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ON STEADY-STATE TRAVELLING SOLUTIONS OF AN EVOLUTION EQUATION DESCRIBING THE BEHAVIOUR OF DISTURBANCES IN ACTIVE DISSIPATIVE MEDIA

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A nonlinear evolution equation is considered which is often encountered in modelling the behaviour of perturbations in various active dissipative media, e.g. in problems of fluid film flow hydrodynamics. Periodic steady-state travelling solutions have been found numerically for it. Stability of these solutions has been investigated and bifurcation analysis has been carried out. The analysis has demonstrated that decrease of the wave number causes more and more new families of steady-state travelling solutions. A countable set of such solutions is formed in the limit when the wave number tends to zero. It is also shown that time-oscillating solutions can be generated from steady-state ones due to bifurcation of the Landau–Hopf type.

1. Introduction

Since recently along with the well-known Kuramoto–Sivashinsky equation

$$y_t + 2y_x^2 + y_{xx} + y_{xxxx} = 0 \qquad (1.1)$$

investigators take great interest in the equation

$$H_t + 4HH_x + H_{xx} + H_{xxxx} = 0. \qquad (1.2)$$

This interest arises from the fact that being one of the simplest equations it appears in modelling nonlinear behaviour of disturbances for a sufficiently large class of active dissipative media. Thus this equation was obtained in [1, 2] while describing waves on the surface of a liquid film freely falling down an inclined plane, it is given in [3] for the countercurrent flow of film and gas, and in [4] for disturbances at the interface between two viscous liquids in a channel.

It should be mentioned that the authors of papers dealing with eq. (1.2) or the Kuramoto–Sivashinsky equation often do not discriminate between them. For example, the Kuramoto–Sivashinsky equation is said to be valid for describing disturbances on the surface of vertically falling liquid films [4, 5], and eq. (1.2) is said to hold for the case of chemical diffusion [6]. Though, actually, eq. (1.2) holds true for falling films, and the Kuramoto–Sivashinsky equation is satisfactorily suitable for modelling the propagation of flame front. Despite outward likeness and the coincidence of linear parts in eqs. (1.1) and (1.2) their solutions differ considerably, though there exists a rather simple relation between them:

$$y_x = H. \qquad (1.3)$$

Linear analysis of stability shows that the trivial solution of eqs. (1.1) and (1.2),

$$H = 0,$$

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is unstable with respect to perturbations of the form

$$\exp[\mathrm{i}\alpha(x-ct)]$$

with wave numbers $\alpha < 1$. (Disturbances with $\alpha > 1$ dissipate.) Increase of such disturbances with time can be limited due to nonlinear effects. As a result, steady-state nonlinear regimes may form.

The present paper is devoted to bifurcation analysis of periodic steady-state travelling solutions of eq. (1.2). It is shown how a countable set of such solutions appears. Owing to relation (1.3), the obtained results yield rather much information on the behaviour of the solution of the Kuramoto–Sivashinsky equation and here we shall not dwell on it in detail.

2. Theory and method of solution

For a steady-state travelling wave

$$H(\xi), \quad \xi = x - ct,$$

eq. (1.2) becomes

$$-cH_\xi + 4HH_\xi + H_{\xi\xi} + H_{\xi\xi\xi\xi} = 0. \tag{2.1}$$

We will consider the periodic solutions of (2.1) with the wavelength $\lambda = 2\pi/\alpha$.

Since (2.1) is invariant under the transformations

$$H \to -H, \quad \xi \to -\xi, \quad c \to -c,$$

and (2.2)

$$H \to H + \text{const}, \quad c \to c + 4\text{const},$$

only such solutions will be considered further for which

$$c \geq 0, \quad \int_0^\lambda H\,\mathrm{d}\xi = 0. \tag{2.3}$$

Strictly speaking, in the case $c = 0$ the solution of eq. (1.2) represents not only a travelling wave but a stationary one as well. However there is no sense in separating these cases as for many situations eq. (1.2) is written in a moving reference system. Thus, for example, while describing the perturbation evolution on the surface of a liquid film falling down an inclined plane, it is given in a system whose velocity is twice the velocity of a free surface in the absence of disturbances. Therefore, solutions with $c = 0$ are also the travelling ones in a stationary reference system. Thus we have a boundary value problem in which the wave velocity c is an eigenvalue and the wave number α is a parameter.

Using the nonlinear stability theory, it is easy to show that a periodic solution with infinitesimal amplitude branches from the trivial solution of eq. (2.1) at $\alpha = 1$. It continues into the region of linear instability ($\alpha < 1$). This is a supercritical type of branching.

Periodic solutions of eq. (2.1) with finite amplitude are found numerically. For this purpose, they are represented as a Fourier series:

$$H = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} H_n \exp[\mathrm{i}\alpha n\xi]. \tag{2.4}$$

Since H is a real function, then

$$H_{-n} = \overline{H}_n.$$

The overbar denotes the complex conjugate.

Taking into account the first N harmonics in the series (2.4), let us substitute it into eq. (2.1). Putting the coefficients equal to zero at the same exponents, we obtain a system of N complex equations for the unknown real c and N complex $H_1, \ldots, H_N$:

$$(-\mathrm{i}\alpha nc - \alpha^2 n^2 + \alpha^4 n^4)H_n + 2\mathrm{i}\alpha n \sum_{m=n-N}^{N} H_m H_{n-m} = 0, \quad n = 1, \ldots, N. \tag{2.5}$$

Since eq. (2.1) is invariant to the coordinate shift

$$\xi \to \xi + \text{const} \tag{2.6}$$

the origin of coordinates was chosen such that

$$\mathrm{Re}(H_1) = 0. \tag{2.7}$$

Thus the system (2.5) is complete due to (2.7). The Newton–Kantorovich method was used to solve it numerically. While reducing the series (2.4), the number of harmonics was taken so as to satisfy the relation

$$|H_N|/\sup|H_n| \lesssim 10^{-3}.$$

For this purpose, the number N had to be varied depending on the value α in the range from 8 to 64.

The main difficulty in solving (2.1) is to determine the initial approximation that is close enough to the solution. An analytic solution is used as such an approximation for the first solution family branching from the trivial one at $\alpha = 1$. To determine the solution for $\alpha < 1$, the step of the wave number is selected in such a way that using the solution previously found as an initial approximation, the numerical process converges to the solution of this family at a new α too.

Let $H_0(\xi)$ be a periodic solution of eq. (1.2) with the wave number α. Substituting

$$H = H_0(\xi) + h(\xi, t) \tag{2.8}$$

into (1.1) and linearizing it with respect to the disturbance $h(\xi, t)$, we obtain the equation [1]

$$h_t - ch_\xi + 4(H_0 h)_\xi + h_{\xi\xi} + h_{\xi\xi\xi\xi} = 0 \tag{2.9}$$

to investigate the stability of the solution H_0 with respect to infinitesimal disturbances.

Since the variable t is not explicitly incorporated in (2.9), we search for a solution of the form

$$h = \mathrm{e}^{-\gamma t} h_1(\xi) + \text{K.C.} \tag{2.10}$$

We obtain a linear ordinary differential equation with periodic coefficients for $h_1(\xi)$:

$$-\gamma h_1 - ch_{1\xi} + 4(H_0 h_1)_\xi + h_{1\xi\xi} + h_{1\xi\xi\xi\xi} = 0. \tag{2.11}$$

Since, while investigating stability, disturbances in (2.8) are limited at the initial moment of time for all values of ξ, it is clear that we are interested in the solutions of eq. (2.11) which are also limited for all ξ. From Floquet's theorem it follows that such solutions are of the form

$$h_1 = \varphi(\xi)\,\mathrm{e}^{\mathrm{i}\alpha Q \xi} \tag{2.12}$$

where $\varphi(\xi)$ is a periodic function of the same period as $H_0(\xi)$, and Q is a real parameter. Substituting (2.12) into (2.11), we obtain

$$A\varphi + B\varphi' + (1 - 6\alpha^2 Q^2)\varphi'' + 4\mathrm{i}\alpha Q\varphi''' + \varphi'''' = \gamma\varphi, \tag{2.13}$$

where

$$A = 4H_0' + 4\mathrm{i}\alpha Q H_0 - \alpha^2 Q^2 + \alpha^4 Q^4 - \mathrm{i}\alpha Q c,$$
$$B = 4H_0 + 2\mathrm{i}\alpha Q - 4\mathrm{i}\alpha^3 Q^3 - c.$$

and the prime means differentiation with respect to ξ.

Thus the investigation of stability of periodic steady-state travelling wave solutions of eq. (1.2) is reduced to studying the spectrum of such eigenvalues γ at different values Q for which eq. (2.13) has periodic solutions of the same period as for H_0. The wave is stable if for any Q all γ have

$$\mathrm{Re}(\gamma) > 0.$$

From (2.12) it is clear that it is sufficient to consider Q in any interval of unit length, for example, $[-0.5, 0.5]$. Performing the operation of complex conjugation on (2.13), it is easy to become convinced that

$$\gamma(-Q) = \bar{\gamma}(Q). \tag{2.14}$$

Thus it is sufficient to consider the solutions (2.13) for

$$0 \le Q \le 0.5.$$

At $Q = 0$ one of the solutions of eq. (2.13) is readily found analytically:

$$\gamma = 0, \quad \varphi = H_0'.$$

This result is a consequence of the Andronov–Vitt theorem [6] concerning the presence of at least one zero Lyapunov index for a closed trajectory. Since for eq. (2.1) an integral exists,

$$-cH + 2H^2 + H_\xi + H_{\xi\xi\xi} = \text{const},$$

then any such trajectory has among Lyapunov indices one more index which is equal to zero [7]; in other words, at $Q = 0$ the root of $\gamma = 0$ is doubly degenerate.

In the general case the boundary value problem (2.13) was solved numerically. Performing the Fourier transformation on (2.13) we obtain an infinite system of linear algebraic homogeneous equations to determine φ_n. Setting all φ_n, $|n| \ge N$, equal to zero, we obtain its finite approximation:

$$\left[-\mathrm{i}\alpha(Q+n)c - \alpha^2(Q+n)^2 + \alpha^4(Q+n)^4\right]\varphi_n + 4\mathrm{i}\alpha(Q+n)\sum_{m=\max(-N,\,n-N)}^{\min(N,\,n+N)} \varphi_{n-m} = \gamma\varphi_n.$$

From (2.9)–(2.11) it follows that if the real part of some eigenvalue vanishes at some point (α, Q), then a new wave regime branches from the original one. If $\mathrm{Im}(\gamma) \ne 0$, time-dependent regimes may be generated. New steady-state travelling regimes arise from the solution H_0 if

$$\gamma(\alpha, Q) = 0. \tag{2.15}$$

A double periodic regime is generated, if Q is an irrational number, and if $Q = p/r$ is a rational number, a regime with a new wave number $\alpha_{\text{new}} = \alpha/r$ is generated which is periodic with respect to ξ.

Relation (2.15) means that numbers $\pm\mathrm{i}Q$ in (2.12) are Lyapunov indices of periodic solution H_0. Since the third derivative is not incorporated in eq. (2.1), the divergence of phase flow of this equation is equal to zero. Hence the sum of all four Lyapunov indices for any periodic solution of eq. (2.1) is also equal to zero [8]. As shown above, among these indices two are zero and, hence, two remaining indices are either imaginary and have the form $\pm\mathrm{i}Q$ where $Q \ge 0$, or are nonzero real and have the form $\pm\Lambda$ where $\Lambda > 0$. Therefore, though eq. (2.15) is a system of two independent conditions,

$$\mathrm{Re}(\gamma(\alpha, Q)) = 0$$

and

$$\mathrm{Im}(\gamma(\alpha, Q)) = 0,$$

it determines not a set of points but curves on the plane (α, Q). These curves may be almost parallel to the Q-axis (see figs. 1 and 13) but taking into account (2.14) it is clear that within a considered interval Q $(0 \le Q \le 0.5)$ any straight line $\alpha = \alpha_0 = \text{const}$ crosses them at not more than two points, one of which is a point $(\alpha_0, 0)$. In other words, for any periodic solution H_0 of eq. (2.1) a unique value of Q different from zero out of the considered interval can exist at which (2.15) can be fulfilled. Otherwise, such a solution H_0 with the wave number α_0 would have more than four Lyapunov indices.

In the present paper we shall be limited to considering only steady-state travelling periodic waves. To find these new solutions of eq. (2.1) with the wave numbers from the neighbourhood of α_{new}, as an initial approximation the expression (2.8) is used. The function h in it is taken to be proportional to the eigenfunction φ satisfying eq. (2.13) for the values of wave number α and $Q = p/r$ at eigenvalue $\gamma = 0$. The calculations show that it is more convenient to use the value of some harmonic as a varying parameter in the neighbourhoods of these origin points. Further, passing to the parameter α we find solution for the values α, which are far from origin point α_{new} of this family.

3. Results

Nepomnyashchy has shown in his paper [1] that the first family of periodic solutions of eq. (2.1) branching from the trivial solution at $\alpha = 1$ can be continued to the range of smaller wave numbers up to the value $\alpha_* = 0.4979$. Each odd harmonic of the series (2.4) becomes equal to zero at this point. As a result we obtain a solution with the wave number $\alpha = 2\alpha_* = 0.9958$. It appears that this is the same solution that has been previously obtained for this α. Thus this family is closed on itself. Note that for all the values α from the region of solution existence the value of phase velocity is equal to zero,

$$c = 0. \tag{2.16}$$

In accordance with (2.2) such solutions are antisymmetric. Further, up to fig. 11 the results concerning only antisymmetric families of solutions will be discussed. The real parts of all harmonics become equal to zero while fulfilling (2.7).

Fig. 1 shows some results of the investigation into stability of the first solution family. Here on the plane (α, Q) curves 1–4 are plotted on which correlation (2.15) is satisfied for some of the eigenvalues γ. Discussing subsequently the solutions branching from some of these curves, we shall call them generating curves. By means of these curves it is easy to find new wave numbers α_{new}, with which new periodic steady-state solutions are generated. The calculations show that these solutions are intricately interlaced with each other.

Fig. 2 demonstrates some mutual transformations of new solutions onto each other. The amplitude of the first harmonic $\mathrm{Im}(H_1)$ depending on α is presented here. Since (2.6) holds for (2.1), it is

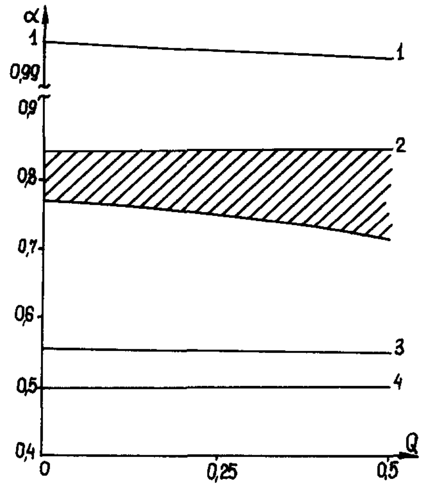

Fig. 1. Bifurcation lines (2.15) (curves 1–4) and stability zone (dashed region) for solutions of the first family.

clear that in this figure and the analogous ones below, all the curves have their mirror doubles with respect to the axis $\mathrm{Im}(H_1) = 0$, which correspond to the same solution shifted by half a period. These curves are not shown to prevent the figures from getting overcrowded. Though for all families one could take $\mathrm{Im}(H_1) \geq 0$, the represen-

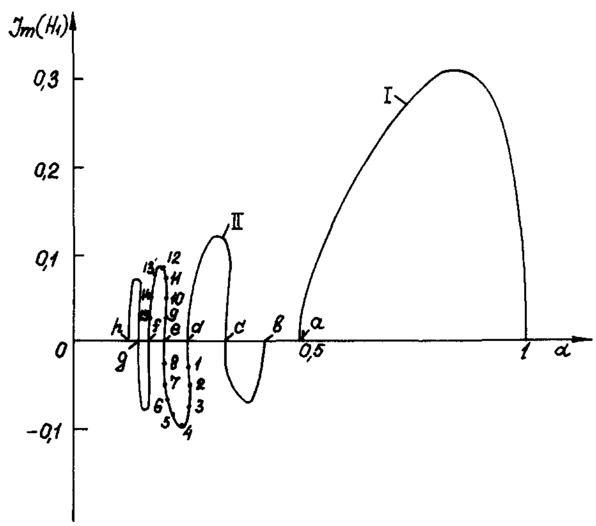

Fig. 2. The first harmonic vs. α for solutions of the first family (curve I) and solution families generating from the first one along curve 1 of fig. 1 (curve II).

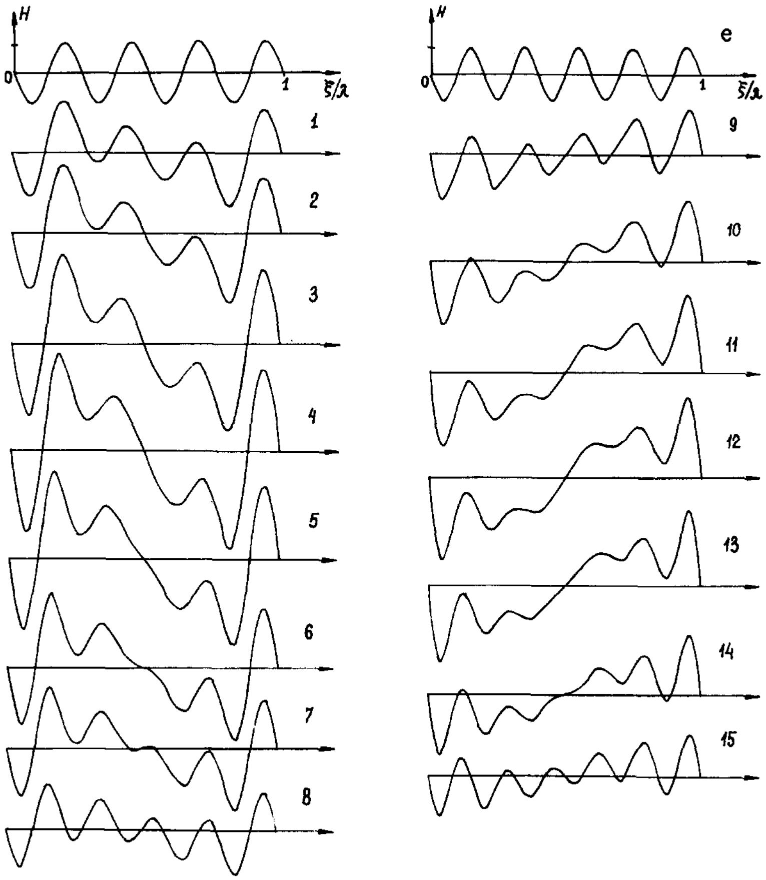

Fig. 3. Profiles of families shown in fig. 2 by curve II.

tation used allows one to see graphically how the solutions pass into each other.

In all analogous figures solutions are generated just in the points where $\mathrm{Im}(H_1) = 0$ (except for some specified cases). A single new solution is generated (see, for example, the points a and b in fig. 2) if these points are related to branching from $Q = 1/2$, and two new solutions are generated if $Q = 1/r$, $r \geq 3$.

Curve I in fig. 2 demonstrates how the first family disappears closing on the family generated from the first one with a new maximally possible wave number – point a ($\alpha_a = 0.4979$). Solution branching from curve 1 in fig. 1, with $Q = 1/2$, actually appears in this point. While describing below new solution families generated directly from the first family from curves 1–4 in fig. 1, and speaking about the corresponding generating curve, we shall not emphasize every time that the curves from fig. 1 are meant.

Curve II (fig. 2) shows that new solution families can pass into each other. Its boundary point b is related to branching from curve 2 with $Q = 1/2$. Moving from the origin point we shall be sequentially passing the points c, d, e, f, g and h generating new solution families related to branching from curve 1 (fig. 1) with $Q = 1/3$, $1/4$, $1/5$, $1/6$, $1/7$ and $1/8$, respectively.

Fig. 3 shows variations of the wave form $H(\xi)$ with variation of the wave number values from α_d to α_f. The numbers of the profiles presented are unambiguously related to the numbered points on curve II in fig. 2. Here and further the waves are presented on one period $\lambda = 2\pi/\alpha$ according to the given wave number α. Since at the points d, e and f of new family generation we actually have a wave with the wave number $r\alpha$ (belonging to curve 1 in fig. 1), respective r wavelengths are presented in one period of these families. Fig. 3 graphically shows how new wave regimes change into each other.

Based on the results presented one could state that new solutions bifurcating from curve 1 with $Q = 1/r$ pass into solution branching from the same curve with $Q = 1/(r+1)$, if α decreases. A similar statement is given in the paper by Demekhin [9]. Using the general theory of nonlinear equation solution branching and taking into account that periodic solutions of the first family are represented well enough by two terms of the series (2.4) at $\alpha \in \{0.75, 1\}$, he has actually investigated some bifurcations from curves 1 and 2 with $Q = 1/r$. Thus he has semianalytically found the branching points a, b, c, d (and j – see fig. 4). The wave numbers $\alpha_a = 0.4979$, $\alpha_b = 0.4352$, $\alpha_c = 0.3323$, $\alpha_d = 0.2994$ and $\alpha_j = 0.2923$ found by him are close enough to $\alpha_a = 0.4979$, $\alpha_b = 0.4211$, $\alpha_c = 0.3323$, $\alpha_d = 0.2494$ and $\alpha_j = 0.2803$ found by means of our method. Development of these solutions was followed by Demekhin only up to $\alpha \approx 0.25$. He stated that the solution of eq. (2.1) with period $2\pi r$ bifurcating from the solution of the first family with period 2π changes to the solution with period $2\pi(r+1)$. However, our investigations of branching from curves 2–4 show that this is not a general statement.

Thus, for example, at branching from curve 2 with $Q = 1/3$, one of the solutions monotonically continues along α up to the minimal value of the wave number, $\alpha_{\min} = 0.1$, which was only considered in the present paper. The second solution branches to the side of large α, then it quickly turns and merges with the solution generated by curve 4 with $Q = 1/2$. These three families are represented by line I in fig. 4. The corresponding wave profiles are given in fig. 5. Identification of profiles here and further is analogous to that presented above for figs. 2 and 3.

Note that at the moment of generation, the wave numbers of the families generated by curve 1 with $Q = 1/4$ and curve 4 with $Q = 1/2$ are very close to each other: $\alpha_d = 0.2494$ and $\alpha_k = 0.2505$, respectively. Nevertheless the given method of bifurcation analysis discriminates them unambiguously enough (compare the first curves in figs. 3 and 5).

Solution families generated from curve 2 with $Q = 1/4$ evolve with changing of α similar to the families generated from the same curve with $Q = 1/3$. Indeed, one of them continues from the point $\alpha_e = 0.2090$ of generation of these families to the side of smaller α and exists up to $\alpha_{\min}$. The second one, having branched to the side of larger α turns to the region of smaller α and may be continued up to $\alpha_m = 0.1992$. This point, as the analysis shows, is, in its turn, the point where two new families are generated from curve 1 with $Q = 2/5$. Thus we have merging of two different families again.

Peculiarity of the point α_m in comparison with the similar points considered above consists in the following: according to eqs. (2.8), (2.10) and (2.12) the second harmonic $\mathrm{Im}(H_2)$ increases faster in the neighbourhood of this point, but not the first harmonic as at branching with $Q = 1/r$.

The second family appearing at the point α_m also continues into the region of small α at least up to $\alpha_{\min}$. On the plane $(\mathrm{Im}(H_1), \alpha)$ this solution first goes to the region of large α, intersects the horizontal axis at the point $\alpha_+ = 0{,}211$ and only after that monotonically goes into the region of small α. Unlike all the other points lying on the intersection of the curves presented in figs. 2–4 with the horizontal axis, the point α_+ is not related with generating any new solution families. Only one harmonic $\mathrm{Im}(H_1)$ vanishes at this point. In particular, the imaginary part of the second harmonic $\mathrm{Im}(H_2)$ monotonically increases in absolute magnitude on passing this point in the given direction.

In fig. 4 the solution families generated at the points α_e and α_m are represented by line II. The behaviour of curve II in the neighbourhood of points α_+, α_e and α_m is shown on a larger scale in

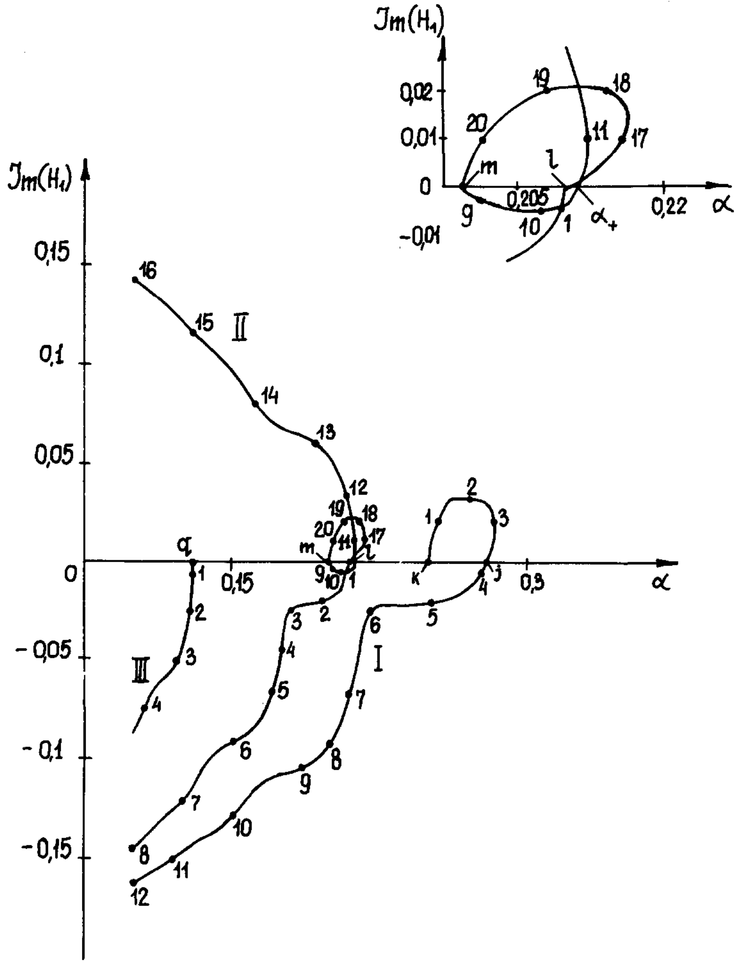

Fig. 4. The first harmonic vs. α for families generating from same points of curves 1–4 (fig. 1).

the right upper part of this figure. Figs. 6 and 7 show profiles of the families which do not pass into each other. Fig. 8 demonstrates how the second families generated at these points merge.

Lines I and II in fig. 9 show the families generated by curve 3 with $Q = 1/2$ and $1/3$. The points of their appearance are $\alpha_n = 0.275$ and $\alpha_p = 0.1836$, respectively. These solutions extend to the region of small α without incidents.

For all the families of periodic solutions of eq. (2.1) considered above, the values of phase velocity satisfy the relation (2.16). Therefore in the limit $\alpha \to 0$, they cannot pass into soliton solutions. Hence these solutions either exist on the finite interval of wave numbers, closing on each other, or can continue up to any small α. In this case more and more local maxima and minima appear on the wave period, if α decreases (see, for example, figs. 6, 7). As a result, such solution may pass into a stochastic one in the limit $\alpha \to 0$.

The above examples of branchings from curves 1–4 (fig. 1) show how newer and newer solutions appear with the decrease of α. Distances between the points of new regime generation decrease in this case, and we have a countable set of solutions in the limit $\alpha \to 0$.

Investigating stability of the obtained solutions, one can obtain new generating curves (2.15) on

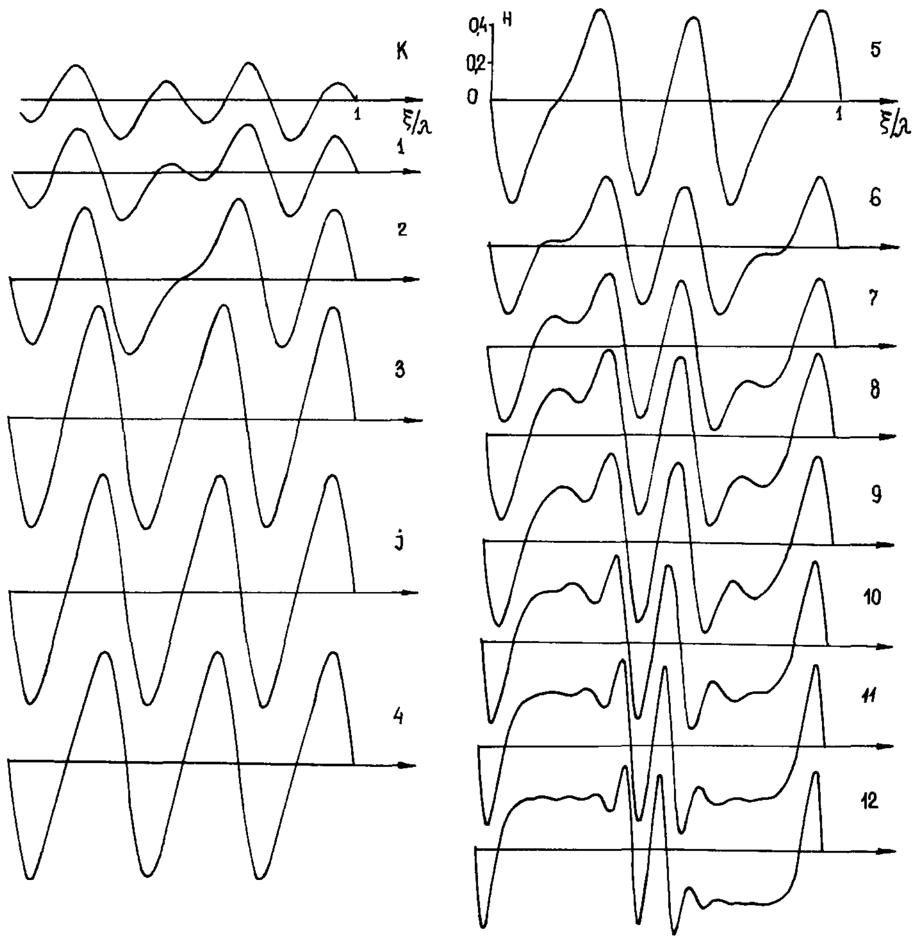

Fig. 5. Profiles of families shown by curve I (fig. 4).

the plane (α, Q) which are analogous to curves 1–4 in fig. 1. It is clear that in principle this procedure can be performed an unlimited number of times.

To illustrate what is said above, fig. 10 presents a family appearing as a result of such "secondary" branching. This regime has branched from the family presented in fig. 4 by a part of curve I lying above the horizontal axis. Remember that this family appears at branching from curve 4 with $Q = 1/2$. The new regime branched from the point $\alpha = 0.2602$ also with $Q = 1/2$. The first harmonic amplitude corresponding to it is represented by line III in fig. 4.

The shading in fig. 1 shows stability region for the first family. Only a real part of one of the eigenvalues goes through zero on its lower boundary. Therefore, corresponding unsteady-state time-dependent regimes are generated from this line, e.g. bifurcation of the Landau–Hopf type occurs here.

This figure shows that in accordance with the results of the paper by Nepomnyashchy [1], regimes with the wave numbers belonging to the interval [0.837–0.778] are stable to all infinitesimal flat disturbances.

The calculations show that the range of stability to special but important type of disturbances (disturbances of the same period, i.e. with $Q=0$) is wider (from the upper boundary of instability of the trivial solution $\alpha=1$ up to $\alpha_{**}=0.554$). For this point all Lyapunov indices are equal to zero and a new solution of the same period with the phase velocity $c>0$ bifurcates from the first family. Recall that we shall not consider solutions with $c<0$ by virtue of (2.2) and (2.3). It is clear, in particular, that they appear at the same point α_{**} too.

Tsvelodub [10] was the first to obtain this family except for the neighbourhood of the bifurcation point α_{**} and Nepomnyashchy [11] was the first to obtain this family in the neighbourhood of this point though the value of this bifurcation wave number, $\alpha=0.58$, given in his paper slightly differs from that obtained by us. In a later paper by Nepomnyashchy [12], this value is revised and it coincides with the value found by us. The same value for α_{**} is given in the papers by Chang and Chen [5] and Michelson [13].

The curve I in fig. 11 represents the values of the amplitudes $A \equiv H_{max} - H_{min}$ depending on the wave number, for the waves of the first antisymmetric family, and curve II represents the amplitude values of a new family bifurcating from the first one at the point α_{**} and having the phase velocity differing from zero. Evolution of the profiles of this family is given in fig. 12. It is evident that for sufficiently small α a region of comparatively fast variation of the wave profile H and a region with nearly constant function can be singled out on the wavelength.

The wave form in the region of fast variation remains practically unchanged on further wave number decrease for the values $\alpha \lesssim 0.3$, only the region of slow variation of the function H increases. In the limit $\alpha \to 0$, the family passes into a positive soliton, for which

$$\int_{-\infty}^{\infty} H \, d\xi > 0.$$

In contrast to soliton solution of the KdV equation it represents "a hump" which monotonically

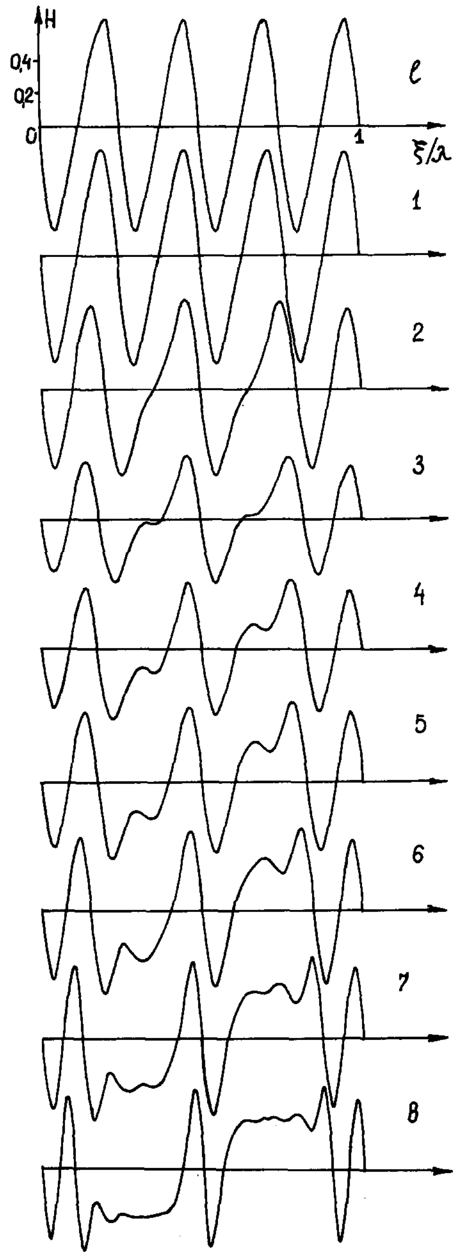

Fig. 6. Profiles of families generating in α_e (fig. 4) and extending to the region of small α.

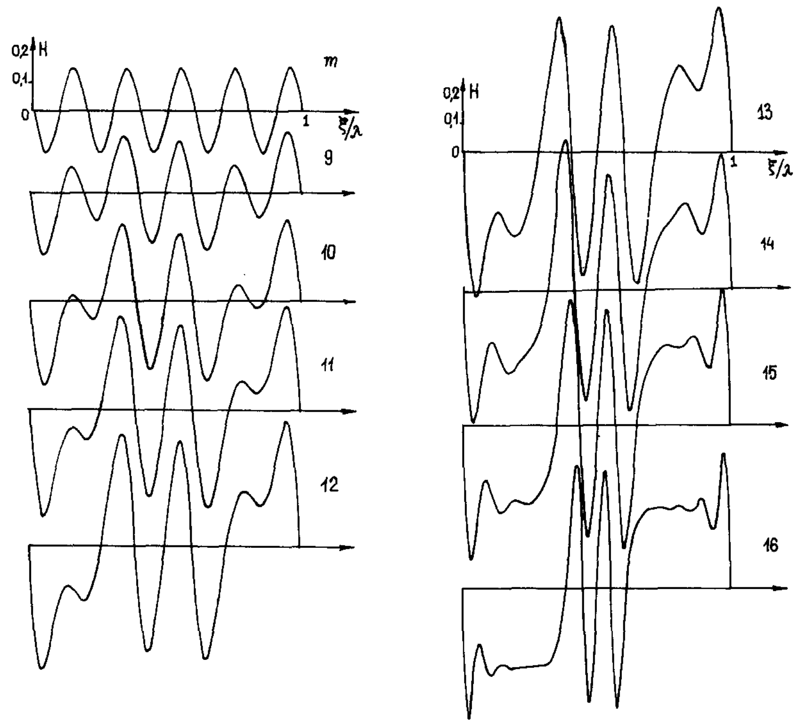

Fig. 7. Profiles of families generating in α_m (fig. 4) and extending to the region of small α.

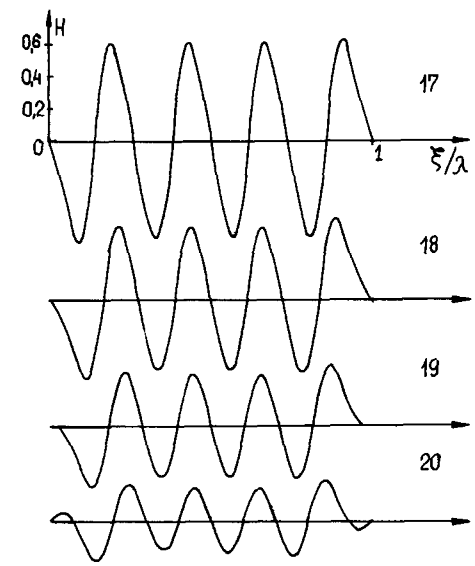

Fig. 8. Profiles of families generating in α_e and α_m (fig. 4) and merging with each other.

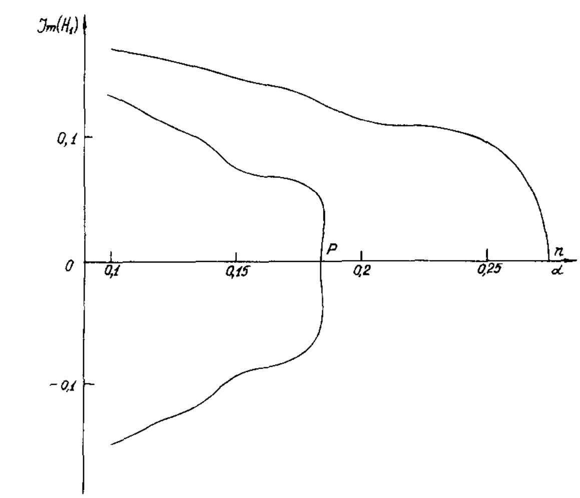

Fig. 9. The first harmonic vs. α for families generating from curve 3 (fig. 1) with $Q = 1/2$ (curve I) and $Q = 1/3$ (curve II).

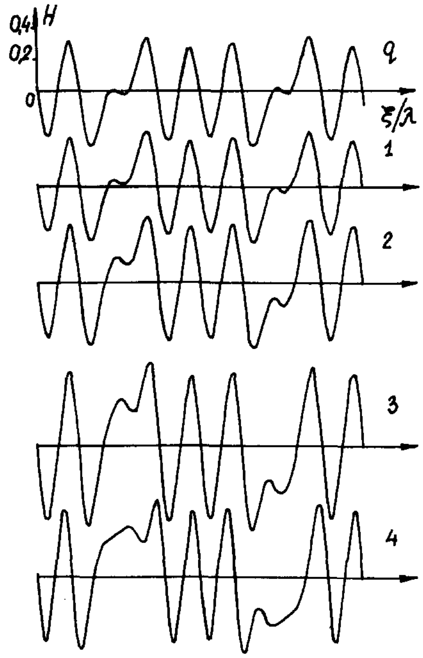

Fig. 10. Profiles of the family generating as a result of "secondary" branching.

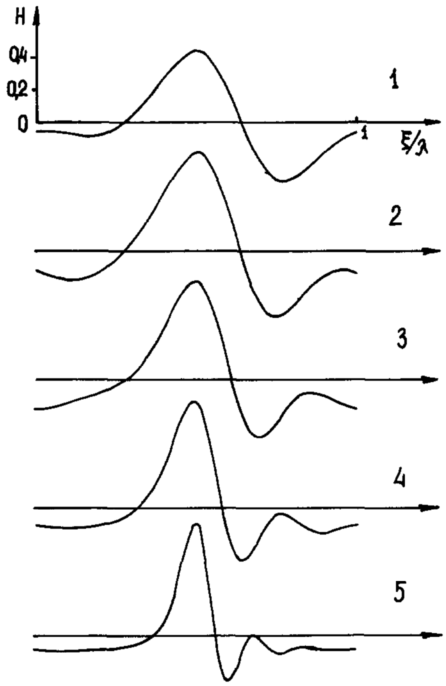

Fig. 12. Profiles of the first family with $c > 0$.

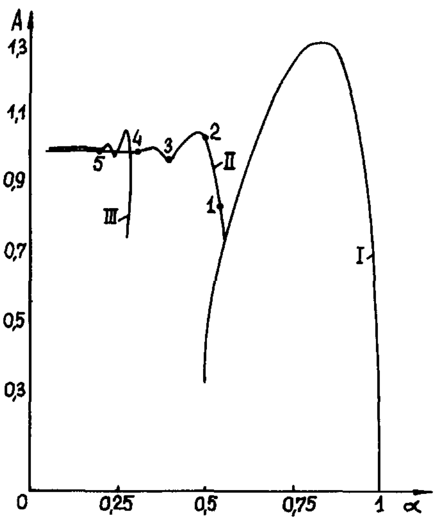

Fig. 11. Amplitude vs. α for the first antisymmetric family (curve I), for the first family with $c > 0$ (curve II) and the family branching from curve II with $Q = 1/2$ (curve III).

decays along the exponent in the tailing front and possesses characteristic attenuating oscillations in the leading front [10]. The value of its phase velocity is $c = 1.2161$.

Investigation of this family stability has shown that it is unstable at any value of the wave number α. There are many lines (2.15) for it crowding on decrease of α. These lines are almost parallel to the Q axis on the plane (α, Q). The first eight of

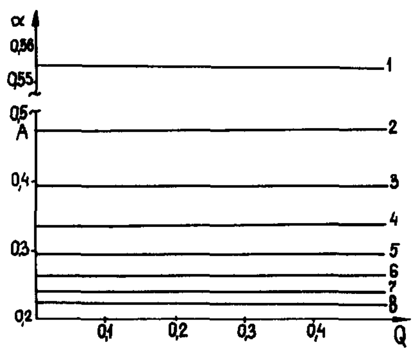

Fig. 13. Bifurcation lines (2.15) (curves 1–8) for solutions of the first family with $c > 0$.

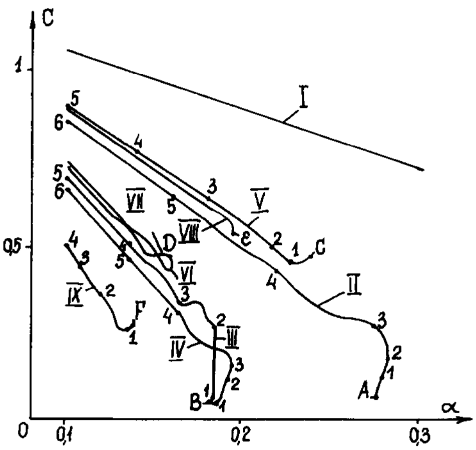

Fig. 14. Phase velocity c vs. α for families branching from the first one with $c > 0$.

them are shown in fig. 13. A part of them is related to the oscillations of the some eigenvalue γ, another part is related to a single passing of the eigenvalue through zero. It is clear that all the above ideology on new family appearance is applicable in this case too.

We have calculated several new families generated from the original family at a few points of the curves (2.15) presented in fig. 13, i.e. the regimes branching from curves 1 and 2 at $Q = 1/2$ and $1/3$, and from curve 3 with $Q = 1/2$. The dependence of velocities of these families on the wave number α is illustrated in fig. 14 by the following lines: for the family generating from curve 1 with $Q = 1/2$ – line II, with $Q = 1/3$ – lines III, IV; for the family generating from curve 2 with $Q = 1/2$ – V, with $Q = 1/3$ – VI, VII; for the family generating from curve 3 with $Q = 1/2$ – VIII. The points of family generation are designated by letters. The first family with $c > 0$ is shown by line I for comparison.

Curve III in fig. 11 represents dependence of the wave amplitude on α for the family branching from curve I with $Q = 1/2$. Amplitudes of other families also quickly approach curve II representing the principal generating family on decrease of α, and consequently they are not shown here.

As seen from the results presented, all these families continue into the region of small α in contrast to the antisymmetric solutions described above.

If the region of the original family (fig. 12) with fast variation of H consists of one hump, then according to the calculations the corresponding new families possess r such humps at branching from one of the lines (2.15) with $Q = 1/r$ and at α small enough. In this case these humps practically reproduce the soliton from the paper by

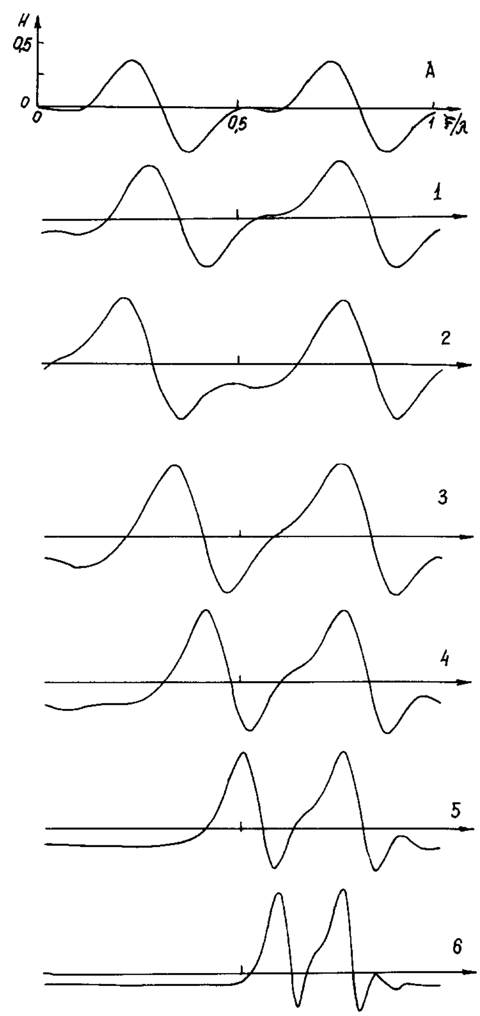

Fig. 15. Profiles of the family shown by line II in fig. 14.

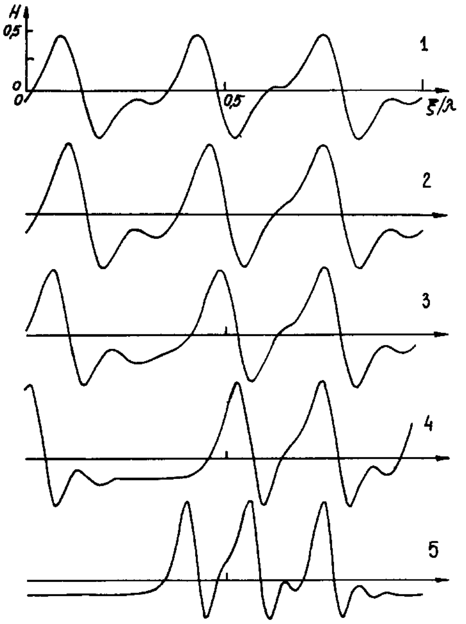

Fig. 16. Profiles of the family shown by line III in fig. 14.

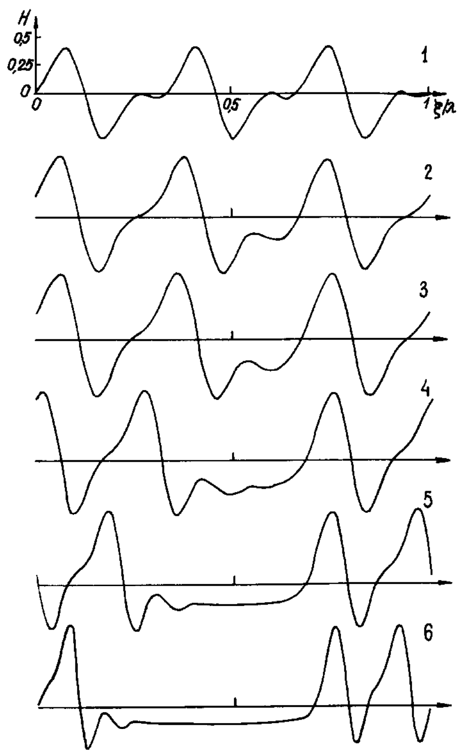

Fig. 17. Profiles of the family shown by line IV in fig. 14.

Tsvelodub [10]. Thus at small α these new solutions differ from each other by the distances between humps and their numbers. This is illustrated in figs. 15–18. These figures show evolution of the profiles of the families represented in fig. 14 by the lines II–V respectively. In the limit $\alpha \to 0$ these families are very likely to pass into corresponding solitons with r humps. This assumption agrees with the results of the paper by Demekhin and Shkadov [14], in which similar multi-hump solitons have been calculated by a completely different method.

It is clear that new sets of curves (2.15) can be obtained exactly as while analyzing branchings of antisymmetric solutions of eq. (2.1), i.e. by means of investigating the family of solutions with $c > 0$ for stability. New regimes are generated in their turn from these solutions along these curves (2.15). These regimes also have velocities differing from zero. And if branching takes place at a rational Q, then new periodic families appearing at such "secondary" branching can also produce solitons in the limit $\alpha \to 0$, and these solitons are of a more complex form.

For example, curve III in fig. 14 represents dependence of the phase velocity value on α for the family branched from the family represented by curve II in the same figure and by curve III in fig. 11. The evolution of wave profiles of this family is shown in fig. 19. As one can see, if seems already to have a 4-hump soliton in the limit at $\alpha \to 0$. The wave number of the point where the new family appears is $\alpha_F = 0.1373$.

Besides, concurrent with the first family (existing in the interval $0.4979 \leq \alpha \leq 1$), other antisymmetric families (if not all then at least some of them) are also sources of solutions with the phase velocity value $c \neq 0$. They also have points analogous to α_{**}, at which new families with $c \neq 0$ bifurcate from the given family. Thus, this bifurcation is illustrated in fig. 20 by the dependences of phase velocity (curve I) and amplitude (curve II) for the family branched with $Q = 0$ from the antisymmetric family represented by curve I in fig.

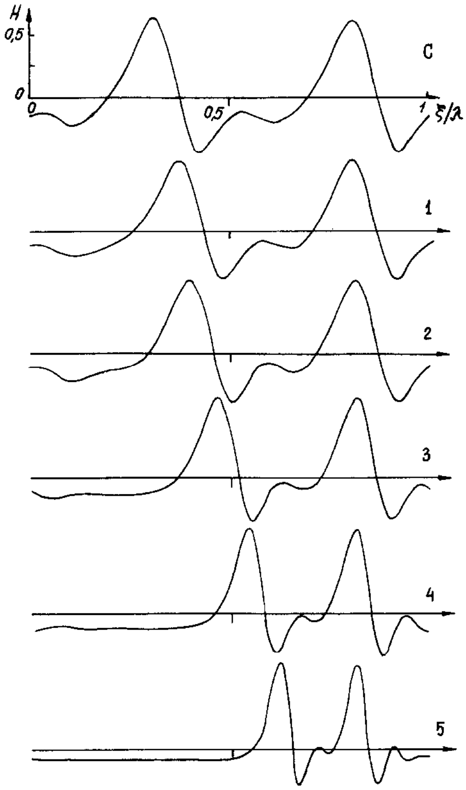

Fig. 18. Profiles of the family shown by line V in fig. 14.

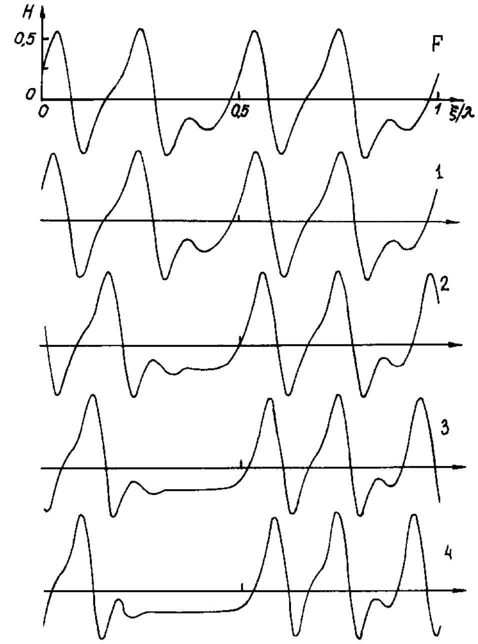

Fig. 19. Profiles of the family shown by line IX in fig. 14.

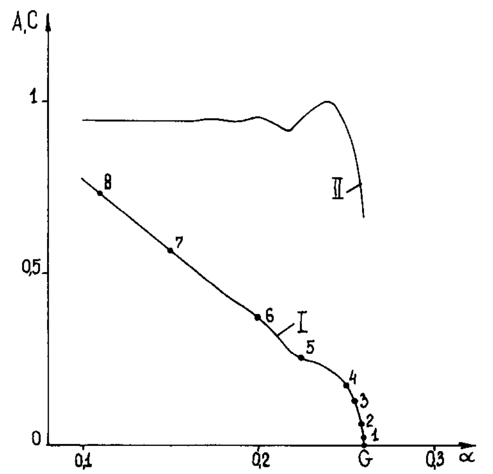

Fig. 20. Velocity (curve I) and amplitude (curve II) vs. α for the family bifurcating from antisymmetric families shown in fig. 14 by the upper part of curve I.

4. The corresponding profiles are shown in fig. 21. Here, $\alpha_G = 0.26018$.

From the above results it follows that eq. (1.2) possesses a multitude of families of steady-state travelling solutions quaintly related to each other. Their number is finite for any fixed wave number and increases in an avalanche on decrease of α, producing a countable multitude of periodic and soliton solutions in the limit $\alpha \to 0$.

In particular, the corresponding bifurcations from periodic solutions with $c \neq 0$ at irrational Q probably will also give regimes possessing the same characteristic humps as a structural element, the distance between the humps will vary irregularly. An arbitrary periodic solution of eq. (1.2) is to become stochastic in the time evolution process due to instability of the regimes considered. In an approximate modelling of such regimes we shall observe a complex behaviour, that will be closer to the real one when the larger number of the wavelengths λ of the original disturbance will be taken as a principal wavelength $r\lambda$. In the language of the Fourier series it means that r zero harmonics will be taken between the original nonzero harmonics in (2.4).

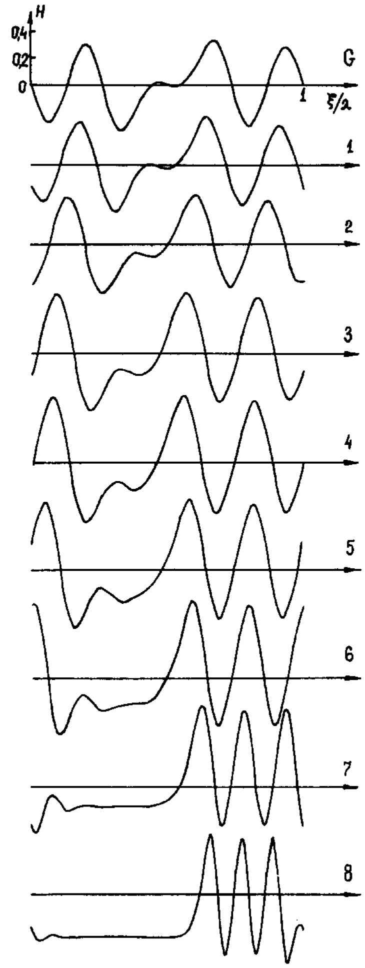

Fig. 21. Profiles of the family generating at point $\alpha_G = 0.2602$ with $c > 0$.

Such complex behaviour of solutions was actually observed in a number of investigations (see, e.g., [15, 16]) on direct numerical calculation of eq. (1.2). With certain care these solutions can be apparently interpreted as the stochastic ones.

Characteristic soliton-like structures are to be discriminated in stochastical regimes, as it follows from the results presented.

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