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OBSERVATIONS OF ORDER AND CHAOS IN NONLINEAR SYSTEMS

Harry L. SWINNEY
Department of Physics, University of Texas, Austin, Texas 78712, USA

Experiments on nonlinear electrical oscillators, the Belousov–Zhabotinskii reaction, Rayleigh–Bénard convection, and Couette–Taylor flow have revealed several common routes to chaos that have also been found in numerical studies of models with a few degrees of freedom. Experimental results are presented illustrating the following transition sequences: period doubling and the U-sequence, intermittency, the periodic–quasiperiodic–chaotic sequence, frequency locking, and an alternating periodic–chaotic sequence.

1. Introduction

We will describe some recent experimental studies of order and chaos in nonlinear systems. Although no attempt at completeness will be made, we will mention most of the transition sequences that have been found to be common to diverse systems.

Noisy (nonperiodic) behavior arising from stochastic driving forces such as thermal fluctuations and fluctuations in a system's environment has long been studied in laboratory experiments, but the experiments to be discussed here concern the nonperiodic (chaotic) behavior that arises primarily from the nonlinear nature of the systems rather than stochastic driving forces. The distinction between stochastic and deterministic noise in experiments is difficult, but the papers of Guckenheimer [55], and Farmer, Ott, and Yorke [53] in this volume suggest that the distinction can be made in systems with a few active degrees of freedom. We will show that the Poincaré sections and maps obtained in experiments on some rather complex nonlinear systems indicate that these systems (for some control parameter ranges) exhibit a dynamical behavior that can be described accurately by deterministic models with a few degrees of freedom.

Four well-studied nonlinear systems are described in section 2, and methods used to characterize their dynamical behavior are outlined in section 3. Some transition sequences that have been observed for a number of different systems are described in section 4. Section 5 is a discussion.

2. Four nonlinear systems

Nonlinear electrical circuits. The characteristic frequencies of electrical circuits can easily be made about 10^7 times higher than the typical oscillation frequencies of the chemical and hydrodynamic systems described in the following paragraphs. Such high information production rates make nonlinear electrical circuits (analog computers) ideal for examining different types of dynamical behavior, developing methods of data analysis, and studying the dependence of behavior on several control parameters [1–4, 51, 57, 63, 64, 68, 70]. An example of a simple nonlinear circuit is shown in fig. 1a; this series circuit has three degrees of freedom–q (the charge across the varactor), $\dot{q}$, and the angle θ, where the driving voltage is $V(t) = V_0 \sin\theta$ with $\theta = \omega t$. The behavior of this circuit is usually studied as a function of V_0, but it can also be studied as a function of other control parameters–ω, R, L, C_0, and β, where the nonlinear capacitance under reverse voltage is given by $C \approx C_0/[1 + \beta V_c]^{1/2}$.

The Belousov–Zhabotinskii reaction [5–15]. This reaction, the most thoroughly studied oscillating

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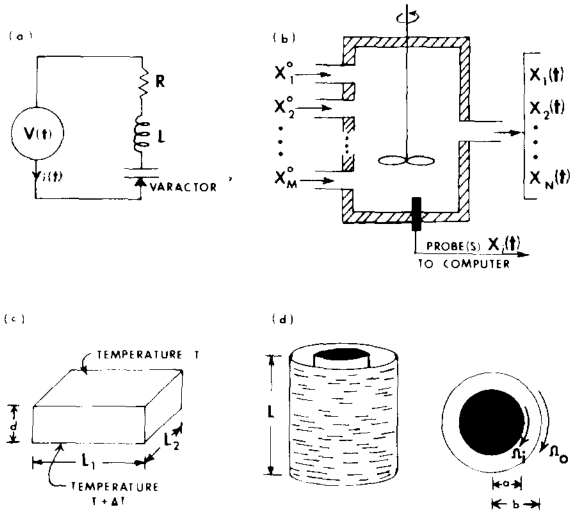

Fig. 1. Four nonlinear systems. (a) A series circuit with a varactor diode which conducts for a forward voltage and has a nonlinear capacitance for a reverse voltage [3, 4]. (b) A stirred flow chemical reactor. In the Belousov–Zhabotinskii reaction $M = 4$ (malonic acid, potassium bromate, cerium sulfate, and sulfuric acid) and $N > 30$. (c) Rayleigh–Bénard convection in a finite box. (d) Couette–Taylor system.

chemical system, involves the cerium-catalyzed bromination and oxidation of malonic acid by a sulfuric acid solution of bromate. (See the papers of Roux [9] and Epstein [5] in this volume.) The reaction can be maintained in a steady state away from equilibrium by continuously pumping the chemicals into a stirred flow reactor, as shown in fig. 1b. In a vigorously stirred reactor the system is essentially homogeneous so the reaction can be modeled by a set of coupled nonlinear ordinary differential equations. For example, the reaction

$$A + B \underset{k_b}{\overset{k_f}{\rightleftharpoons}} C$$

is described by the equations

$$\frac{dA}{dt} = -k_f AB + k_b C - r(A - A^0),$$

$$\frac{dB}{dt} = -k_f AB + k_b C - r(B - B^0),$$

$$\frac{dC}{dt} = k_f AB - k_b C - rC,$$

where A^0, B^0, and C^0 (with $C^0 = 0$) are the concentrations of the chemicals in the input to the reactor and r is the flow rate. Generalizing, the reactions among N chemical species of concentration $X_i(t)$ are described by

$$\frac{dX_i}{dt} = g_i(X_j) - r(X_i - X_i^0) \quad [i, j = 1, \ldots, N],$$

where the functions $g_i(X_j)$ involve nonlinear terms of the form X_i^2 and X_iX_j (i.e., three-body interactions can be neglected). Transitions in the dynamical behavior are studied as a function of the flow rate: as $r \to 0$, the system approaches thermodynamic equilibrium, while for large r the chemicals have no time to react as they pass through the reactor; the interesting dynamics occurs for r between these extremes. The behavior can also be studied as a function of other control parameters–the reactor temperature and the input concentrations X_i^0.

Rayleigh–Bénard convection [16–30]. In contrast to the nonlinear oscillator and stirred flow reactor, which presumably have a well-defined finite number of degrees of freedom, the next two examples, the Rayleigh–Bénard and Couette–Taylor systems, are continuum hydrodynamic systems which can in principle have an infinite number of degrees of freedom (although just beyond the onset of chaos there are presumably only a few degrees of freedom that are excited).

In a Rayleigh–Bénard system a fluid is contained between parallel plates heated from below, as shown in fig. 1c. (Also see the papers of Libchaber [26] and Maeno and Haucke [28] in this volume.) The behavior is usually studied as function of the (dimensionless) Rayleigh number $R_a = (g\alpha d^3/\kappa\nu)\Delta T$, where g is the gravitational acceleration, α the thermal expansion coefficient, d the separation between the plates, κ the thermal diffusivity, and ν the kinematic viscosity. Other control variables are the Prandtl number, $P = \nu/\kappa$, the aspect ratios, $\Gamma_1 = L_1/d$ and $\Gamma_2 = L_2/d$, and the boundary conditions at the side walls.

Couette–Taylor system [31–41]. In this system a fluid is contained between concentric cylinders that rotate independently with angular velocities Ω_i (inner) and Ω_o (outer); see fig. 1d. The (dimensionless) Reynolds numbers are then $R_i = (b-a)a\Omega_i/\nu$ and $R_o = (b-a)b\Omega_o/\nu$, where a and b are the radii of the inner and outer cylinders, respectively. Most experiments including those to be described here have been conducted with $R_o = 0$. The behavior is quite different and much richer when both cylinders are rotated (Andereck, Liu, and Swinney [31]), because the instabilities do not depend simply on the differential rotation rate of the cylinders, but on a subtle interplay between the radial pressure gradient and the centrifugal force $r(\Omega_{\text{fluid}})^2$. (There is no equivalence principle for rotating reference frames!) Other control parameters for this system are the radius ratio a/b, the aspect ratio $\Gamma = L/(b-a)$, and the boundary conditions at the ends.

Other systems. Some results from experiments on a few other systems [42–50] will be mentioned in section 4.

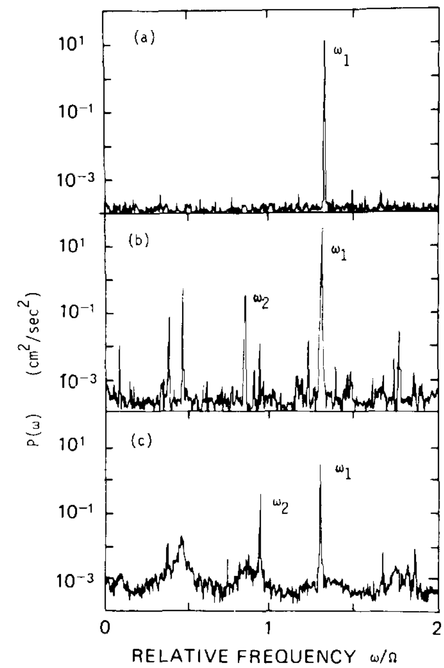

Fig. 2. Power spectra for different dynamical regimes in the Couette–Taylor system (from [35, 41]). (a) $R/R_c = 9.6$, periodic; the spectrum consists of a single fundamental frequency, ω_1. (b) $R/R_c = 11.0$, quasiperiodic; the spectrum consists of two fundamental frequencies, ω_1, and ω_2, and integer combinations. (c) $R/R_c = 18.9$, chaotic; the spectrum contains broadband noise in addition to the sharp components ω_1 and ω_2. The noise in (a) and (b) is instrumental, while in (c) the fluid noise is well above the instrumental noise level. These spectra illustrate the periodic–quasiperiodic–chaotic transition sequence discussed in section 4.3.

3. Analysis of dynamical behavior [51–72]

In experiments the time dependence of a dynamical variable $V(t)$ is determined in sequential time intervals $t_k = k(\Delta t)$, where $k = 1, \ldots, n$ (typically $n = 8192$). The time series $V(t_k)$ is recorded in a computer and its power spectral density $P(\omega)$ (the modulus squared of the Fourier transform) is calculated using the Cooley–Tukey fast Fourier transform algorithm.

Power spectra make it possible to distinguish between periodic, quasiperiodic, and chaotic regimes, as fig. 2 illustrates. However, the broadband noise or broadened spectral lines that indicate nonperiodic (chaotic) behavior could arise from stochastic as well as deterministic processes. A better method of analysis is needed to determine if the nonperiodic behavior is characteristic of a deterministic nonlinear system.

Before the turn of the century Poincaré showed that much can be learned about dynamical behavior from an analysis of trajectories in a multidimensional phase space in which a single point characterizes the entire system at an instant of time. The experimenters' dilemma has been that for a system with N degrees of freedom it seemed that it would be necessary to measure N independent variables, an almost impossible chore for complex systems.

A much simpler alternative was suggested several years ago by Ruelle [65] and Packard et al. [63]. Their idea, which is justified by embedding theorems [69, 71], was that a multi-dimensional phase portrait can be constructed from measurements of a *single* variable, as follows: For almost every observable $V(t)$ and time delay T the m-dimensional portrait constructed from the vectors $\{V(t_k), V(t_k+T), \ldots, V(t_k+(m-1)T)\}$, $k=1,\ldots,\infty$, will have many of the same properties (strictly speaking, will give an embedding of the original manifold) as one constructed from measurements of the N independent variables, if $m \geqslant 2N+1$. In practice m is increased by one at a time until additional structure fails to appear in the phase portrait when an extra dimension is added. Phase portraits constructed for a periodic state and a chaotic state in the Belousov–Zhabotinskii reaction are shown in fig. 3 [11, 14].

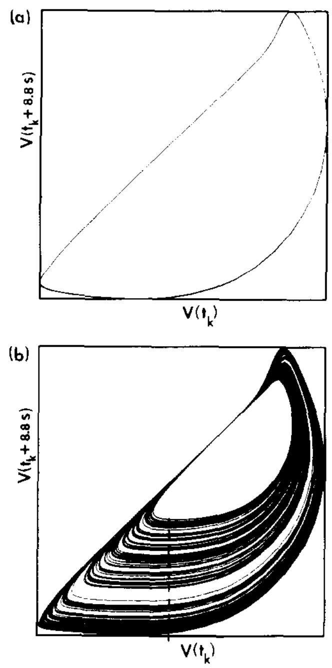

Fig. 3. (a) A two-dimensional phase portrait for a periodic state observed in experiments on the Belousov–Zhabotinskii reaction; the corresponding power spectrum has a single sharp fundamental component and its harmonics. (b) A two-dimensional projection of a three-dimensional phase portrait [with the third axis, $V(t_k+17.6\,\text{s})$, normal to the page] for a chaotic state observed in the Belousov–Zhabotinskii reaction; the corresponding power spectrum contains broadband noise. The attractor in (a) is a limit cycle and in (b) a strange attractor. (From [11, 14].)

Rather than analyze the phase portrait directly it is easier to analyze Poincaré sections and maps. A Poincaré section is formed by the intersection of "positively" directed orbits with a $(m-1)$-dimensional hypersurface. For example, fig. 4(a) shows a Poincaré section constructed for the 3-dimensional phase portrait in fig. 3b. The orbits for this chaotic attractor clearly lie essentially along a sheet. Thus intersections of this sheet-like attractor with a plane lie to a good approximation along a parameterizable curve, not on a higher dimensional set. (Actually, the Poincaré section must have a dimension at least slightly greater than unity because of the fractal nature of the attractor [52, 53, 61]). The parameter values at successive intersections provide a sequence $\{X_n\}$ which defines a one-dimensional map, $X_{n+1}=f(X_n)$, as shown in fig. 4b. The data appear to fall on a single-valued curved. This indicates that the system is deterministic: for any X_n, the map *determines* X_{n+1}.

The power spectrum for the data in fig. 3b contains broadband noise [11, 14], indicating that the state is nonperiodic, but the phase portrait must be analyzed to determine if the system is really characterized by a *strange attractor*. To demonstrate that the phase space trajectories define a strange attractor it must be shown that the post-transient subset described by the trajectories is:

(1) An *attractor*–orbits rapidly return to this subset after finite perturbations. However, perturbations too large could send the orbit out of the basin of attraction for the attractor; see [59, 66, 67].)

(2) *Strange*–nearby orbits diverge exponentially on the average ("sensitive dependence on initial conditions" [66]) [59, 62, 67, 68].

Studies of the effect of perturbations on the state characterized by the phase portrait in fig. 3b show that the trajectories lie on an *attractor* [11, 12]). In addition, an analysis of the corresponding map,

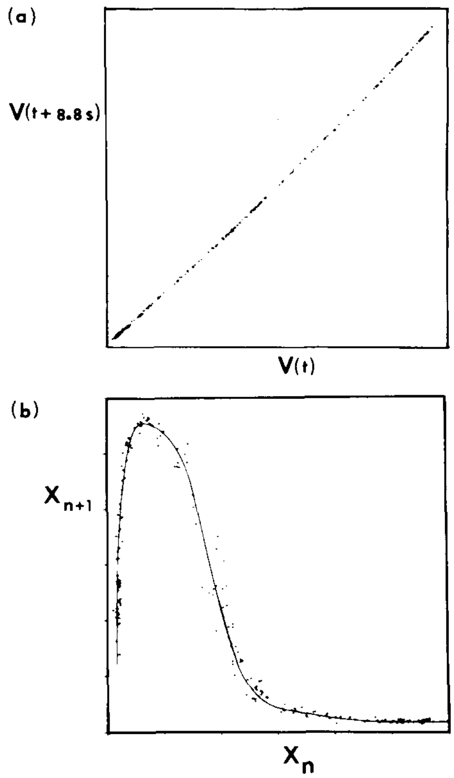

Fig. 4. (a) A Poincaré section formed by the intersection of trajectories in a three-dimensional phase space with the plane (normal to the page) passing through the dashed line in fig. 3b. (b) A one-dimensional map constructed from the data in fig. 4a. (From [11, 14].)

fig. 4b, shows that the attractor is *strange*–the largest Lyapunov exponent, given by

$$\lambda = \int_0^1 P(X)\left[\ln\left|\frac{df}{dX}\right|\right] dX,$$

where $P(X)\,dX$ is the probability of finding an iterate of the map in the interval $(X, X + dX)$, is positive [7, 11, 12].

The one-dimensional map [fig. 4b] indicates that the attracting sheet seen in cross-section in fig. 4a must exhibit the stretching and folding that is characteristic of strange attractors. This stretching and folding has been directly observed by analyzing Poincaré sections through the different parts of the attractor [12]; in fig. 3b the folding occurs in the part of the attractor where the orbits appear (at the resolution of this figure) to narrow down to a line.

Other methods of analysis of phase portraits include the determination of the following properties (see the papers in this volume by Farmer, Ott and Yorke [53], Guckenheimer [55], Mandelbrot [61], Packard [63], and Shaw [68]); (1) attractor dimension (the terms capacity, Hausdorff, fractal, information or Renyi, and Lyapunov dimension correspond to different definitions of dimension); (2) entropy (topological, metric or Kolmogorov–Sinai); (3) the spectrum of Lyapunov exponents; and (4) probability distribution functions for the Poincaré sections and maps.

4. Transition sequences

4.1. *Intermittency*

Some systems exhibit a transition from periodic behavior (for $R < R_T$) to a chaotic behavior (for $R > R_T$) characterized by occasional bursts of noise [73, 74]. For R only slightly greater than R_T there are long intervals of periodic behavior between the short bursts, but with increasing R the intervals between the bursts decrease; it becomes more and more diffcult and finally impossible to recognize the regular oscillations of the periodic state. Examples of intermittency transitions are shown in fig. 5.

Pomeau and Manneville [74] have shown that intermittency appears at a tangent bifurcation where a stable fixed point of a map disappears. Direct evidence of the tangent bifurcation has been observed in the experiments by Pomeau et al. [8] on the Belousov–Zhabotinskii reaction; by Jeffries and Perez [2] on a nonlinear oscillator; and by Bergé et al. [17] on convection. Also, the predicted behavior of the mean time T between bursts, $T \propto (R - R_T)^{-1/2}$, has been observed in the experiments of Jeffries et al.

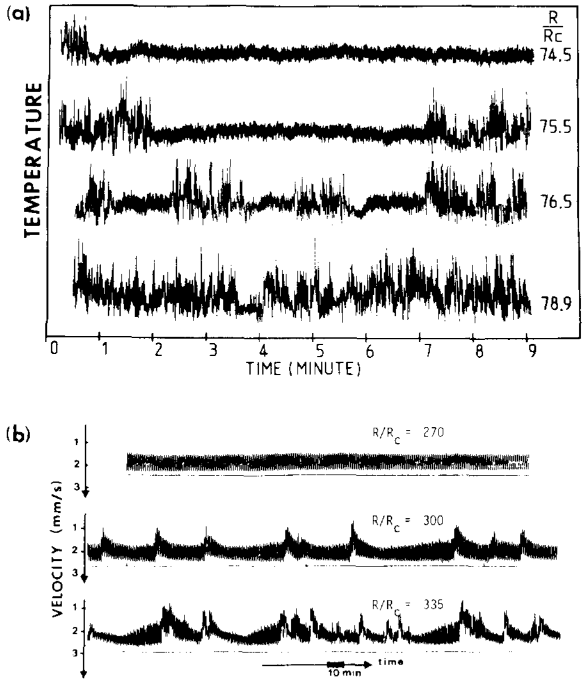

Fig. 5. Intermittency in convection. The turbulent bursts occur with increasing frequency with increasing Rayleigh number. (a) Temperature measurements of Maurer and Libchaber [30]; Prandtl number, 0.62, and aspect ratios $\Gamma_1 = 2.4$ and $\Gamma_2 = 2.0$. (b) Velocity measurements of Bergé et al. [17]; Prandtl number, 130, and aspect ratios $\Gamma_1 = 2.0$ and $\Gamma_2 = 1.2$.

4.2. *Frequency locking*

In some experiments a transition from a quasiperiodic state to a frequency-locked (periodic) state has been observed with increasing control parameter. The periodic state persists for some range in control parameter and then, in some cases, there is a well-defined transition to a chaotic state, as illustrated by data from a Rayleigh–Bénard experiment shown in fig. 6. This quasiperiodic→locked→chaotic sequence has been discussed theoretically [75–78].

4.3. *Periodic–quasiperiodic–chaotic sequence*

This transition sequence, first suggested by Ruelle and Takens [82] more than a decade ago, has been observed in many experiments since it was first observed by Gollub and Swinney in 1975 [36]. The sequence is illustrated by data for the Couette–Taylor system in fig. 2.

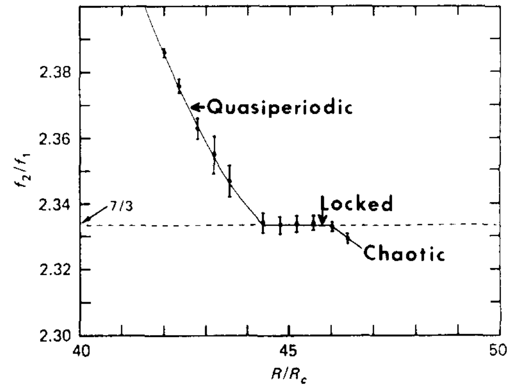

Fig. 6. Frequency locking in the convection experiments of Gollub and Benson [23]. The Prandtl number was 2.5 and the aspect ratios were $\Gamma_1 = 3.5$ and $\Gamma_2 = 2.0$. The curve through the data is drawn to guide the eye.

In the Ruelle–Takens picture [80, 82], when a system makes a transition from a quasiperiodic state with two incommensurate frequencies (a flow on a 2-torus) to quasiperiodic state with three incommensurate frequencies (a flow on a 3-torus), there is in every suitably differentiable neighborhood of the vector field on the 3-torus a vector field which has a strange attractor. Since chaos could thus arise from infinitesimal perturbations of a 3-frequency state, states with three independent frequencies would usually not be observed. (Three independent frequencies have, in fact, been seen in only a few experiments; see [20] and [24].)

An alternative theoretical picture of the quasiperiodic–chaotic transition has recently been developed by Rand et al. [81], Shenker [84], and Feigenbaum et al. [79], and is described in the papers of Shenker [83] and Siggia [85] in this volume. In this theory the 2-torus develops wrinkles as the onset of chaos is approached, and the corresponding power spectrum has a self-similar structure, at least for a system with the ratio of frequencies near the Golden Mean, $(5^{1/2} - 1)/2$. Although the detailed predictions have been developed for frequencies in the ratio of the Golden Mean, the breakdown of the torus is predicted to occur for other irrational frequency ratios. Thus far there have been no experimental observations of the predicted self-similar spectrum near the

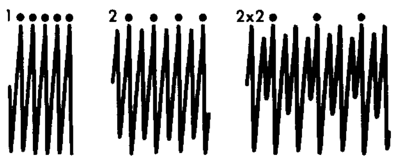

Fig. 7. Period doubling sequence time series with periods τ(115 s), 2τ, and $2^2\tau$, obtained in experiments on the Belousov–Zhabotinskii reaction; period $2^3\tau$ was also observed. The quantity measured was the bromide ion potential. The dots above the time series are separated by one period. (From [13].)

onset of chaos, but experiments are underway in several laboratories on periodically driven oscillating systems where the frequency ratio can be adjusted to the Golden Mean.

4.4 *Period doubling*

The period doubling route to chaos [87–90] has been observed in experiments on Rayleigh–Bénard convection [21, 24, 26, 29], nonlinear electrical oscillators [3, 4], acoustics [48–50], shallow water waves [46–47], a hybrid optical system [45], and the Belousov–Zhabotinskii reaction [13]. At least two or three period doublings were observed in each of these experiments; for example, see fig. 7. The measured values of Feigenbaum's universal number δ [87–90] (which describes asymptotically the ratio of successive intervals in the bifurcation parameter between period doubling transitions) and the scaling parameter α are consistent with the theory for one-dimensional maps with a single extremum. However, the experimental values of δ and α are accurate to only about 5% at best [4] because the rapid convergence rate of the doubling sequence makes it very difficult to observe many doublings.

In systems with many active degrees of freedom, departures from the period doubling sequence are observed. In Rayleigh–Bénard convection Arneodo et al. [86] have shown that this departure can be understood in terms of a two-dimensional Hénon-like map.

4.5. *The U-sequence*

Universality in the period doubling sequence for one-dimensional maps is now well known. Perhaps less well known is the U (universal)-sequence that occurs beyond the accumulation point (2^∞-cycle) of the 2^n-sequence. Metropolis, Stein, and Stein [93] found, several years before the universal scaling properties of one-dimensional quadratic maps were discovered by Feigenbaum, that one-dimensional maps with a single extremum (not necessarily quadratic) exhibit universal dynamics as a function of the bifurcation parameter. Beyond the period doubling sequence, which is an infinite sequence of doublings of a periodic state with one oscillation per period, periodic states with K oscillations per period appear for all natural numbers K, and each of these "K-cycles" undergoes its own infinite period doubling sequence, 2^nK [87, 92, 93]. Fig. 8a shows examples of a fundamental 5-cycle, 6-cycle, and 3-cycle (and the first doubling of the 3-cycle) observed in the Belousov–Zhabotinskii reaction.

The order in which the periodic states appear as a function of bifurcation parameter and the iteration patterns of the corresponding maps are all deduced in the theory using only the single-extrenum property of the one-dimensional map [92]. Table I shows all U-sequence states with $K \leqslant 6$. The full U-sequence consists of the (infinitely long) extension of table I to include all the periodic states allowed by the theory. The larger the fundamental

Table I
The U-sequence states with periods up to 6 (in order of occurrence as a function of bifurcation parameter) [93]

Period	Map iteration pattern
1	0
2	0–1
2 × 2	2–0–3–1
6	2–0–4–3–5–1
5	2–0–3–4–1
3	2–0–1
2 × 3	2–5–3–0–4–1
5	2–3–0–4–1
6	2–3–0–4–5–1
4	2–3–0–1
6	2–3–4–0–5–1
5	2–3–4–0–1
6	2–3–4–5–0–1

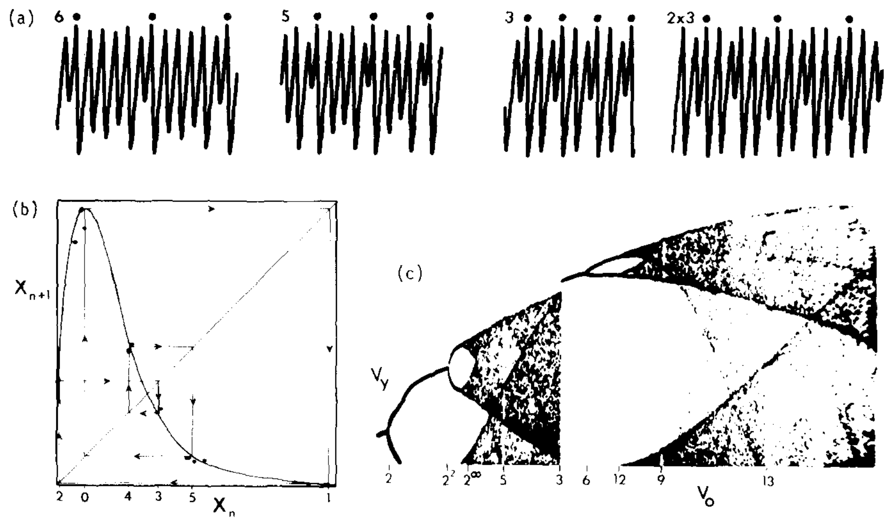

Fig. 8. Observations of the U-sequence. (a) Time series with periods 6τ (where $\tau = 115$ s), 5τ, 3τ, and $2 \times 3\tau$, observed in the Belousov–Zhabotinskii reaction [13]. The dots above the time series are separated by one period. (b) The one-dimensional map constructed from data for the state with period 6τ shown in (a); the iteration pattern is 2–0–4–3–5–1, as predicted (see table I) [13]. (c) A bifurcation diagram obtained in experiments of Testa et al. [4] on a nonlinear electrical oscillator. The vertical axis is the voltage across a varactor [see fig. 1a] and the horizontal axis is the control parameter, the amplitude V_0 of the driving voltage. The onsets of some of the U-sequence states are indicated at the bottom of the diagram.

period K, the larger the number of allowed states; there are three distinct allowed 5-cycles, four distinct 6-cycles (see table I), and 27 distinct 9-cycles.

Table I also shows the predicted map iteration patterns–the order of visitation of points on the X-axis–for periodic states with $K \leqslant 6$. Each iteration pattern occurs only once, and for a given value of the bifurcation parameter not more than one periodic state is stable. An experimentally determined map illustrating the iteration pattern for a period-six state is shown in fig. 8b.

Beyond the $2^\infty K$-cycle of each period doubling sequence there is a chaotic reverse bifurcation sequence, as discussed by Lorenz [60]; although the chaotic states do not exist for intervals in bifurcation parameter, the set of bifurcation parameter values for which the behavior is chaotic has positive measure [56]. Both chaotic and periodic states can be seen in the bifurcation diagram obtained for a nonlinear oscillator shown in fig. 8c.

Many states of the U-sequence have been observed in experiments on nonlinear electrical oscillators [4], the Belousov–Zhabotinskii reaction [13], and Rayleigh–Bénard convection in a magnetic field [26]. The observed iteration patterns and ordering of the states are in accord with the theory for one-dimensional maps.

4.6. *Alternating periodic–chaotic sequences*

Fig. 9a shows an alternating periodic–chaotic transition sequence observed in an experiment on the Belousov–Zhabotinskii reaction [11, 14]; time series for the first three periodic states (P_1^0, P_1^1, and P_1^2) are shown in figs. 9b–d, respectively, and the time series for the third chaotic state ($C_1^{2,3}$), which occurs between P_1^2 and P_1^3, is shown in fig. 9e. [Notation: P = periodic, C = chaotic. The subscript (superscript) is the number of large (small) amplitude oscillations per period; see fig. 9.] Alternating periodic–chaotic transition sequences similar to that in fig. 9 have been observed in other experiments on the Belousov–Zhabotinskii reaction [6, 7, 10, 15] (for rather different control parameters) and in an experiment on a driven Josephson junction [42]. In addition, alternating

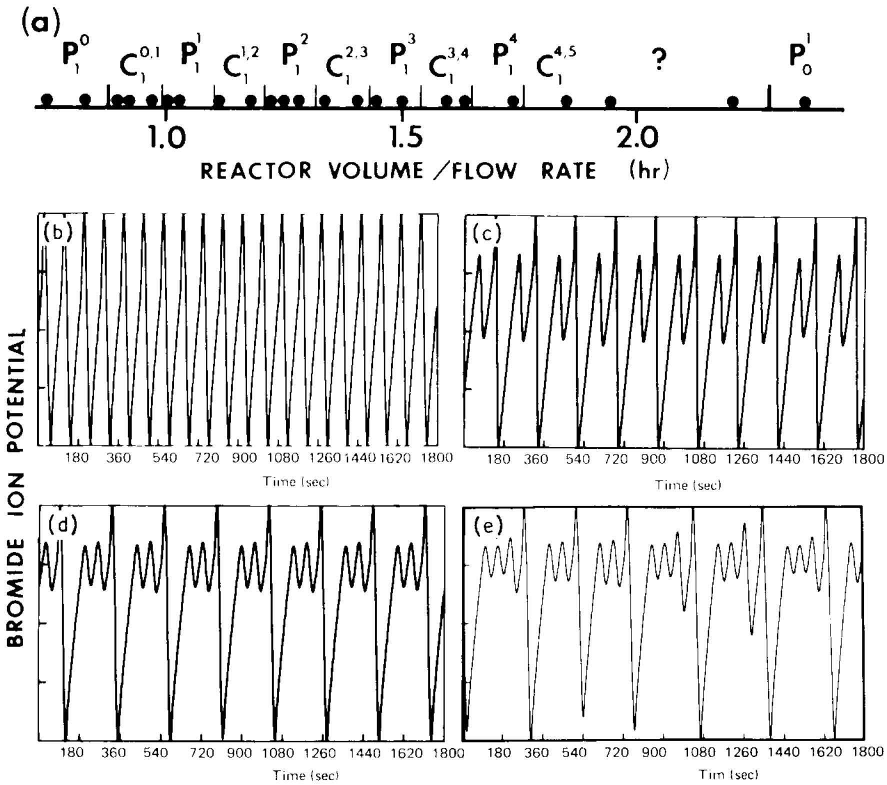

Fig. 9. (a) An alternating periodic-chaotic sequence observed in the Belousov–Zhabotinskii reaction. (b) A time series for the first periodic state, P_1^0. (c) P_1^1. (d) P_1^2. (e) A time series for the third chaotic state, $C_1^{2,3}$, where the number of small amplitude oscillations following each large amplitude oscillation is either two or three but is unpredictable. (From [11, 14].)

periodic–chaotic sequences have been found in studies of models of the Belousov–Zhabotinskii reaction [14, 97, 98] and the Josephson junction [94] (or, equivalently, the forced pendulum [95]) and in a symbolic dynamics analysis of a driven van der Pol oscillator [96]. A one-dimensional map that has an alternating periodic–chaotic sequence is described in the paper by Roux [9] in this volume.

Perhaps these alternating periodic–chaotic sequences have different mathematical descriptions; nevertheless, several common features can be noted: (1) The sequences are finite, not infinite; successive states exist for comparable ranges in control parameter. (2) Successive periodic states are simply related; for example, the states P_1^0, P_1^1, and P_1^2 in figs. 9b–d have (in each period) one large amplitude oscillation and, respectively 0, 1, and 2 small amplitude oscillations. (3) The chaotic states are mixtures of nearby periodic states; for example, $C_1^{n,n+1}$ is a nonperiodic mixture of states P_1^n and P_1^{n+1} (and perhaps occasional cycles of P_1^{n-1} and P_1^{n+2}), as fig. 9e illustrates. (4) The route by which a periodic state becomes chaotic has not been established in most cases, but presumably the transition occurs through period doubling or intermittency (see [9]). For the data in fig. 9 the P_1^0 to $C_1^{0,1}$ transition occurs through period doubling, as fig. 7 illustrates. (5) Each "chaotic" regime can contain many subintervals that are periodic. For example, the U-sequence states shown in fig. 8 occur within the $C_1^{0,1}$ regime.

4.7. *Soft mode instability*

Langford et al. [101] predicted that an instability associated with a low frequency mode could result from the nonlinear competition between a symmetry-breaking linear instability and oscillatory instability (see also [99, 100]); just above the instability the soft-mode frequency would increase linearly as a function of the bifurcation parameter. Such an instability has been observed in the convection experiments of Libchaber [26] described in this volume; he found that the system gradually became chaotic with increasing Rayleigh number beyond the instability.

5. Discussion

In section 4 we have considered some common features of transitions observed in experiments on diverse systems. While it is natural to focus on common features, it should be emphasized that the range of dynamical behavior that has been observed is quite large. We consider now some observations not mentioned in section 4.

Aspect ratio dependence. In experiments on Rayleigh–Bénard convection Ahlers and coworkers [16] found that as the aspect ratio was increased, nonperiodic behavior occurred at lower and lower Rayleigh numbers. A similar dependence on aspect ratio was subsequently observed in other hydrodynamic experiments [27, 32, 34, 37]. The number of accessible modes and the equilibration time both increase rapidly with increasing aspect ratio; therefore, a large aspect ratio system is especially susceptible to small external perturbations and apparently never settles down into an ideal ordered state. Thus the nonperiodic behavior observed in large aspect ratio systems at small Rayleigh or Reynolds numbers may not correspond to deterministic chaos.

Nonuniqueness. It is widely recognized that nonlinear systems can have two or more stable states at a given set of values of the control parameters, but the extreme degree of nonuniqueness in real systems is not often appreciated. For example, experiments in our laboratory indicate that a Couette–Taylor system with $R_i = 10R_c$, $R_o = 0$, $a/b = 0.88$ and $\Gamma = 30$ has more than 100 different stable states, some periodic, some quasiperiodic, and some chaotic! Each of these states corresponds to a phase space attractor which has its own basin of attraction (set of initial conditions for which the system will asymptotically approach that attractor). There is no systematic way to determine if all basins of attraction have been discovered, even at particular values of the control parameters. In fact, two independent investigators working on the same kind of system at the same control parameters could observe quite different phenomena because of different Reynolds number histories.

Multiple control parameters. Transition sequences are usually investigated as a function of a single control parameter (e.g., voltage, flow rate, Rayleigh number, or Reynolds number; see section 1). New kinds of bifurcations are possible when the dependence on two control parameters is considered [56]. The bifurcations that can occur with more than two control parameters have not been classified, but the experiments of Andereck et al. [31] and King and Swinney [38] on the Couette–Taylor system as a function of R_i, R_o, Γ, and a/b reveal an incredible richness in dynamical behavior.

Summary. In view of the great variety of behavior observed in experiments on nonlinear systems, it would be premature at this time to make sweeping generalizations about routes to chaos. Nevertheless, it is encouraging that a small number of common transition scenarios, as described in section 4, are beginning to emerge from theory and experiment.

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