Chapter 17

Complex Analysis I

Although this chapter is called complex *analysis*, we will try to develop the subject as complex *calculus* — meaning that we shall follow the calculus course tradition of telling you how to do things, and explaining why theorems are true, with arguments that would not pass for rigorous proofs in a course on real analysis. We try, however, to tell no lies.

This chapter will focus on the basic ideas that need to be understood before we apply complex methods to evaluating integrals, analysing data, and solving differential equations.

17.1 Cauchy-Riemann equations

We focus on functions, $f(z)$, of a single complex variable, z, where $z = x+iy$. We can think of these as being complex valued functions of two real variables, x and y. For example

$$\begin{aligned} f(z) = \sin z \equiv \sin(x+iy) &= \sin x \cos iy + \cos x \sin iy \\ &= \sin x \cosh y + i \cos x \sinh y. \end{aligned} \tag{17.1}$$

Here, we have used

$$\begin{aligned} \sin x &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right), \quad \sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right), \\ \cos x &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \quad \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right), \end{aligned}$$

to make the connection between the circular and hyperbolic functions. We shall often write $f(z) = u + iv$, where u and v are real functions of x and y. In the present example, $u = \sin x \cosh y$ and $v = \cos x \sinh y$.

If all four partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial u}{\partial y}, \tag{17.2}$$

exist and are continuous then $f = u + iv$ is differentiable as a complex-valued function of two real variables. This means that we can approximate the variation in f as

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \cdots, \tag{17.3}$$

where the dots represent a remainder that goes to zero faster than linearly as δx, δy go to zero. We now regroup the terms, setting $\delta z = \delta x + i\delta y$, $\delta\overline{z} = \delta x - i\delta y$, so that

$$\delta f = \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial \overline{z}}\delta\overline{z} + \cdots, \tag{17.4}$$

where we have defined

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \\ \frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \end{aligned} \tag{17.5}$$

Now our function $f(z)$ does not depend on $\overline{z}$, and so it must satisfy

$$\frac{\partial f}{\partial \overline{z}} = 0. \tag{17.6}$$

Thus, with $f = u + iv$,

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = 0 \tag{17.7}$$

i.e.

$$\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0. \tag{17.8}$$

Since the vanishing of a complex number requires the real and imaginary parts to be separately zero, this implies that

$$\begin{aligned}\frac{\partial u}{\partial x} &= +\frac{\partial v}{\partial y},\\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}.\end{aligned} \tag{17.9}$$

These two relations between u and v are known as the *Cauchy-Riemann equations*, although they were probably discovered by Gauss. If our continuous partial derivatives satisfy the Cauchy-Riemann equations at $z_0 = x_0 + iy_0$ then we say that the function is *complex differentiable* (or just differentiable) at that point. By taking $\delta z = z - z_0$, we have

$$\delta f \stackrel{\text{def}}{=} f(z) - f(z_0) = \frac{\partial f}{\partial z}(z - z_0) + \cdots, \tag{17.10}$$

where the remainder, represented by the dots, tends to zero faster than $|z - z_0|$ as $z \to z_0$. This validity of this linear approximation to the variation in $f(z)$ is equivalent to the statement that the ratio

$$\frac{f(z) - f(z_0)}{z - z_0} \tag{17.11}$$

tends to a definite limit as $z \to z_0$ from any direction. It is the direction-independence of this limit that provides a proper meaning to the phrase "does not depend on $\overline{z}$." Since we are not allowing dependence on $\bar{z}$, it is natural to drop the partial derivative signs and write the limit as an ordinary derivative

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{df}{dz}. \tag{17.12}$$

We will also use Newton's fluxion notation

$$\frac{df}{dz} = f'(z). \tag{17.13}$$

The complex derivative obeys exactly the same calculus rules as ordinary real derivatives:

$$\begin{aligned}\frac{d}{dz} z^n &= n z^{n-1},\\ \frac{d}{dz} \sin z &= \cos z,\\ \frac{d}{dz}(fg) &= \frac{df}{dz} g + f \frac{dg}{dz}, \quad \textit{etc.}\end{aligned} \tag{17.14}$$

If the function is differentiable at all points in an arcwise-connected[1] open set, or *domain*, D, the function is said to be *analytic* there. The words *regular* or *holomorphic* are also used.

17.1.1 Conjugate pairs

The functions u and v comprising the real and imaginary parts of an analytic function are said to form a pair of *harmonic conjugate functions*. Such pairs have many properties that are useful for solving physical problems.

From the Cauchy-Riemann equations we deduce that

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u &= 0, \\ \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v &= 0. \end{aligned} \tag{17.15}$$

and so both the real and imaginary parts of $f(z)$ are automatically *harmonic* functions of x, y.

Further, from the Cauchy-Riemann conditions, we deduce that

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}=0. \tag{17.16}$$

This means that $\nabla u \cdot \nabla v = 0$. We conclude that, provided that neither of these gradients vanishes, the pair of curves $u = const.$ and $v = const.$ intersect at right angles. If we regard u as the potential ϕ solving some electrostatics problem $\nabla^2\phi = 0$, then the curves $v = const.$ are the associated field lines.

Another application is to fluid mechanics. If $\mathbf{v}$ is the velocity field of an irrotational ($\text{curl}\mathbf{v} = \mathbf{0}$) flow, then we can (perhaps only locally) write the flow field as a gradient

$$\begin{aligned} v_x &= \partial_x\phi, \\ v_y &= \partial_y\phi, \end{aligned} \tag{17.17}$$

where ϕ is a *velocity potential*. If the flow is incompressible ($\text{div}\mathbf{v} = 0$), then we can (locally) write it as a curl

$$\begin{aligned} v_x &= \partial_y\chi, \\ v_y &= -\partial_x\chi, \end{aligned} \tag{17.18}$$

[1] *Arcwise connected* means that any two points in D can be joined by a continuous path that lies wholely within D.

where χ is a *stream function*. The curves $\chi = const.$ are the flow streamlines. If the flow is both irrotational and incompressible, then we may use either ϕ or χ to represent the flow, and, since the two representations must agree, we have

$$\begin{aligned} \partial_x \phi &= +\partial_y \chi, \\ \partial_y \phi &= -\partial_x \chi. \end{aligned} \tag{17.19}$$

Thus ϕ and χ are harmonic conjugates, and so the complex combination $\Phi = \phi + i\chi$ is an analytic function called the *complex stream function*.

A conjugate v exists (at least locally) for any harmonic function u. To see why, assume first that we have a (u, v) pair obeying the Cauchy-Riemann equations. Then we can write

$$\begin{aligned} dv &= \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \\ &= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy. \end{aligned} \tag{17.20}$$

This observation suggests that if we are given a harmonic function u in some simply connected domain D, we can *define* a v by setting

$$v(z) = \int_{z_0}^{z} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + v(z_0), \tag{17.21}$$

for some real constant $v(z_0)$ and point z_0. The integral does not depend on choice of path from z_0 to z, and so $v(z)$ is well defined. The path independence comes about because the curl

$$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} \right) = -\nabla^2 u \tag{17.22}$$

vanishes, and because in a simply connected domain all paths connecting the same endpoints are homologous.

We now verify that this candidate $v(z)$ satisfies the Cauchy-Riemann realtions. The path independence, allows us to make our final approach to $z = x + iy$ along a straight line segment lying on either the x or y axis. If we approach along the x axis, we have

$$v(z) = \int^{x} \left(-\frac{\partial u}{\partial y} \right) dx' + \text{ rest of integral}, \tag{17.23}$$

and may use

$$\frac{d}{dx}\int^x f(x',y)\,dx' = f(x,y) \tag{17.24}$$

to see that

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{17.25}$$

at (x, y). If, instead, we approach along the y axis, we may similarly compute

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}. \tag{17.26}$$

Thus $v(z)$ does indeed obey the Cauchy-Riemann equations.

Because of the utility the harmonic conjugate it is worth giving a practical recipe for finding it, and so obtaining $f(z)$ when given only its real part $u(x, y)$. The method we give below is one we learned from John d'Angelo. It is more efficient than those given in most textbooks. We first observe that if f is a function of z only, then $\overline{f(z)}$ depends only on $\overline{z}$. We can therefore define a function $\overline{f}$ of $\overline{z}$ by setting $\overline{f(z)} = \overline{f}(\overline{z})$. Now

$$\frac{1}{2}\left(f(z) + \overline{f(z)}\right) = u(x,y). \tag{17.27}$$

Set

$$x = \frac{1}{2}(z+\overline{z}), \quad y = \frac{1}{2i}(z-\overline{z}), \tag{17.28}$$

so

$$u\left(\frac{1}{2}(z+\overline{z}), \frac{1}{2i}(z-\overline{z})\right) = \frac{1}{2}\left(f(z)+\overline{f}(\overline{z})\right). \tag{17.29}$$

Now set $\overline{z} = 0$, while keeping z fixed! Thus

$$f(z) + \overline{f}(0) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right). \tag{17.30}$$

The function f is not completely determined of course, because we can always add a constant to v, and so we have the result

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) + iC, \quad C \in \mathbb{R}. \tag{17.31}$$

For example, let $u = x^2 - y^2$. We find

$$f(z) + \overline{f}(0) = 2\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2i}\right)^2 = z^2, \tag{17.32}$$

or

$$f(z) = z^2 + iC, \quad C \in \mathbb{R}. \tag{17.33}$$

The business of setting setting $\overline{z} = 0$, while keeping z fixed, may feel like a dirty trick, but it can be justified by the (as yet to be proved) fact that f has a convergent expansion as a power series in $z = x + iy$. In this expansion it is meaningful to let x and y themselves be complex, and so allow z and $\overline{z}$ to become two independent complex variables. Anyway, you can always check *ex post facto* that your answer is correct.

17.1.2 Conformal mapping

An analytic function $w = f(z)$ maps subsets of its domain of definition in the "z" plane on to subsets in the "w" plane. These maps are often useful for solving problems in two dimensional electrostatics or fluid flow. Their simplest property is geometrical: such maps are *conformal*.

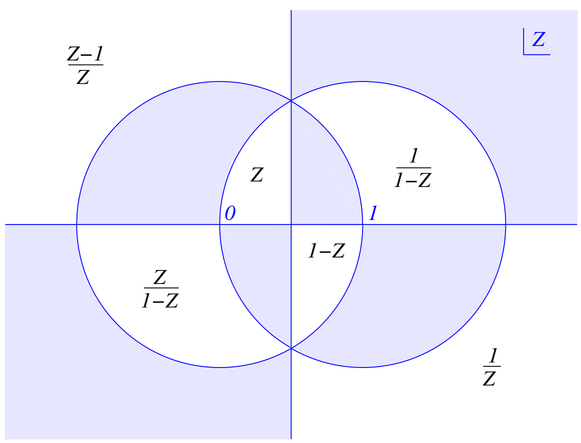

Figure 17.1: *An illustration of conformal mapping. The unshaded "triangle" marked z is mapped into the other five unshaded regions by the functions labeling them. Observe that although the regions are distorted, the angles of the "triangle" are preserved by the maps (with the exception of those corners that get mapped to infinity).*

Suppose that the derivative of $f(z)$ at a point z_0 is non-zero. Then, for z near z_0 we have

$$f(z) - f(z_0) \approx A(z - z_0), \tag{17.34}$$

where

$$A = \left.\frac{df}{dz}\right|_{z_0}. \tag{17.35}$$

If you think about the geometric interpretation of complex multiplication (multiply the magnitudes, add the arguments) you will see that the "f" image of a small neighbourhood of z_0 is stretched by a factor $|A|$, and rotated through an angle $\arg A$ — but relative angles are not altered. The map $z \mapsto f(z) = w$ is therefore *isogonal*. Our map also preserves orientation (the sense of rotation of the relative angle) and these two properties, isogonality and orientation-preservation, are what make the map conformal.[2] The conformal property fails at points where the derivative vanishes or becomes infinite.

If we can find a conformal map $z\ (\equiv x + iy) \mapsto w\ (\equiv u + iv)$ of some domain D to another D' then a function $f(z)$ that solves a potential theory problem (a Dirichlet boundary-value problem, for example) in D will lead to $f(z(w))$ solving an analogous problem in D'.

Consider, for example, the map $z \mapsto w = z + e^z$. This map takes the strip $-\infty < x < \infty$, $-\pi \le y \le \pi$ to the entire complex plane with cuts from $-\infty + i\pi$ to $-1 + i\pi$ and from $-\infty - i\pi$ to $-1 - i\pi$. The cuts occur because the images of the lines $y = \pm\pi$ get folded back on themselves at $w = -1 \pm i\pi$, where the derivative of $w(z)$ vanishes. (See figure 17.2)

In this case, the imaginary part of the function $f(z) = x + iy$ trivially solves the Dirichlet problem $\nabla^2_{x,y}\, y = 0$ in the infinite strip, with $y = \pi$ on the upper boundary and $y = -\pi$ on the lower boundary. The function $y(u, v)$, now quite non-trivially, solves $\nabla^2_{u,v}\, y = 0$ in the entire w plane, with $y = \pi$ on the half-line running from $-\infty + i\pi$ to $-1 + i\pi$, and $y = -\pi$ on the half-line running from $-\infty - i\pi$ to $-1 - i\pi$. We may regard the images of the lines $y = const.$ (solid curves) as being the streamlines of an irrotational and incompressible flow out of the end of a tube into an infinite region, or as the equipotentials near the edge of a pair of capacitor plates. In the latter case, the images of the lines $x = const.$ (dotted curves) are the corresponding field-lines

Example: The Joukowski map. This map is famous in the history of aeronautics because it can be used to map the exterior of a circle to the exterior of an aerofoil-shaped region. We can use the *Milne-Thomson circle theorem* (see 17.3.2) to find the streamlines for the flow past a circle in the z plane,

[2] If f were a function of $\overline{z}$ only, then the map would still be isogonal, but would reverse the orientation. We call such maps *antiholomorphic* or *anti-conformal*.

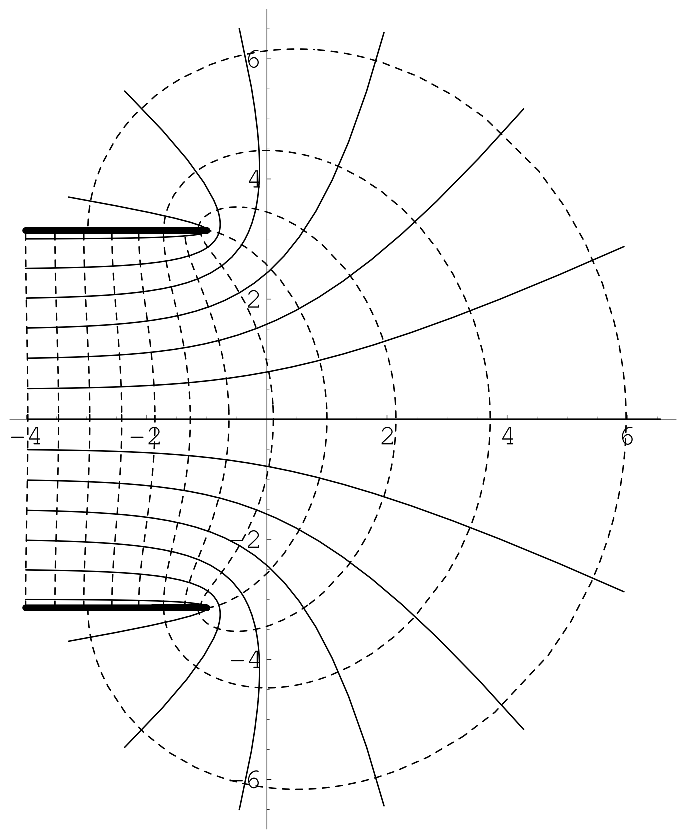

Figure 17.2: *Image of part of the strip* $-\pi \leq y \leq \pi$, $-\infty < x < \infty$ *under the map* $z \mapsto w = z + e^z$.

and then use Joukowski's transformation,

$$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right), \tag{17.36}$$

to map this simple flow to the flow past the aerofoil. To produce an aerofoil shape, the circle must go through the point $z = 1$, where the derivative of f vanishes, and the image of this point becomes the sharp trailing edge of the aerofoil.

The Riemann mapping theorem

There are tables of conformal maps for D, D' pairs, but an underlying principle is provided by the Riemann mapping theorem:
Theorem: The interior of any simply connected domain D in $\mathbb{C}$ whose boundary consists of more that one point can be mapped conformally one-to-one and onto the interior of the unit circle. It is possible to choose an arbitrary interior point w_0 of D and map it to the origin, and to take an arbitrary direction through w_0 and make it the direction of the real axis. With these two choices the mapping is unique.

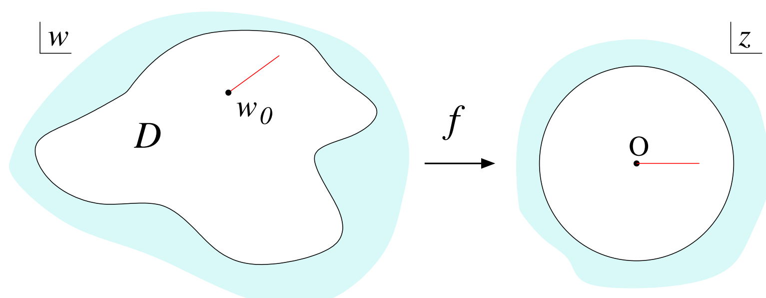

Figure 17.3: *The Riemann mapping theorem.*

This theorem was first stated in Riemann's PhD thesis in 1851. He regarded it as "obvious" for the reason that we will give as a physical "proof." Riemann's argument is not rigorous, however, and it was not until 1912 that a real proof was obtained by Constantin Carathéodory. A proof that is both shorter and more in spirit of Riemann's ideas was given by Leopold Fejér and Frigyes Riesz in 1922.

For the physical "proof," observe that in the function

$$-\frac{1}{2\pi}\ln z = -\frac{1}{2\pi}\left\{\ln|z| + i\theta\right\}, \tag{17.37}$$

the real part $\phi = -\frac{1}{2\pi}\ln|z|$ is the potential of a unit charge at the origin, and with the additive constant chosen so that $\phi = 0$ on the circle $|z| = 1$. Now imagine that we have solved the two-dimensional electrostatics problem of finding the potential for a unit charge located at $w_0 \in D$, also with the boundary of D being held at zero potential. We have

$$\nabla^2\phi_1 = -\delta^2(w - w_0), \quad \phi_1 = 0 \quad \text{on} \quad \partial D. \tag{17.38}$$

Now find the ϕ_2 that is harmonically conjugate to ϕ_1. Set

$$\phi_1 + i\phi_2 = \Phi(w) = -\frac{1}{2\pi}\ln(ze^{i\alpha}) \tag{17.39}$$

where α is a real constant. We see that the transformation $w \mapsto z$, or

$$z = e^{-i\alpha}e^{-2\pi\Phi(w)}, \tag{17.40}$$

does the job of mapping the interior of D into the interior of the unit circle, and the boundary of D to the boundary of the unit circle. Note how our freedom to choose the constant α is what allows us to "take an arbitrary direction through w_0 and make it the direction of the real axis."
Example: To find the map that takes the upper half-plane into the unit circle, with the point $z = i$ mapping to the origin, we use the method of images to solve for the complex potential of a unit charge at $w = i$:

$$\begin{aligned}\phi_1 + i\phi_2 &= -\frac{1}{2\pi}\left(\ln(w - i) - \ln(w + i)\right)\\ &= -\frac{1}{2\pi}\ln(e^{i\alpha}z).\end{aligned}$$

Therefore

$$z = e^{-i\alpha}\frac{w - i}{w + i}. \tag{17.41}$$

We immediately verify that that this works: we have $|z| = 1$ when w is real, and $z = 0$ at $w = i$.

The difficulty with the physical argument is that it is not clear that a solution to the point-charge electrostatics problem exists. In three dimensions, for example, there is no solution when the boundary has a sharp inward directed spike. (We cannot physically realize such a situation either: the electric field becomes unboundedly large near the tip of a spike, and boundary charge will leak off and neutralize the point charge.) There might well be analogous difficulties in two dimensions if the boundary of D is pathological. However, the fact that there *is* a proof of the Riemann mapping theorem shows that the two-dimensional electrostatics problem does always have a solution, at least in the *interior* of D — even if the boundary is an infinite-length fractal. However, unless ∂D is reasonably smooth the resulting Riemann map cannot be continuously extended to the boundary. When the boundary of D *is* a smooth closed curve, then the the boundary of D *will* map one-to-one and continuously onto the boundary of the unit circle.

Exercise 17.1: *Van der Pauw's Theorem.*[3] This problem explains a practical method of for determining the conductivity σ of a material, given a sample in the form of of a wafer of uniform thickness d, but of irregular shape. In practice at the Phillips company in Eindhoven, this was a wafer of semiconductor cut from an unmachined boule.

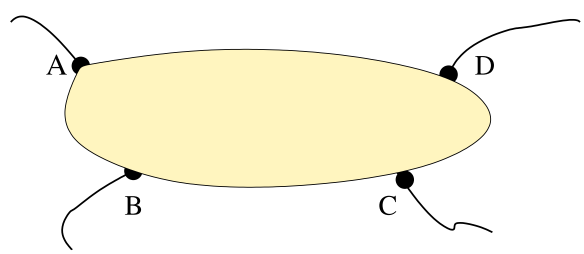

Figure 17.4: A thin semiconductor wafer with attached leads.

We attach leads to point contacts A, B, C, D, taken in anticlockwise order, on the periphery of the wafer and drive a current I_{AB} from A to B. We record the potential difference $V_D - V_C$ and so find $R_{AB,DC} = (V_D - V_C)/I_{AB}$. Similarly we measure $R_{BC,AD}$. The current flow in the wafer is assumed to be two dimensional, and to obey

$$\mathbf{J} = -(\sigma d)\nabla V, \qquad \nabla \cdot \mathbf{J} = 0,$$

[3]L. J. Van der Pauw, *Phillips Research Reps.* **13** (1958) 1. See also A. M. Thompson, D. G. Lampard, *Nature* **177** (1956) 888, and D. G. Lampard. *Proc. Inst. Elec. Eng. C.* **104** (1957) 271, for the "Calculable Capacitor."

and $\mathbf{n}\cdot\mathbf{J}=0$ at the boundary (except at the current source and drain). The potential V is therefore harmonic, with Neumann boundary conditions.

Van der Pauw claims that

$$\exp\{-\pi\sigma d R_{AB,DC}\}+\exp\{-\pi\sigma d R_{BC,AD}\}=1.$$

From this σd can be found numerically.

a) First show that Van der Pauw's claim is true if the wafer were the entire upper half-plane with A,B,C,D on the real axis with $x_A<x_B<x_C<x_D$.
b) Next, taking care to consider the transformation of the current source terms and the Neumann boundary conditions, show that the claim is invariant under conformal maps, and, by mapping the wafer to the upper half-plane, show that it is true in general.

17.2 Complex integration: Cauchy and Stokes

In this section we will define the integral of an analytic function, and make contact with the exterior calculus from chapters 11-13. The most obvious difference between the real and complex integral is that in evaluating the definite integral of a function in the complex plane we must specify the path along which we integrate. When this path of integration is the boundary of a region, it is often called a *contour* from the use of the word in the graphic arts to describe the outline of something. The integrals themselves are then called *contour integrals*.

17.2.1 The complex integral

The complex integral

$$\int_\Gamma f(z)dz \tag{17.42}$$

over a path Γ may be defined by expanding out the real and imaginary parts

$$\int_\Gamma f(z)dz \stackrel{\text{def}}{=} \int_\Gamma (u+iv)(dx+idy)=\int_\Gamma (udx-vdy)+i\int_\Gamma (vdx+udy). \tag{17.43}$$

and treating the two integrals on the right hand side as standard vector-calculus line-integrals of the form $\int \mathbf{v}\cdot d\mathbf{r}$, one with $\mathbf{v}\to(u,-v)$ and and one with $\mathbf{v}\to(v,u)$.

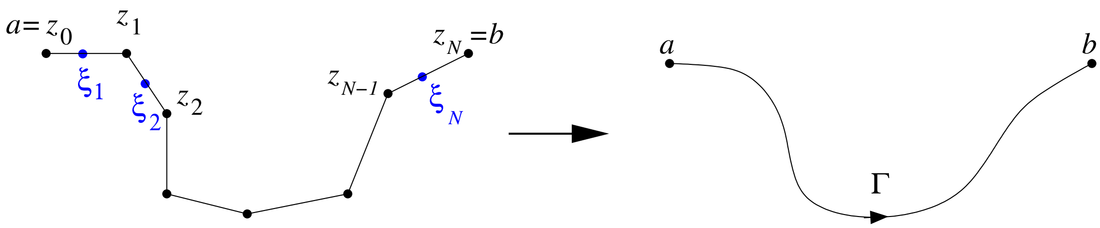

Figure 17.5: *A chain approximation to the curve* Γ*.*

The complex integral can also be constructed as the limit of a Riemann sum in a manner parallel to the definition of the real-variable Riemann integral of elementary calculus. Replace the path Γ with a chain composed of of N line-segments z_0-to-z_1, z_1-to-z_2, all the way to z_{N-1}-to-z_N. Now let ξ_m lie on the line segment joining z_{m-1} and z_m. Then the integral $\int_\Gamma f(z)dz$ is the limit of the (Riemann) sum

$$S = \sum_{m=1}^{N} f(\xi_m)(z_m - z_{m-1}) \tag{17.44}$$

as N gets large and all the $|z_m - z_{m-1}| \to 0$. For this definition to make sense and be useful, the limit must be independent of both how we chop up the curve and how we select the points ξ_m. This will be the case when the integration path is smooth and the function being integrated is continuous.

The Riemann-sum definition of the integral leads to a useful inequality: combining the triangle inequality $|a + b| \leq |a| + |b|$ with $|ab| = |a|\,|b|$ we deduce that

$$\begin{aligned}\left|\sum_{m=1}^{N} f(\xi_m)(z_m - z_{m-1})\right| &\leq \sum_{m=1}^{N} |f(\xi_m)(z_m - z_{m-1})| \\ &= \sum_{m=1}^{N} |f(\xi_m)|\,|(z_m - z_{m-1})|. \end{aligned} \tag{17.45}$$

For sufficiently smooth curves the last sum converges to the real integral $\int_\Gamma |f(z)|\,|dz|$, and we deduce that

$$\left|\int_\Gamma f(z)\,dz\right| \leq \int_\Gamma |f(z)|\,|dz|. \tag{17.46}$$

For curves Γ that are smooth enough to have a well-defined length $|\Gamma|$, we will have $\int_\Gamma |dz| = |\Gamma|$. From this identification we conclude that if $|f| \le M$ on Γ, then we have the *Darboux inequality*

$$\left| \int_\Gamma f(z)\, dz \right| \le M|\Gamma|. \tag{17.47}$$

We shall find many uses for this inequality.

The Riemann sum definition also makes it clear that if $f(z)$ is the derivative of another analytic function $g(z)$, *i.e.*

$$f(z) = \frac{dg}{dz}, \tag{17.48}$$

then, for Γ a smooth path from $z = a$ to $z = b$, we have

$$\int_\Gamma f(z) dz = g(b) - g(a). \tag{17.49}$$

This claim is established by approximating $f(\xi_m) \approx (g(z_m) - g(z_{m-1}))/(z_m - z_{m-1})$, and observing that the resulting Riemann sum

$$\sum_{m=1}^{N} \Big(g(z_m) - g(z_{m-1}) \Big) \tag{17.50}$$

telescopes. The approximation to the derivative will become accurate in the limit $|z_m - z_{m-1}| \to 0$. Thus, when $f(z)$ is the derivative of another function, the integral is independent of the route that Γ takes from a to b.

We shall see that any analytic function is (at least locally) the derivative of another analytic function, and so this path independence holds generally — provided that we do not try to move the integration contour over a place where f ceases to be differentiable. This is the essence of what is known as *Cauchy's Theorem* — although, as with much of complex analysis, the result was known to Gauss.

17.2.2 Cauchy's theorem

Before we state and prove Cauchy's theorem, we must introduce an orientation convention and some traditional notation. Recall that a p-chain is a finite formal sum of p-dimensional oriented surfaces or curves, and that a

p-cycle is a p-chain Γ whose boundary vanishes: $\partial\Gamma = 0$. A 1-cycle that consists of only a single connected component is a closed curve. We will mostly consider integrals over *simple* closed curves — these being curves that do not self intersect — or 1-cycles consisting of finite formal sums of such curves. The orientation of a simple closed curve can be described by the sense, clockwise or anticlockwise, in which we traverse it. We will adopt the convention that a positively oriented curve is one such that the integration is performed in a *anticlockwise* direction. The integral over a chain Γ of oriented simple closed curves will be denoted by the symbol $\oint_\Gamma f\,dz$.

We now establish Cauchy's theorem by relating it to our previous work with exterior derivatives: Suppose that f is analytic with domain D, so that $\partial_{\overline{z}} f = 0$ within D. We therefore have that the exterior derivative of f is

$$df = \partial_z f\, dz + \partial_{\overline{z}} f\, d\overline{z} = \partial_z f\, dz. \tag{17.51}$$

Now suppose that the simple closed curve Γ is the boundary of a region $\Omega \subset D$. We can exploit Stokes' theorem to deduce that

$$\oint_{\Gamma=\partial\Omega} f(z)dz = \int_\Omega d(f(z)dz) = \int_\Omega (\partial_z f)\, dz \wedge dz = 0. \tag{17.52}$$

The last integral is zero because $dz \wedge dz = 0$. We may state our result as: *Theorem (Cauchy, in modern language): The integral of an analytic function over a 1-cycle that is homologous to zero vanishes.*

The zero result is only guaranteed if the function f is analytic throughout the region Ω. For example, if Γ is the unit circle $z = e^{i\theta}$ then

$$\oint_\Gamma \left(\frac{1}{z}\right) dz = \int_0^{2\pi} e^{-i\theta}\, d\left(e^{i\theta}\right) = i\int_0^{2\pi} d\theta = 2\pi i. \tag{17.53}$$

Cauchy's theorem is not applicable because $1/z$ is *singular*, *i.e.* not differentiable, at $z = 0$. The formula (17.53) will hold for Γ any contour homologous to the unit circle in $\mathbb{C} \setminus 0$, the complex plane punctured by the removal of the point $z = 0$. Thus

$$\oint_\Gamma \left(\frac{1}{z}\right) dz = 2\pi i \tag{17.54}$$

for any contour Γ that encloses the origin. We can deduce a rather remarkable formula from (17.54): Writing $\Gamma = \partial\Omega$ with anticlockwise orientation, we use Stokes' theorem to obtain

$$\oint_{\partial\Omega} \left(\frac{1}{z}\right) dz = \int_\Omega \partial_{\overline{z}} \left(\frac{1}{z}\right) d\overline{z} \wedge dz = \begin{cases} 2\pi i, & 0 \in \Omega, \\ 0, & 0 \notin \Omega. \end{cases} \tag{17.55}$$

Since $d\overline{z} \wedge dz = 2i dx \wedge dy$, we have established that

$$\partial_{\overline{z}} \left(\frac{1}{z} \right) = \pi \delta(x) \delta(y). \tag{17.56}$$

This rather cryptic formula encodes one of the most useful results in mathematics.

Perhaps perversely, functions that are more singular than $1/z$ have vanishing integrals about their singularities. With Γ again the unit circle, we have

$$\oint_\Gamma \left(\frac{1}{z^2} \right) dz = \int_0^{2\pi} e^{-2i\theta} \, d\left(e^{i\theta}\right) = i \int_0^{2\pi} e^{-i\theta} \, d\theta = 0. \tag{17.57}$$

The same is true for all higher integer powers:

$$\oint_\Gamma \left(\frac{1}{z^n} \right) dz = 0, \quad n \geq 2. \tag{17.58}$$

We can understand this vanishing in another way, by evaluating the integral as

$$\oint_\Gamma \left(\frac{1}{z^n} \right) dz = \oint_\Gamma \frac{d}{dz} \left(-\frac{1}{n-1} \frac{1}{z^{n-1}} \right) dz = \left[-\frac{1}{n-1} \frac{1}{z^{n-1}} \right]_\Gamma = 0, \quad n \neq 1. \tag{17.59}$$

Here, the notation $[A]_\Gamma$ means the difference in the value of A at two ends of the integration path Γ. For a closed curve the difference is zero because the two ends are at the same point. This approach reinforces the fact that the complex integral can be computed from the "anti-derivative" in the same way as the real-variable integral. We also see why $1/z$ is special. It is the derivative of $\ln z = \ln |z| + i \arg z$, and $\ln z$ is not really a function, as it is multivalued. In evaluating $[\ln z]_\Gamma$ we must follow the continuous evolution of $\arg z$ as we traverse the contour. As the origin is within the contour, this angle increases by 2π, and so

$$[\ln z]_\Gamma = [i \arg z]_\Gamma = i \left(\arg e^{2\pi i} - \arg e^{0i} \right) = 2\pi i. \tag{17.60}$$

Exercise 17.2: Suppose $f(z)$ is analytic in a simply-connected domain D, and $z_0 \in D$. Set $g(z) = \int_{z_0}^z f(z)\, dz$ along some path in D from z_0 to z. Use the path-independence of the integral to compute the derivative of $g(z)$ and show that

$$f(z) = \frac{dg}{dz}.$$

This confirms our earlier claim that any analytic function is the derivative of some other analytic function.

Exercise 17.3:The "D-bar" problem: Suppose we are given a simply-connected domain Ω, and a function $f(z,\overline{z})$ defined on it, and wish to find a function $F(z,\overline{z})$ such that

$$\frac{\partial F(z,\overline{z})}{\partial \overline{z}} = f(z,\overline{z}), \quad (z,\overline{z}) \in \Omega.$$

Use (17.56) to argue formally that the general solution is

$$F(\zeta,\bar{\zeta}) = -\frac{1}{\pi}\int_\Omega \frac{f(z,\overline{z})}{z-\zeta} dx \wedge dy + g(\zeta),$$

where $g(\zeta)$ is an arbitrary analytic function. This result can be shown to be correct by more rigorous reasoning.

17.2.3 The residue theorem

The essential tool for computations with complex integrals is provided by the *residue theorem*. With the aid of this theorem, the evaluation of contour integrals becomes easy. All one has to do is identify points at which the function being integrated blows up, and examine just how it blows up.

If, near the point z_i, the function can be written

$$f(z) = \left\{ \frac{a_N^{(i)}}{(z-z_i)^N} + \cdots + \frac{a_2^{(i)}}{(z-z_i)^2} + \frac{a_1^{(i)}}{(z-z_i)} \right\} g^{(i)}(z), \tag{17.61}$$

where $g^{(i)}(z)$ is analytic and non-zero at z_i, then $f(z)$ has a *pole* of order N at z_i. If $N=1$ then $f(z)$ is said to have a *simple pole* at z_i. We can normalize $g^{(i)}(z)$ so that $g^{(i)}(z_i)=1$, and then the coefficient, $a_1^{(i)}$, of $1/(z-z_i)$ is called the *residue* of the pole at z_i. The coefficients of the more singular terms do not influence the result of the integral, but N must be finite for the singularity to be called a pole.

Theorem: Let the function $f(z)$ be analytic within and on the boundary $\Gamma = \partial D$ of a simply connected domain D, with the exception of finite number of points at which $f(z)$ has poles. Then

$$\oint_\Gamma f(z)\,dz = \sum_{\text{poles} \in D} 2\pi i\,(\text{residue at pole}), \tag{17.62}$$

the integral being traversed in the positive (anticlockwise) sense.

We prove the residue theorem by drawing small circles C_i about each singular point z_i in D.

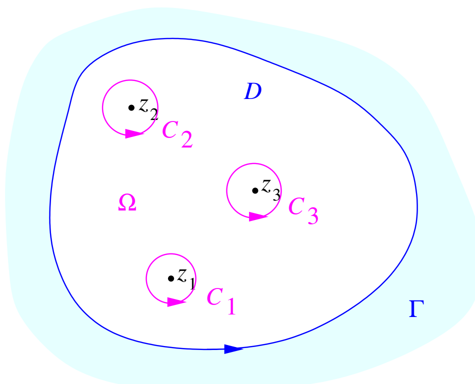

Figure 17.6: *Circles for the residue theorem.*

We now assert that

$$\oint_\Gamma f(z)\,dz = \sum_i \oint_{C_i} f(z)\,dz, \tag{17.63}$$

because the 1-cycle

$$C \equiv \Gamma - \sum_i C_i = \partial\Omega \tag{17.64}$$

is the boundary of a region Ω in which f is analytic, and hence C is homologous to zero. If we make the radius R_i of the circle C_i sufficiently small, we may replace each $g^{(i)}(z)$ by its limit $g^{(i)}(z_i) = 1$, and so take

$$\begin{aligned} f(z) \quad &\to \quad \left\{\frac{a_1^{(i)}}{(z-z_i)} + \frac{a_2^{(i)}}{(z-z_i)^2} + \cdots + \frac{a_N^{(i)}}{(z-z_i)^N}\right\} g^{(i)}(z_i) \\ &= \quad \frac{a_1^{(i)}}{(z-z_i)} + \frac{a_2^{(i)}}{(z-z_i)^2} + \cdots + \frac{a_N^{(i)}}{(z-z_i)^N}, \end{aligned} \tag{17.65}$$

on C_i. We then evaluate the integral over C_i by using our previous results to get

$$\oint_{C_i} f(z)\,dz = 2\pi i a_1^{(i)}. \tag{17.66}$$

The integral around Γ is therefore equal to $2\pi i\sum_i a_1^{(i)}$.

The restriction to contours containing only finitely many poles arises for two reasons: Firstly, with infinitely many poles, the sum over i might not converge; secondly, there may be a point whose every neighbourhood contains infinitely many of the poles, and there our construction of drawing circles around each individual pole would not be possible.

Exercise 17.4: *Poisson's Formula.* The function $f(z)$ is analytic in $|z| < R'$. Prove that if $|a| < R < R'$,

$$f(a) = \frac{1}{2\pi i}\oint_{|z|=R} \frac{R^2 - \bar{a}a}{(z-a)(R^2 - \bar{a}z)} f(z) dz.$$

Deduce that, for $0 < r < R$,

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \phi) + r^2} f(Re^{i\phi}) d\phi.$$

Show that this formula solves the boundary-value problem for Laplace's equation in the disc $|z| < R$.

Exercise 17.5: *Bergman Kernel.* The Hilbert space of analytic functions on a domain D with inner product

$$\langle f, g\rangle = \int_D \bar{f} g \, dxdy$$

is called the Bergman[4] space of D.

a) Suppose that $\varphi_n(z)$, $n = 0, 1, 2, \ldots$, are a complete set of orthonormal functions on the Bergman space. Show that

$$K(\zeta, z) = \sum_{m=0}^{\infty} \varphi_m(\zeta)\overline{\varphi_m(z)}.$$

has the property that

$$g(\zeta) = \iint_D K(\zeta, z) g(z) \, dxdy.$$

[4]This space should not be confused with Bargmann-Fock space which is the space analytic functions on the entirety of $\mathbb{C}$ with inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} e^{-|z|^2} \bar{f} g \, d^2z.$$

Stefan Bergman and Valentine Bargmann are two different people.

for any function g analytic in D. Thus $K(\zeta, z)$ plays the role of the delta function on the space of analytic functions on D. This object is called the *reproducing* or *Bergman kernel*. By taking $g(z) = \varphi_n(z)$, show that it is the unique integral kernel with the reproducing property.

b) Consider the case of D being the unit circle. Use the Gramm-Schmidt procedure to construct an orthonormal set from the functions z^n, $n = 0, 1, 2, \ldots$. Use the result of part a) to conjecture (because we have not proved that the set is complete) that, for the unit circle,

$$K(\zeta, z) = \frac{1}{\pi}\frac{1}{(1-\zeta\overline{z})^2}.$$

c) For any smooth, complex valued, function g defined on a domain D and its boundary, use Stokes' theorem to show that

$$\iint_D \partial_{\overline{z}} g(z, \overline{z}) dxdy = \frac{1}{2i}\oint_{\partial D} g(z, \overline{z}) dz.$$

Use this to verify that this the $K(\zeta, z)$ you constructed in part b) is indeed a (and hence "the") reproducing kernel.

d) Now suppose that D is a simply connected domain whose boundary ∂D is a smooth curve. We know from the Riemann mapping theorem that there exists an analytic function $f(z) = f(z;\zeta)$ that maps D onto the interior of the unit circle in such a way that $f(\zeta) = 0$ and $f'(\zeta)$ is real and non-zero. Show that if we set $K(\zeta, z) = \overline{f'(z)}f'(\zeta)/\pi$, then, by using part c) together with the residue theorem to evaluate the integral over the boundary, we have

$$g(\zeta) = \iint_D K(\zeta, z) g(z)\, dxdy.$$

This $K(\zeta, z)$ must therefore be the reproducing kernel. We see that if we know K we can recover the map f from

$$f'(z;\zeta) = \sqrt{\frac{\pi}{K(\zeta,\zeta)}}K(z,\zeta).$$

e) Apply the formula from part d) to the unit circle, and so deduce that

$$f(z;\zeta) = \frac{z-\zeta}{1-\bar{\zeta}z}$$

is the unique function that maps the unit circle onto itself with the point ζ mapping to the origin and with the horizontal direction through ζ remaining horizontal.

17.3 Applications

We now know enough about complex variables to work through some interesting applications, including the mechanism by which an aeroplane flies.

17.3.1 Two-dimensional vector calculus

It is often convenient to use complex co-ordinates for vectors and tensors. In these co-ordinates the standard metric on $\mathbb{R}^2$ becomes

$$\begin{aligned} \text{“}ds^2\text{”} &= dx \otimes dx + dy \otimes dy \\ &= d\overline{z} \otimes dz \\ &= g_{zz} dz \otimes dz + g_{\overline{z}z} d\overline{z} \otimes dz + g_{z\overline{z}} dz \otimes d\overline{z} + g_{\overline{z}\overline{z}} d\overline{z} \otimes d\overline{z}, \end{aligned} \tag{17.67}$$

so the complex co-ordinate components of the metric tensor are $g_{zz} = g_{\overline{z}\overline{z}} = 0$, $g_{z\overline{z}} = g_{\overline{z}z} = \frac{1}{2}$. The inverse metric tensor is $g^{z\overline{z}} = g^{\overline{z}z} = 2$, $g^{zz} = g^{\overline{z}\overline{z}} = 0$.

In these co-ordinates the Laplacian is

$$\nabla^2 = g^{ij}\partial^2_{ij} = 2(\partial_z \partial_{\overline{z}} + \partial_{\overline{z}} \partial_z). \tag{17.68}$$

When f has singularities, it is not safe to assume that $\partial_z \partial_{\overline{z}} f = \partial_{\overline{z}} \partial_z f$. For example, from

$$\partial_{\overline{z}} \left(\frac{1}{z} \right) = \pi \delta^2(x, y), \tag{17.69}$$

we deduce that

$$\partial_{\overline{z}} \partial_z \ln z = \pi \delta^2(x, y). \tag{17.70}$$

When we evaluate the derivatives in the opposite order, however, we have

$$\partial_z \partial_{\overline{z}} \ln z = 0. \tag{17.71}$$

To understand the source of the non-commutativity, take real and imaginary parts of these last two equations. Write $\ln z = \ln|z| + i\theta$, where $\theta = \arg z$, and add and subtract. We find

$$\begin{aligned} \nabla^2 \ln|z| &= 2\pi \delta^2(x, y), \\ (\partial_x \partial_y - \partial_y \partial_x)\theta &= 2\pi \delta^2(x, y). \end{aligned} \tag{17.72}$$

The first of these shows that $\frac{1}{2\pi} \ln|z|$ is the Green function for the Laplace operator, and the second reveals that the vector field $\nabla\theta$ is singular, having a delta function “curl” at the origin.

If we have a vector field $\mathbf{v}$ with contravariant components (v^x, v^y) and (numerically equal) covariant components (v_x, v_y) then the covariant components in the complex co-ordinate system are $v_z = \frac{1}{2}(v_x - iv_y)$ and $v_{\overline{z}} = \frac{1}{2}(v_x + iv_y)$. This can be obtained by a using the change of co-ordinates rule, but a quicker route is to observe that

$$\mathbf{v} \cdot d\mathbf{r} = v_x dx + v_y dy = v_z dz + v_{\overline{z}} d\overline{z}. \tag{17.73}$$

Now

$$\partial_{\overline{z}} v_z = \frac{1}{4}(\partial_x v_x + \partial_y v_y) + i\frac{1}{4}(\partial_y v_x - \partial_x v_y). \tag{17.74}$$

Thus the statement that $\partial_{\overline{z}} v_z = 0$ is equivalent to the vector field $\mathbf{v}$ being both solenoidal (incompressible) and irrotational. This can also be expressed in form language by setting $\eta = v_z\, dz$ and saying that $d\eta = 0$ means that the corresponding vector field is both solenoidal and irrotational.

17.3.2 Milne-Thomson circle theorem

As we mentioned earlier, we can describe an irrotational and incompressible fluid motion either by a velocity potential

$$v_x = \partial_x \phi, \quad v_y = \partial_y \phi, \tag{17.75}$$

where $\mathbf{v}$ is automatically irrotational but incompressibilty requires $\nabla^2\phi = 0$, or by a stream function

$$v_x = \partial_y \chi, \quad v_y = -\partial_x \chi, \tag{17.76}$$

where $\mathbf{v}$ is automatically incompressible but irrotationality requires $\nabla^2\chi = 0$. We can combine these into a single *complex stream function* $\Phi = \phi + i\chi$ which, for an irrotational incompressible flow, satisfies the Cauchy-Riemann equations and is therefore an analytic function of z. We see that

$$2v_z = \frac{d\Phi}{dz}, \tag{17.77}$$

ϕ and χ making equal contributions.

The Milne-Thomson theorem says that if Φ is the complex stream function for a flow in unobstructed space, then

$$\widetilde{\Phi} = \Phi(z) + \overline{\Phi}\left(\frac{a^2}{z}\right) \tag{17.78}$$

is the stream function after the cylindrical obstacle $|z| = a$ is inserted into the flow. Here $\overline{\Phi}(z)$ denotes the analytic function defined by $\overline{\Phi}(z) = \overline{\Phi(\overline{z})}$. To see that this works, observe that $a^2/z = \overline{z}$ on the curve $|z| = a$, and so on this curve $\mathrm{Im}\,\widetilde{\Phi} = \chi = 0$. The surface of the cylinder has therefore become a streamline, and so the flow does not penetrate into the cylinder. If the original flow is created by souces and sinks exterior to $|z| = a$, which will be singularities of Φ, the additional term has singularites that lie only within $|z| = a$. These will be the "images" of the sources and sinks in the sense of the "method of images."

Example: A uniform flow with speed U in the x direction has $\Phi(z) = Uz$. Inserting a cylinder makes this

$$\tilde{\Phi}(z) = U\left(z + \frac{a^2}{z}\right). \tag{17.79}$$

Because v_z is the derivative of this, we see that the perturbing effect of the obstacle on the velocity field falls off as the square of the distance from the cylinder. This is a general result for obstructed flows.

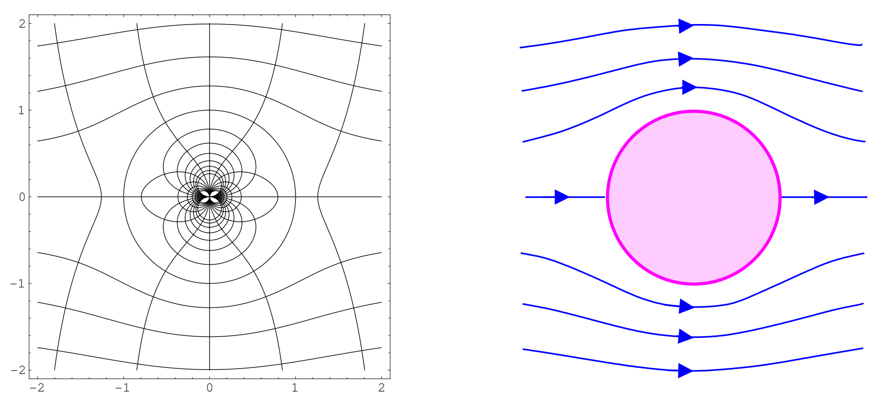

Figure 17.7: *The real and imaginary parts of the function* $z + z^{-1}$ *provide the velocity potentials and streamlines for irrotational incompressible flow past a cylinder of unit radius.*

17.3.3 Blasius and Kutta-Joukowski theorems

We now derive the celebrated result, discovered independently by Martin Wilhelm Kutta (1902) and Nikolai Egorovich Joukowski (1906), that the lift per unit span of an aircraft wing is equal to the product of the density of the air ρ, the circulation $\kappa \equiv \oint \mathbf{v} \cdot d\mathbf{r}$ about the wing, and the forward velocity U of the wing through the air. Their theory treats the air as being incompressible—a good approximation unless the flow-velocities approach the speed of sound—and assumes that the wing is long enough that the flow can be regarded as being two dimensional.

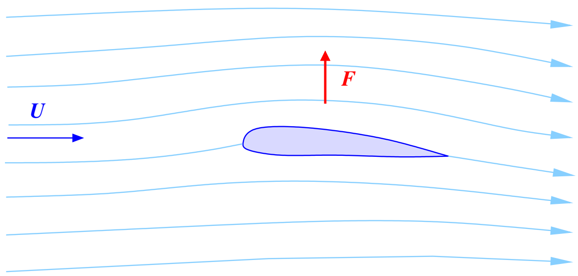

Figure 17.8: *Flow past an aerofoil.*

Begin by recalling how the momentum flux tensor

$$T_{ij} = \rho v_i v_j + g_{ij} P \tag{17.80}$$

enters fluid mechanics. In cartesian co-ordinates, and in the presence of an external body force f_i acting on the fluid, the Euler equation of motion for the fluid is

$$\rho(\partial_t v_i + v^j \partial_j v_i) = -\partial_i P + f_i. \tag{17.81}$$

Here P is the pressure and we are distinguishing between co and contravariant components, although at the moment $g_{ij} \equiv \delta_{ij}$. We can combine Euler's equation with the law of mass conservation,

$$\partial_t \rho + \partial^i (\rho v_i) = 0, \tag{17.82}$$

to obtain

$$\partial_t (\rho v_i) + \partial^j (\rho v_j v_i + g_{ij} P) = f_i. \tag{17.83}$$

This momemtum-tracking equation shows that the external force acts as a source of momentum, and that for steady flow f_i is equal to the divergence of the momentum flux tensor:

$$f_i = \partial^l T_{li} = g^{kl} \partial_k T_{li}. \tag{17.84}$$

As we are interested in steady, irrotational motion with uniform density we may use Bernoulli's theorem, $P + \frac{1}{2}\rho|v|^2 = const.$, to substitute $-\frac{1}{2}\rho|v|^2$ in place of P. (The constant will not affect the momentum flux.) With this substitution T_{ij} becomes a traceless symmetric tensor:

$$T_{ij} = \rho(v_i v_j - \frac{1}{2}g_{ij}|v|^2). \tag{17.85}$$

Using $v_z = \frac{1}{2}(v_x - iv_y)$ and

$$T_{zz} = \frac{\partial x^i}{\partial z}\frac{\partial x^j}{\partial z}T_{ij}, \tag{17.86}$$

together with

$$x \equiv x^1 = \frac{1}{2}(z + \overline{z}), \qquad y \equiv x^2 = \frac{1}{2i}(z - \overline{z}) \tag{17.87}$$

we find

$$T \equiv T_{zz} = \frac{1}{4}(T_{xx} - T_{yy} - 2iT_{xy}) = \rho(v_z)^2. \tag{17.88}$$

This is the only component of T_{ij} that we will need to consider. $T_{\overline{z}\overline{z}}$ is simply $\overline{T}$, whereas $T_{z\overline{z}} = 0 = T_{\overline{z}z}$ because T_{ij} is traceless.

In our complex co-ordinates, the equation

$$f_i = g^{kl}\partial_k T_{li} \tag{17.89}$$

reads

$$f_z = g^{\overline{z}z}\partial_{\overline{z}}T_{zz} + g^{z\overline{z}}\partial_z T_{\overline{z}z} = 2\partial_{\overline{z}}T. \tag{17.90}$$

We see that in steady flow the net momentum flux $\dot{P}_i$ out of a region Ω is given by

$$\dot{P}_z = \int_\Omega f_z\,dxdy = \frac{1}{2i}\int_\Omega f_z\,d\overline{z}dz = \frac{1}{i}\int_\Omega \partial_{\overline{z}}T\,d\overline{z}dz = \frac{1}{i}\oint_{\partial\Omega} T\,dz. \tag{17.91}$$

We have used Stokes' theorem at the last step. In regions where there is no external force, T is analytic, $\partial_{\overline{z}}T = 0$, and the integral will be independent of the choice of contour $\partial\Omega$. We can subsititute $T = \rho v_z^2$ to get

$$\dot{P}_z = -i\rho\oint_{\partial\Omega} v_z^2\,dz. \tag{17.92}$$

To apply this result to our aerofoil we take can take $\partial\Omega$ to be its boundary. Then $\dot{P}_z$ is the total force exerted on the fluid by the wing, and, by Newton's third law, this is minus the force exerted by the fluid on the wing. The total force on the aerofoil is therefore

$$F_z = i\rho \oint_{\partial\Omega} v_z^2 \, dz. \tag{17.93}$$

The result (17.93) is often called *Blasius' theorem*.

Evaluating the integral in (17.93) is not immediately possible because the velocity $\mathbf{v}$ on the boundary will be a complicated function of the shape of the body. We can, however, exploit the contour independence of the integral and evaluate it over a path encircling the aerofoil at large distance where the flow field takes the asymptotic form

$$v_z = U_z + \frac{\kappa}{4\pi i}\frac{1}{z} + O\left(\frac{1}{z^2}\right). \tag{17.94}$$

The $O(1/z^2)$ term is the velocity perturbation due to the air having to flow round the wing, as with the cylinder in a free flow. To confirm that this flow has the correct circulation we compute

$$\oint \mathbf{v}\cdot d\mathbf{r} = \oint v_z dz + \oint v_{\overline{z}}\, d\overline{z} = \kappa. \tag{17.95}$$

Substituting v_z in (17.93) we find that the $O(1/z^2)$ term cannot contribute as it cannot affect the residue of any pole. The only part that does contribute is the cross term that arises from multiplying U_z by $\kappa/(4\pi i z)$. This gives

$$F_z = i\rho\left(\frac{U_z\kappa}{2\pi i}\right)\oint \frac{dz}{z} = i\rho\kappa U_z \tag{17.96}$$

so that

$$\frac{1}{2}(F_x - iF_y) = i\rho\kappa\frac{1}{2}(U_x - iU_y). \tag{17.97}$$

Thus, in conventional co-ordinates, the reaction force on the body is

$$\begin{aligned} F_x &= \rho\kappa U_y, \\ F_y &= -\rho\kappa U_x. \end{aligned} \tag{17.98}$$

The fluid therefore provides a lift force proportional to the product of the circulation with the asymptotic velocity. The force is at right angles to the incident airstream, so there is no *drag*.

The circulation around the wing is determined by the *Kutta condition* that the velocity of the flow at the sharp trailing edge of the wing be finite. If the wing starts moving into the air and the requisite circulation is not yet established then the flow under the wing does not leave the trailing edge smoothly but tries to whip round to the topside. The velocity gradients become very large and viscous forces become important and prevent the air from making the sharp turn. Instead, a *starting vortex* is shed from the trailing edge. Kelvin's theorem on the conservation of vorticity shows that this causes a circulation of equal and opposite strength to be induced about the wing.

For finite wings, the path independence of $\oint \mathbf{v}\cdot d\mathbf{r}$ means that the wings must leave a pair of trailing *wingtip vortices* of strength κ that connect back to the starting vortex to form a closed loop. The velocity field induced by the trailing vortices cause the airstream incident on the aerofoil to come from a slighly different direction than the asymptotic flow. Consequently, the lift is not quite perpendicular to the motion of the wing. For finite-length wings, therefore, lift comes at the expense of an inevitable *induced drag* force. The work that has to be done against this drag force in driving the wing forwards provides the kinetic energy in the trailing vortices.

17.4 Applications of Cauchy's theorem

Cauchy's theorem provides the Royal Road to complex analysis. It is possible to develop the theory without it, but the path is harder going.

17.4.1 Cauchy's integral formula

If $f(z)$ is analytic within and on the boundary of a simply connected domain Ω, with $\partial\Omega = \Gamma$, and if ζ is a point in Ω, then, noting that the the integrand has a simple pole at $z=\zeta$ and applying the residue formula, we have *Cauchy's integral formula*

$$f(\zeta) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{z-\zeta}\,dz, \quad \zeta\in\Omega. \tag{17.99}$$

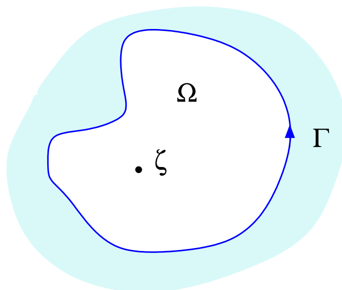

Figure 17.9: *Cauchy contour.*

This formula holds only if ζ lies within Ω. If it lies outside, then the integrand is analytic everywhere inside Ω, and so the integral gives zero.

We may show that it is legitimate to differentiate under the integral sign in Cauchy's formula. If we do so n times, we have the useful corollary that

$$f^{(n)}(\zeta) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-\zeta)^{n+1}}\, dz. \tag{17.100}$$

This shows that being *once* differentiable (analytic) in a region automatically implies that $f(z)$ is differentiable *arbitrarily many times*!

Exercise 17.6: *The generalized Cauchy formula.* Suppose that we have solved a D-bar problem (see exercise 17.3), and so found an $F(z,\overline{z})$ with $\partial_{\overline{z}} F = f(z,\overline{z})$ in a region Ω. Compute the exterior derivative of

$$\frac{F(z,\overline{z})}{z-\zeta}$$

using (17.56). Now, manipulating formally with delta functions, apply Stokes' theorem to show that, for $(\zeta,\bar{\zeta})$ in the interior of Ω, we have

$$F(\zeta,\bar{\zeta}) = \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{F(z,\overline{z})}{z-\zeta}\, dz - \frac{1}{\pi}\int_\Omega \frac{f(z,\overline{z})}{z-\zeta} dx\, dy.$$

This is called the *generalized Cauchy formula*. Note that the first term on the right, unlike the second, is a function only of ζ, and so is analytic.

Liouville's theorem

A dramatic corollary of Cauchy's integral formula is provided by

Liouville's theorem: *If $f(z)$ is analytic in all of $\mathbb{C}$, and is bounded there, meaning that there is a positive real number K such that $|f(z)| < K$, then $f(z)$ is a constant.*

This result provides a powerful strategy for proving that two formulæ, $f_1(z)$ and $f_2(z)$, represent the same analytic function. If we can show that the difference $f_1 - f_2$ is analytic and tends to zero at infinity then Liouville's theorem tells us that $f_1 = f_2$.

Because the result is perhaps unintuitive, and because the methods are typical, we will spell out in detail how Liouville's theorem works. We select any two points, z_1 and z_2, and use Cauchy's formula to write

$$f(z_1) - f(z_2) = \frac{1}{2\pi i}\oint_\Gamma \left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right) f(z)\, dz. \tag{17.101}$$

We take the contour Γ to be circle of radius ρ centered on z_1. We make $\rho > 2|z_1 - z_2|$, so that when z is on Γ we are sure that $|z - z_2| > \rho/2$.

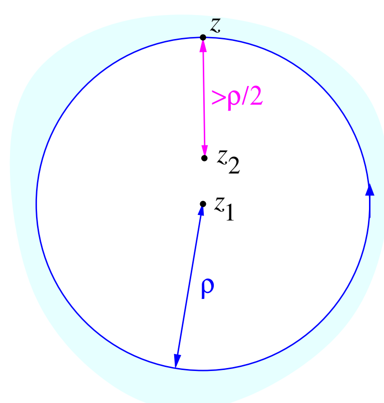

Figure 17.10: *Contour for Liouville' theorem.*

Then, using $|\int f(z)dz| \le \int |f(z)||dz|$, we have

$$\begin{aligned} |f(z_1) - f(z_2)| &= \frac{1}{2\pi}\left|\oint_\Gamma \frac{(z_1 - z_2)}{(z-z_1)(z-z_2)} f(z)\, dz\right| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{|z_1 - z_2| K}{\rho/2}\, d\theta = \frac{2|z_1 - z_2|K}{\rho}. \end{aligned} \tag{17.102}$$

The right hand side can be made arbitrarily small by taking ρ large enough, so we we must have $f(z_1) = f(z_2)$. As z_1 and z_2 were any pair of points, we deduce that $f(z)$ takes the same value everywhere.

Exercise 17.7: Let $a_1, \ldots, a_N$ be N distinct complex numbers. Use Liouville's theorem to prove that

$$\sum_{k\neq j}\sum_{j=1}^{N} \frac{1}{(z-a_j)}\frac{1}{(z-a_k)^2} = \sum_{k\neq j}\sum_{j=1}^{N} \frac{1}{(a_k-a_j)}\frac{1}{(z-a_k)^2}.$$

17.4.2 Taylor and Laurent series

We have defined a function to be analytic in a domain D if it is (once) complex differentiable at all points in D. It turned out that this apparently mild requirement automatically implied that the function is differentiable *arbitrarily many times* in D. In this section we shall see that knowledge of all derivatives of $f(z)$ at any single point in D is enough to completely determine the function at any other point in D. Compare this with functions of a real variable, for which it is easy to construct examples that are once but not twice differentiable, and where complete knowledge of function at a point, or in even in a neighbourhood of a point, tells us absolutely nothing of the behaviour of the function away from the point or neighbourhood.

The key ingredient in these almost magical properties of complex analytic functions is that any analytic function has a Taylor series expansion that actually converges to the function. Indeed an alternative definition of analyticity is that $f(z)$ be representable by a convergent power series. For real variables this is the definition of a *real analytic* function.

To appreciate the utility of power series representations we do need to discuss some basic properties of power series. Most of these results are extensions to the complex plane of what we hope are familiar notions from real analysis.

Consider the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \equiv \lim_{N\to\infty} S_N, \tag{17.103}$$

where S_N are the *partial sums*

$$S_N = \sum_{n=0}^{N} a_n(z-z_0)^n. \tag{17.104}$$

Suppose that this limit exists (i.e the series is convergent) for some $z = \zeta$;

then it turns out that the series is *absolutely convergent*[5] for any $|z - z_0| < |\zeta - z_0|$.

To establish this absolute convergence we may assume, without loss of generality, that $z_0 = 0$. Then, convergence of the sum $\sum a_n \zeta^n$ requires that $|a_n \zeta^n| \to 0$, and thus $|a_n \zeta^n|$ is bounded. In other words, there is a B such that $|a_n \zeta^n| < B$ for any n. We now write

$$|a_n z^n| = |a_n \zeta^n| \left|\frac{z}{\zeta}\right|^n < B \left|\frac{z}{\zeta}\right|^n. \tag{17.105}$$

The sum $\sum |a_n z^n|$ therefore converges for $|z/\zeta| < 1$, by comparison with a geometric progression.

This result, that if a power series in $(z - z_0)$ converges at a point then it converges at all points closer to z_0, shows that a power series possesses some *radius of convergence* R. The series converges for all $|z - z_0| < R$, and diverges for all $|z - z_0| > R$. What happens *on* the circle $|z - z_0| = R$ is usually delicate, and harder to establish. A useful result, however, is *Abel's theorem*, which we will not try to prove. Abel's theorem says that if the sum $\sum a_n$ is convergent, and if $A(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$, then

$$\lim_{z \to 1_-} A(z) = \sum_{n=0}^{\infty} a_n. \tag{17.106}$$

The converse is not true: if $A(z)$ has a finite limit as we approach the circle of convergence, the corresponding sum need not converge

By comparison with a geometric progression, we may establish the following useful formulæ giving R for the series $\sum a_n z^n$:

$$\begin{aligned} R &= \lim_{n \to \infty} \frac{|a_{n-1}|}{|a_n|} \\ &= \lim_{n \to \infty} |a_n|^{1/n}. \end{aligned} \tag{17.107}$$

The proof of these formulæ is identical the real-variable version.

[5]Recall that absolute convergence of $\sum a_n$ means that $\sum |a_n|$ converges. Absolute convergence implies convergence, and also allows us to rearrange the order of terms in the series without changing the value of the sum. Compare this with *conditional convergence*, where $\sum a_n$ converges, but $\sum |a_n|$ does not. You may remember that Riemann showed that the terms of a conditionally convergent series can be rearranged so as to *get any answer whatsoever*!

We soon show that the radius of convergence of a power series is the distance from z_0 to the nearest singularity of the function that it represents.

When we differentiate the terms in a power series, and thus take $a_n z^n \to n a_n z^{n-1}$, this does not alter R. This observation suggests that it is legitimate to evaluate the derivative of the function represented by the powers series by differentiating term-by-term. As step on the way to justifying this, observe that if the series converges at $z = \zeta$ and D_r is the domain $|z| < r < |\zeta|$ then, using the same bound as in the proof of absolute convergence, we have

$$|a_n z^n| < B\frac{|z^n|}{|\zeta|^n} < B\frac{r^n}{|\zeta|^n} = M_n \tag{17.108}$$

where $\sum M_n$ is convergent. As a consequence $\sum a_n z^n$ is *uniformly convergent* in D_r by the Weierstrass "M" test. You probably know that uniform convergence allows the interchange the order of sums and *integrals*: $\int(\sum f_n(x))dx = \sum \int f_n(x)dx$. For real variables, uniform convergence is *not* a strong enough a condition for us to to safely interchange order of sums and *derivatives*: $(\sum f_n(x))'$ is not necessarily equal to $\sum f_n'(x)$. For complex analytic functions, however, Cauchy's integral formula reduces the operation of differentiation to that of integration, and so this interchange *is* permitted. In particular we have that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{17.109}$$

and R is defined by $R = |\zeta|$ for any ζ for which the series converges, then $f(z)$ is analytic in $|z| < R$ and

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}, \tag{17.110}$$

is also analytic in $|z| < R$.

Morera's theorem

There is is a partial converse of Cauchy's theorem:
Theorem (Morera): If $f(z)$ is defined and continuous in a domain D, and if $\oint_\Gamma f(z)\,dz = 0$ for all closed contours, then $f(z)$ is analytic in D. To prove this we set $F(z) = \int_P^z f(\zeta)\,d\zeta$. The integral is path-independent by the

hypothesis of the theorem, and because $f(z)$ is continuous we can differentiate with respect to the integration limit to find that $F'(z) = f(z)$. Thus $F(z)$ is complex differentiable, and so analytic. Then, by Cauchy's formula for higher derivatives, $F''(z) = f'(z)$ exists, and so $f(z)$ itself is analytic.

A corollary of Morera's theorem is that if $f_n(z) \to f(z)$ uniformly in D, with all the f_n analytic, then

i) $f(z)$ is analytic in D, and

ii) $f_n'(z) \to f'(z)$ uniformly.

We use Morera's theorem to prove (i) (appealing to the uniform convergence to justify the interchange the order of summation and integration), and use Cauchy's theorem to prove (ii).

Taylor's theorem for analytic functions

Theorem: Let Γ be a circle of radius ρ centered on the point a. Suppose that $f(z)$ is analytic within and on Γ, and and that the point $z = \zeta$ is within Γ. Then $f(\zeta)$ can be expanded as a Taylor series

$$f(\zeta) = f(a) + \sum_{n=1}^{\infty} \frac{(\zeta - a)^n}{n!} f^{(n)}(a), \tag{17.111}$$

meaning that this series converges to $f(\zeta)$ for all ζ such that $|\zeta - a| < \rho$.

To prove this theorem we use identity

$$\frac{1}{z-\zeta} = \frac{1}{z-a} + \frac{(\zeta - a)}{(z-a)^2} + \cdots + \frac{(\zeta-a)^{N-1}}{(z-a)^N} + \frac{(\zeta-a)^N}{(z-a)^N}\frac{1}{z-\zeta} \tag{17.112}$$

and Cauchy's integral, to write

$$\begin{aligned} f(\zeta) &= \frac{1}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-\zeta)}\,dz \\ &= \sum_{n=0}^{N-1} \frac{(\zeta-a)^n}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}}\,dz + \frac{(\zeta-a)^N}{2\pi i}\oint \frac{f(z)}{(z-a)^N(z-\zeta)}\,dz \\ &= \sum_{n=0}^{N-1} \frac{(\zeta-a)^n}{n!} f^{(n)}(a) + R_N, \end{aligned} \tag{17.113}$$

where

$$R_N \stackrel{\text{def}}{=} \frac{(\zeta-a)^N}{2\pi i}\oint_\Gamma \frac{f(z)}{(z-a)^N(z-\zeta)}\,dz. \tag{17.114}$$

This is Taylor's theorem with remainder. For real variables this is as far as we can go. Even if a real function is differentiable infinitely many times, there is no reason for the remainder to become small. For analytic functions, however, we can show that $R_N \to 0$ as $N \to \infty$. This means that the complex-variable Taylor series is convergent, and its limit is actually equal to $f(z)$. To show that $R_N \to 0$, recall that Γ is a circle of radius ρ centered on $z = a$. Let $r = |\zeta - a| < \rho$, and let M be an upper bound for $f(z)$ on Γ. (This exists because f is continuous and Γ is a compact subset of $\mathbb{C}$.) Then, estimating the integral using methods similar to those invoked in our proof of Liouville's Theorem, we find that

$$R_N < \frac{r^N}{2\pi}\left(\frac{2\pi\rho M}{\rho^N(\rho - r)}\right). \tag{17.115}$$

As $r < \rho$, this tends to zero as $N \to \infty$.

We can take ρ as large as we like provided there are no singularities of f end up within, or on, the circle. This confirms the claim made earlier: the radius of convergence of the powers series representation of an analytic functionis the distance to the nearest singularity.

Laurent series

Theorem (Laurent): Let Γ_1 and Γ_2 be two anticlockwise circlular paths with centre a, radii ρ_1 and ρ_2, and with $\rho_2 < \rho_1$. If $f(z)$ is analytic on the circles and within the annulus between them, then, for ζ in the annulus:

$$f(\zeta) = \sum_{n=0}^{\infty} a_n(\zeta - a)^n + \sum_{n=1}^{\infty} b_n(\zeta - a)^{-n}. \tag{17.116}$$

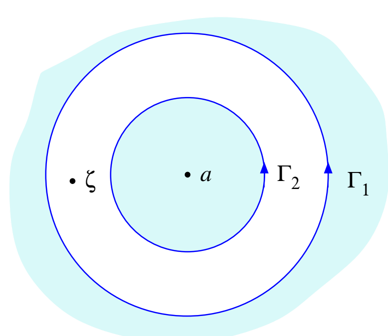

Figure 17.11: *Contours for Laurent's theorem.*

The coefficients a_n and b_n are given by

$$a_n = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(z)}{(z-a)^{n+1}}\,dz, \quad b_n = \frac{1}{2\pi i}\oint_{\Gamma_2} f(z)(z-a)^{n-1}\,dz. \qquad (17.117)$$

Laurent's theorem is proved by observing that

$$f(\zeta) = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(z)}{(z-\zeta)}\,dz - \frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f(z)}{(z-\zeta)}\,dz, \qquad (17.118)$$

and using the identities

$$\frac{1}{z-\zeta} = \frac{1}{z-a} + \frac{(\zeta-a)}{(z-a)^2} + \cdots + \frac{(\zeta-a)^{N-1}}{(z-a)^N} + \frac{(\zeta-a)^N}{(z-a)^N}\frac{1}{z-\zeta}, \quad (17.119)$$

and

$$-\frac{1}{z-\zeta} = \frac{1}{\zeta-a} + \frac{(z-a)}{(\zeta-a)^2} + \cdots + \frac{(z-a)^{N-1}}{(\zeta-a)^N} + \frac{(z-a)^N}{(\zeta-a)^N}\frac{1}{\zeta-z}. \quad (17.120)$$

Once again we can show that the remainder terms tend to zero.
Warning: Although the coefficients a_n are given by the same integrals as in Taylor's theorem, they are not interpretable as derivatives of f unless $f(z)$ is analytic within the inner circle, in which case all the b_n are zero.

17.4.3 Zeros and singularities

This section is something of a *nosology* — a classification of diseases — but you should study it carefully as there is some tight reasoning here, and the conclusions are the essential foundations for the rest of subject.

First a review and some definitions:

a) If $f(z)$ is analytic with a domain D, we have seen that f may be expanded in a Taylor series about any point $z_0 \in D$:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n. \qquad (17.121)$$

If $a_0 = a_1 = \cdots = a_{n-1} = 0$, and $a_n \neq 0$, so that the first non-zero term in the series is $a_n(z-z_0)^n$, we say that $f(z)$ has a *zero* of order n at z_0.

b) A *singularity* of $f(z)$ is a point at which $f(z)$ ceases to be differentiable. If $f(z)$ has no singularities at finite z (for example, $f(z) = \sin z$) then it is said to be an *entire* function.

c) If $f(z)$ is analytic in D except at $z = a$, an *isolated singularity*, then we may draw two concentric circles of centre a, both within D, and in the annulus between them we have the Laurent expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}. \tag{17.122}$$

The second term, consisting of negative powers, is called the *principal part* of $f(z)$ at $z = a$. It may happen that $b_m \neq 0$ but $b_n = 0$, $n > m$. Such a singularity is called a pole of order m at $z = a$. The coefficient b_1, which may be 0, is called the residue of f at the pole $z = a$. If the series of negative powers does not terminate, the singularity is called an *isolated essential singularity*

Now some observations:

i) Suppose $f(z)$ is analytic in a domain D containing the point $z = a$. Then we can expand: $f(z) = \sum a_n (z-a)^n$. If $f(z)$ is zero at $z = 0$, then there are exactly two possibilities: a) all the a_n vanish, and then $f(z)$ is identically zero; b) there is a first non-zero coefficient, a_m say, and so $f(z) = z^m \varphi(z)$, where $\varphi(a) \neq 0$. In the second case f is said to possess a *zero of order* m at $z = a$.

ii) If $z = a$ is a zero of order m, of $f(z)$ then the zero is *isolated* – *i.e.* there is a neighbourhood of a which contains no other zero. To see this observe that $f(z) = (z-a)^m \varphi(z)$ where $\varphi(z)$ is analytic and $\varphi(a) \neq 0$. Analyticity implies continuity, and by continuity there is a neighbourhood of a in which $\varphi(z)$ does not vanish.

iii) Limit points of zeros I: Suppose that we know that $f(z)$ is analytic in D and we know that it vanishes at a sequence of points $a_1, a_2, a_3, \ldots \in D$. If these points have a limit point[6] that is interior to D then $f(z)$ must, by continuity, be zero there. But this would be a non-isolated zero, in contradiction to item ii), unless $f(z)$ actually vanishes identically in D. This, then, is the only option.

iv) From the definition of poles, they too are isolated.

[6] A point z_0 is a limit point of a set S if for every $\epsilon > 0$ there is some $a \in S$, other than z_0 itself, such that $|a - z_0| \leq \epsilon$. A sequence need not have a limit for it to possess one or more limit points.

v) If $f(z)$ has a pole at $z = a$ then $f(z) \to \infty$ as $z \to a$ in any manner.

vi) Limit points of zeros II: Suppose we know that f is analytic in D, except possibly at $z = a$ which is limit point of zeros as in iii), but we also know that f is not identically zero. Then $z = a$ must be singularity of f — but not a pole (because f would tend to infinity and could not have arbitrarily close zeros) — so a must be an isolated essential singularity. For example $\sin 1/z$ has an isolated essential singularity at $z = 0$, this being a limit point of the zeros at $z = 1/n\pi$.

vii) A limit point of poles or other singularities would be a *non-isolated essential singularity*.

17.4.4 Analytic continuation

Suppose that $f_1(z)$ is analytic in the (open, arcwise-connected) domain D_1, and $f_2(z)$ is analytic in D_2, with $D_1 \cap D_2 \neq \emptyset$. Suppose further that $f_1(z) = f_2(z)$ in $D_1 \cap D_2$. Then we say that f_2 is an analytic continuation of f_1 to D_2. Such analytic continuations are *unique*: if f_3 is also analytic in D_2, and $f_3 = f_1$ in $D_1 \cap D_2$, then $f_2 - f_3 = 0$ in $D_1 \cap D_2$. Because the intersection of two open sets is also open, $f_1 - f_2$ vanishes on an open set and, so by observation iii) of the previous section, it vanishes everywhere in D_2.

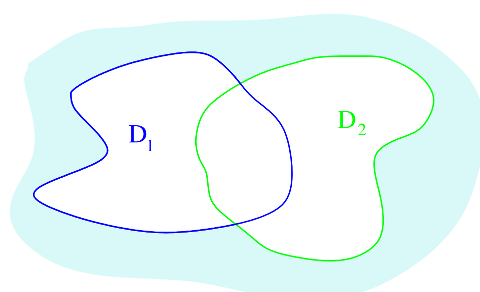

Figure 17.12: *Intersecting domains.*

We can use this uniqueness result, coupled with the circular domains of convergence of the Taylor series, to extend the definition of analytic functions beyond the domain of their initial definition.

The distribution $x_+^{\alpha-1}$

An interesting and useful example of analytic continuation is provided by the distribution $x_+^{\alpha-1}$, which, for real positive α, is defined by its evaluation on a test function $\varphi(x)$ as

$$(x_+^{\alpha-1}, \varphi) = \int_0^\infty x^{\alpha-1}\varphi(x)\, dx. \tag{17.123}$$

The pairing $(x_+^{\alpha-1}, \varphi)$ extends to an complex analytic function of α provided the integral converges. Test functions are required to decrease at infinity faster than any power of x, and so the integral always converges at the upper limit. It will converge at the lower limit provided $\mathrm{Re}\,(\alpha) > 0$. Assume that this is so, and integrate by parts using

$$\frac{d}{dx}\left(\frac{x^\alpha}{\alpha}\varphi(x)\right) = x^{\alpha-1}\varphi(x) + \frac{x^\alpha}{\alpha}\varphi'(x). \tag{17.124}$$

We find that, for $\epsilon > 0$,

$$\left[\frac{x^\alpha}{\alpha}\varphi(x)\right]_\epsilon^\infty = \int_\epsilon^\infty x^{\alpha-1}\varphi(x)\, dx + \int_\epsilon^\infty \frac{x^\alpha}{\alpha}\varphi'(x)\, dx. \tag{17.125}$$

The integrated-out part on the left-hand-side of (17.125) tends to zero as we take ϵ to zero, and both of the integrals converge in this limit as well. Consequently

$$I_1(\alpha) \equiv -\frac{1}{\alpha}\int_0^\infty x^\alpha \varphi'(x)\, dx \tag{17.126}$$

is equal to $(x_+^{\alpha-1}, \varphi)$ for $0 < \mathrm{Re}\,(\alpha) < \infty$. However, the integral defining $I_1(\alpha)$ converges in the larger region $-1 < \mathrm{Re}\,(\alpha) < \infty$. It therefore provides an analytic continuation to this larger domain. The factor of $1/\alpha$ reveals that the analytically-continued function possesses a pole at $\alpha = 0$, with residue

$$-\int_0^\infty \varphi'(x)\, dx = \varphi(0). \tag{17.127}$$

We can repeat the integration by parts, and find that

$$I_2(\alpha) \equiv \frac{1}{\alpha(\alpha+1)}\int_0^\infty x^{\alpha+1}\varphi''(x)\, dx \tag{17.128}$$

provides an analytic continuation to the region $-2 < \mathrm{Re}\,(\alpha) < \infty$. By proceeding in this manner, we can continue $(x_+^{\alpha-1}, \varphi)$ to a function analytic in the entire complex α plane with the exception of zero and the negative integers, at which it has simple poles. The residue of the pole at $\alpha = -n$ is $\varphi^{(n)}(0)/n!$.

There is another, much more revealing, way of expressing these analytic continuations. To obtain this, suppose that $\phi \in C^\infty[0,\infty]$ and $\phi \to 0$ at infinity as least as fast as $1/x$. (Our test function φ decreases much more rapidly than this, but $1/x$ is all we need for what follows.) Now the function

$$I(\alpha) \equiv \int_0^\infty x^{\alpha-1}\phi(x)\,dx \tag{17.129}$$

is convergent and analytic in the strip $0 < \mathrm{Re}\,(\alpha) < 1$. By the same reasoning as above, $I(\alpha)$ is there equal to

$$-\int_0^\infty \frac{x^\alpha}{\alpha}\phi'(x)\,dx. \tag{17.130}$$

Again this new integral provides an analytic continuation to the larger strip $-1 < \mathrm{Re}\,(\alpha) < 1$. But in the left-hand half of this strip, where $-1 < \mathrm{Re}\,(\alpha) < 0$, we can write

$$\begin{aligned}
-\int_0^\infty \frac{x^\alpha}{\alpha}\phi'(x)\,dx &= \lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty x^{\alpha-1}\phi(x)\,dx - \left[\frac{x^\alpha}{\alpha}\phi(x)\right]_\epsilon^\infty\right\}\\
&= \lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty x^{\alpha-1}\phi(x)\,dx + \phi(\epsilon)\frac{\epsilon^\alpha}{\alpha}\right\}\\
&= \lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty x^{\alpha-1}[\phi(x)-\phi(\epsilon)]\,dx\right\},\\
&= \int_0^\infty x^{\alpha-1}[\phi(x)-\phi(0)]\,dx.
\end{aligned} \tag{17.131}$$

Observe how the integrated out part, which tends to zero in $0 < \mathrm{Re}\,(\alpha) < 1$, becomes divergent in the strip $-1 < \mathrm{Re}\,(\alpha) < 0$. This divergence is there craftily combined with the integral to cancel *its* divergence, leaving a finite remainder. As a consequence, for $-1 < \mathrm{Re}\,(\alpha) < 0$, the analytic continuation is given by

$$I(\alpha) = \int_0^\infty x^{\alpha-1}[\phi(x)-\phi(0)]\,dx. \tag{17.132}$$

Next we observe that $\chi(x) = [\phi(x) - \phi(0)]/x$ tends to zero as $1/x$ for large x, and at $x = 0$ can be defined by its limit as $\chi(0) = \phi'(0)$. This $\chi(x)$ then satisfies the same hypotheses as $\phi(x)$. With $I(\alpha)$ denoting the analytic continuation of the original I, we therefore have

$$\begin{aligned} I(\alpha) &= \int_0^\infty x^{\alpha-1}[\phi(x) - \phi(0)]\,dx, \qquad -1 < \mathrm{Re}\,(\alpha) < 0 \\ &= \int_0^\infty x^{\beta-1}\left[\frac{\phi(x) - \phi(0)}{x}\right] dx, \quad \text{where } \beta = \alpha + 1, \\ &\to \int_0^\infty x^{\beta-1}\left[\frac{\phi(x) - \phi(0)}{x} - \phi'(0)\right] dx, \quad -1 < \mathrm{Re}\,(\beta) < 0 \\ &= \int_0^\infty x^{\alpha-1}[\phi(x) - \phi(0) - x\phi'(0)]\,dx, \quad -2 < \mathrm{Re}\,(\alpha) < -1, \end{aligned} \tag{17.133}$$

the arrow denoting the same analytic continuation process that we used with ϕ.

We can now apply this machinary to our original $\varphi(x)$, and so deduce that the analytically-continued distribution is given by

$$(x_+^{\alpha-1}, \varphi) = \begin{cases} \displaystyle\int_0^\infty x^{\alpha-1}\varphi(x)\,dx, & 0 < \mathrm{Re}\,(\alpha) < \infty, \\ \displaystyle\int_0^\infty x^{\alpha-1}[\varphi(x) - \varphi(0)]\,dx, & -1 < \mathrm{Re}\,(\alpha) < 0, \\ \displaystyle\int_0^\infty x^{\alpha-1}[\varphi(x) - \varphi(0) - x\varphi'(0)]\,dx, & -2 < \mathrm{Re}\,(\alpha) < -1, \end{cases} \tag{17.134}$$

and so on. The analytic continuation automatically subtracts more and more terms of the Taylor series of $\varphi(x)$ the deeper we penetrate into the left-hand half-plane. This property, that analytic continuation covertly subtracts the minimal number of Taylor-series terms required ensure convergence, lies behind a number of physics applications, most notably the method of *dimensional regularization* in quantum field theory.

The following exercise illustrates some standard techniques of reasoning *via* analytic continuation.

Exercise 17.8: Define the *dilogarithm* function by the series

$$\mathrm{Li}_2(z) = \frac{z}{1^2} + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \cdots.$$

The radius of convergence of this series is unity, but the domain of $\mathrm{Li}_2(z)$ can be extended to $|z|>1$ by analytic continuation.

a) Observe that the series converges at $z=\pm 1$, and at $z=1$ is

$$\mathrm{Li}_2(1) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Rearrange the series to show that

$$\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}.$$

b) Identify the derivative of the power series for $\mathrm{Li}_2(z)$ with that of an elementary function. Exploit your identification to extend the definition of $[\mathrm{Li}_2(z)]'$ outside $|z|<1$. Use the properties of this derivative function, together with part a), to prove that the extended function obeys

$$\mathrm{Li}_2(-z) + \mathrm{Li}_2\left(-\frac{1}{z}\right) = -\frac{1}{2}(\ln z)^2 - \frac{\pi^2}{6}.$$

This formula allows us to calculate values of the dilogarithm for $|z|>1$ in terms of those with $|z|<1$.

Many weird identities involving dilogarithms exist. Some, such as

$$\mathrm{Li}_2\left(-\frac{1}{2}\right) + \frac{1}{6}\mathrm{Li}_2\left(\frac{1}{9}\right) = -\frac{1}{18}\pi^2 + \ln 2 \ln 3 - \frac{1}{2}(\ln 2)^2 - \frac{1}{3}(\ln 3)^2,$$

were found by Ramanujan. Others, originally discovered by sophisticated numerical methods, have been given proofs based on techniques from quantum mechanics. *Polylogarithms*, defined by

$$\mathrm{Li}_k(z) = \frac{z}{1^k} + \frac{z^2}{2^k} + \frac{z^3}{3^k} + \cdots,$$

occur frequently when evaluating Feynman diagrams.

17.4.5 Removable singularities and the Weierstrass-Casorati theorem

Sometimes we are given a definition that makes a function analytic in a region with the exception of a single point. Can we extend the definition to make the function analytic in the entire region? Provided that the function is well enough behaved near the point, the answer is yes, and the extension is unique. Curiously, the proof that this is so gives us insight into the wild behaviour of functions near essential singularities.

Removable singularities

Suppose that $f(z)$ is analytic in $D \backslash a$, but that $\lim_{z\to a}(z-a)f(z) = 0$, then f may be extended to a function analytic in all of D — *i.e.* $z = a$ is a *removable singularity*. To see this, let ζ lie between two simple closed contours Γ_1 and Γ_2, with a within the smaller, Γ_2. We use Cauchy to write

$$f(\zeta) = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(z)}{z-\zeta}\,dz - \frac{1}{2\pi i}\oint_{\Gamma_2} \frac{f(z)}{z-\zeta}\,dz. \tag{17.135}$$

Now we can shrink Γ_2 down to be very close to a, and because of the condition on $f(z)$ near $z = a$, we see that the second integral vanishes. We can also arrange for Γ_1 to enclose any chosen point in D. Thus, if we set

$$\tilde{f}(\zeta) = \frac{1}{2\pi i}\oint_{\Gamma_1} \frac{f(z)}{z-\zeta}\,dz \tag{17.136}$$

within Γ_1, we see that $\tilde{f} = f$ in $D \backslash a$, and is analytic in all of D. The extension is unique because any two analytic functions that agree everywhere except for a single point, must also agree at that point.

Weierstrass-Casorati

We apply the idea of removable singularities to show just how pathological a beast is an isolated essential singularity:
Theorem (Weierstrass-Casorati): Let $z = a$ be an isolated essential singularity of $f(z)$, then in any neighbourhood of a the function $f(z)$ comes arbitrarily close to any assigned valued in $\mathbb{C}$.

To prove this, define $N_\delta(a) = \{z \in \mathbb{C} : |z-a| < \delta\}$, and $N_\epsilon(\zeta) = \{z \in \mathbb{C} : |z-\zeta| < \epsilon\}$. The claim is then that there is an $z \in N_\delta(a)$ such that $f(z) \in N_\epsilon(\zeta)$. Suppose that the claim is *not* true. Then we have $|f(z)-\zeta| > \epsilon$ for all $z \in N_\delta(a)$. Therefore

$$\left|\frac{1}{f(z)-\zeta}\right| < \frac{1}{\epsilon} \tag{17.137}$$

in $N_\delta(a)$, while $1/(f(z)-\zeta)$ is analytic in $N_\delta(a) \setminus a$. Therefore $z = a$ is a removable singularity of $1/(f(z)-\zeta)$, and there is an an analytic $g(z)$ which coincides with $1/(f(z)-\zeta)$ at all points except a. Therefore

$$f(z) = \zeta + \frac{1}{g(z)} \tag{17.138}$$

except at a. Now $g(z)$, being analytic, may have a zero at $z = a$ giving a pole in f, but it cannot give rise to an essential singularity. The claim is true, therefore.

Picard's theorems

Weierstrass-Casorati is elementary. There are much stronger results:
Theorem (Picard's little theorem): Every nonconstant entire function attains every complex value with at most one *exception.*
Theorem (Picard's big theorem): In any neighbourhood of an isolated essential singularity, $f(z)$ takes every complex value with at most one *exception.*
The proofs of these theorems are hard.

As an illustration of Picard's little theorem, observe that the function $\exp z$ is entire, and takes all values except 0. For the big theorem observe that function $f(z) = \exp(1/z)$. has an essential singularity at $z = 0$, and takes all values, with the exception of 0, in any neighbourhood of $z = 0$.

17.5 Meromorphic functions and the winding-number

A function whose only singularities in D are poles is said to be *meromorphic* there. These functions have a number of properties that are essentially topological in character.

17.5.1 Principle of the argument

If $f(z)$ is meromorphic in D with $\partial D = \Gamma$, and $f(z) \neq 0$ on Γ, then

$$\frac{1}{2\pi i} \oint_\Gamma \frac{f'(z)}{f(z)}\, dz = N - P \tag{17.139}$$

where N is the number of zero's in D and P is the number of poles. To show this, we note that if $f(z) = (z-a)^m \varphi(z)$ where φ is analytic and non-zero near a, then

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{\varphi'(z)}{\varphi(z)} \tag{17.140}$$

so f'/f has a simple pole at a with residue m. Here m can be either positive or negative. The term $\varphi'(z)/\varphi(z)$ is analytic at $z = a$, so collecting all the residues from each zero or pole gives the result.

Since $f'/f = \frac{d}{dz} \ln f$ the integral may be written

$$\oint_\Gamma \frac{f'(z)}{f(z)}\, dz = \Delta_\Gamma \ln f(z) = i\Delta_\Gamma \arg f(z), \tag{17.141}$$

the symbol Δ_Γ denoting the total change in the quantity after we traverse Γ. Thus

$$N - P = \frac{1}{2\pi} \Delta_\Gamma \arg f(z). \tag{17.142}$$

This result is known as the principle of the argument.

Local mapping theorem

Suppose the function $w = f(z)$ maps a region Ω holomorphicly onto a region Ω', and a simple closed curve $\gamma \subset \Omega$ onto another closed curve $\Gamma \subset \Omega'$, which will in general have self intersections. Given a point $a \in \Omega'$, we can ask ourselves how many points within the simple closed curve γ map to a. The answer is given by the *winding number* of the image curve Γ about a.

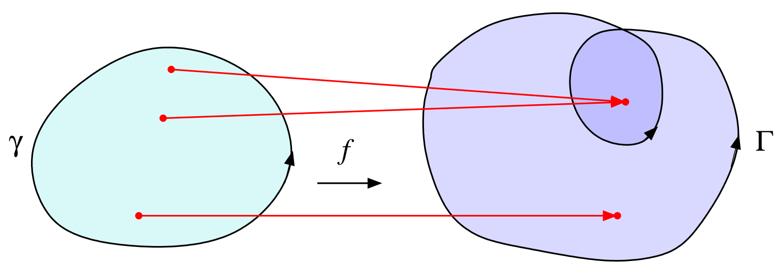

Figure 17.13: *An analytic map is one-to-one where the winding number is unity, but two-to-one at points where the image curve winds twice.*

To that this is so, we appeal to the principal of the argument as

$$\begin{aligned} \#\text{ of zeros of } (f-a) \text{ within } \gamma &= \frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)-a}\, dz, \\ &= \frac{1}{2\pi i}\oint_\Gamma \frac{dw}{w-a}, \\ &= n(\Gamma, a), \end{aligned} \tag{17.143}$$

where $n(\Gamma, a)$ is called the winding number of the image curve Γ about a. It is equal to

$$n(\Gamma, a) = \frac{1}{2\pi}\Delta_\gamma \arg(w - a), \tag{17.144}$$

and is the number of times the image point w encircles a as z traverses the original curve γ.

Since the number of pre-image points cannot be negative, these winding numbers must be positive. This means that the holomorphic image of curve winding in the anticlockwise direction is also a curve winding anticlockwise.

For mathematicians, another important consequence of this result is that a holomorphic map is *open*– *i.e.* the holomorphic image of an open set is itself an open set. The local mapping theorem is therefore sometime called the *open mapping theorem*.

17.5.2 Rouché's theorem

Here we provide an effective tool for locating zeros of functions.
Theorem (Rouché): Let $f(z)$ and $g(z)$ be analytic within and on a simple closed contour γ. Suppose further that $|g(z)| < |f(z)|$ everywhere on γ, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros within γ.

Before giving the proof, we illustrate Rouché's theorem by giving its most important corollary: the algebraic completeness of the complex numbers, a result otherwise known as the *fundamental theorem of algebra*. This asserts that, if R is sufficiently large, a polynomial $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ has exactly n zeros, when counted with their multiplicity, lying within the circle $|z| = R$. To prove this note that we can take R sufficiently big that

$$\begin{aligned} |a_n z^n| &= |a_n| R^n \\ &> |a_{n-1}|R^{n-1} + |a_{n-2}|R^{n-2} \cdots + |a_0| \\ &> |a_{n-a}z^{n-1} + a_{n-2}z^{n-2} \cdots + a_0|, \end{aligned} \tag{17.145}$$

on the circle $|z| = R$. We can therefore take $f(z) = a_n z^n$ and $g(z) = a_{n-a}z^{n-1} + a_{n-2}z^{n-2} \cdots + a_0$ in Rouché. Since $a_n z^n$ has exactly n zeros, all lying at $z = 0$, within $|z| = R$, we conclude that so does $P(z)$.

The proof of Rouché is a corollary of the principle of the argument. We observe that

$$\#\text{ of zeros of } f + g = n(\Gamma, 0)$$

$$\begin{aligned}
&= \frac{1}{2\pi}\Delta_\gamma \arg(f+g) \\
&= \frac{1}{2\pi i}\Delta_\gamma \ln(f+g) \\
&= \frac{1}{2\pi i}\Delta_\gamma \ln f + \frac{1}{2\pi i}\Delta_\gamma \ln(1+g/f) \\
&= \frac{1}{2\pi}\Delta_\gamma \arg f + \frac{1}{2\pi}\Delta_\gamma \arg(1+g/f). \quad (17.146)
\end{aligned}$$

Now $|g/f| < 1$ on γ, so $1 + g/f$ cannot circle the origin as we traverse γ. As a consequence $\Delta_\gamma \arg(1 + g/f) = 0$. Thus the number of zeros of $f + g$ inside γ is the same as that of f alone. (Naturally, they are not usually in the same places.)

The geometric part of this argument is often illustrated by a dog on a lead. If the lead has length L, and the dog's owner stays a distance $R > L$ away from a lamp post, then the dog cannot run round the lamp post unless the owner does the same.

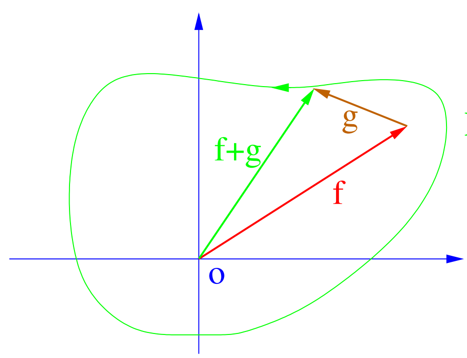

Figure 17.14: *The curve* Γ *is the image of* γ *under the map* $f+g$. *If* $|g| < |f|$, *then, as* z *traverses* γ, $f+g$ *winds about the origin the same number of times that* f *does.*

Exercise 17.9: *Jacobi Theta Function.* The function $\theta(z|\tau)$ is defined for $\operatorname{Im}\tau > 0$ by the sum

$$\theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi\tau n^2} e^{2\pi i n z}.$$

Show that $\theta(z+1|\tau) = \theta(z|\tau)$, and $\theta(z+\tau|\tau) = e^{-i\pi\tau - 2\pi i z}\theta(z|\tau)$. Use this information and the principle of the argument to show that $\theta(z|\tau)$ has exactly one zero in each unit cell of the Bravais lattice comprising the points $z = m + n\tau$; $m, n \in \mathbb{Z}$. Show that these zeros are located at $z = (m + 1/2) + (n + 1/2)\tau$.

Exercise 17.10: Use Rouché's theorem to find the number of roots of the equation $z^5 + 15z + 1 = 0$ lying within the circles, i) $|z| = 2$, ii) $|z| = 3/2$.

17.6 Analytic functions and topology

17.6.1 The point at infinity

Some functions, $f(z) = 1/z$ for example, tend to a fixed limit (here 0) as z become large, independently of in which direction we set off towards infinity. Others, such as $f(z) = \exp z$, behave quite differently depending on what direction we take as $|z|$ becomes large.

To accommodate the former type of function, and to be able to legitimately write $f(\infty) = 0$ for $f(z) = 1/z$, it is convenient to add "∞" to the set of complex numbers. Technically, we are constructing the *one-point compactification* of the locally compact space $\mathbb{C}$. We often portray this extended complex plane as a sphere S^2 (the Riemann sphere), using stereographic projection to locate infinity at the north pole, and 0 at the south pole.

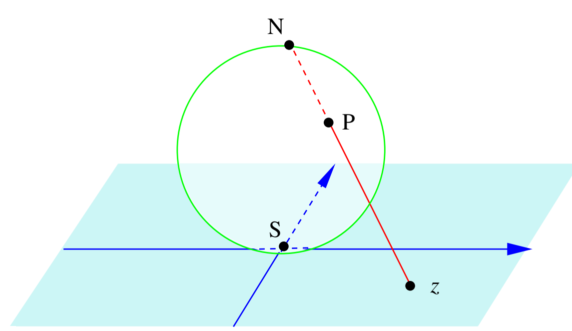

Figure 17.15: *Stereographic mapping of the complex plane to the 2-Sphere.*

By the phrase a open *neighbourhood* of z, we mean an open set containing z. We use the stereographic map to define an open *neighbourhood of infinity* as the stereographic image of a open neighbourhood of the north pole. With this definition, the extended complex plane $\mathbb{C} \cup \{\infty\}$ becomes topologically a sphere, and in particular, becomes a compact set.

If we wish to study the behaviour of a function "at infinity," we use the map $z \mapsto \zeta = 1/z$ to bring ∞ to the origin, and study the behaviour of the function there. Thus the polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_N z^N \tag{17.147}$$

becomes

$$f(\zeta) = a_0 + a_1 \zeta^{-1} + \cdots + a_N \zeta^{-N}, \tag{17.148}$$

and so has a pole of order N at infinity. Similarly, the function $f(z) = z^{-3}$ has a zero of order three at infinity, and $\sin z$ has an isolated essential singularity there.

We must be a careful about defining *residues* at infinity. The residue is more a property of the 1-form $f(z)\,dz$ than of the function $f(z)$ alone, and to find the residue we need to transform the dz as well as $f(z)$. For example, if we set $z = 1/\zeta$ in dz/z we have

$$\frac{dz}{z} = \zeta\, d\left(\frac{1}{\zeta}\right) = -\frac{d\zeta}{\zeta}, \tag{17.149}$$

so the 1-form $(1/z)\,dz$ has a pole at $z = 0$ with residue 1, and has a pole with residue -1 at infinity—even though the *function* $1/z$ has no pole there. This 1-form viewpoint is required for compatability with the residue theorem: The integral of $1/z$ around the positively oriented unit circle is simultaneously minus the integral of $1/z$ about the oppositely oriented unit circle, now regarded as a a positively oriented circle enclosing the point at infinity. Thus if $f(z)$ has of pole of order N at infinity, and

$$\begin{aligned} f(z) &= \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + \cdots + A_N z^N \\ &= \cdots + a_{-2}\zeta^2 + a_{-1}\zeta + a_0 + a_1\zeta^{-1} + a_2\zeta^{-2} + \cdots + A_N \zeta^{-N} \end{aligned} \tag{17.150}$$

near infinity, then the residue at infinity must be defined to be $-a_{-1}$, and not a_1 as one might naïvely have thought.

Once we have allowed ∞ as a point in the set we map *from*, it is only natural to add it to the set we map *to* — in other words to allow ∞ as a possible value for $f(z)$. We will set $f(a) = \infty$, if $|f(z)|$ becomes unboundedly large as $z \to a$ in any manner. Thus, if $f(z) = 1/z$ we have $f(0) = \infty$.

The map

$$w = \left(\frac{z - z_0}{z - z_\infty}\right)\left(\frac{z_1 - z_\infty}{z_1 - z_0}\right) \tag{17.151}$$

takes

$$\begin{aligned} z_0 &\to 0, \\ z_1 &\to 1, \\ z_\infty &\to \infty, \end{aligned} \tag{17.152}$$

for example. Using this language, the Möbius maps

$$w = \frac{az+b}{cz+d} \tag{17.153}$$

become one-to-one maps of $S^2 \to S^2$. They are the only such globally conformal one-to-one maps. When the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{17.154}$$

is an element of SU(2), the resulting one–to-one map is a rigid rotation of the Riemann sphere. Stereographic projection is thus revealed to be the geometric origin of the spinor representations of the rotation group.

If an analytic function $f(z)$ has no essential singularities anywhere on the Riemann sphere then f is *rational*, meaning that it can be written as $f(z) = P(z)/Q(z)$ for some polynomials P, Q.

We begin the proof of this fact by observing that $f(z)$ can have only a finite number of poles. If, to the contrary, f had an infinite number of poles then the compactness of S^2 would ensure that the poles would have a limit point somewhere. This would be a non-isolated singularity of f, and hence an essential singularity. Now suppose we have poles at $z_1, z_2, \ldots, z_N$ with principal parts

$$\sum_{m=1}^{m_n} \frac{b_{n,m}}{(z-z_n)^m}.$$

If one of the z_n is ∞, we first use a Möbius map to move it to some finite point. Then

$$F(z) = f(z) - \sum_{n=1}^{N}\sum_{m=1}^{m_n} \frac{b_{n,m}}{(z-z_n)^m} \tag{17.155}$$

is everywhere analytic, and therefore continuous, on S^2. But S^2 being compact and $F(z)$ being continuous implies that F is bounded. Therefore, by

Liouville's theorem, it is a constant. Thus

$$f(z) = \sum_{n=1}^{N} \sum_{m=1}^{m_n} \frac{b_{n,m}}{(z - z_n)^m} + C, \tag{17.156}$$

and this is a rational function. If we made use of a Möbius map to move a pole at infinity, we use the inverse map to restore the original variables. This manoeuvre does not affect the claimed result because Möbius maps take rational functions to rational functions.

The map $z \mapsto f(z)$ given by the rational function

$$f(z) = \frac{P(z)}{Q(z)} = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots a_0}{b_n z^n + b_{n-1} z^{n-1} + \cdots b_0} \tag{17.157}$$

wraps the Riemann sphere n times around the target S^2. In other words, it is a n-to-one map.

17.6.2 Logarithms and branch cuts

The function $y = \ln z$ is defined to be the solution to $z = \exp y$. Unfortunately, since $\exp 2\pi i = 1$, the solution is not unique: if y is a solution, so is $y + 2\pi i$. Another way of looking at this is that if $z = \rho \exp i\theta$, with ρ real, then $y = \ln \rho + i\theta$, and the angle θ has the same $2\pi i$ ambiguity. Now there is no such thing as a "many valued function." By definition, a function is a machine into which we plug something and get a unique output. To make $\ln z$ into a legitimate function we must select a unique $\theta = \arg z$ for each z. This can be achieved by cutting the z plane along a curve extending from the the *branch point* at $z = 0$ all the way to infinity. Exactly where we put this *branch cut* is not important; what *is* important is that it serve as an impenetrable fence preventing us from following the continuous evolution of the function along a path that winds around the origin.

Similar branch cuts serve to make fractional powers single valued. We define the power z^α for for non-integral α by setting

$$z^\alpha = \exp\{\alpha \ln z\} = |z|^\alpha e^{i\alpha\theta}, \tag{17.158}$$

where $z = |z| e^{i\theta}$. For the square root $z^{1/2}$ we get

$$z^{1/2} = \sqrt{|z|} e^{i\theta/2}, \tag{17.159}$$

where $\sqrt{|z|}$ represents the *positive* square root of $|z|$. We can therefore make this single-valued by a cut from 0 to ∞. To make $\sqrt{(z-a)(z-b)}$ single valued we only need to cut from a to b. (Why? — think this through!).

We can get away without cuts if we imagine the functions being maps *from* some set other than the complex plane. The new set is called a *Riemann surface*. It consists of a number of copies of the complex plane, one for each possible value of our "multivalued function." The map from this new surface is then single-valued, because each possible value of the function is the value of the function evaluated at a point on a different copy. The copies of the complex plane are called *sheets*, and are connected to each other in a manner dictated by the function. The cut plane may now be thought of as a drawing of one level of the multilayered Riemann surface. Think of an architect's floor plan of a spiral-floored multi-story car park: If the architect starts drawing at one parking spot and works her way round the central core, at some point she will find that the floor has become the ceiling of the part already drawn. The rest of the structure will therefore have to be plotted on the plan of the next floor up — but exactly where she draws the division between one floor and the one above is rather arbitrary. The spiral car-park is a good model for the Riemann surface of the $\ln z$ function. See figure 17.16.

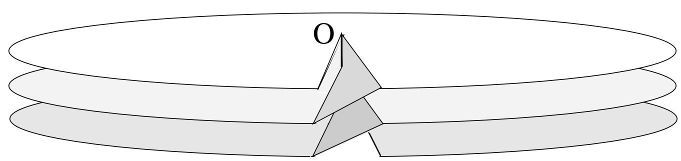

Figure 17.16: *Part of the Riemann surface for* $\ln z$*. Each time we circle the origin, we go up one level.*

To see what happens for a square root, follow $z^{1/2}$ along a curve circling the branch point singularity at $z = 0$. We come back to our starting point with the function having changed sign; A second trip along the same path would bring us back to the original value. The square root thus has only two sheets, and they are cross-connected as shown in figure 17.17.

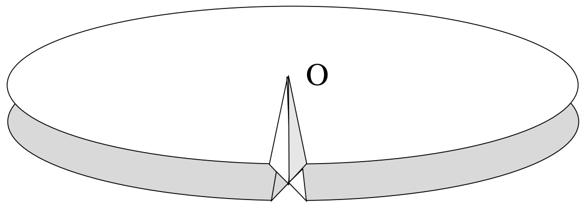

Figure 17.17: *Part of the Riemann surface for $\sqrt{z}$. Two copies of $\mathbb{C}$ are cross-connected. Circling the origin once takes you to the lower level. A second circuit brings you back to the upper level.*

In figures 17.16 and 17.17, we have shown the cross-connections being made rather abruptly along the cuts. This is not necessary —there is no singularity in the function at the cut — but it is often a convenient way to think about the structure of the surface. For example, the surface for $\sqrt{(z-a)(z-b)}$ also consists of two sheets. If we include the point at infinity, this surface can be thought of as two spheres, one inside the other, and cross connected along the cut from a to b.

17.6.3 Topology of Riemann surfaces

Riemann surfaces often have interesting topology. Indeed much of modern algebraic topology emerged from the need to develop tools to understand multiply-connected Riemann surfaces. As we have seen, the complex numbers, with the point at infinity included, have the topology of a sphere. The $\sqrt{(z-a)(z-b)}$ surface is still topologically a sphere. To see this imagine continuously deforming the Riemann sphere by pinching it at the equator down to a narrow waist. Now squeeze the front and back of the waist together and (imagining that the the surface can pass freely through itself) fold the upper half of the sphere inside the lower. The result is the precisely the two-sheeted $\sqrt{(z-a)(z-b)}$ surface described above. The Riemann surface of the function $\sqrt{(z-a)(z-b)(z-c)(z-d)}$, which can be thought of a two spheres, one inside the other and connected along two cuts, one from a to b and one from c to d, is, however, a *torus*. Think of the torus as a bicycle inner tube. Imagine using the fingers of your left hand to pinch the front and back of the tube together and the fingers of your right hand to do the same on the diametrically opposite part of the tube. Now fold the tube about the pinch lines through itself so that one half of the tube is inside the other, and connected to the outer half through two square-root cross-connects. If

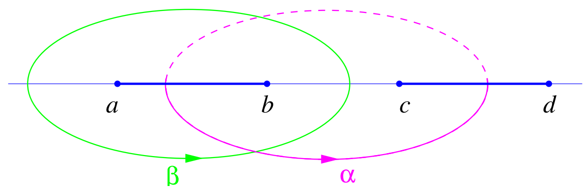

Figure 17.18: *The 1-cycles α and β on the plane with two square-root branch cuts. The dashed part of α lies hidden on the second sheet of the Riemann surface.*

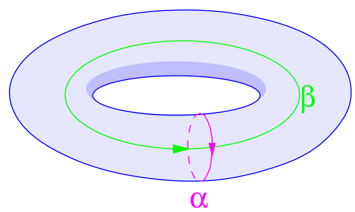

Figure 17.19: *The 1-cycles α and β on the torus.*

you have difficulty visualizing this process, figures 17.18 and 17.19 show how the two 1-cycles, α and β, that generate the homology group $H_1(T^2)$ appear when drawn on the plane cut from a to b and c to d, and then when drawn on the torus. Observe, in figure 17.18, how the curves in the two-sheeted plane manage to intersect in only one point, just as they do when drawn on the torus in figure 17.19.

That the topology of the twice-cut plane is that of a torus has important consequences. This is because the *elliptic integral*

$$w = I^{-1}(z) = \int_{z_0}^{z} \frac{dt}{\sqrt{(t-a)(t-b)(t-c)(t-d)}} \tag{17.160}$$

maps the twice-cut z-plane 1-to-1 onto the torus, the latter being considered as the complex w-plane with the points w and $w + n\omega_1 + m\omega_2$ identified. The two numbers $\omega_{1,2}$ are given by

$$\begin{aligned} \omega_1 &= \oint_\alpha \frac{dt}{\sqrt{(t-a)(t-b)(t-c)(t-d)}}, \\ \omega_2 &= \oint_\beta \frac{dt}{\sqrt{(t-a)(t-b)(t-c)(t-d)}}, \end{aligned} \tag{17.161}$$

and are called the *periods* of the *elliptic function* $z = I(w)$. The map $w \mapsto z = I(w)$ is a genuine function because the original z is uniquely determined by w. It is *doubly periodic* because

$$I(w + n\omega_1 + m\omega_2) = I(w), \quad n, m \in \mathbb{Z}. \tag{17.162}$$

The inverse "function" $w = I^{-1}(z)$ is not a genuine function of z, however, because w increases by ω_1 or ω_2 each time z goes around a curve deformable into α or β, respectively. The periods are complicated functions of a, b, c, d.

If you recall our discussion of de Rham's theorem from chapter 4, you will see that the ω_i are the results of pairing the closed holomorphic 1-form.

$$\text{“}dw\text{”} = \frac{dz}{\sqrt{(z-a)(z-b)(z-c)(z-d)}} \in H^1(T^2) \tag{17.163}$$

with the two generators of $H_1(T^2)$. The quotation marks about dw are there to remind us that dw is not an exact form, *i.e.* it is not the exterior derivative of a single-valued function w. This cohomological interpretation of the periods of the elliptic function is the origin of the use of the word "period" in the context of de Rham's theorem. (See section 19.5 for more information on elliptic functions.)

More general Riemann surfaces are oriented 2-manifolds that can be thought of as the surfaces of doughnuts with g holes. The number g is called the *genus* of the surface. The sphere has $g = 0$ and the torus has $g = 1$. The Euler character of the Riemann surface of genus g is $\chi = 2(1-g)$. For example, figure 17.20 shows a surface of genus three. The surface is in one piece, so $\dim H_0(M) = 1$. The other Betti numbers are $\dim H_1(M) = 6$ and $\dim H_2(M) = 1$, so

$$\chi = \sum_{p=0}^{2} (-1)^p \dim H_p(M) = 1 - 6 + 1 = -4, \tag{17.164}$$

in agreement with $\chi = 2(1-3) = -4$. For complicated functions, the genus may be infinite.

If we have two complex variables z and w then a polynomial relation $P(z, w) = 0$ defines a *complex algebraic curve*. Except for degenerate cases, this one (complex) dimensional curve is simultaneously a two (real) dimensional Riemann surface. With

$$P(z, w) = z^3 + 3w^2 z + w + 3 = 0, \tag{17.165}$$

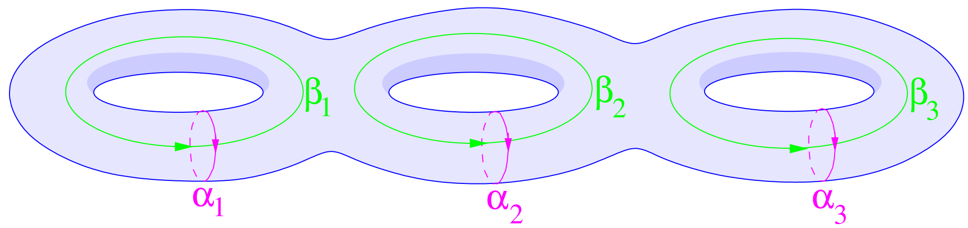

Figure 17.20: *A surface M of genus 3. The non-bounding 1-cycles α_i and β_i form a basis of $H_1(M)$. The entire surface forms the single 2-cycle that spans $H_2(M)$.*

for example, we can think of $z(w)$ being a three-sheeted function of w defined by solving this cubic. Alternatively we can consider $w(z)$ to be the two-sheeted function of z obtained by solving the quadratic equation

$$w^2 + \frac{1}{3z}w + \frac{(3+z^3)}{3z} = 0. \tag{17.166}$$

In each case the branch points will be located where two or more roots coincide. The roots of (17.166), for example, coincide when

$$1 - 12z(3+z^3) = 0. \tag{17.167}$$

This quartic equation has four solutions, so there are four square-root branch points. Although constructed differently, the Riemann surface for $w(z)$ and the Riemann surface for $z(w)$ will have the same genus (in this case $g = 1$) because they are really are one and the same object — the algebraic curve defined by the original polynomial equation.

In order to capture all its points at infinity, we often consider a complex algebraic curve as being a subset of $\mathbb{C}P^2$. To do this we make the defining equation homogeneous by introducing a third co-ordinate. For example, for (17.165) we make

$$P(z,w) = z^3+3w^2z+w+3 \to P(z,w,v) = z^3+3w^2z+wv^2+3v^3. \tag{17.168}$$

The points where $P(z,w,v) = 0$ define[7] a *projective curve* lying in $\mathbb{C}P^2$. Places on this curve where the co-ordinate v is zero are the added points at

[7] A homogeneous polynomial $P(z,w,v)$ of degree n does not provide a map from $\mathbb{C}P^2 \to \mathbb{C}$ because $P(\lambda z, \lambda w, \lambda v) = \lambda^n P(z,w,v)$ usually depends on λ, while the co-ordinates $(\lambda z, \lambda w, \lambda v)$ and (z,w,v) correspond to the same point in $\mathbb{C}P^2$. The *zero set* where $P = 0$ is, however, well-defined in $\mathbb{C}P^2$.

infinity. Places where v is non-zero (and where we may as well set $v = 1$) constitute the original *affine curve*.

A generic (non-singular) curve

$$P(z,w) = \sum_{r,s} a_{rs} z^r w^s = 0, \tag{17.169}$$

with its points at infinity included, has genus

$$g = \frac{1}{2}(d-1)(d-2). \tag{17.170}$$

Here $d = \max(r+s)$ is the *degree* of the curve. This *degree-genus* relation is due to Plücker. It is not, however, trivial to prove. Also not easy to prove is Riemann's theorem of 1852 that *any* finite genus Riemann surface is the complex algebraic curve associated with some two-variable polynomial.

The two assertions in the previous paragraph seem to contradict each other. "Any" finite genus, must surely include $g = 2$, but how can a genus two surface be a complex algebraic curve? There is no integer value of d such that $(d-1)(d-2)/2 = 2$. This is where the "non-singular" caveat becomes important. An affine curve $P(z,w) = 0$ is said to be *singular* at $\mathrm{P} = (z_0, w_0)$ if all of

$$P(z,w), \quad \frac{\partial P}{\partial z}, \quad \frac{\partial P}{\partial w},$$

vanish at P. A projective curve is singular at $\mathrm{P} \in \mathbb{C}P^2$ if all of

$$P(z,w,v), \quad \frac{\partial P}{\partial z}, \quad \frac{\partial P}{\partial w}, \quad \frac{\partial P}{\partial v}$$

are zero there. If the curve has a singular point then then it degenerates and ceases to be a manifold. Now Riemann's construction does not guarantee an *embedding* of the surface into $\mathbb{C}P^2$, only an *immersion*. The distinction between these two concepts is that an immersed surface is allowed to self-intersect, while an embedded one is not. Being a double root of the defining equation $P(z,w) = 0$, a point of self-intersection is necessarily a singular point.

As an illustration of a singular curve, consider our earlier example of the curve

$$w^2 = (z-a)(z-b)(z-c)(z-d) \tag{17.171}$$

whose Riemann surface we know to be a torus once two some points are added at infinity, and when a, b, c, d are all distinct. The degree-genus formula applied to this degree four curve gives, however, $g = 3$ instead of the expected $g = 1$. This is because the corresponding projective curve

$$w^2 v^2 = (z - av)(z - bv)(z - cv)(z - dv) \tag{17.172}$$

has a *tacnode* singularity at the point $(z, w, v) = (0, 1, 0)$. Rather than investigate this rather complicated singularity at infinity, we will consider the simpler case of what happens if we allow b to coincide with c. When b and c merge, the finite point $\mathrm{P} = (w_0, z_0) = (0, b)$ becomes a singular. Near the singularity, the equation defining our curve looks like

$$0 = w^2 - ad\,(z - b)^2, \tag{17.173}$$

which is the equation of two lines, $w = \sqrt{ad}\,(z - b)$ and $w = -\sqrt{ad}\,(z - b)$, that intersect at the point $(w, z) = (0, b)$. To understand what is happening topologically it is first necessary to realize that a *complex* line is a copy of $\mathbb{C}$ and hence, after the point at infinity is included, is topologically a sphere. A pair of intersecting complex lines is therefore topologically a pair of spheres sharing a common point. Our degenerate curve only looks like a pair of lines near the point of intersection however. To see the larger picture, look back at the figure of the twice-cut plane where we see that as b approaches c we have an α cycle of zero total length. A zero length cycle means that the circumference of the torus becomes zero at P, so that it looks like a bent sausage with its two ends sharing the common point P. Instead of two separate spheres, our sausage is equivalent to a single two-sphere with two points identified.

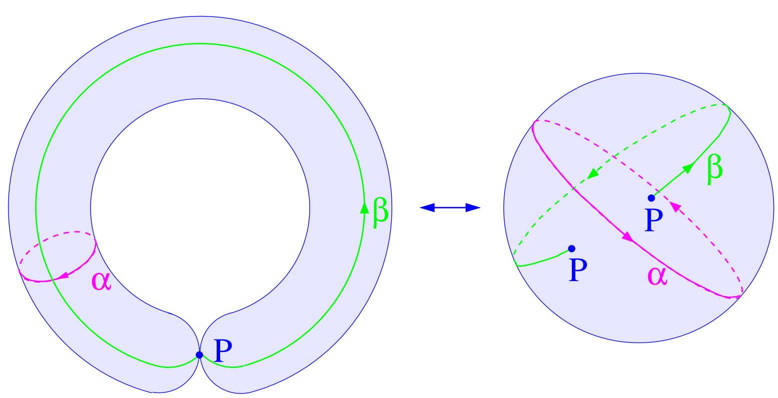

Figure 17.21: *A degenerate torus is topologically the same as a sphere with two points identified.*

As it stands, such a set is no longer a manifold because any neighbourhood of P will contain bits of both ends of the sausage, and therefore cannot be given co-ordinates that make it look like a region in $\mathbb{R}^2$. We can, however, simply agree to delete the common point, and then plug the resulting holes in the sausage ends with two distinct points. The new set is again a manifold, and topologically a sphere. From the viewpoint of the pair of intersecting lines, this construction means that we stay on one line, and ignore the other as it passes through.

A similar *resolution of singularities* allows us to regard immersed surfaces as non-singular manifolds, and it is this sense that Riemann's theorem is to be understood. When n such self-intersection double points are deleted and replaced by pairs of distinct points The degree-genus formula becomes

$$g = \frac{1}{2}(d-1)(d-2) - n, \tag{17.174}$$

and this can take any integer value.

17.6.4 Conformal geometry of Riemann surfaces

In this section we recall Hodge's theory of harmonic forms from section 13.7.1, and see how it looks from a complex-variable perspective. This viewpoint reveals a relationship between Riemann surfaces and Riemann manifolds that forms an important ingredient in string and conformal field theory.

Isothermal co-ordinates and complex structure

Suppose we have a two-dimensional orientable Riemann manifold M with metric

$$ds^2 = g_{ij}\, dx^i dx^j. \tag{17.175}$$

In two dimensions g_{ij} has three independent components. When we make a co-ordinate transformation we have two arbitrary functions at our disposal, and so we can use this freedom to select local co-ordinates in which only one independent component remains. The most useful choice is *isothermal* (also called *conformal*) co-ordinates x, y in which the metric tensor is diagonal, $g_{ij} = e^{\sigma}\delta_{ij}$, and so

$$ds^2 = e^{\sigma}(dx^2 + dy^2). \tag{17.176}$$

The e^σ is called the *scale factor* or *conformal factor*. If we set $z = x + iy$ and $\overline{z} = x - iy$ the metric becomes

$$ds^2 = e^{\sigma(z,\overline{z})} d\overline{z}\, dz. \tag{17.177}$$

We can construct isothermal co-ordinates for some open neighbourhood of any point in M. If in an overlapping isothermal co-ordinate patch the metric is

$$ds^2 = e^{\tau(\zeta,\overline{\zeta})} d\overline{\zeta}\, d\zeta, \tag{17.178}$$

and if the co-ordinates have the same orientation, then in the overlap region ζ must be a function only of z and $\overline{\zeta}$ a function only of $\overline{z}$. This is so that

$$e^{\tau(\zeta,\overline{\zeta})} d\overline{\zeta}\, d\zeta = e^{\sigma(z,\overline{z})} \left|\frac{dz}{d\zeta}\right|^2 d\overline{\zeta}\, d\zeta \tag{17.179}$$

without any $d\zeta^2$ or $d\overline{\zeta}^2$ terms appearing. A manifold with an atlas of complex charts whose change-of-co-ordinate formulae are holomorphic in this way is said to be a *complex manifold*, and the co-ordinates endow it with a *complex structure*. The existence of a global complex structure allows to us to define the notion of meromorphic and rational functions on M. Our Riemann manifold is therefore also a Riemann surface.

While any compact, orientable, two-dimensional Riemann manifold has a complex structure that is determined by the metric, the mapping: *metric* $\rightarrow$ *complex structure* is not one-to-one. Two metrics g_{ij}, $\tilde{g}_{ij}$ that are related by a conformal scale factor

$$g_{ij} = \lambda(x^1, x^2)\tilde{g}_{ij} \tag{17.180}$$

give rise to the same complex structure. Conversely, a pair of two-dimensional Riemann manifolds having the same complex structure have metrics that are related by a scale factor.

The use of isothermal co-ordinates simplifies many computations. Firstly, observe that $g^{ij}/\sqrt{g} = \delta_{ij}$, the conformal factor having cancelled. If you look back at its definition, you will see that this means that when the Hodge "$\star$" map acts on one-forms, the result is independent of the metric. If ω is a one-form

$$\omega = p\, dx + q\, dy, \tag{17.181}$$

then

$$\star\omega = -q\, dx + p\, dy. \tag{17.182}$$

Note that, on one-forms,

$$\star\star = -1. \tag{17.183}$$

With $z = x + iy$, $\overline{z} = x - iy$, we have

$$\omega = \frac{1}{2}(p - iq)\,dz + \frac{1}{2}(p + iq)\,d\overline{z}. \tag{17.184}$$

Let us focus on the dz part:

$$A = \frac{1}{2}(p - iq)\,dz = \frac{1}{2}(p - iq)(dx + idy). \tag{17.185}$$

Then

$$\star A = \frac{1}{2}(p - iq)(dy - idx) = -iA. \tag{17.186}$$

Similarly, with

$$B = \frac{1}{2}(p + iq)\,d\overline{z} \tag{17.187}$$

we have

$$\star B = iB. \tag{17.188}$$

Thus the dz and $d\overline{z}$ parts of the original form are separately eigenvectors of $\star$ with different eigenvalues. We use this observation to construct a resolution of the identity Id into the sum of two projection operators

$$\begin{aligned} Id &= \frac{1}{2}(1 + i\star) + \frac{1}{2}(1 - i\star), \\ &= \quad P \qquad + \qquad \overline{P}, \end{aligned} \tag{17.189}$$

where P projects on the dz part and $\overline{P}$ onto the $d\overline{z}$ part of the form.

The original form is harmonic if it is both closed $d\omega = 0$, and co-closed $d\star\omega = 0$. Thus, in two dimensions, the notion of being harmonic (*i.e.* a solution of Laplace's equation) is independent of what metric we are given. If ω is a harmonic form, then $(p-iq)dz$ and $(p+iq)d\overline{z}$ are separately closed. Observe that $(p-iq)dz$ being closed means that $\partial_{\overline{z}}(p-iq) = 0$, and so $p-iq$ is a holomorphic (and hence harmonic) function. Since both $(p-iq)$ and dz depend only on z, we will call $(p-iq)dz$ a holomorphic 1-form. The complex conjugate form

$$\overline{(p - iq)dz} = (p + iq)d\overline{z} \tag{17.190}$$

then depends only on $\overline{z}$ and is anti-holomorphic.

Riemann bilinear relations

As an illustration of the interplay of harmonic forms and two-dimensional topology, we derive some famous formuæ due to Riemann. These formulæ have applications in string theory and in conformal field theory.

Suppose that M is a Riemann surface of genus g, with α_i, β_i ,$i = 1, \ldots, g$, the representative generators of $H_1(M)$ that intersect as shown in figure 17.20. By applying Hodge-de Rham to this surface, we know that we can select a set of $2g$ independent, real, harmonic, 1-forms as a basis of $H^1(M, \mathbb{R})$. With the aid of the projector P we can assemble these into g holomorphic closed 1-forms ω_i, together with g anti-holomorphic closed 1-forms $\overline{\omega}_i$, the original $2g$ real forms being recovered from these as $\omega_i + \overline{\omega}_i$ and $\star(\omega_i + \overline{\omega}_i) = i(\overline{\omega}_i - \omega_i)$. A physical interpretation of these forms is as the z and $\overline{z}$ components of irrotational and incompressible fluid flows on the surface M. It is not surprising that such flows form a $2g$ real dimensional, or g complex dimensional, vector space because we can independently specify the circulation $\oint \mathbf{v} \cdot d\mathbf{r}$ around each of the $2g$ generators of $H_1(M)$. If the flow field has (covariant) components v_x, v_y, then $\omega = v_z dz$ where $v_z = (v_x - iv_y)/2$, and $\overline{\omega} = v_{\overline{z}} d\overline{z}$ where $v_{\overline{z}} = (v_x + iv_y)/2$.

Suppose now that a and b are closed 1-forms on M. Then, either by exploiting the powerful and general intersection-form formula (13.77) or by cutting open the surface along the curves α_i, β_i and using the more direct strategy that gave us (13.79), we find that

$$\int_M a \wedge b = \sum_{i=1}^{g} \left\{ \int_{\alpha_i} a \int_{\beta_i} b - \int_{\beta_i} a \int_{\alpha_i} b \right\}. \tag{17.191}$$

We use this formula to derive two *bilinear relations* associated with a closed holomorphic 1-form ω. Firstly we compute its Hodge inner-product norm

$$\begin{aligned} \|\omega\|^2 \equiv \int_M \omega \wedge \star\overline{\omega} &= \sum_{i=1}^{g} \left\{ \int_{\alpha_i} \omega \int_{\beta_i} \star\overline{\omega} - \int_{\beta_i} \omega \int_{\alpha_i} \star\overline{\omega} \right\} \\ &= i \sum_{i=1}^{g} \left\{ \int_{\alpha_i} \omega \int_{\beta_i} \overline{\omega} - \int_{\beta_i} \omega \int_{\alpha_i} \overline{\omega} \right\} \\ &= i \sum_{i=1}^{g} \left\{ A_i \overline{B}_i - B_i \overline{A}_i \right\}, \end{aligned} \tag{17.192}$$

where $A_i = \int_{\alpha_i} \omega$ and $B_i = \int_{\beta_i} \omega$. We have used the fact that $\overline{\omega}$ is an anti-holomorphic 1 form and thus an eigenvector of $\star$ with eigenvalue i. It follows, therefore, that if all the A_i are zero then $\|\omega\| = 0$ and so $\omega = 0$.

Let $A_{ij} = \int_{\alpha_i} \omega_j$. The determinant of the matrix A_{ij} is non-zero: If it were zero, then there would be numbers λ_i, not all zero, such that

$$0 = A_{ij}\lambda_j = \int_{\alpha_i} (\omega_j \lambda_j), \tag{17.193}$$

but, by (17.192), this implies that $\|\omega_j \lambda_j\| = 0$ and hence $\omega_j \lambda_j = 0$, contrary to the linear independence of the ω_i. We can therefore solve the equations

$$A_{ij}\lambda_{jk} = \delta_{ik} \tag{17.194}$$

for the numbers λ_{jk} and use these to replace each of the ω_i by the linear combination $\omega_j \lambda_{ji}$. The new ω_i then obey $\int_{\alpha_i} \omega_j = \delta_{ij}$. From now on we suppose that this has be done.

Define $\tau_{ij} = \int_{\beta_i} \omega_j$. Observe that $dz \wedge dz = 0$ forces $\omega_i \wedge \omega_j = 0$, and therefore we have a second relation

$$\begin{aligned} 0 = \int_M \omega_m \wedge \omega_n &= \sum_{i=1}^{g} \left\{ \int_{\alpha_i} \omega_m \int_{\beta_i} \omega_n - \int_{\beta_i} \omega_m \int_{\alpha_i} \omega_n \right\} \\ &= \sum_{i=1}^{g} \{\delta_{im}\tau_{in} - \tau_{im}\delta_{in}\} \\ &= \tau_{mn} - \tau_{nm}. \end{aligned} \tag{17.195}$$

The matrix τ_{ij} is therefore symmetric. A similar compuation shows that

$$\|\lambda_i \omega_i\|^2 = 2\overline{\lambda}_i (\mathrm{Im}\, \tau_{ij}) \lambda_j \tag{17.196}$$

so the matrix $(\mathrm{Im}\, \tau_{ij})$ is positive definite. The set of such symmetric matrices whose imaginary part is positive definite is called the *Siegel upper half-plane*. Not every such matrix correponds to a Riemann surface, but when it does it encodes all information about the shape of the Riemann manifold M that is left invariant under conformal rescaling.

17.7 Further exercises and problems

Exercise 17.11: *Harmonic partners.* Show that the function

$$u = \sin x \cosh y + 2 \cos x \sinh y$$

is harmonic. Determine the corresponding analytic function $u + iv$.

Exercise 17.12: *Möbius Maps.* The Map

$$z \mapsto w = \frac{az+b}{cz+d}$$

is called a Möbius transformation. These maps are important because they are the only one-to-one conformal maps of the Riemann sphere onto itself.

a) Show that two successive Möbius transformations

$$z' = \frac{az+b}{cz+d}, \qquad z'' = \frac{Az'+B}{Cz'+D}$$

give rise to another Möbius transformation, and show that the rule for combining them is equivalent to matrix multiplication.

b) Let z_1, z_2, z_3, z_4 be complex numbers. Show that a necessary and sufficient condition for the four points to be concyclic is that their *cross-ratio*

$$\{z_1, z_2, z_3, z_4\} \stackrel{\text{def}}{=} \frac{(z_1-z_4)(z_3-z_2)}{(z_1-z_2)(z_3-z_4)}$$

be real (Hint: use a well-known property of opposite angles of a cyclic quadrilateral). Show that Möbius transformations leave the cross-ratio invariant, and thus take circles into circles.

Exercise 17.13: *Hyperbolic geometry*. The Riemann metric for the Poincaré-disc model of Lobachevski's hyperbolic plane (See exercises 1.7 and 12.13) can be taken to be

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}, \quad |z|^2 < 1.$$

a) Show that the Möbius transformation

$$z \mapsto w = e^{i\lambda}\frac{z-a}{\bar{a}z-1}, \qquad |a| < 1, \quad \lambda \in \mathbb{R}$$

provides a 1-1 map of the interior of the unit disc onto itself. Show that these maps form a group.

b) Show that the hyperbolic-plane metric is left invariant under the group of maps in part (a). Deduce that such maps are orientation-preserving *isometries* of the hyperbolic plane.

c) Use the circle-preserving property of the Möbius maps to deduce that circles in hyperbolic geometry are represented in the Poincaré disc by Euclidean circles that lie entirely within the disc.

The conformal maps of part (a) are in fact the *only* orientation preserving isometries of the hyperbolic plane. With the exception of circles centered at $z = 0$, the center of the hyperbolic circle does not coincide with the center of its representative Euclidean circle. Euclidean circles that are internally tangent to the boundary of the unit disc have infinite hyperbolic radius and their hyperbolic centers lie on the boundary of the unit disc and hence at hyperbolic infinity. They are known as *horocycles.*

Exercise 17.14: *Rectangle to Ellipse.* Consider the map $w \mapsto z = \sin w$. Draw a picture of the image, in the z plane, of the interior of the rectangle with corners $u = \pm\pi/2$, $v = \pm\lambda$. $(w = u + iv)$. Show which points correspond to the corners of the rectangle, and verify that the vertex angles remain $\pi/2$. At what points does the isogonal property fail?

Exercise 17.15: The part of the negative real axis where $x < -1$ is occupied by a conductor held at potential $-V_0$. The positive real axis for $x > +1$ is similarly occupied by a conductor held at potential $+V_0$. The conductors extend to infinity in both directions perpendicular to the $x - y$ plane, and so the potential V satisfies the two-dimensional Laplace equation.

a) Find the image in the ζ plane of the cut z plane where the cuts run from -1 to $-\infty$ and from $+1$ to $+\infty$ under the map $z \mapsto \zeta = \sin^{-1} z$
b) Use your answer from part a) to solve the electrostatic problem and show that the field lines and equipotentials are conic sections of the form $ax^2 + by^2 = 1$. Find expressions for a and b for the both the field lines and the equipotentials and draw a labelled sketch to illustrate your results.

Exercise 17.16: Draw the image under the map $z \mapsto w = e^{\pi z/a}$ of the infinite strip S, consisting of those points $z = x + iy \in \mathbb{C}$ for which $0 < y < a$. Label enough points to show which point in the w plane corresponds to which in the z plane. Hence or otherwise show that the Dirichlet Green function $G(x, y; x_0, y_0)$ that obeys

$$\nabla^2 G = \delta(x - x_0)\delta(y - y_0)$$

in S, and $G(x, y; x_0, y_0) = 0$ for (x, y) on the boundary of S, can be written as

$$G(x, y; x_0, y_0) = \frac{1}{2\pi} \ln |\sinh(\pi(z - z_0)/2a)| + \dots$$

The dots indicate the presence of a second function, similar to the first, that you should find. Assume that $(x_0, y_0) \in S$.

Exercise 17.17: State Laurent's theorem for functions analytic in an annulus. Include formulae for the coefficients of the expansion. Show that, suitably interpreted, this theorem reduces to a form of Fourier's theorem for functions analytic in a neighbourhood of the unit circle.

Exercise 17.18: *Laurent Paradox.* Show that in the annulus $1 < |z| < 2$ the function

$$f(z) = \frac{1}{(z-1)(2-z)}$$

has a Laurent expansion in powers of z. Find the coefficients. The part of the series with negative powers of z does not terminate. Does this mean that $f(z)$ has an essential singularity at $z = 0$?

Exercise 17.19: Assuming the following series

$$\frac{1}{\sinh z} = \frac{1}{z} - \frac{1}{6}z + \frac{7}{16}z^3 + \dots,$$

evaluate the integral

$$I = \oint_{|z|=1} \frac{1}{z^2 \sinh z} dz.$$

Now evaluate the integral

$$I = \oint_{|z|=4} \frac{1}{z^2 \sinh z} dz.$$

(Hint: The zeros of $\sinh z$ lie at $z = n\pi i$.)

Exercise 17.20: State the theorem relating the difference between the number of poles and zeros of $f(z)$ in a region to the winding number of argument of $f(z)$. Hence, or otherwise, evaluate the integral

$$I = \oint_C \frac{5z^4 + 1}{z^5 + z + 1} dz$$

where C is the circle $|z| = 2$. Prove, including a statement of any relevent theorem, any assertions you make about the locations of the zeros of $z^5 + z + 1$.

Exercise 17.21: *Arcsine branch cuts.* Let $w = \sin^{-1} z$. Show that

$$w = n\pi \pm i \ln\{iz + \sqrt{1 - z^2}\}$$

with the $\pm$ being selected depending on whether n is odd or even. Where would you put cuts to ensure that w is a single-valued function?

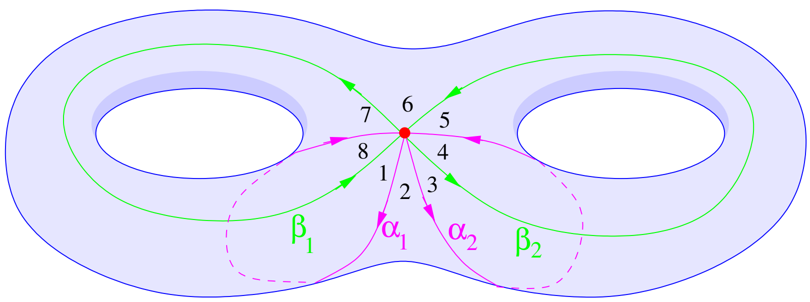

Figure 17.22: *Concurrent 1-cycles on a genus-2 surface.*

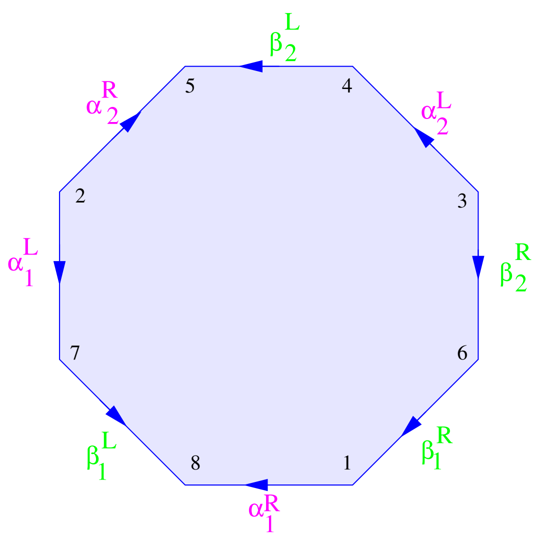

Figure 17.23: *The cut-open genus-2 surface. The superscripts L and R denote respectively the left and right sides of each 1-cycle, viewed from the direction of the arrow orienting the cycle.*

Problem 17.22: *Cutting open a genus-2 surface.* The Riemann surface for the function

$$y = \sqrt{(z-a_1)(z-a_2)(z-a_3)(z-a_4)(z-a_5)(z-a_6)}$$

has genus $g = 2$. Such a surface M is sketched in figure 17.22, where the four independent 1-cycles $\alpha_{1,2}$ and $\beta_{1,2}$ that generate $H_1(M)$ have been drawn so that they share a common vertex.

a) Realize the genus-2 surface as two copies of $\mathbb{C} \cup \{\infty\}$ cross-connected by three square-root branch cuts. Sketch how the 1-cycles α_i and β_i, $i = 1, 2$ of figure 17.22 appear when drawn on your thrice-cut plane.
b) Cut the surface open along the four 1-cycles, and convince yourself that resulting surface is homeomorphic to the octagonal region appearing in figure 17.23.
c) Apply the direct method that gave us (13.79) to the octagonal region of part b). Hence show that for closed 1-forms a, b, on the surface we have

$$\int_M a \wedge b = \sum_{i=1}^{2} \left\{ \int_{\alpha_i} a \int_{\beta_i} b - \int_{\beta_i} a \int_{\alpha_i} b \right\}.$$