Three-dimensional traveling-wave solutions in plane Couette flow

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Nonlinear three-dimensional time-dependent solution branches are obtained in plane Couette flow modified by plane Poiseuille flow component. It is found that as the Poiseuille component is added a branch of *time-dependent* solutions is produced from the *time-independent* solution branch in plane Couette flow, and that there exists a second branch of time-dependent solutions in the form of a closed loop inside the primary time-dependent solution branch. The second branch intersects the line of zero plane Poiseuille flow component at two points with nonvanishing phase velocity for higher Reynolds numbers, creating shape preserving nonlinear traveling-wave solutions in plane Couette flow. [S1063-651X(97)00902-1]

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In order to understand transition mechanisms from laminar state to turbulence for the simplest form of shear motion, plane Couette flow has been studied a great deal both theoretically [1–3] and experimentally [4–9] in the last few years. In experiments, turbulent spots are triggered by injecting a liquid jet into a stable laminar state and streamwise vortex structures are observed. On the theoretical side, mainly because of the lack of the linear instability mechanism, this flow had been defying proper nonlinear investigations for decades, until Nagata [10] discovered a branch of nonlinear time-independent three-dimensional solutions, known as the Nagata solution, numerically. The solution originates from the Taylor vortex flow in a circular Couette system. Although the accuracy of the Nagata solution has been improved a great deal [11], the stability of the solution is not yet conclusive due to the lack of sufficient computational power [12]. However, the existence of the subcritical solution itself is expected to play an important role in the phase space dynamics. Recently, other types of finite amplitude steady solutions [2,3] in plane Couette flow are found numerically by extending a two-dimensional solution branch bifurcating from a laminar plane Poiseuille flow to the plane Couette flow region. None of the experimental counterparts of the finite amplitude steady solutions has been detected yet.

As for plane Poiseuille flow, the importance of streamwise vortex structures has been recognized [13] in the transition process. Since the Nagata solution does not originate from the spanwise vortex flow, it is quite natural to extend the solution to a mixed flow situation with two Reynolds numbers: $R=U_0L/\nu$ based on the total translational boundary motion U_0, and $Q=(G/2\rho)L^3/\nu^2$ based on the pressure gradient G imposed along the channel. In the definitions of the two Reynolds numbers, ρ and ν are the density and the kinematic viscosity of the fluid, respectively, and L is the whole width of the channel.

The basic flow with the appropriate boundary conditions on the channel walls, $z=\pm\frac{1}{2}$, is given by the exact solution of the Navier-Stokes equation for an incompressible fluid:

$$U(z)=-Rz+Q(z^2-\tfrac{1}{4}). \tag{1}$$

The equations for the velocity deviation,

$$\mathbf{u}=\check{U}(t,z)\hat{\mathbf{i}}+\nabla\times\nabla\times(\phi\hat{\mathbf{k}})+\nabla\times(\psi\hat{\mathbf{k}}), \tag{2}$$

from Eq. (1), where $\check{U}(t,z)$ is the modification of the mean flow through the action of the Reynolds stress, and $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ are the unit vectors in x (the streamwise direction) and z, respectively, are solved by the Galerkin projection method [10].

No-slip boundary conditions

$$\phi=\phi'=\psi=\check{U}=0 \tag{3}$$

are applied on $z=\pm\frac{1}{2}$.

Assuming a traveling-wave type of solution propagating in the streamwise direction x and periodic in the spanwise direction y, we expand ϕ, ψ, and $\check{U}$ in the form

$$\phi=\sum_{\ell=1}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}a_{\ell mn}\exp i[m\alpha(x-\gamma t)+n\beta y]f_{\ell}(z), \tag{4}$$

$$\psi=\sum_{\ell=1}^{\infty}\sum_{m=\infty}^{\infty}\sum_{n=-\infty}^{\infty}b_{\ell mn}\exp i[m\alpha(x-\gamma t)+n\beta y]\times\sin\ell\pi(z+\tfrac{1}{2}), \tag{5}$$

$$\check{U}(z)=\sum_{k=1}^{\infty}c_k\sin 2k\pi z+\sum_{k=1}^{\infty}d_k\sin(2k-1)\pi(z+\tfrac{1}{2}), \tag{6}$$

where α and β are the wave numbers in the x and the y directions and γ is the phase velocity. The Chandrasekhar function f_{ℓ}(z) satisfies $f_{\ell}(\pm1/2)=f'_{\ell}(\pm1/2)=0$. Because of the contribution from the symmetric basic velocity profile for $Q\neq0$, the amplitudes $a_{\ell\mathrm{mn}}$ and $b_{\ell\mathrm{mn}}$ are complex in general, whereas the mean flow distortion components c_k and d_k are real.

For numerical purposes we truncate the expansions (4), (5), and (6) so that only those terms satisfying

$$\ell+|m|+|n|<N,\quad k<N' \tag{7}$$

are taken into account. In order to determine the phase velocity γ, the real (or imaginary) part of one of the amplitudes is set to zero without losing generality [14]. The resulting

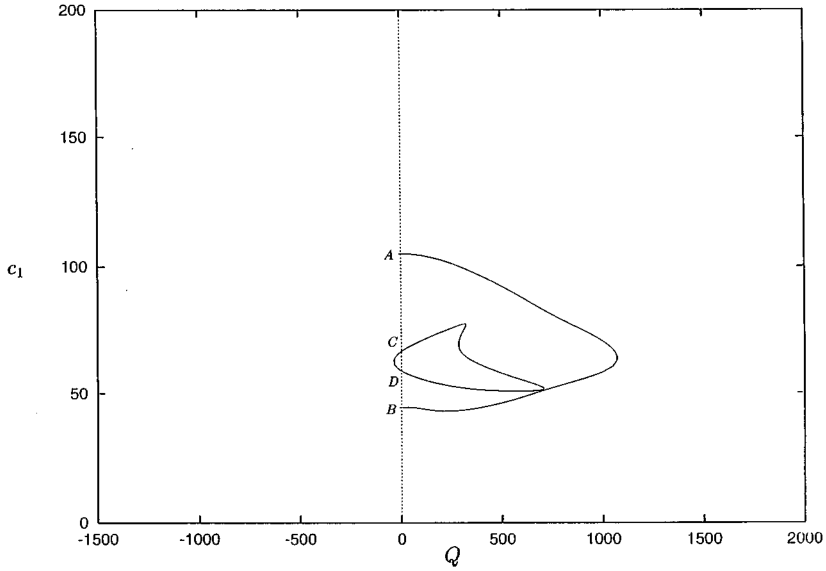

FIG. 1. The amplitude c_1 of the time-dependent solutions in plane Couette/Poiseuille flow at $R=600$. The primary branch is continued from Nagata solution at A and B on $Q=0$. The second branch in the form of a loop inside is not connected to the primary branch. The points C and D correspond to the traveling-wave solutions in plane Couette flow.

finite system of nonlinear algebraic equations for the amplitudes $a_{\ell mn}$, $b_{\ell mn}$, c_k, and d_k and the phase velocity γ are solved by the Newton-Raphson method.

In this paper, a rather low truncation level $(N,N')=(12,8)$ is taken. Also, the wave number pair (α,β) is limited to (1.3,2.6), which is situated near the center of the existence region for the steady three-dimensional solutions in the $\alpha\beta$ plane when $R=600$ [see Fig. 3(b) in [11]].

We start with the Nagata solution at $R=600$. Both the amplitudes d_k and the phase velocity γ depart from zero, as Q is gradually added. The primary time-dependent solution branch for $Q\neq 0$ is connected to the upper and the lower branches of the Nagata solution at $Q=0$. It is found that there exists a second branch of time-dependent solutions in the form of a closed loop inside the primary branch as shown in Fig. 1. (The primary branch merges into the second for $R=800$. The loop at $R=600$ in Fig. 1 was actually continued from the solution on the second branch at $R=800$ by keeping Q constant.) Note that the loop at $R=600$ intersects the line of $Q=0$ at the two points, C and D in Fig. 1, where the phase velocities turn out to be nonzero, corresponding to three-dimensional traveling-wave solutions in plane Couette flow.

Figure 2 shows the continuation of the time-dependent solutions at higher values of R in plane Couette flow, together with the Nagata solution. The time-dependent solutions appear suddenly at R slightly higher than R_c for the abrupt bifurcation of the Nagata solution. The phase velocity of the time-dependent solutions is plotted in Fig. 3.

Three-dimensional traveling-wave solutions in plane Couette flow are found in the process of extending the three-dimensional steady solutions obtained previously to a region mixed with plane Poiseuille flow. This process is not simply the inverse of those successfully used in finding other types of steady solutions [2,3] in plane Couette flow, where the two-dimensional traveling-wave solution branch in plane Poiseuille flow is extended to the mixed flow region until it attains either two-dimensional spatially localized steady flows [2] or a three-dimensional equilibrium state [3] in the plane Couette flow limit with vanishing phase velocity. Since

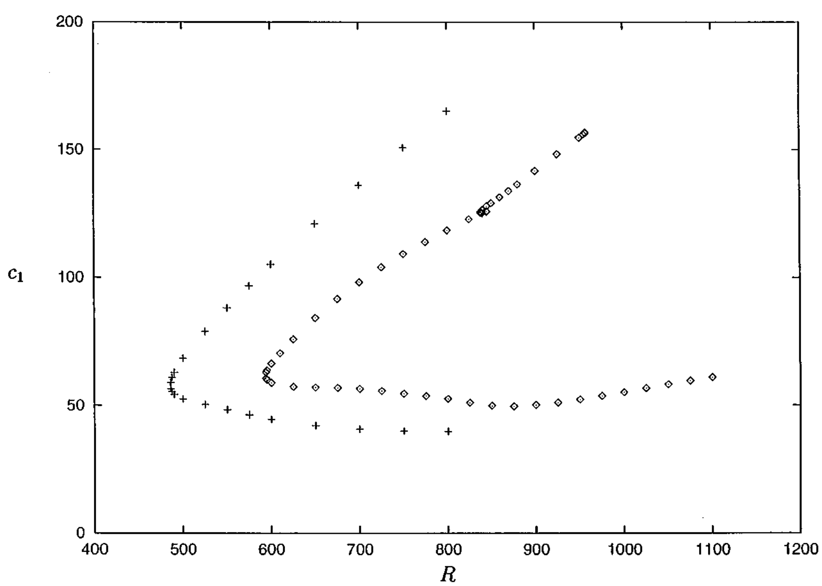

FIG. 2. The amplitude c_1 of the nonlinear three-dimensional solutions in plane Couette flow ($Q=0$). Nagata solutions are indicated by the crosses and new time-dependent solutions are indicated by the diamonds.

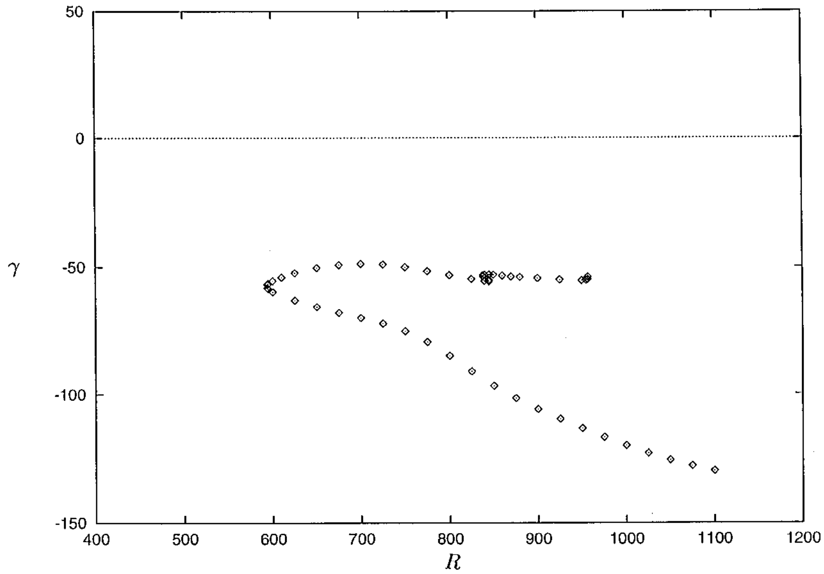

FIG. 3. The phase velocity γ of the time-dependent three-dimensional solutions in plane Couette flow ($Q=0$).

the three-dimensional steady solutions that have been extended to the traveling-wave solution in the present analysis originate from the streamwise vortex flow and do not possess the spanwise vortex structure responsible for the instability of plane Poiseuille flow, they are structurally independent of those steady solutions originating from plane Poiseuille flow. This structural independence also holds between the traveling-wave solutions extended by the two processes in the mixed flow region. It would be interesting to see whether the present form of time-dependent solutions exists even in the plane Poiseuille flow limit. Conventionally, only those solutions bifurcating primarily at the linear critical Reynolds number Q_c have been considered in the nonlinear stability analysis of plane Poiseuille flow [14], where the secondary vortex flows are independent of the spanwise direction deduced from the Squire theorem.

Because of the symmetry of the problem in the plane Couette flow limit, the flow with the opposite phase velocity is also a solution. Although the superposition of two traveling-wave solutions propagating in opposite directions is not permitted in nonlinear analyses, some form of a standing wave solution might be a possibility. Time periodic flows with a small scale deviation from the Nagata solution have been reported in the numerical simulation [15]. In contrast to the traveling wave solutions found in the current study, they are not shape preserving. Calculations at higher truncation levels are under way.

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