From Schwichtenberg textbook (2015)

Now we want to take a look at an example of how one can derive the Lie algebra of a given group.

[42] For group elements $g, h \in G$ we have $g \circ h \in G$. For elements of the Lie algebra $X, Y \in \mathfrak{g}$ we have $[X, Y] \in \mathfrak{g}$ and in general $X \circ Y \notin \mathfrak{g}$

3.4.1 The Generators and Lie Algebra of $SO(3)$

The defining conditions of the $SO(3)$ group are (Eq. 3.10)

$$O^T O \overset{!}{=} I \qquad \text{and} \qquad \det(O) \overset{!}{=} 1. \tag{3.58}$$

We can write every group element O in terms of a generator J:

$$O = \mathrm{e}^{\Phi J}. \tag{3.59}$$

Putting this into the first defining condition yields

$$O^T O = \mathrm{e}^{\Phi J^T} \mathrm{e}^{\Phi J} \overset{!}{=} 1 \rightarrow J^T + J \overset{!}{=} 0. \tag{3.60}$$

Using the second condition in Eq. 3.58 and the identity[43] $\det(\mathrm{e}^A) = \mathrm{e}^{\mathrm{tr}(A)}$ for the matrix exponential function, we see

$$\det(O) \overset{!}{=} 1 \qquad \rightarrow \qquad \det(\mathrm{e}^{\Phi J}) = \mathrm{e}^{\Phi \mathrm{tr}(J)} \overset{!}{=} 1$$

$$\rightarrow \mathrm{tr}(J) \overset{!}{=} 0 \tag{3.61}$$

[43] $\mathrm{tr}(A)$ denotes the trace of the matrix A, which means the sum of all elements on the main diagonal. For example for $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ we have $\mathrm{tr}(A) = A_{11} + A_{22}$.

Three linearly independent matrices fulfilling the conditions Eq. 3.60 and Eq. 3.61 are

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.62}$$

These matrices form a **basis** for the generators of the group $SO(3)$. This means any generator of the group can be written as a linear combination of these basis generators: $J = aJ_1 + bJ_2 + cJ_3$, where a, b, c denote real constants. These generators can be written more compactly by using the Levi-Civita symbol[45]

$$(J_i)_{jk} = -\epsilon_{ijk}, \tag{3.63}$$

where j, k denote the components of the generator J_i. For example,

$$(J_1)_{jk} = -\epsilon_{1jk} \leftrightarrow \begin{pmatrix} (J_1)_{11} & (J_1)_{12} & (J_1)_{13} \\ (J_1)_{21} & (J_1)_{22} & (J_1)_{23} \\ (J_1)_{31} & (J_1)_{32} & (J_1)_{33} \end{pmatrix} = - \begin{pmatrix} \epsilon_{111} & \epsilon_{112} & \epsilon_{113} \\ \epsilon_{121} & \epsilon_{122} & \epsilon_{123} \\ \epsilon_{131} & \epsilon_{132} & \epsilon_{133} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.64}$$

Let's see what finite transformation matrix we get from the first of these basis generators. We can focus on the lower right 2×2 matrix[46] j_1 and ignore the zeroes for a moment:

[46] This is exactly the two-dimensional Levi-Civita symbol $(j_1)_{ij} = \epsilon_{ijk}$ in matrix form, which is the generator of rotations in two dimensions (of $SO(2)$).

$$J_1 = \begin{pmatrix} 0 & \\ & \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\equiv j_1} \end{pmatrix}. \tag{3.65}$$

We can immediately compute

$$(j_1)^2 = -1, \tag{3.66}$$

therefore

$$(j_1)^3 = \underbrace{(j_1)^2}_{=-1} j_1 = -j_1 \quad , \quad (j_1)^4 = +1 \quad , \quad (j_1)^5 = +j. \tag{3.67}$$

In general, we have

$$(j_1)^{2n} = (-1)^n I \qquad \text{and} \qquad (j_1)^{2n+1} = (-1)^n j_1, \tag{3.68}$$

which we can use when we evaluate the exponential function as series expansion[47]

$$
\begin{aligned}
R_{1\ \text{fin}} &= \mathrm{e}^{\Phi j_1} = \sum_{n=0}^{\infty} \frac{\Phi^n j_1^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\Phi^{2n}}{(2n)!} \underbrace{(j_1)^{2n}}_{(-1)^n I} + \sum_{n=0}^{\infty} \frac{\Phi^{2n+1}}{(2n+1)!} \underbrace{(j_1)^{2n+1}}_{(-1)^n j_1} \\
&= \underbrace{\left(\sum_{n=0}^{\infty} \frac{\Phi^{2n}}{(2n)!} (-1)^n \right)}_{=\cos(\phi)} I + \underbrace{\left(\sum_{n=0}^{\infty} \frac{\Phi^{2n+1}}{(2n+1)!} (-1)^n \right)}_{=\sin(\phi)} j_1 \\
&= \cos(\phi) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin(\phi) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}
\end{aligned}
\tag{3.69}
$$

Therefore the complete, finite transformation matrix is, using $\mathrm{e}^0 = 1$ for the upper-left component

$$
R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}, \tag{3.70}
$$

which we can recognize as one of the well-known rotation matrices in 3-dimensions that were quoted at the beginning of this chapter (Eq. 3.23). Following the same steps, we can derive the matrices for rotations around the other axes.

We now have the generators of the group in explicit matrix form (Eq. 3.62) and we can compute the corresponding Lie bracket[48] by brute force. This yields[49]

$$
[J_i, J_j] = \epsilon_{ijk} J_k, \tag{3.71}
$$

where ϵ_{ijk} is again the Levi-Civita symbol.

In physics it's conventional to define the generators of $SO(3)$ with an extra "i", that is instead of $\mathrm{e}^{\phi J}$, we write $\mathrm{e}^{i\phi J}$ and our generators are then

$$
J_1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad J_2 = i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad J_3 = i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.72}
$$

and the Lie algebra[50] reads

$$
[J_i, J_j] = i\epsilon_{ijk} J_k. \tag{3.73}
$$

We do this in physics to get Hermitian generators, which means[51]

[48] As explained above, the natural product of the Lie algebra is the Lie bracket. Here we compute how the basis generators behave, when put into the Lie bracket. All other generators can be constructed by linear combination of these basis generators. Therefore, if we know the result of the Lie bracket of the basis generators, we know automatically the result for all other generators. This behaviour of the basis generators in the Lie bracket, will become incredibly important in the next section. Everything that is important about a Lie algebra, is encoded in the Lie bracket relation of the basis generators.

[49] For example, we have

$$
\begin{aligned}
&[J_1, J_2] = J_1 J_2 - J_1 J_2 \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} - \\
&\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \\
&\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \\
&\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underbrace{\epsilon_{12k}}_{=0 \text{ except for } k=3} J_k \\
&= \epsilon_{123} J_3 = J_3
\end{aligned}
$$

[50] We will call the Lie bracket relation of the basis generators **the** Lie algebra, because everything important is encoded here.

[51] For example now we have $J_1^\star = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ and therefore

$$
J_1^\dagger = (J_1^\star)^T = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = J
$$

$J^\dagger = (J^\star)^T = J$, because Hermitian matrices have real eigenvalues and this becomes important in quantum mechanics when the eigenvalues of the generators become the values we can expect to measure in experiments, which will be discussed in Sec. 8.3. Otherwise, that is without the "i", the generators are anti-Hermitian $J^\dagger = (J^\star)^T = -J$ and the corresponding eigenvalues are complex.

We can derive the basis generators in another way, by starting with the well known rotation matrices and using from Eq. 3.55 that $X = \frac{dh}{d\theta}|_{\theta=0}$. For the first rotation matrix, as quoted in Eq. 3.23 and derived in Eq. 3.70, this yields

$$J_1 = \frac{dR_1}{d\theta}|_{\theta=0} = \frac{d}{d\theta}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}\Big|_{\theta=0}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin(\theta) & -\cos(\theta) \\ 0 & \cos(\theta) & \sin(\theta) \end{pmatrix}\Big|_{\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (3.74)$$

which is exactly the first generator in Eq. 3.62. Nevertheless, the first method is more general, because we will not always start with given finite transformation matrices. For the Lorentz group we will start with the definition of the group, derive the basis generators and compute only afterwards the explicit matrix form for the Lorentz transformations. If you already have explicit transformation matrices, you can always use Eq. 3.55 to derive the corresponding generators.

Before we move on, we will have a look at the modern definition of a Lie algebra.

3.4.2 The Abstract Definition of a Lie Algebra

Up to this point we used a simplified definition: The Lie algebra consists of all elements X that result in an element of the corresponding group G, when put into the exponential function $e^X \in G$. Later we learned that an important part of a group, the rule for the combination of group elements, is encoded in the Lie algebra in form of the Lie bracket. As we did for groups, we distil the defining features of this idea into precise mathematical axioms:

A Lie algebra is a vector space $\mathfrak{g}$ equipped with a binary operation $[,]$: $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. The binary operation satisfies the following axioms:

- Bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and $[Z, aX + bY] = a[Z, X] + b[Z, Y]$, for arbitrary number a, b and $\forall X, Y, Z \in \mathfrak{g}$
- Anticommutativity: $[X, Y] = -[Y, X] \ \forall \quad X, Y \in \mathfrak{g}$

- The Jacobi Identity: $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$ $\forall X,Y,Z \in \mathfrak{g}$

You can check that the commutator of matrices fulfils all these conditions and of course this standard commutator was used historically to get to these axioms. Nevertheless, there are quite different binary operations that fulfil these axioms, for example, the famous Poisson bracket of classical mechanics.

The important point is that this definition makes no reference to any Lie group. The definition of a Lie algebra stands on its own and we will see that this makes sense. In the next section we will have a look at the generators of $SU(2)$ and find that the basis generators, which is the set of generators we can use to construct all other generators by linear combination, fulfil the same Lie bracket relation as the basis generators of $SO(3)$ (Eq. 3.73). This is interpreted as $SU(2)$ and $SO(3)$ having the **same** Lie algebra. This is an incredibly important result and it will tell us a lot about $SU(2)$ and $SO(3)$.

3.4.3 The Generators and Lie Algebra of $SU(2)$

We stumbled upon $SU(2)$ while trying to describe rotations in three dimensions and discovered that $SU(2)$ is the double cover[52] of $SO(3)$.

[52] Recall that this means that the map from $SU(2)$ to $SO(3)$ identifies two elements of $SU(2)$ with the same element of $SO(3)$.

Remember that $SU(2)$ is the group of unitary 2×2 matrices with unit determinant[53] :

[53] This is what the "S" stands for: Special = unit determinant.

$$U^\dagger U = UU^\dagger = 1 \tag{3.75}$$

$$\det(U) = 1. \tag{3.76}$$

The first thing we want to take a look at is the Lie algebra of this group. Writing the defining conditions of the group in terms of the generators $J_1, J_2, \dots$ yields[54]

[54] As discussed above, we now work with an extra "i" in the exponent, in order to get Hermitian matrices, which guarantees that we get real numbers as predictions for experiments in quantum mechanics.

$$U^\dagger U = (e^{iJ_i})^\dagger e^{iJ_i} \overset{!}{=} 1 \tag{3.77}$$

$$\det(U) = \det(e^{iJ_i}) \overset{!}{=} 1 \tag{3.78}$$

The first condition tells us, using the Baker-Champell-Hausdorf Theorem (Eq. 3.56) and $[J_i, J_i] = 0$

$$
\begin{aligned}
&(\mathrm{e}^{iJ_i})^\dagger \mathrm{e}^{iJ_i} = \mathrm{e}^{-iJ_i^\dagger}\mathrm{e}^{iJ_i} \stackrel{!}{=} 1 \\
&\rightarrow \mathrm{e}^{-iJ_i^\dagger + iJ_i + \frac{1}{2}[J_i^\dagger, J_i] + \ldots} \stackrel{!}{=} 1 \\
&\underbrace{\rightarrow}_{\mathrm{e}^0 = 1} J_i^\dagger \stackrel{!}{=} J_i.
\end{aligned}
\tag{3.79}
$$

A matrix fulfilling the condition $J_i^\dagger = J_i$ is called Hermitian and we therefore learn here that the generators of $SU(2)$ must be Hermitian.

Using the identity $\det(\mathrm{e}^A) = \mathrm{e}^{\mathrm{tr}(A)}$, we see from the second condition:

$$\det(\mathrm{e}^{iJ_i}) = \mathrm{e}^{i\mathrm{tr}(J_i)} = 1 \underbrace{\rightarrow}_{\mathrm{e}^0 = 1} \mathrm{tr}(J_i) \stackrel{!}{=} 0. \tag{3.80}$$

We conclude the generators of $SU(2)$ must be Hermitian traceless matrices. A basis for Hermitian traceless 2×2 matrices is given by 3 matrices[55]:

[55] A complex 2×2 matrix has 4 complex entries and therefore 8 degrees of freedom. Because of the two conditions only three degrees of freedom remain.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.81}$$

This means every Hermitian traceless 2×2 matrix can be written as a linear combination of these matrices that are called **Pauli matrices**.

We can put these explicit matrices for the basis generators into the Lie bracket, which yields

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \tag{3.82}$$

where ϵ_{ijk} is again the Levi-Civita symbol. To get rid of the nasty 2 it is conventional to define the generators of $SU(2)$ as $J_i \equiv \frac{1}{2}\sigma_i$. The Lie algebra then reads

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{3.83}$$

Take note that this is exactly the same Lie bracket relation we derived for $SO(3)$ (Eq. 3.73)! Therefore one says that $SU(2)$ and $SO(3)$ have the same Lie algebra, because we define Lie algebras by their Lie bracket. We will use the abstract definition of this Lie algebra, to get different descriptions for the transformations described by $SU(2)$. We will learn that an $SU(2)$ transformation doesn't need to be described by 2×2 matrices. To make sense of things like this, we need a more abstract definition of a Lie group. At this point $SU(2)$ is defined as a set of 2×2 matrices, and a description of $SU(2)$ by, for example, 3×3 matrices, makes little sense. The abstract definition of a Lie group will enable us to see the connection between different descriptions of the same transformation. We will identify with each Lie group a geometrical object (a manifold) and use this abstract object to define a group. This may seem like a strange thought, but will make a lot of sense after taking a second look at two examples we already encountered in earlier chapters.

3.4.4 The Abstract Definition of a Lie Group

One of the first Lie groups we discussed was $U(1)$, the unit complex numbers. These are defined by $z^\star z = 1$, which reads if we write $z = a + ib$:

$$z^\star z = (a+ib)^\star(a+ib) = (a-ib)(a+ib) = a^2 + b^2 = 1. \tag{3.84}$$

This is exactly the defining condition of the unit circle[56]. The set of unit-complex numbers **is** the unit circle in the complex plane. Furthermore, we found that there is a one-to-one map[57] between elements of $U(1)$ and $SO(2)$. Therefore, for these groups it is easy to identify them with a geometric object: The unit circle. Instead of talking about different descriptions for $SO(2)$ or $U(1)$, which are defined by objects of given dimension, it can help to think about this group as the unit-circle. Rotations in two-dimensions are, as a Lie group, the unit-circle and we can represent these transformations by elements of $SO(2)$, i.e. 2×2 matrices or elements of $U(1)$, i.e. unit-complex numbers.

The next groups we discussed were $SO(3)$ and $SU(2)$. Remember that we found a one-to-one map between $SU(2)$ and the unit quaternions. The unit quaternions are defined as those quaternions $q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ that satisfy the condition (Eq. 3.29)

$$a^2 + b^2 + c^2 + d^2 \stackrel{!}{=} 1, \tag{3.85}$$

which is the same condition that defines[58] the three sphere S^3! Therefore this map provides us with a map between $SU(2)$ and the three sphere S^3. This map is an isomorphism (1-1 and onto) and therefore we can really think of $SU(2)$ as a the three sphere S^3.

These observations motivate the modern definition of a Lie group[59]:

A Lie group is a group, which is also a differentiable manifold[60]. Furthermore, the group operation $\circ$ must induce a differentiable map of the manifold into itself. This is a compatibility requirement that ensures that the group property is compatible with the manifold property. Concretely this means that every group element, say a induces a map that takes any element of the group b to another element of the group $c = ab$ and this map must be differentiable. Using coordinates this means that the coordinates of ab must be differentiable functions of the coordinates of b.

[56] The unit circle S^1 is the set of all points in two dimensions with distance 1 from the origin. In mathematical terms this means all points (x_1, x_2) fulfilling $x_1^2 + x_2^2 = 1$.

[57] To be precise: An isomorphism. To say two things are isomorphic is the mathematical way of saying that they are "the same thing" and two things are called isomorphic if there exists an isomorphism between them.

[58] Recall that the unit circle S^1 is defined as the set of points that satisfy the condition $x_1^2 + x_2^2 = 1$. Equally, the two-sphere S^2 is defined by the condition $x_1^2 + x_2^2 + x_3^2 = 1$ and analogously the three sphere S^3 by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. The number that follows the S denotes the dimension. In two dimensions, with one condition we get a one-dimensional object: S^1. Equally we get in four dimensions, with one condition $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ a three dimensional object S^3. S^3 is the surface of the four-dimensional sphere.

[59] The technical details that follow aren't important for what we want to do in this book. The important message to take away is: Lie group = manifold.

[60] A manifold is a set of points, for example a sphere that looks locally like flat Euclidean space R^n. Another way of thinking about a n-dimensional manifold is that it's a set which can be given n independent coordinates in some neighborhood of any point. For some more information about manifolds, see the appendix in Sec. 3.11 at the end of this chapter.

By the abstract definition of a Lie algebra we say that $SO(3)$ and $SU(2)$ have the same Lie algebra (Eq. 3.83). Now it's time to talk about the remark at the end of Sec. 3.4:

"... there is precisely one **distinguished** Lie group for each Lie algebra."

We can now understand a bit better, why this one group is distinguished. The distinguished group has the property of being **simply connected**. This means that, if we use the modern definition of a Lie group as a manifold, any closed curve on this manifold can be shrunk smoothly to a point[61].

[61] We will not discuss this any further, but you are encouraged to read about it, for example in the books recommended at the end of this chapter. For the purpose of this book it suffices to know that there is always **one** distinguished group.

To emphasize this important point:[62]

[62] A proof can be found, for example, in Michael Spivak. *A Comprehensive Introduction to Differential Geometry, Vol. 1, 3rd Edition*. Publish or Perish, 3rd edition, 1 1999. ISBN 9780914098706

There is precisely one simply-connected Lie group corresponding to each Lie algebra.

This simply-connected group can be thought of as the "mother" of all those groups having the same Lie algebra, because there are maps to all other groups with the same Lie algebra from the simply connected group, but not vice versa. We could call it the mother group of this particular Lie algebra, but mathematicians tend to be less dramatic and call it the covering group. All other groups having the same Lie algebra are said to be covered by the simply connected one. We already stumbled upon an example of this: $SU(2)$ is the double cover of $SO(3)$. This means there is a two-to-one map from $SU(2)$ to $SO(3)$.

Furthermore, $SU(2)$ is the three sphere, which is a simply connected manifold. Therefore, we have already found the "most important" group belonging to this Lie algebra, i.e. Eq. 3.83. We can get all other groups belonging to this Lie algebra through maps from $SU(2)$.

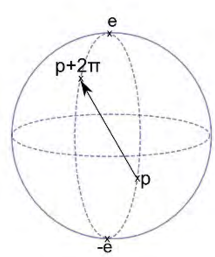

Fig. 3.7: Two-dimensional slice of the three Sphere S^3 (which is a three dimensional surface and therefore not drawable itself). We can see that the top half of the sphere is $SO(3)$, because to get from $SU(2)$ to $SO(3)$ we identify two points, for example, p and $p + 2\pi$, with each other.

We can now see what manifold $SO(3)$ is. The map from $SU(2)$ to $SO(3)$ identifies with **two** points of $SU(2)$, **one** point of $SO(3)$. Therefore, $SO(3)$ is one half of the unit sphere.

We can see, from the point of view that Lie groups are manifolds that $SU(2)$ is a more complete object than $SO(3)$. $SO(3)$ is just part of the complete object.

I want to take the view in this book that in order to describe nature at the most fundamental level, we need to use the most fundamental groups. For rotations in three dimensions this group is $SU(2)$ and not $SO(3)$. We will discover something similar when considering the symmetry group of special symmetry.

We will see that Nature agrees with such lines of thought! To describe elementary particles one uses the representations of the covering group of the Poincare group, instead of just the usual representation one uses to transform four-vectors. To describe nature at the most fundamental level, one must use the covering group, instead of any of the other groups one can map to from the covering group.

We are able to derive the representations[63] of the most fundamental group, belonging to a given Lie algebra, by deriving representations of the Lie algebra. We can then put the matrices representing the Lie algebra elements (the generators) into the exponential function to get matrices representing group elements.

[63] This notion will be made precise in the next section.

Herein lies the strength of Lie theory. By using pure mathematics we are able to reveal something fundamental about nature. The standard symmetry group of special relativity hides[64] something from us, because it is not the most fundamental group belonging to this symmetry. The covering group of the Poincare group[65] is the fundamental group and therefore we will use it to describe nature.

[64] For those who already know some quantum mechanics: The standard symmetry group hides spin from us!

[65] For brevity, we will avoid writing "double cover of" or "covering group of" most of the time. We will use one representation of the Poincare group to derive the corresponding Lie algebra. Then we will use this Lie algebra to derive the representations of the one distinguished group that belongs to this Lie algebra. In other words: The representations of the double cover of the Poincare group.

To summarize[66]

- $S^1 \hat{=} U(1) \underbrace{\leftrightarrow}_{\text{one-to-one}} SO(2)$
- $S^3 \hat{=} SU(2) \underbrace{\rightarrow}_{\text{two-to-one}} SO(3) \hat{=}$ half of S^3
  $\Rightarrow$ $SU(2)$ is the distinguished group belonging to the Lie algebra $[J_i, J_j] = i\epsilon_{ijk}J_k$ (Eq. 3.83), because S^3 is simply connected.

[66] Maybe you wonder why S^2, the surface of the sphere in three dimensions, is missing. S^2 is not a Lie group and this is closely related to the fact that there are no three-dimensional complex numbers. Recall that we had to move from two-dimensional complex numbers with just **i** to the four-dimensional quaternions with **i,j,k**.

Next, we will introduce another important branch of Lie theory, called representation theory. It is representation theory that enables us to derive from a given Lie group the tools we need to describe nature at the most fundamental level.

3.5 Representation Theory

The important thing about group theory is that it is able to describe transformations without referring to any objects in the real world.

For theoretical considerations it is often useful to regard any group as an abstract group. This means defining the group by its manifold structure and the group operation. For example $SU(2)$ is the three sphere S^3, the elements of the group are points of the manifold and the rule associating a product point ab with any two points b and a satisfies the usual group axioms. In physical applications one is more interested in what the group actually does, i.e. the group action.

An important idea is that **one** group can act on **many** different kinds of objects[67]. This idea motivates the definition of a representation: A representation is a map[68] between any group element g of a group G and a linear transformation[69] $R(g)$ of some vector-space V

[67] This will make much more sense in a moment.

[68] The mathematical term for a map with these special properties is **homomorphism**. The definition of an isomorphism is then a homomorphism, which is in addition one-to-one.

[69] In the context of this book this will always mean that we map each group element to a matrix. Each group element is then given by a matrix that acts by usual matrix multiplication on the elements of some vector space.

$$g \underbrace{\rightarrow}_{R} R(g) \tag{3.86}$$

in such a way that the group properties are preserved:

- $R(e) = I$ (The identity element of the group transforms nothing at all)
- $R(g^{-1}) = \big(R(g)\big)^{-1}$ (Every inverse element is mapped to the corresponding inverse transformation)
- $R(g) \circ R(h) = R(gh)$ (The combination of transformations corresponding to g and h is the same as the transformation corresponding to the point gh)

A representation[70] identifies with each point (abstract group element) of the group manifold (the abstract group) a linear transformation of a vector space. Although we define a representation as a map, most of the time we will call a set of matrices a representation. For example, the usual rotation matrices are a representation of the group $SO(3)$ on the vector space[71] R^3. The rotation matrices are linear transformations on R^3. But take note that we can examine the group action on other vector spaces, too.

[70] This concept can be formulated more generally if one accepts arbitrary (not linear) transformations of an arbitrary (not necessarily a vector) space. Such a map is called a realization. In physics one is concerned most of the time with linear transformations of objects living in some vector space (for example Hilpert space in quantum mechanics or Minkowski space for special relativity), therefore the concept of a representation is more relevant to physics than the general concept called realization.

[71] R^3 denotes three dimensional Euclidean space, where elements are ordinary 3 component vectors, as we use them for example in appendix A.1.

Using representation theory, we able to investigate systematically how a given group acts on very different vector spaces and that is were things start to get really interesting.

One of the most important examples in physics is $SU(2)$. For example, we can examine how $SU(2)$ acts on the complex vector space of dimension one C^1, which is especially easy, as we will see later, or two: C^2, which we will discuss in detail in the following sections. The objects living in C^2 are complex vectors of dimension two and therefore $SU(2)$ acts on them as 2×2 matrices. The matrices (=linear transformations) acting on C^2 are just the usual matrices we already know for $SU(2)$. In addition, we can examine how $SU(2)$ acts on C^3. There is a well defined framework for constructing such representations and as a result, $SU(2)$ acts on complex vectors of dimension three as 3×3 matrices. For example, a basis for the $SU(2)$ generators on C^3 is given by[72]

[72] We will learn later in this chapter how to derive these. At this point just take notice that it is possible.

$$J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.87}$$

As usual, we can then compute $SU(2)$ matrices in this representation by putting linear combinations of these generators into the exponential function.

One can go on and inspect how $SU(2)$ acts on higher dimensional vectors. This can be quite confusing and it would be better to call[73] this group S^3 instead of $SU(2)$, because usually $SU(2)$ is defined as the set of complex 2×2 (!) matrices satisfying (Eq. 3.33)

$$U^\dagger U = 1 \quad \text{and} \quad \det(U) = 1 \tag{3.88}$$

and now we write $SU(2)$ as 3×3 matrices. Therefore one must always keep in mind that we mean the abstract group, instead of the 2×2 definition, when we talk about higher dimensional representation of $SU(2)$ or any other group.

Typically a group is defined in the first place by a representation. For example, for $SU(2)$ we started with 2×2 matrices. This enables us to study the group properties concretely, as we did in the preceding chapters. After this initial study it's often more helpful to regard the group as an abstract group[74], because it's possible to find other, useful representations of the group.

Before we move on to examples we need to define some abstract, but useful, notions. These notions will clarify the hierarchy of representations, because not every possible representation is equally fundamental.

The first notion we want to talk about is **similarity transformation**. Given a matrix D and an invertible[75] matrix S then a transformation of the form

$$R \to R' = S^{-1} R S \tag{3.89}$$

is called a similarity transformation. The usefulness of this kind of transformation in this context lies in the fact that if we have a representation $R(G)$ of a group G, then $S^{-1}RS$ is also a representation. This follows directly from the definition of a representation: Suppose we have two group elements g_1, g_2 and a map R: $G \Rightarrow GL(V)$, i.e. $R(g_1)$ and $R(g_2)$. This is a representation if

$$R(g_1)R(g_2) = R(g_1 g_2) \tag{3.90}$$

[73] In an early draft version of this book the group was consequently called S^3. Unfortunately, such a non-standard name makes it hard for beginners to dive deeper into the subject using the standard textbooks.

[74] For $SU(2)$ this means using S^3.

[75] A matrix S is called invertible, if there exists a matrix T, such that $ST = TS = 1$. The inverse matrix is usually denoted S^{-1}.

If we now look at the similarity transformation of the representation

$$S^{-1}R(g_1)\underbrace{SS^{-1}}_{=1}R(g_2)S = S^{-1}R(g_1)R(g_2)S = S^{-1}R(g_1g_2)S \tag{3.91}$$

we see that this is a representation, too. Speaking colloquially, this means that if we have a representation, we can transform its elements wildly with literally any non-singular matrix S to get nicer matrices.

The next notion we want to introduce is **invariant subspace**. If we have a representation R of a group G on a vector space V we call $V' \subseteq V$ an invariant subspace if for[76] $v \in V'$ we have $R(g)v \in V'$ for all $g \in G$. This means, if we have a vector in the subspace V' and we act on it with arbitrary group elements, the transformed vector will always be again part of the subspace V'. If we find such an invariant subspace we can define a representation R' of G on V', called a subrepresentation of R, by

[76] Of course $v \in V$, too. The vector space V' must be part of the vector space V, which is mathematically denoted by $V' \subseteq V$. In other words this means that every element of V' is at the same time an element of V.

$$R'(g)v = R(g)v \tag{3.92}$$

for all $v \in V'$. Therefore, one is led to the thought that the representation R, we talked about in the first place, is not fundamental, but a composite of smaller building blocks, called subrepresentations.

This leads us to the very important notion of an **irreducible representation**. An irreducible representation is a representation of a group G on a vector space V that has no invariant subspaces besides 0 and V itself. Such representations can be thought of as truly fundamental, because they are not made up by smaller representations. The irreducible representations of a group are the building blocks from which we can build up all other representations. There is another way to think about irreducible representation: A irreducible representation cannot be rewritten, using a similarity transformation, in block diagonal form. In contrast to a reducible representation, which can be rewritten in block-diagonal form by similarity transformations. This notion is important because nature uses irreducible representations[77] to describe elementary particles. We will see later that the behaviour of elementary particles under transformations is described by irreducible representations of the corresponding symmetry group.

[77] What else?

There are many possible representations[78] for each group, how do we know which one to choose to describe nature? There is an idea that is based on the Casimir elements. A Casimir element C is build from generators of the Lie algebra and its defining feature is that it

[78] For example we already know two different representations for rotations in two-dimensions. One using complex numbers and one using 2×2 matrices. Both are representations of S^1 as a group.

commutes with every generator X of the group

$$[C, X] = 0. \tag{3.93}$$

What does this mean? A famous Lemma, called Schur's Lemma[79], tells us that if we have an irreducible representation $R : \mathfrak{g} \to GL(V)$, any linear operator $T : V \to V$ that commutes with all operators $R(X)$ must be a scalar multiple of the identity operator. Therefore, the Casimir elements give us linear operators with constant values for each representation. As we will see, these values provide us with a way of labelling representations naturally.[80] We can then start to investigate the irreducible representations, starting with the representation with the lowest possible scalar value for the Casimir element.

[79] A basic result of group theory, which you can look up in any book about group theory.

[80] This will become much clearer as soon as we look at an example.

Is there anything we can say about the vector space V mentioned in the definition of a representation above? The definition states that a representation is a map from the abstract group to the space of linear operators on a vector space. Now, from linear algebra we know that the eigenvectors of a linear operator always form a basis for the vector space. We can use this to inspect the vector space. For any Lie group, one or more of the generators of a Lie group can be diagonalized[81] using similarity transformations and we will use these diagonalized generators to get a basis of our vector space.

We will now start deriving the irreducible representations of the Lie algebra of $SU(2)$ because, as we will see, the Lie algebra of the Lorentz group can be thought of as two copies of the $SU(2)$ algebra. The Lorentz group is part of the Poincare group and we will talk about these groups in this order.

[81] The set of diagonal generators is called Cartan subalgebra, and the corresponding generators Cartan generators. These generators play a big role in quantum field theory, because the eigenvalues of the Cartan generators are used to give charge labels to elementary particles. For example, to derive quantum chromodynamics, we use the group $SU(3)$, as we will see later, and there are two Cartan generators. Therefore, each particle that interacts via chromodynamics, carries two charge labels. Conventionally instead of writing two numbers, one uses the words red, blue, green, and calls the corresponding charge colour. Analogous, the theory of weak interactions uses the group $SU(2)$, which has only one Cartan generator. Therefore, each particle is labelled by the corresponding eigenvalues of this Cartan generator.

3.6 $SU(2)$

We used in Sec. 3.4.3 specific matrices (=a specific representation) to identify how the generators of $SU(2)$ behave, when put into the Lie bracket[82]. We can use this knowledge to find further representations. We will arrive again at the representation we started with, which means the set of unitary $2\times$ matrices with unit determinant and are then able to see that it is just one special case. Before we are going to tackle this task, we want to take a moment to think about what representations we can expect.

[82] Recall that this is what we use to define the Lie algebra of a group in abstract terms. The final result was Eq. 3.83.

3.6.1 The Finite-dimensional Irreducible Representations of $SU(2)$

To learn something about what finite-dimensional, irreducible representations of $SU(2)$ are possible, we define new operators from the ones we used in Sec. 3.4.3, by linear combination[83]

$$J_+ = \frac{1}{\sqrt{2}}(J_1 + iJ_2) \tag{3.94}$$

$$J_- = \frac{1}{\sqrt{2}}(J_1 - iJ_2) \tag{3.95}$$

[83] We can always diagonalize **one** of the generators. Following the convention we choose J_3 as diagonal and therefore yielding the basis vectors for our vector space. Furthermore, it is conventional to introduce the new operators $J_\pm$ in the way we do here.

These new operators obey the following commutation relations, as you can check by using the commutator relations in Eq. 3.83

$$[J_3, J_\pm] = \pm J_\pm \tag{3.96}$$

$$[J_+, J_-] = J_3. \tag{3.97}$$

If we now investigate how these operators act on an eigenvector v of J_3 with eigenvalue[84] b we discover something remarkable:

[84] This means $J_3 v = bv$ as explained in appendix C.4.

$$\begin{aligned} J_3(J_\pm v) &= J_3(J_\pm v) + \underbrace{J_\pm J_3 v - J_\pm J_3 v}_{=0} \\ &= \underbrace{J_\pm J_3 v}_{=J_\pm bv} + \underbrace{J_3 J_\pm v - J_\pm J_3 v}_{=[J_3, J_\pm]v} \\ &\underbrace{=}_{\text{Eq. 3.96}} (b \pm 1) J_\pm v \end{aligned} \tag{3.98}$$

We conclude that $J_\pm v$ is again an eigenvector, let's call him w, of J_3 with eigenvalue $(b \pm 1)$:

$$J_3 w = (b \pm 1) w \quad \text{with} \quad w = J_\pm v. \tag{3.99}$$

The operators J_- and J_+ are called raising and lowering or **ladder operators**. We can construct more and more eigenvectors of J_3 using the operators the ladder operators $J_\pm$ repeatedly. This process must come to an end, because eigenvectors with different eigenvalues are linearly independent and we are dealing with finite-dimensional representations. This means that the corresponding vector space is finite-dimensional and therefore we can only find a finite number of linearly independent vectors.

We conclude there must be an eigenvector with a maximum eigenvalue v_{max}. After a finite number N of applications of J_+ we reach the maximum eigenvector v_{max}

$$v_{max} = J_+^N v \tag{3.100}$$

We have

$$J_+ v_{max} = 0, \tag{3.101}$$

because v_{max} is, by definition, the eigenvector with the highest eigenvalue. We call the maximum eigenvalue $j := b + N$. The same must be true for the other direction: There must be an eigenvector with minimum eigenvalue v_{min} for which the following relation holds

$$J_- v_{min} = 0 \tag{3.102}$$

Let us say we reach the minimum after operating M times with J_- on v_{max}

$$v_{min} = J_-^M v_{max}. \tag{3.103}$$

Therefore, v_{min} has eigenvalue j-M. To go further we need to know how exactly $J_\pm$ acts on eigenvectors. The computation above shows that $J_- v_k$ is, in general, a scalar multiplied by an eigenvector with eigenvalue $k-1$:

$$J_- v_k = \alpha_k v_{k-1}. \tag{3.104}$$

If we inspect in detail how J_- acts on v_{max} we get[85] the general rule for the scalar factor

$$\alpha_{j-k} = \frac{1}{\sqrt{2}} \sqrt{(2j-k)(k+1)} \tag{3.105}$$

Take note that this scalar factor becomes zero for $k = 2j$ and therefore, we have reached the end of the ladder after $2j$ steps if we start at the top. Therefore v_{min} has eigenvalue $j - 2j = -j$. We conclude that we have in general $2j+1$ eigenstates with eigenvalues

$$\{-j, -j+1, \ldots, j-1, j\} \tag{3.106}$$

This is only possible if j is an integer or an half-integer[86]. Now we know that our vector space V has $2j+1$ dimensions[87], because we have $2j+1$ linearly independent eigenvectors. Those eigenvectors of J_3 span the complete vector space V because J_1 and J_2 can be expressed in terms of J_+ and J_- and therefore take any linear combination $\sum_i a_i v_i$ into a possibly different linear combination $\sum_i b_i v_i$, with scalar factors a_i, b_i. Therefore, the span of the eigenvectors of J_3 is a non-zero invariant subspace of V and because we are looking for irreducible representations they span the complete vector space V.

We can use the construction above to define representations of $SU(2)$ on a vector space V_j with $2j+1$ dimensions and basis given by the eigenvectors v_k of J_3. Furthermore, it's possible to show that every irreducible representation of $SU(2)$ must be equivalent to one of these[88].

[85] See, for example, page 90 in Matthew Robinson. *Symmetry and the Standard Model*. Springer, 1st edition, August 2011. ISBN 978-1-4419-8267-4

[86] Try it with other fractions if you don't believe this!

[87] See, for example, page 189 in Nadir Jeevanjee. *An Introduction to Tensors and Group Theory for Physicists*. Birkhaeuser, 1st edition, August 2011. ISBN 978-0817647148

[88] See page 190 in: Nadir Jeevanjee. *An Introduction to Tensors and Group Theory for Physicists*. Birkhaeuser, 1st edition, August 2011. ISBN 978-0817647148

3.6.2 The Casimir Operator of $SU(2)$

As described in Sec. 3.5, we can naturally label representations by using the Casimir operators[89] of the group. $SU(2)$ has exactly one Casimir operator:

$$J^2 := (J_1)^2 + (J_2)^2 + (J_3)^2 \tag{3.107}$$

that fulfils the defining condition:

$$[J^2, J_i] = 0. \tag{3.108}$$

We can re-express J^2 in terms of $J_\pm$ by using the definition of $J_\pm$ in Eq. 3.95 and Eq. 3.94:

$$\begin{aligned}
J^2 &= J_+J_- + J_-J_+ + (J_3)^2 \\
&= \frac{1}{2}(J_1 + iJ_2)(J_1 - iJ_2) + \frac{1}{2}(J_1 - iJ_2)(J_1 + iJ_2) + (J_3)^2 \\
&= \frac{1}{2}\left((J_1)^2 - iJ_1J_2 + iJ_2J_1 + (J_2)^2\right) + \frac{1}{2}\left((J_1)^2 + iJ_1J_2 - iJ_2J_1 + (J_2)^2\right) \\
&\quad + (J_3)^2 \\
&= (J_1)^2 + (J_2)^2 + (J_3)^2 \quad \checkmark
\end{aligned} \tag{3.109}$$

If we now use[90]

$$J_+v_k = \frac{1}{\sqrt{2}}\sqrt{(j+k+1)(j-k)}v_{k+1} \tag{3.110}$$

and

$$J_-v_k = \frac{1}{\sqrt{2}}\sqrt{(j+k)(j-k+1)}v_{k-1} \tag{3.111}$$

we can compute the fixed scalar value for each representation:

$$\begin{aligned}
J^2v_k &= \left(\frac{1}{2}(J_+J_- + J_-J_+) + (J_3)^2\right)v_k \\
&= J_+J_-v_k + J_-J_+v_k + k^2v_k \\
&= J_+\frac{1}{\sqrt{2}}\sqrt{(j+k)(j-k+1)}v_{k-1} + J_-\frac{1}{\sqrt{2}}\sqrt{(j+k+1)(j-k)}v_{k+1} + k^2v_k \\
&= \frac{1}{\sqrt{2}}\sqrt{(j+k)(j-k+1)}J_+v_{k-1} + \frac{1}{\sqrt{2}}\sqrt{(j+k+1)(j-k)}J_-v_{k+1} + k^2v_k \\
&= \frac{1}{\sqrt{2}}\sqrt{(j+k)(j-k+1)}\frac{1}{\sqrt{2}}\sqrt{(j+(k-1)+1)(j-(k-1))}v_k \\
&\quad + \frac{1}{\sqrt{2}}\sqrt{(j+k+1)(j-k)}\frac{1}{\sqrt{2}}\sqrt{(j+(k+1))(j-(k+1)+1)}v_k + k^2v_k \\
&= \frac{1}{2}(j+k)(j-k+1) + \frac{1}{2}(j-k)(j+k+1)v_k + k^2v_k \\
&= (j^2+j)v_k = j(j+1)v_k
\end{aligned} \tag{3.112}$$

Now we look at specific examples for the representations. We start, of course, with the lowest dimensional representations.

[89] Recall that Casimir operators are defined as operators C, built from the generators of the group that commute with every generator X of the group: $[C, X] = 0$.

[90] These are just the normalization constants. If we act with $J_\pm$ onto a normalized state, the resulting state will in general not be normalized, too. Nevertheless, in physics we always prefer working with normalized states, for reasons that will become clear in the following chapters. The derivation is a bit tedious, but simply starts with $J_\pm v_k = cv_{k\pm1}$ where c is the normalization constant in question. The complete computation can be found in most books about quantum mechanics in the chapter about angular momentum and angular momentum ladder operators. If this is new to you, do not waste too much time here because the result of this section is not too important for everything that follows.

3.6.3 The Representation of $SU(2)$ in one Dimension

The lowest possible value for j is zero. In this case our representation acts on a $2j + 1 = 2 \cdot 0 + 1 = 1$ dimensional vector space. We can see that this representation is trivial, because the only 1×1 matrices fulfilling the commutation relations of the $SU(2)$ Lie algebra
$[J_l, J_m] = i\epsilon_{lmn} J_n$, are trivially 0. If we exponentiate the generator 0 we always get the transformation $U = e^0 = 1$ which changes nothing at all.

3.6.4 The Representation of $SU(2)$ in two Dimensions

We now take a look at the next lowest possible value $j = \frac{1}{2}$. This representation is $2\frac{1}{2} + 1 = 2$ dimensional. The generator J_3 has eigenvalues $\frac{1}{2}$ and $\frac{1}{2} - 1 = -\frac{1}{2}$, as can be seen from Eq. 3.106 and is therefore given by

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.113}$$

because we choose J_3 to be the diagonal generator[91]. The eigenvectors corresponding to the eigenvalues $+\frac{1}{2}$,$-\frac{1}{2}$ are:

[91] For $SU(2)$ only one generator is diagonal, because of the commutation relations. Furthermore, remember that we are able to transform the generators using similarity transformations and could therefore easily make another generator diagonal.

$$v_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{3.114}$$

We can find the explicit matrix form of the other two generators of $SU(2)$ in this basis by rewriting them using the ladder operators

$$J_1 = \frac{1}{\sqrt{2}}(J_- + J_+) \tag{3.115}$$

$$J_2 = \frac{i}{\sqrt{2}}(J_- - J_+), \tag{3.116}$$

which we get directly from inverting the definitions of $J_\pm$ in Eq. 3.95 and Eq. 3.94. Recall that a basis four the vector space of this representation is given by the eigenvectors of J_3 and we therefore express the generators J_1 and J_2 in this basis. In other words: In this basis J_1 and J_2 are defined by their action on the eigenvectors of J_3. We compute

$$J_1 v_{\frac{1}{2}} = \frac{1}{\sqrt{2}}(J_- + J_+) v_{\frac{1}{2}} = \frac{1}{\sqrt{2}}(J_- v_{\frac{1}{2}} + \underbrace{J_+ v_{\frac{1}{2}}}_{=0}) = \frac{1}{\sqrt{2}} J_- v_{\frac{1}{2}} = \frac{1}{2} v_{-\frac{1}{2}}, \tag{3.117}$$

where we used that $\frac{1}{2}$ is already the maximum value for $v_{\frac{1}{2}}$ and we cannot go higher. The factor $\frac{1}{2}$ is the scalar factor we get from Eq. 3.105. Similarly we get

$$J_1 v_{-\frac{1}{2}} = \frac{1}{\sqrt{2}}(J_- + J_+) v_{-\frac{1}{2}} = \frac{1}{2} v_{\frac{1}{2}} \tag{3.118}$$

Written in matrix form, where our basis is given by $v_{\frac{1}{2}} = (1,0)^T$ and $v_{-\frac{1}{2}} = (0,1)^T$:

$$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.119}$$

You can check that this matrix has the action on the basis vectors we derived above[92]. In the same way, we find

$$J_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3.120}$$

[92] We derived in Eq. 3.117: $J_1 v_{\frac{1}{2}} = \frac{1}{2} v_{-\frac{1}{2}}$. Using the explicit matrix form of J_1 we get $J_1 v_{\frac{1}{2}} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2} v_{-\frac{1}{2}} \quad \checkmark$.

These are the same generators $J_i = \frac{1}{2}\sigma_i$, with the Pauli matrices σ_i, we found while investigating Lie algebra of $SU(2)$ at the beginning of this chapter (Eq. 3.81). We can now see that the representation we used there was exactly this two dimensional representation. Nevertheless, there are many more, for example, in three-dimensions as we will see in the next section[93].

[93] Again, don't get confused by the name $SU(2)$, which we originally defined as the set of unitary 2×2 matrices with unit determinant. Here we mean the abstract group, defined by the corresponding manifold S^3 and we are going to talk about higher dimensional representations of this group, which result in, for example, a representation with 3×3 matrices. It would help if we could give this structure a different name (For example, using the name of the corresponding manifold S^3), but unfortunately $SU(2)$ is the conventional name.

3.6.5 The Representation of $SU(2)$ in three Dimensions

Following the same procedure[94] as in two-dimensions, we find:

$$J_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.121}$$

[94] We start again with the diagonal generator J_3, which we can write down immediately because we know its eigenvalues $(1, 0, -1)$. Afterwards, the other two generators J_1, J_2 can be derived by their action, where we again use that we can write them in terms of $J_\pm$, on the eigenvectors of J_3.

This is the representation of the generators of $SU(2)$ in three dimensions. If you're interested, you can derive the corresponding representation for the group elements of $SU(2)$ in three dimensions, by putting these generators into the exponential function. We will not go any further and deriving even higher dimensional representations.

> "To arrive at abstraction, it is always necessary to begin with a concrete reality ... You must always start with something. Afterward you can remove all traces of reality."
>
> **- Pablo Picasso**[95]

[95] As quoted in Robert S. Root-Bernstein and Michele M. Root-Bernstein. *Sparks of Genius*. Mariner Books, 1st edition, 8 2001. ISBN 9780618127450