Appendix B

Fourier Series and Integrals.

Fourier series and Fourier integral representations are the most important examples of the expansion of a function in terms of a complete orthonormal set. The material in this appendix reviews features peculiar to these special cases, and is intended to complement the the general discussion of orthogonal series in chapter 2.

B.1 Fourier Series

A function defined on a finite interval may be expanded as a Fourier *series*.

B.1.1 Finite Fourier series

Suppose we have measured $f(x)$ in the interval $[0, L]$, but only at the discrete set of points $x = na$, where a is the sampling interval and $n = 0, 1, \ldots, N-1$, with $Na = L$. We can then represent our data $f(na)$ by a *finite* Fourier series. This representation is based on the geometric sum

$$\sum_{m=0}^{N-1} e^{ik_m(n'-n)a} = \frac{e^{2\pi i(n-n')a} - 1}{e^{2\pi i(n'-n)a/N} - 1}, \tag{B.1}$$

where $k_m \equiv 2\pi m/Na$. For integer n, and n', the expression on the right hand side of (B.1) is zero unless $n' - n'$ is an integer multiple of N, when it becomes indeterminate. In this case, however, each term on the left hand side is equal to unity, and so their sum is equal to N. If we restrict n and n'

to lie between 0 and $N-1$, we have

$$\sum_{m=0}^{N-1} e^{ik_m(n'-n)a} = N\delta_{n'n}\,. \tag{B.2}$$

Inserting (B.2) into the formula

$$f(na) = \sum_{n'=0}^{N-1} f(n'a)\,\delta_{n'n}, \tag{B.3}$$

shows that

$$f(na) = \sum_{m=0}^{N-1} a_m e^{-ik_m na}, \quad \text{where} \quad a_m \equiv \frac{1}{N}\sum_{n=0}^{N-1} f(na)e^{ik_m na}. \tag{B.4}$$

This is the finite Fourier representation.

When $f(na)$ is real, it is convenient to make the k_m sum symmetric about $k_m = 0$ by taking $N = 2M+1$ and setting the summation limits to be $\pm M$. The finite geometric sum then becomes

$$\sum_{m=-M}^{M} e^{im\theta} = \frac{\sin(2M+1)\theta/2}{\sin\theta/2}. \tag{B.5}$$

We set $\theta = 2\pi(n'-n)/N$ and use the same tactics as before to deduce that

$$f(na) = \sum_{m=-M}^{M} a_m\, e^{-ik_m na}, \tag{B.6}$$

where again $k_m = 2\pi m/L$, with $L = Na$, and

$$a_m = \frac{1}{N}\sum_{n=0}^{2M} f(na)\, e^{ik_m na}. \tag{B.7}$$

In this form it is manifest that f being real both implies and is implied by $a_{-m} = a_m^*$.

These finite Fourier expansions are algebraic identities. No limits have to be taken, and so no restrictions need be placed on $f(na)$ for them to be valid. They are all that is needed for processing experimental data.

Although the initial $f(na)$ was defined only for the finite range $0 \le n \le N-1$, the Fourier sum (B.4) or (B.7) is defined for any n, and so extends f to a periodic function of n with period N.

B.1.2 Continuum limit

Now we wish to derive a Fourier representation for functions defined everywhere on the interval $[0, L]$, rather just at the sampling points. The natural way to proceed is to build on the results from the previous section by replacing the interval $[0, L]$ with a discrete lattice of $N = 2M + 1$ points at $x = na$, where a is a small lattice spacing which we ultimately take to zero. For any non-zero a the continuum function $f(x)$ is thus replaced by the finite set of numbers $f(na)$. If we stand back and blur our vision so that we can no longer perceive the individual lattice points, a plot of this discrete function will look little different from the original continuum $f(x)$. In other words, provided that f is slowly varying on the scale of the lattice spacing, $f(an)$ can be regarded as a smooth function of $x = an$.

The basic "integration rule" for such smooth functions is that

$$a \sum_n f(an) \to \int f(an)\, a\, dn \to \int f(x)\, dx\,, \tag{B.8}$$

as a becomes small. A sum involving a Kronecker δ will become an integral containing a Dirac δ-function:

$$a \sum_n f(na) \frac{1}{a} \delta_{nm} = f(ma) \to \int f(x)\, \delta(x-y)\, dx = f(y). \tag{B.9}$$

We can therefore think of the δ function as arising from

$$\frac{\delta_{nn'}}{a} \to \delta(x - x'). \tag{B.10}$$

In particular, the divergent quantity $\delta(0)$ (in x space) is obtained by setting $n = n'$, and can therefore be understood to be the reciprocal of the lattice spacing, or, equivalently, the number of lattice points per unit volume.

Now we take the formal continuum limit of (B.7) by letting $a \to 0$ and $N \to \infty$ while keeping their product $Na = L$ fixed. The *finite* Fourier representation

$$f(na) = \sum_{m=-M}^{M} a_m e^{-\frac{2\pi i m}{Na} na} \tag{B.11}$$

now becomes an *infinite* series

$$f(x) = \sum_{m=-\infty}^{\infty} a_m\, e^{-2\pi i m x/L}, \tag{B.12}$$

whereas

$$a_m = \frac{a}{Na} \sum_{n=0}^{N-1} f(na) e^{\frac{2\pi i m}{Na} na} \to \frac{1}{L} \int_0^L f(x) e^{2\pi i m x / L} \, dx. \tag{B.13}$$

The series (B.12) is the Fourier expansion for a function on a finite interval. The sum is equal to $f(x)$ in the interval $[0, L]$. Outside, it produces L-periodic translates of the original f.

This Fourier expansion (B.12,B.13) is same series that we would obtain by using the $L^2[0, L]$ orthonormality

$$\frac{1}{L} \int_0^L e^{2\pi i m x / L} \, e^{-2\pi i n x / L} \, dx = \delta_{nm}, \tag{B.14}$$

and using the methods of chapter two. The arguments adduced there, however, guarantee convergence only in the L^2 sense. While our present "continuum limit" derivation is only heuristic, it does suggest that for reasonably-behaved periodic functions f the Fourier series (B.12) converges *pointwise* to $f(x)$. A continuous periodic function possessing a continuous first derivative is sufficiently "well-behaved" for pointwise convergence. Furthermore, if the function f is smooth then the convergence is uniform. This is useful to know, but we often desire a Fourier representation for a function with discontinuities. A stronger result is that if f is *piecewise continuous* in $[0, L]$ — *i.e.*, continuous with the exception of at most finite number of discontinuities — and its first derivative is *also* piecewise continuous, then the Fourier series will converge pointwise (but not uniformly[1]) to $f(x)$ at points where $f(x)$ is continuous, and to its average

$$F(x) = \frac{1}{2} \lim_{\epsilon \to 0} \{ f(x + \epsilon) + f(x - \epsilon) \} \tag{B.15}$$

at those points where $f(x)$ has jumps. In the section B.3.2 we shall explain why the series converges to this average, and examine the nature of this convergence.

Most functions of interest to engineers are piecewise continuous, and this result is then all that they require. In physics, however, we often have to work with a broader class of functions, and so other forms of of convergence

[1] If a sequence of continuous functions converges uniformly, then its limit function is continuous.

become relevant. In quantum mechanics, in particular, the probability interpretation of the wavefunction requires only convergence in the L^2 sense, and this demands no smoothness properties at all—the Fourier representation converging to f whenever the L^2 norm $\|f\|^2$ is finite.

Half-range Fourier series

The exponential series

$$f(x) = \sum_{m=-\infty}^{\infty} a_m \, e^{-2\pi i m x/L}. \tag{B.16}$$

can be re-expressed as the trigonometric sum

$$f(x) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty} \left\{ A_m \cos(2\pi m x/L) + B_m \sin(2\pi m x/L) \right\}, \tag{B.17}$$

where

$$\begin{aligned} A_m &= \begin{cases} 2a_0 & m = 0, \\ a_m + a_{-m}, & m > 0, \end{cases} \\ B_m &= i(a_{-m} - a_m). \end{aligned} \tag{B.18}$$

This is called a *full-range* trigonometric Fourier series for functions defined on $[0, L]$. In chapter 2 we expanded functions in series containing only sines. We can expand any function $f(x)$ defined on a finite interval as such a *half-range* Fourier series. To do this, we regard the given domain of $f(x)$ as being the half interval $[0, L/2]$ (hence the name). We then extend $f(x)$ to a function on the whole of $[0, L]$ and expand as usual. If we extend $f(x)$ by setting $f(x + L/2) = -f(x)$ then the A_m are all zero and we have

$$f(x) = \sum_{m=1}^{\infty} B_m \sin(2\pi m x/L), \quad x \in [0, L/2], \tag{B.19}$$

where,

$$B_m = \frac{4}{L} \int_0^{L/2} f(x) \sin(2\pi m x/L) \, dx. \tag{B.20}$$

Alternatively, we may extend the range of definition by setting $f(x+L/2) = f(L/2 - x)$. In this case it is the B_m that become zero and we have

$$f(x) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty} A_m \cos(2\pi m x/L), \quad x \in [0, L/2], \tag{B.21}$$

with

$$A_m = \frac{4}{L}\int_0^{L/2} f(x)\cos(2\pi m x/L)\,dx. \tag{B.22}$$

The difference between a full-range and a half-range series is therefore seen principally in the continuation of the function outside its initial interval of definition. A full range series repeats the function periodically. A half-range sine series changes the sign of the continued function each time we pass to an adjacent interval, whilst the half-range cosine series reflects the function as if each interval endpoint were a mirror.

B.2 Fourier Integral Transforms

When the function we wish to represent is defined on the entirety of $\mathbb{R}$ then we must use the Fourier *integral* representation.

B.2.1 Inversion formula

We formally obtain the Fourier integral representation from the Fourier series for a function defined on $[-L/2, L/2]$. Start from

$$f(x) = \sum_{m=-\infty}^{\infty} a_m\, e^{-\frac{2\pi i m}{L}x}, \tag{B.23}$$

$$a_m = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{\frac{2\pi i m}{L}x}\,dx, \tag{B.24}$$

and let L become large. The discrete $k_m = 2\pi m/L$ then merge into the continuous variable k and

$$\sum_{m=-\infty}^{\infty} \to \int_{-\infty}^{\infty} dm = L\int_{-\infty}^{\infty}\frac{dk}{2\pi}. \tag{B.25}$$

The product La_m remains finite, and becomes a function $\widetilde{f}(k)$. Thus

$$f(x) = \int_{-\infty}^{\infty} \widetilde{f}(k)\, e^{-ikx}\,\frac{dk}{2\pi}, \tag{B.26}$$

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\,dx\,. \tag{B.27}$$

This is the Fourier integral transform and its inverse.

It is good practice when doing Fourier transforms in physics to treat x and k asymmetrically: always put the 2π's with the dk's. This is because, as (B.25) shows, $dk/2\pi$ has the physical meaning of the number of Fourier modes per unit (spatial) volume with wavenumber between k and $k + dk$.

The Fourier representation of the Dirac delta-function is

$$\delta(x - x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')}. \tag{B.28}$$

Suppose we put $x = x'$. Then "$\delta(0)$", which we earlier saw can be interpreted as the inverse lattice spacing, and hence the density of lattice points, is equal to $\int_{-\infty}^{\infty} \frac{dk}{2\pi}$. This is the total number of Fourier modes per unit length.

Exchanging x and k in the integral representation of $\delta(x - x')$ gives us the Fourier representation for $\delta(k - k')$:

$$\int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = 2\pi\, \delta(k - k'). \tag{B.29}$$

Thus $2\pi\delta(0)$ (in k space), although mathematically divergent, has the physical meaning $\int dx$, the volume of the system. It is good practice to put a 2π with each $\delta(k)$ because this combination has a direct physical interpretation.

Take care to note that the symbol $\delta(0)$ has a very different physical interpretation depending on whether δ is a delta-function in x or in k space.

Parseval's identity

Note that with the Fourier transform pair defined as

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} e^{ikx}\, f(x)\, dx \tag{B.30}$$

$$f(x) = \int_{-\infty}^{\infty} e^{-ikx}\, \widetilde{f}(k)\, \frac{dk}{2\pi}, \tag{B.31}$$

Pareseval's theorem takes the form

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\widetilde{f}(k)|^2\, \frac{dk}{2\pi}. \tag{B.32}$$

Parseval's theorem tells us that the Fourier transform is a unitary map from $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

B.2.2 The Riemann-Lebesgue lemma

There is a reciprocal relationship between the rates at which a function and its Fourier transform decay at infinity. The more rapidly the function decays, the more high frequency modes it must contain—and hence the slower the decay of its Fourier transform. Conversely, the smoother a function the fewer high frequency modes it contains and the faster the decay of its transform. Quantitative estimates of this version of Heisenberg's uncertainty principle are based on the *Riemann-Lebesgue lemma*.

Recall that a function f is in $L^1(\mathbb{R})$ if it is integrable (this condition excludes the delta function) and goes to zero at infinity sufficiently rapidly that

$$\|f\|_1 \equiv \int_{-\infty}^{\infty} |f|\, dx < \infty. \tag{B.33}$$

If $f \in L^1(\mathbb{R})$ then its Fourier transform

$$\widetilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\, dx \tag{B.34}$$

exists, is a continuous function of k, and

$$|\widetilde{f}(k)| \le \|f\|_1. \tag{B.35}$$

The Riemann-Lebesgue lemma asserts that if $f \in L^1(\mathbb{R})$ then

$$\lim_{k\to\infty} \widetilde{f}(k) = 0. \tag{B.36}$$

We will not give the proof. For f integrable in the Riemann sense, it is not difficult, being almost a corollary of the definition of the Riemann integral. We must point out, however, that the "$|\ldots|$" modulus sign is essential in the $L^1(\mathbb{R})$ condition. For example, the integral

$$I = \int_{-\infty}^{\infty} \sin(x^2)\, dx \tag{B.37}$$

is convergent, but only because of extensive cancellations. The $L^1(\mathbb{R})$ norm

$$\int_{-\infty}^{\infty} |\sin(x^2)|\, dx \tag{B.38}$$

is *not* finite, and whereas the Fourier transform of $\sin(x^2)$, *i.e.*

$$\int_{-\infty}^{\infty} \sin(x^2)\, e^{ikx}\, dx = \sqrt{\pi}\, \cos\left(\frac{k^2+\pi}{4}\right), \tag{B.39}$$

is also convergent, it does not decay to zero as k grows large.

The Riemann-Lebesgue lemma tells us that the Fourier transform maps $L^1(\mathbb{R})$ into $C_\infty(\mathbb{R})$, the latter being the space of continuous functions vanishing at infinity. Be careful: this map is only *into* and not *onto*. The inverse Fourier transform of a function vanishing at infinity does not necessariliy lie in $L^1(\mathbb{R})$.

We link the smoothness of $f(x)$ to the rapid decay of $\widetilde{f}(k)$, by combining Riemann-Lebesgue with integration by parts. Suppose that both f and f' are in $L^1(\mathbb{R})$. Then

$$\widetilde{[f']}(k) \equiv \int_{-\infty}^{\infty} f'(x)\, e^{ikx}\, dx = -ik \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx = -ik\widetilde{f}(k) \tag{B.40}$$

tends to zero. (No boundary terms arise from the integration by parts because for both f and f' to be in $L^1(\mathbb{R})$ the function f must tend to zero at infinity.) Since $k\widetilde{f}(k)$ tends to zero, $\widetilde{f}(k)$ itself must go to zero faster than $1/k$. We can continue in this manner and see that each additional derivative of f that lies in $L^1(\mathbb{R})$ buys us an extra power of $1/k$ in the decay rate of $\widetilde{f}$ at infinity. If any derivative possesses a jump discontinuity, however, *its* derivative will contain a delta-function, and a delta-function is not in $L^1(\mathbb{R})$. Thus, if n is the largest integer for which $k^n\widetilde{f}(k) \to 0$ we may expect $f^{(n)}(x)$ to be somewhere discontinuous. For example, the function $f(x) = e^{-|x|}$ has a first derivative that lies in $L^1(\mathbb{R})$, but this derivative is discontinuous. The Fourier transform $\widetilde{f}(k) = 2/(1+k^2)$ therefore decays as $1/k^2$, but no faster.

B.3 Convolution

Suppose that $f(x)$ and $g(x)$ are functions on the real line $\mathbb{R}$. We define their *convolution* $f * g$, when it exists, by

$$[f * g](x) \equiv \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi\,. \tag{B.41}$$

A change of variable $\xi \to x-\xi$ shows that, despite the apparently asymmetric treatment of f and g in the definition, the $*$ product obeys $f * g = g * f$.

B.3.1 The convolution theorem

Now, let $\tilde{f}(k)$ denote the Fourier transforms of f, *i.e.*

$$\tilde{f}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx. \tag{B.42}$$

We claim that

$$\widetilde{[f * g]} = \tilde{f}\,\tilde{g}. \tag{B.43}$$

The following computation shows that this claim is correct:

$$\begin{aligned}
\widetilde{[f * g]}(k) &= \int_{-\infty}^{\infty} e^{ikx} \left(\int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi \right) dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ikx} f(x-\xi)\, g(\xi)\, d\xi\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik(x-\xi)}\, e^{ik\xi}\, f(x-\xi)\, g(\xi)\, d\xi\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ikx'} e^{ik\xi}\, f(x')\, g(\xi)\, d\xi\, dx' \\
&= \left(\int_{-\infty}^{\infty} e^{ikx'}\, f(x')\, dx' \right) \left(\int_{-\infty}^{\infty} e^{ik\xi} g(\xi)\, d\xi \right) \\
&= \tilde{f}(k) \tilde{g}(k).
\end{aligned} \tag{B.44}$$

Note that we have freely interchanged the order of integrations. This is not always permissible, but it is allowed if $f, g \in L^1(\mathbb{R})$, in which case $f * g$ is also in $L^1(\mathbb{R})$.

B.3.2 Apodization and Gibbs' phenomenon

The convolution theorem is useful for understanding what happens when we truncate a Fourier series at a finite number of terms, or cut off a Fourier integral at a finite frequency or wavenumber.

Consider, for example, the cut-off Fourier integral representation

$$f_\Lambda(x) \equiv \frac{1}{2\pi} \int_{-\Lambda}^{\Lambda} \tilde{f}(k) e^{-ikx}\, dk, \tag{B.45}$$

where $\widetilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$ is the Fourier transform of f. We can write this as

$$f_\Lambda(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta_\Lambda(k)\widetilde{f}(k)\, e^{-ikx}\, dk \tag{B.46}$$

where $\theta_\Lambda(k)$ is unity if $|k| < \Lambda$ and zero otherwise. Written this way, the Fourier transform of f_Λ becomes the product of the Fourier transform of the original f with θ_Λ. The function f_Λ itself is therefore the convolution

$$f_\Lambda(x) = \int_{-\infty}^{\infty} \delta_\Lambda^{\rm D}(x-\xi) f(\xi)\, d\xi \tag{B.47}$$

of f with

$$\delta_\Lambda^{\rm D}(x) \equiv \frac{1}{\pi}\frac{\sin(\Lambda x)}{x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \theta_\Lambda(k) e^{-ikx}\, dk, \tag{B.48}$$

which is the inverse Fourier transform of $\theta_\Lambda(x)$. We see that $f_\Lambda(x)$ is a kind of local average of the values of $f(x)$ smeared by the approximate delta-function $\delta_\Lambda^{\rm D}(x)$. The superscript D stands for "Dirichlet," and $\delta_\Lambda^{\rm D}(x)$ is known as the *Dirichlet kernel*.

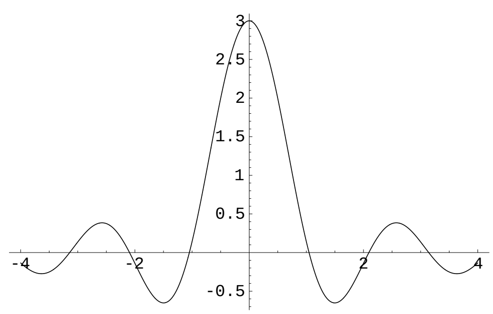

Figure B.1: *A plot of* $\pi\delta_\Lambda^{\rm D}(x)$ *for* $\Lambda = 3$.

When $f(x)$ can be treated as a constant on the scale ($\approx 2\pi/\Lambda$) of the oscillation in $\delta_\Lambda^{\rm D}(x)$, all that matters is that $\int_{-\infty}^{\infty} \delta_\Lambda^{\rm D}(x)\, dx = 1$, and so $f_\Lambda(x) \approx f(x)$. This is case if $f(x)$ is smooth and Λ is sufficiently large. However, if $f(x)$ possesses a discontinuity at x_0, say, then we can never treat it as a constant and the rapid oscillations in $\delta_\Lambda^{\rm D}(x)$ cause a "ringing" in $f_\Lambda(x)$ whose amplitude does not decrease (although the *width* of the region surrounding x_0 in which the effect is noticeable will decrease) as Λ grows. This ringing is known as *Gibbs' phenomenon*.

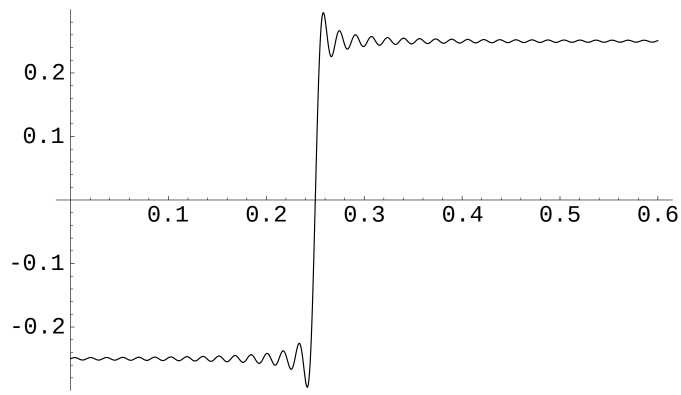

Figure B.2: *The Gibbs phenomenon: A Fourier reconstruction of a piecewise constant function that jumps discontinuously from* $y = -0.25$ *to* $+0.25$ *at* $x = 0.25$.

The amplitude of the ringing is largest immediately on either side of the the point of discontinuity, where it is about 9% of the jump in f. This magnitude is determined by the area under the central spike in $\delta^{\rm D}_\Lambda(x)$, which is

$$\frac{1}{\pi}\int_{-\pi/\Lambda}^{\pi/\Lambda} \frac{\sin(\Lambda x)}{x}\,dx = 1.18\ldots, \tag{B.49}$$

independent of Λ. For x *exactly* at the point of discontinuity, $f_\Lambda(x)$ receives equal contributions from both sides of the jump and hence converges to the average

$$\lim_{\Lambda\to\infty} f_\Lambda(x) = \frac{1}{2}\Big\{f(x_+) + f(x_-)\Big\}, \tag{B.50}$$

where $f(x_\pm)$ are the limits of f taken from the the right and left, respectively. When $x = x_0 - \pi/\Lambda$, however, the central spike lies entirely to the left of the point of discontinuity and

$$\begin{aligned} f_\Lambda(x) &\approx \frac{1}{2}\{(1+1.18)f(x_-) + (1-1.18)f(x_+)\} \\ &\approx f(x_-) + 0.09\{f(x_-) - f(x_+)\}. \end{aligned} \tag{B.51}$$

Consequently, $f_\Lambda(x)$ overshoots its target $f(x_-)$ by approximately 9% of the discontinuity. Similarly when $x = x_0 + \pi/\Lambda$

$$f_\Lambda(x) \approx f(x_+) + 0.09\{f(x_+) - f(x_-)\}. \tag{B.52}$$

The ringing is a consequence of the abrupt truncation of the Fourier sum. If, instead of a sharp cutoff, we gradually de-emphasize the higher frequencies by the replacement

$$\widetilde{f}(k) \to \widetilde{f}(k)\, e^{-\alpha k^2/2} \tag{B.53}$$

then

$$\begin{aligned} f_\alpha(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(k) e^{-\alpha k^2} e^{-ikx}\, dk \\ &= \int_{-\infty}^{\infty} \delta_\alpha^{\mathrm{G}}(x-\xi) f(y)\, d\xi \end{aligned} \tag{B.54}$$

where

$$\delta_\alpha^{\mathrm{G}}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{-x^2/2\alpha}, \tag{B.55}$$

is a non-oscillating Gaussian approximation to a delta function. The effect of this convolution is to smooth out, or *mollify*, the original f, resulting in a C^∞ function. As α becomes small, the limit of $f_\alpha(x)$ will again be the local average of $f(x)$, so at a discontinuity f_α will converge to the mean $\frac{1}{2}\{f(x_+) + f(x_-)\}$.

When reconstructing a signal from a finite range of its Fourier components—for example from the output of an aperture-synthesis radio-telescope—it is good practice to smoothly suppress the higher frequencies in such a manner. This process is called *apodizing* (*i.e.* cutting off the feet of) the data. If we fail to apodize then any interesting sharp feature in the signal will be surrounded by "diffraction ring" artifacts.

Exercise B.1: Suppose that we exponentially suppress the higher frequencies by multiplying the Fourier amplitude $\widetilde{f}(k)$ by $e^{-\epsilon|k|}$. Show that the original signal is smoothed by convolution with a *Lorentzian* approximation to a delta function

$$\delta_\epsilon^{\mathrm{L}}(x-\xi) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (x-\xi)^2}.$$

Observe that

$$\lim_{\epsilon\to 0} \delta_\epsilon^{\mathrm{L}}(x) = \delta(x).$$

Exercise B.2: Consider the apodized Fourier series

$$f_r(\theta) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta},$$

where the parameter r lies in the range $0 < r < 1$, and the coefficients are

$$a_n \equiv \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta.$$

Assuming that it is legitimate to interchange the order of the sum and integral, show that

$$\begin{aligned} f_r(\theta) &= \int_0^{2\pi} \delta_r^{\mathrm{P}}(\theta-\theta') f(\theta') d\theta' \\ &\equiv \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2}\right) f(\theta') d\theta'. \end{aligned}$$

Here the superscript P stands for for Poisson because $\delta_r^{\mathrm{P}}(\theta)$ is the *Poisson kernel* that solves the Dirichlet problem in the unit disc. Show that $\delta_r^{\mathrm{P}}(\theta)$ tends to a delta function as $r \to 1$ from below.

Exercise B.3: *The periodic Hilbert transform.* Show that in the limit $r \to 1$ the sum

$$\sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) e^{in\theta} r^{|n|} = \frac{re^{i\theta}}{1-re^{i\theta}} - \frac{re^{-i\theta}}{1-re^{-i\theta}}, \quad 0 < r < 1$$

becomes the principal-part distribution

$$P\left(i\cot\left(\frac{\theta}{2}\right)\right).$$

Let $f(\theta)$ be a smooth function on the unit circle, and define its *Hilbert transform* $\mathcal{H}f$ to be

$$(\mathcal{H}f)(\theta) = \frac{1}{2\pi} P\int_0^{2\pi} f(\theta')\cot\left(\frac{\theta-\theta'}{2}\right) d\theta'$$

Show the original function can be recovered from $(\mathcal{H}f)(\theta)$, together with knowledge of the angular average $\bar{f} = \int_0^{2\pi} f(\theta)\, d\theta/2\pi$, as

$$\begin{aligned} f(\theta) &= -\frac{1}{2\pi} P\int_0^{2\pi} (\mathcal{H}f)(\theta')\cot\left(\frac{\theta-\theta'}{2}\right) d\theta' + \frac{1}{2\pi}\int_0^{2\pi} f(\theta')\, d\theta' \\ &= -(\mathcal{H}^2 f))(\theta) + \bar{f}. \end{aligned}$$

Exercise B.4: Find a closed-form expression for the sum

$$\sum_{n=-\infty}^{\infty} |n|\, e^{in\theta} r^{2|n|}, \quad 0 < r < 1.$$

Now let $f(\theta)$ be a smooth function defined on the unit circle and

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta}\, d\theta$$

its n-th Fourier coefficient. By taking a limit $r \to 1$, show that

$$\pi \sum_{n=-\infty}^{\infty} |n|\, a_n a_{-n} = \frac{\pi}{4}\int_0^{2\pi}\!\!\int_0^{2\pi} [f(\theta) - f(\theta')]^2 \operatorname{cosec}^2\left(\frac{\theta-\theta'}{2}\right) \frac{d\theta}{2\pi}\frac{d\theta'}{2\pi},$$

both the sum and integral being convergent. Show that these last two expressions are equal to

$$\frac{1}{2}\iint_{r<1} |\nabla\varphi|^2\, r dr d\theta$$

where $\varphi(r,\theta)$ is the function harmonic in the unit disc, whose boundary value is $f(\theta)$.

Exercise B.5: Let $\widetilde{f}(k)$ be the Fourier transform of the smooth real function $f(x)$. Take a suitable limit in the previous problem to show that that

$$S[f] \equiv \frac{1}{4\pi}\int_{-\infty}^{\infty}\!\!\int_{-\infty}^{\infty} \left\{\frac{f(x) - f(x')}{x - x'}\right\}^2 dx dx' = \frac{1}{2}\int_{-\infty}^{\infty} |k| \left|\widetilde{f}(k)\right|^2 \frac{dk}{2\pi}.$$

Exercise B.6: By taking a suitable limit in exercise B.3 show that, when acting on smooth functions f such that $\int_{-\infty}^{\infty} |f|\, dx$ is finite, we have $\mathcal{H}(\mathcal{H}f) = -f$, where

$$(\mathcal{H}f)(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{f(x')}{x - x'}\, dx'$$

defines the Hilbert transform of a function on the real line. (Because $\mathcal{H}$ gives zero when acting on a constant, some condition, such as $\int_{-\infty}^{\infty} |f|\, dx$ being finite, is necessary if $\mathcal{H}$ is to be invertible.)

B.4 The Poisson Summation Formula

Suppose that $f(x)$ is a smooth function that tends rapidly to zero at infinity. Then the series

$$F(x) = \sum_{n=-\infty}^{\infty} f(x+nL) \tag{B.56}$$

converges to a smooth function of period L. It therefore has a Fourier expansion

$$F(x) = \sum_{m=-\infty}^{\infty} a_m \, e^{-2\pi i m x/L}. \tag{B.57}$$

We can compute the Fourier coefficients a_m by integrating term-by-term

$$\begin{aligned} a_m &= \frac{1}{L}\int_0^L F(x)\, e^{2\pi i m x/L}\, dx \\ &= \frac{1}{L}\sum_{n=-\infty}^{\infty}\int_0^L f(x+nL)\, e^{2\pi i m x/L}\, dx \\ &= \frac{1}{L}\int_{-\infty}^{\infty} f(x)\, e^{2\pi i m x/L}\, dx \\ &= \frac{1}{L}\widetilde{f}(2\pi m/L). \end{aligned} \tag{B.58}$$

Thus

$$\sum_{n=-\infty}^{\infty} f(x+nL) = \frac{1}{L}\sum_{m=-\infty}^{\infty} \widetilde{f}(2\pi m/L) e^{-2\pi i m x/L}. \tag{B.59}$$

When we set $x=0$, this last equation becomes

$$\sum_{n=-\infty}^{\infty} f(nL) = \frac{1}{L}\sum_{m=-\infty}^{\infty} \widetilde{f}\,(2\pi m/L)\,. \tag{B.60}$$

The equality of this pair of doubly infinite sums is known as the *Poisson summation formula*.

Example: As the Fourier transform of a Gaussian is another Gaussian, the Poisson formula with $L=1$ applied to $f(x)=\exp(-\kappa x^2)$ gives

$$\sum_{m=-\infty}^{\infty} e^{-\kappa m^2} = \sqrt{\frac{\pi}{\kappa}}\sum_{m=-\infty}^{\infty} e^{-m^2\pi^2/\kappa}, \tag{B.61}$$

and (rather more usefully) applied to $\exp(-\frac{1}{2}tx^2 + ix\theta)$ gives

$$\sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}tn^2+in\theta} = \sqrt{\frac{2\pi}{t}} \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2t}(\theta+2\pi n)^2}. \tag{B.62}$$

The last identity is known as *Jacobi's imaginary transformation*. It reflects the equivalence of the eigenmode expansion and the method-of-images solution of the diffusion equation

$$\frac{1}{2}\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial\varphi}{\partial t} \tag{B.63}$$

on the unit circle. Notice that when t is small the sum on the right-hand side converges very slowly, whereas the sum on the left converges very rapidly. The opposite is true for large t. The conversion of a slowly converging series into a rapidly converging one is a standard application of the Poisson summation formula. It is the prototype of many *duality maps* that exchange a physical model with a large coupling constant for one with weak coupling.

If we take the limit $t \to 0$ in (B.62), the right hand side approaches a sum of delta functions, and so gives us the useful identity

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx} = \sum_{n=-\infty}^{\infty} \delta(x + 2\pi n). \tag{B.64}$$

The right-hand side of (B.64) is sometimes called the "Dirac comb."

Gauss sums

The Poisson sum formula

$$\sum_{m=-\infty}^{\infty} e^{-\kappa m^2} = \sqrt{\frac{\pi}{\kappa}} \sum_{m=-\infty}^{\infty} e^{-m^2\pi^2/\kappa}. \tag{B.65}$$

remains valid for complex κ, provided that $\mathrm{Re}\,\kappa > 0$. We can therefore consider the special case

$$\kappa = i\pi\frac{p}{q} + \epsilon, \tag{B.66}$$

where ϵ is a positive real number and p and q are positive integers whose product pq we assume to be *even*. We investigate what happens to (B.65) as $\epsilon \to 0$.

The left-hand side of (B.65) can be decomposed into the double sum

$$\sum_{m=-\infty}^{\infty} \sum_{r=0}^{q-1} e^{-i\pi(p/q)(r+mq)^2} e^{-\epsilon(r+mq)^2}. \tag{B.67}$$

Because pq is even, each term in $e^{-i\pi(p/q)(r+mq)^2}$ is independent of m. At the same time, the small ϵ limit of the infinite sum

$$\sum_{m=-\infty}^{\infty} e^{-\epsilon(r+mq)^2}, \tag{B.68}$$

being a Riemann sum for the integral

$$\int_{-\infty}^{\infty} e^{-\epsilon q^2 m^2} dm = \frac{1}{q}\sqrt{\frac{\pi}{\epsilon}}, \tag{B.69}$$

becomes independent of r, and so a common factor of all terms in the finite sum over r.

If ϵ is small, we can make the replacement,

$$\kappa^{-1} = \frac{\epsilon - i\pi p/q}{\epsilon^2 + \pi^2 p^2/q^2} \to \frac{\epsilon - i\pi p/q}{\pi^2 p^2/q^2}, \tag{B.70}$$

after which, the right-hand side contains the double sum

$$\sum_{m=-\infty}^{\infty} \sum_{r=0}^{p-1} e^{i\pi(q/p)(r+mp)^2} e^{-\epsilon(q^2/p^2)(r+mp)^2}. \tag{B.71}$$

Again each term in $e^{i\pi(q/p)(r+mp)^2}$ is independent of m, and

$$\sum_{m=-\infty}^{\infty} e^{-\epsilon(q^2/p^2)(r+mp)^2} \to \int_{-\infty}^{\infty} e^{-\epsilon q^2 m^2} dm = \frac{1}{q}\sqrt{\frac{\pi}{\epsilon}}, \tag{B.72}$$

becomes independent of r. Also

$$\lim_{\epsilon \to 0}\left\{\sqrt{\frac{\pi}{\kappa}}\right\} = e^{-i\pi/4}\sqrt{\frac{q}{p}}. \tag{B.73}$$

Thus, after cancelling the common factor of $(1/q)\sqrt{\pi/\epsilon}$, we find that

$$\frac{1}{\sqrt{q}} \sum_{r=0}^{q-1} e^{-i\pi(p/q)r^2} = e^{-i\pi/4}\frac{1}{\sqrt{p}} \sum_{r=0}^{p-1} e^{i\pi(q/p)r^2}, \quad pq \text{ even}. \tag{B.74}$$

This Poisson-summation-like equality of *finite* sums is known as the *Landsberg-Schaar identity.* No purely algebraic proof is known.
Gauss considered the special case $p = 2$, in which case we get

$$\frac{1}{\sqrt{q}}\sum_{r=0}^{q-1} e^{-2\pi i r^2/q} = e^{-i\pi/4}\frac{1}{\sqrt{2}}(1+e^{i\pi q/2}) \tag{B.75}$$

or, more exlicitly

$$\sum_{r=0}^{q-1} e^{-2\pi i r^2/q} = \begin{cases} (1-i)\sqrt{q}, & q = 0 \text{ (mod 4)}, \\ \sqrt{q}, & q = 1 \text{ (mod 4)}, \\ 0, & q = 2 \text{ (mod 4)}, \\ -i\sqrt{q}, & q = 3 \text{ (mod 4)}. \end{cases} \tag{B.76}$$

The complex conjugate result is perhaps slightly prettier:

$$\sum_{r=0}^{q-1} e^{2\pi i r^2/q} = \begin{cases} (1+i)\sqrt{q}, & q = 0 \text{ (mod 4)}, \\ \sqrt{q}, & q = 1 \text{ (mod 4)}, \\ 0, & q = 2 \text{ (mod 4)}, \\ i\sqrt{q}, & q = 3 \text{ (mod 4)}. \end{cases} \tag{B.77}$$

Gauss used these sums to prove the law of quadratic reciprocity.

Exercise B.7: By applying the Poisson summation formula to the Fourier transform pair

$$f(x) = e^{-\epsilon|x|}e^{-ix\theta}, \quad \text{and} \quad \widetilde{f}(k) = \frac{2\epsilon}{\epsilon^2+(k-\theta)^2},$$

where $\epsilon > 0$, deduce that

$$\frac{\sinh\epsilon}{\cosh\epsilon - \cos(\theta-\theta')} = \sum_{n=-\infty}^{\infty}\frac{2\epsilon}{\epsilon^2+(\theta-\theta'+2\pi n)^2}. \tag{B.78}$$

Hence show that the Poisson kernel is equivalent to an infinite periodic sum of Lorentzians

$$\frac{1}{2\pi}\left(\frac{1-r^2}{1-2r\cos(\theta-\theta')+r^2}\right) = -\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{\ln r}{(\ln r)^2+(\theta-\theta'+2\pi n)^2}.$$