the nuclear force enabled him to predict not only the existence

of the particle associated with this field, now called the π meson[1] or the pion, but its mass as well. As you probably know, the pion was eventually discovered with essentially the properties predicted by Yukawa.

Origin of force

That the exchange of a particle can produce a force was one of the most profound conceptual advances in physics. We now associate a particle with each of the known forces: for example, the photon with the electromagnetic force and the graviton with the gravitational force; the former is experimentally well established and the latter while it has not yet been detected experimentally hardly anyone doubts its existence. We will discuss the photon and the graviton in the next chapter, but we can already answer a question smart high school students often ask: Why do Newton's gravitational force and Coulomb's electric force both obey the $1/r^2$ law?

We see from (7) that if the mass m of the mediating particle vanishes, the force produced will obey the $1/r^2$ law. If you trace back over our derivation, you will see that this comes about from the fact that the Lagrangian density for the simplest field theory involves two powers of the spacetime derivative ∂ (since any term involving one derivative such as $\varphi\, \partial\varphi$ is not Lorentz invariant). Indeed, the power dependence of the potential follows simply from dimensional analysis: $\int d^3k (e^{i\vec{k}\cdot\vec{x}}/k^2) \sim 1/r$.

Connected versus disconnected

We end with a couple of formal remarks of importance to us only in Chapter I.7. First, note that we might want to draw a small picture Fig.(I.4.2) to represent the integrand $J(x)D(x-y)J(y)$ in $W(J)$: A disturbance propagates from y to x (or vice versa). In fact, this is the beginning of Feynman diagrams! Second, recall that

$$Z(J) = Z(J=0) \sum_{n=0}^{\infty} \frac{[iW(J)]^n}{n!}$$

For instance, the $n = 2$ term in $Z(J)/Z(J=0)$ is given by

$$\frac{1}{2!}\left(-\frac{i}{2}\right)^2 \iiiint d^4x_1 d^4x_2 d^4x_3 d^4x_4 D(x_1 - x_2)$$
$$D(x_3 - x_4) J(x_1) J(x_2) J(x_3) J(x_4)$$

The integrand is graphically described in Figure I.4.3. The process is said to be disconnected: The propagation from x_1 to x_2 and the propagation from x_3 to x_4

[1] The etymology behind this word is quite interesting (A. Zee, *Fearful Symmetry*: see pp. 169 and 335 to learn, among other things, the French objection and the connection between meson and illusion).

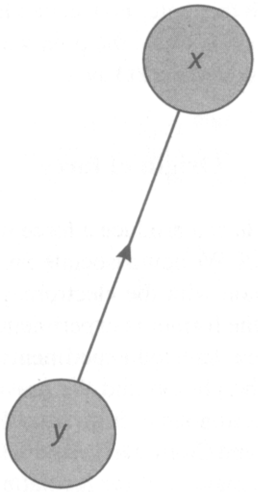

Figure I.4.2

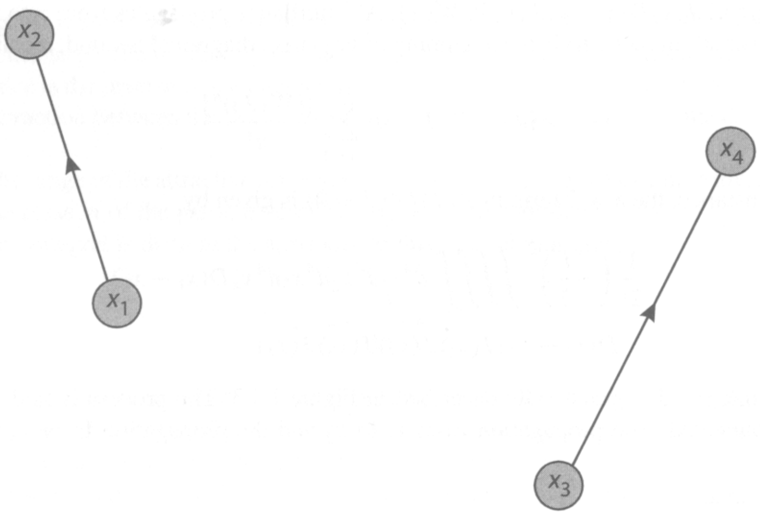

Figure I.4.3

proceed independently. We will come back to the difference between connected and disconnected in Chapter I.7.

Appendix

Writing $\vec{x} \equiv (\vec{x}_1 - \vec{x}_2)$ and $u \equiv \cos\theta$ with θ the angle between $\vec{k}$ and $\vec{x}$, we evaluate the integral in (6) spherical coordinates (with $k = |\vec{k}|$ and $r = |\vec{x}|$) to be

$$I \equiv \frac{1}{(2\pi)^2}\int_0^\infty dk\, k^2 \int_{-1}^{+1} du \frac{e^{ikru}}{k^2+m^2} = \frac{2i}{(2\pi)^2 ir}\int_0^\infty dk\, k \frac{\sin kr}{k^2+m^2} \tag{8}$$

Since the integrand is even, we can extend the integral and write it as

$$\frac{1}{2}\int_{-\infty}^{\infty} dk\, k \frac{\sin kr}{k^2+m^2} = \frac{1}{2i}\int_{-\infty}^{\infty} dk\, k \frac{1}{k^2+m^2} e^{ikr}.$$

Since r is positive, we can close the contour in the upper half-plane and pick up the pole at $+im$, obtaining $(1/2i)(2\pi i)(im/2im)e^{-mr} = (\pi/2)e^{-mr}$. Thus, $I = (1/4\pi r)e^{-mr}$.

Exercise

I.4.1. Calculate the analog of the inverse square law in a $(2+1)$-dimensional universe, and more generally in a $(D+1)$-dimensional universe.

Chapter I.5

Coulomb and Newton: Repulsion and Attraction

Why like charges repel

We suggested that quantum field theory can explain both Newton's gravitational force and Coulomb's electric force naturally. Between like objects Newton's force is attractive while Coulomb's force is repulsive. Is quantum field theory "smart enough" to produce this observational fact, one of the most basic in our understanding of the physical universe? You bet!

We will first treat the quantum field theory of the electromagnetic field, known as quantum electrodynamics or QED for short. In order to avoid complications at this stage associated with gauge invariance (about which much more later) I will consider instead the field theory of a massive spin 1 meson, or vector meson. After all, experimentally all we know is an upper bound on the photon mass, which although tiny is not mathematically zero. We can adopt a pragmatic attitude: Calculate with a photon mass m and set $m = 0$ at the end, and if the result does not blow up in our faces, we will presume that it is OK.[1]

Recall Maxwell's Lagrangian for electromagnetism $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ with $A_\mu(x)$ the vector potential. You can see the reason for the important overall minus sign in the Lagrangian by looking at the coefficient of $(\partial_0 A_i)^2$, which has to be positive, just like the coefficient of $(\partial_0\varphi)^2$ in the Lagrangian for the scalar field. This says simply that time variation should cost a positive amount of action.

I will now give the photon a small mass by changing the Lagrangian to $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu + A_\mu J^\mu$. (The mass term is written in analogy to the mass term $m^2\varphi^2$ in the scalar field Lagrangian; we will see later that it is indeed the photon mass.) I have also added a source $J^\mu(x)$,which in this context is more familiarly known as a current. We will assume that the current is conserved so that $\partial_\mu J^\mu = 0$.

[1] When I took a field theory course as a student with Sidney Coleman this was how he treated QED in order to avoid discussing gauge invariance.

Well, you know that the field theory of our vector meson is defined by the path integral $Z = \int DA\, e^{iS(A)} \equiv e^{iW(J)}$ with the action

$$S(A) = \int d^4x \mathcal{L} = \int d^4x \{\tfrac{1}{2}A_\mu[(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu\partial^\nu]A_\nu + A_\mu J^\mu\} \quad (1)$$

The second equality follows upon integrating by parts [compare (I.3.15)].

By now you have learned that we simply apply (I.3.16). We merely have to find the inverse of the differential operator in the square bracket; in other words, we have to solve

$$[(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu\partial^\nu]D_{\nu\lambda}(x) = \delta^\mu_\lambda \delta^{(4)}(x) \quad (2)$$

As before [compare (I.3.17)] we go to momentum space by defining

$$D_{\nu\lambda}(x) = \int \frac{d^4k}{(2\pi)^4} D_{\nu\lambda}(k) e^{ikx}$$

Plugging in, we find that $[-(k^2 - m^2)g^{\mu\nu} + k^\mu k^\nu]D_{\nu\lambda}(k) = \delta^\mu_\lambda$, giving

$$D_{\nu\lambda}(k) = \frac{-g_{\nu\lambda} + k_\nu k_\lambda/m^2}{k^2 - m^2} \quad (3)$$

This is the photon, or more accurately the massive vector meson, propagator. Thus

$$W(J) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{-g_{\mu\nu} + k_\mu k_\nu/m^2}{k^2 - m^2 + i\varepsilon} J^\nu(k) \quad (4)$$

Since current conservation $\partial_\mu J^\mu(x) = 0$ gets translated into momentum space as $k_\mu J^\mu(k) = 0$, we can throw away the $k_\mu k_\nu$ term in the photon propagator. The effective action simplifies to

$$W(J) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J_\mu(k) \quad (5)$$

No further computation is needed to obtain a profound result. Just compare this result to (I.4.2). The field theory has produced an extra sign. The potential energy between two lumps of charge density $J^0(x)$ is positive. The electromagnetic force between like charges is repulsive!

We can now safely let the photon mass m go to zero thanks to current conservation, [Note that we could not have done that in (3).] Indeed, referring to (I.4.7) we see that the potential energy between like charges is

$$E = \frac{1}{4\pi r}e^{-mr} \to \frac{1}{4\pi r} \quad (6)$$

To accommodate positive and negative charges we can simply write $J^\mu = J^\mu_p - J^\mu_n$. We see that a lump with charge density J^0_p is attracted to a lump with charge density J^0_n.

Bypassing Maxwell

Having done electromagnetism in two minutes flat let me now do gravity. Let us move on to the massive spin 2 meson field. In my treatment of the massive spin 1 meson field I took a short cut. Assuming that you are familiar with the Maxwell Lagrangian, I simply added a mass term to it and took off. But I do not feel comfortable assuming that you are equally familiar with the corresponding Lagrangian for the massless spin 2 field (the so-called linearized Einstein Lagrangian, which I will discuss in a later chapter). So here I will follow another strategy.

I invite you to think physically, and together we will arrive at the propagator for a massive spin 2 field. First, we will warm up with the massive spin 1 case.

In fact, start with something even easier: the propagator $D(k) = 1/(k^2 - m^2)$ for a massive spin 0 field. It tells us that the amplitude for the propagation of a spin 0 disturbance blows up when the disturbance is almost a real particle. The residue of the pole is a property of the particle. The propagator for a spin 1 field $D_{\nu\lambda}$ carries a pair of Lorentz indices and in fact we know what it is from (3):

$$D_{\nu\lambda}(k) = \frac{-G_{\nu\lambda}}{k^2 - m^2} \tag{7}$$

where for later convenience we have defined

$$G_{\nu\lambda}(k) \equiv g_{\nu\lambda} - \frac{k_\nu k_\lambda}{m^2} \tag{8}$$

Let us now understand the physics behind $G_{\nu\lambda}$. I expect you to remember the concept of polarization from your course on electromagnetism. A massive spin 1 particle has three degrees of polarization for the obvious reason that in its rest frame its spin vector can point in three different directions. The three polarization vectors $\varepsilon^{(a)}_\mu$ are simply the three unit vectors pointing along the x, y, and z axes, respectively ($a = 1, 2, 3$): $\varepsilon^{(1)}_\mu = (0, 1, 0, 0)$, $\varepsilon^{(2)}_\mu = (0, 0, 1, 0)$, $\varepsilon^{(3)}_\mu = (0, 0, 0, 1)$. In the rest frame $k^\mu = (m, 0, 0, 0)$ and so

$$k^\mu \varepsilon^{(a)}_\mu = 0 \tag{9}$$

Since this is a Lorentz invariant equation, it holds for a moving spin 1 particle as well. Indeed, with a suitable normalization condition this fixes the three polarization vectors $\varepsilon^{(a)}_\mu(k)$ for a particle with momentum k.

The amplitude for a particle with momentum k and polarization a to be created at the source is proportional to $\varepsilon^{(a)}_\lambda(k)$, and the amplitude for it to be absorbed at the sink is proportional to $\varepsilon^{(a)}_\nu(k)$. We multiply the amplitudes together to get the amplitude for propagation from source to sink, and then sum over the three possible polarizations. Now we understand the residue of the pole in the spin 1 propagator $D_{\nu\lambda}(k)$: It represents $\sum_a \varepsilon^{(a)}_\nu(k)\, \varepsilon^{(a)}_\lambda(k)$. To calculate this quantity, note that by Lorentz invariance it can only be a linear combination of $g_{\nu\lambda}$ and $k_\nu k_\lambda$. The condition $k^\mu \varepsilon^{(a)}_\mu = 0$ fixes the combination as $g_{\nu\lambda} - k_\nu k_\lambda / m^2$. We evaluate the left-hand side for k at rest with $\nu = \lambda = 1$, for instance, and fix the overall and

all-crucial sign to be -1. Thus

$$\sum_a \varepsilon^{(a)}_\nu(k)\varepsilon^{(a)}_\lambda(k) = -G_{\nu\lambda}(k) \equiv -\left(g_{\nu\lambda} - \frac{k_\nu k_\lambda}{m^2}\right) \tag{10}$$

We have thus constructed the propagator $D_{\nu\lambda}(k)$ for a massive spin 1 particle, bypassing Maxwell. Onward to spin 2! We want to similarly bypass Einstein.

Bypassing Einstein

A massive spin 2 particle has 5 ($2 \cdot 2 + 1 = 5$, remember?) degrees of polarization, characterized by the five polarization tensors $\varepsilon^{(a)}_{\mu\nu}$ ($a = 1, 2, \cdots, 5$) symmetric in the indices μ and ν satisfying

$$k^\mu \varepsilon^{(a)}_{\mu\nu} = 0 \tag{11}$$

and the tracelessness condition

$$g^{\mu\nu} \varepsilon^{(a)}_{\mu\nu} = 0 \tag{12}$$

Let's count as a check. A symmetric Lorentz tensor has $4 \cdot 5/2 = 10$ components. The four conditions in (11) and the single condition in (12) cut the number of components down to $10 - 4 - 1 = 5$, precisely the right number. (Just to throw some jargon around, remember how to construct irreducible group representations? If not, read Appendix C.) We fix the normalization of $\varepsilon_{\mu\nu}$ by setting the positive quantity $\sum_a \varepsilon^{(a)}_{12}(k)\varepsilon^{(a)}_{12}(k) = 1$.

So, in analogy with the spin 1 case we now determine $\sum_a \varepsilon^{(a)}_{\mu\nu}(k)\varepsilon^{(a)}_{\lambda\sigma}(k)$. We have to construct this object out of $g_{\mu\nu}$ and k_μ, or equivalently $G_{\mu\nu}$ and k_μ. This quantity must be a linear combination of terms such as $G_{\mu\nu}G_{\lambda\sigma}$, $G_{\mu\nu}k_\lambda k_\sigma$, and so forth. Using (11) and (12) repeatedly (Exercise I.5.1) you will easily find that

$$\sum_a \varepsilon^{(a)}_{\mu\nu}(k)\varepsilon^{(a)}_{\lambda\sigma}(k) = (G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) - \tfrac{2}{3}G_{\mu\nu}G_{\lambda\sigma} \tag{13}$$

The overall sign and proportionality constant are determined by evaluating both sides for $\mu = \lambda = 1$ and $\nu = \sigma = 2$, for instance.

Thus, we have determined the propagator for a massive spin 2 particle

$$D_{\mu\nu,\,\lambda\sigma}(k) = \frac{(G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}}{k^2 - m^2} \tag{14}$$

Why we fall

We are now ready to understand one of the fundamental mysteries of the universe: Why masses attract.

Recall from your courses on electromagnetism and special relativity that the energy or mass density out of which mass is composed is part of a stress-energy tensor $T^{\mu\nu}$. For our purposes, in fact, all you need to remember is that it is a symmetric tensor and that the component T^{00} is the energy density.

To couple to the stress-energy tensor, we need a tensor field $\varphi_{\mu\nu}$ symmetric in its two indices. In other words, the Lagrangian of the world should contain a term like $\varphi_{\mu\nu}T^{\mu\nu}$. This is in fact how we know that the graviton, the particle responsible for gravity, has spin 2, just as we know that the photon, the particle responsible for electromagnetism and hence coupled to the current J^μ, has spin 1. In Einstein's theory, which we will discuss in a later chapter, $\varphi_{\mu\nu}$ is of course part of the metric tensor.

Just as we pretended that the photon has a small mass to avoid having to discuss gauge invariance, we will pretend that the graviton has a small mass to avoid having to discuss general coordinate invariance.[2] Aha, we just found the propagator for a massive spin 2 particle. So let's put it to work.

In precise analogy to (4)

$$W(J) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{-g_{\mu\nu} + k_\mu k_\nu/m^2}{k^2 - m^2 + i\varepsilon} J^\nu(k) \tag{15}$$

describing the interaction between two electromagnetic currents, the interaction between two lumps of stress energy is described by

$$W(T) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{\mu\nu}(k)^* \frac{(G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}}{k^2 - m^2 + i\varepsilon} T^{\lambda\sigma}(k) \tag{16}$$

From the conservation of energy and momentum $\partial_\mu T^{\mu\nu}(x) = 0$ and hence $k_\mu T^{\mu\nu}(k) = 0$, we can replace $G_{\mu\nu}$ in (16) by $g_{\mu\nu}$.

Now comes the punchline. Look at the interaction between two lumps of energy density T^{00}. We have from (16) that

$$W(T) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{00}(k)^* \frac{1 + 1 - \frac{2}{3}}{k^2 - m^2 + i\varepsilon} T^{00}(k) \tag{17}$$

Comparing with (5) and using the well-known fact that $(1 + 1 - \frac{2}{3}) > 0$, we see that while like charges repel, masses attract. Trumpets, please!

[2] For the moment, I ask you to ignore all subtleties and simply assume that in order to understand gravity it is kosher to let $m \to 0$. I will give a precise discussion of Einstein's theory of gravity in Chapter VIII.1.

The universe

It is difficult to overstate the importance (not to speak of the beauty) of what we have learned: The exchange of a spin 0 particle produces an attractive force, of a spin 1 particle a repulsive force, and of a spin 2 particle an attractive force, realized in the hadronic strong interaction, the electromagnetic interaction, and the gravitational interaction, respectively. The universal attraction of gravity produces an instability that drives the formation of structure in the early universe.[3] Denser regions become denser yet. The attractive nuclear force mediated by the spin 0 particle eventually ignites the stars. Furthermore, the attractive force between protons and neutrons mediated by the spin 0 particle is able to overcome the repulsive electric force between protons mediated by the spin 1 particle to form a variety of nuclei without which the world would certainly be rather boring. The repulsion between likes and hence attraction between opposites generated by the spin 1 particle allow electrically neutral atoms to form.

The world results from a subtle interplay among spin 0, 1, and 2.

In this lightning tour of the universe, we did not mention the weak interaction. In fact, the weak interaction plays a crucial role in keeping stars such as our sun burning at a steady rate.

Degrees of freedom

Now for a bit of cold water: Logically and mathematically the physics of a particle with mass $m \neq 0$ could well be different from the physics with $m = 0$. Indeed, we know from classical electromagnetism that an electromagnetic wave has 2 polarizations, that is, 2 degrees of freedom. For a massive spin 1 particle we can go to its rest frame, where the rotation group tells us that there are $2\cdot 1 + 1 = 3$ degrees of freedom. The crucial piece of physics is that we can never bring the massless photon to its rest frame. Mathematically, the rotation group $SO(3)$ degenerates into $SO(2)$, the group of 2-dimensional rotations around the direction of the photon's momentum.

We will see in Chapter II.7 that the longitudinal degree of freedom of a massive spin 1 meson decouples as we take the mass to zero. The treatment given here for the interaction between charges (6) is correct. However, in the case of gravity, the $\frac{2}{3}$ in (17) is replaced by 1 in Einstein's theory, as we will see Chapter VIII.1. Fortunately, the sign of the interaction given in (17) does not change. Mute the trumpets a bit.

[3] A good place to read about gravitational instability and the formation of structure in the universe along the line sketched here is in A. Zee, *Einstein's Universe* (formerly known as *An Old Man's Toy*).

Appendix

Pretend that we never heard of the Maxwell Lagrangian. We want to construct a relativistic Lagrangian for a massive spin 1 meson field. Together we will discover Maxwell. Spin 1 means that the field transforms as a vector under the 3-dimensional rotation group. The simplest Lorentz object that contains the 3-dimensional vector is obviously the 4-dimensional vector. Thus, we start with a vector field $A_\mu(x)$.

That the vector field carries mass m means that it satisfies the field equation

$$(\partial^2 + m^2)A_\mu = 0 \tag{18}$$

A spin 1 particle has 3 degrees of freedom [remember, in fancy language, the representation j of the rotation group has dimension $(2j+1)$; here $j=1$.] On the other hand, the field $A_\mu(x)$ contains 4 components. Thus, we must impose a constraint to cut down the number of degrees from 4 to 3. The only Lorentz covariant possibility (linear in A_μ) is

$$\partial_\mu A^\mu = 0 \tag{19}$$

It may also be helpful to look at (18) and (19) in momentum space, where they read $(k^2 - m^2)A_\mu(k) = 0$ and $k_\mu A^\mu(k) = 0$. The first equation tells us that $k^2 = m^2$ and the second that if we go to the rest frame $k^\mu = (m, \vec{0})$ then A^0 vanishes, leaving us with 3 nonzero components A^i with $i = 1, 2, 3$.

The remarkable observation is that we can combine (18) and (19) into a single equation, namely

$$(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + m^2 A^\mu = 0 \tag{20}$$

Verify that (20) contains both (18) and (19). Act with ∂_μ on (20). We obtain $m^2\partial_\mu A^\mu = 0$, which implies that $\partial_\mu A^\mu = 0$. (At this step it is crucial that $m \neq 0$ and that we are not talking about the strictly massless photon.) We have thus obtained (19); using (19) in (20) we recover (18).

We can now construct a Lagrangian by multiplying the left-hand side of (20) by $+\frac{1}{2}A_\mu$ (the $\frac{1}{2}$ is conventional but the plus sign is fixed by physics, namely the requirement of positive kinetic energy); thus

$$\mathcal{L} = \tfrac{1}{2}A_\mu[(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu\partial^\nu]A_\nu \tag{21}$$

Integrating by parts, we recognize this as the massive version of the Maxwell Lagrangian. In the limit $m \to 0$ we recover Maxwell.

A word about terminology: Some people insist on calling only $F_{\mu\nu}$ a field and A_μ a potential. Conforming to common usage, we will not make this fine distinction. For us, any dynamical function of spacetime is a field.

Exercises

I.5.1. Write down the most general form for $\sum_a \varepsilon^{(a)}_{\mu\nu}(k)\varepsilon^{(a)}_{\lambda\sigma}(k)$ using symmetry repeatedly. For example, it must be invariant under the exchange $\{\mu\nu \leftrightarrow \lambda\sigma\}$. You might

end up with something like

$$\begin{aligned} &AG_{\mu\nu}G_{\lambda\sigma} + B(G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) + C(G_{\mu\nu}k_\lambda k_\sigma + k_\mu k_\nu G_{\lambda\sigma}) \\ &+ D(k_\mu k_\lambda G_{\nu\sigma} + k_\mu k_\sigma G_{\nu\lambda} + k_\nu k_\sigma G_{\mu\lambda} + k_\nu k_\lambda G_{\mu\sigma}) + Ek_\mu k_\nu k_\lambda k_\sigma \end{aligned} \tag{22}$$

with various unknown $A, \cdots, E$. Apply $k^\mu \sum_a \varepsilon^{(a)}_{\mu\nu}(k)\varepsilon^{(a)}_{\lambda\sigma}(k) = 0$ and find out what that implies for the constants. Proceeding in this way, derive (13).