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Cantori for symplectic maps near the anti-integrable limit

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Abstract. We prove the existence of 'cantori' of all incommensurate rotation vectors, for symplectic maps of arbitrary dimension near enough to any non-degenerate anti-integrable limit, and derive an asymptotic form for them. Cantori are invariant Cantor sets which can be thought of as remnants of KAM tori.

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1. Introduction

KAM theory establishes the remarkable fact that for Hamiltonian systems close enough to a non-degenerate integrable one, all the invariant tori of the integrable system with sufficiently incommensurate winding ratio persist, just slightly deformed (see [1, 35, 13] for reviews). A similar result also holds for symplectic maps, the discrete time analogue of Hamiltonian systems.

A key question is what happens to these invariant tori when the perturbation away from the integrable case is large.

In the case of area-preserving twist maps, Aubry–Mather theory provides a very satisfying answer [7, 30] (see [9] for a review). For every $\omega \in \mathbb{R}$ there is a special set M_ω defined as the set of points whose orbits have rotation number ω, are recurrent, and have minimal action. For ω irrational, M_ω is either a circle or a Cantor set, named 'cantorus' by Percival [36]. Recall that a *Cantor set* is a topological space that is compact, totally disconnected and has no isolated points. M_ω satisfies continuity properties with respect to variations in the map and in ω [31]. Whenever there is a rotational (i.e. homotopically non-trivial) invariant circle of rotation number ω, then it is M_ω. Examples are easily constructed for which all the circles become cantori on sufficient perturbation from integrable (e.g. [32]). So the fate of an invariant circle is to form a dense set of gaps and become a cantorus. The transition between the two shows remarkable scaling properties, and a renormalization theory has been proposed to explain it (e.g. [23, 24]).

Extension of Aubry–Mather theory to higher dimensions appears to be difficult. Under suitable conditions, there are periodic orbits of all rational rotation vectors

[18]. There are results about the set of rotation vectors for orbits of minimal action [10, 33]; however, a result of Hedlund shows that in general one cannot hope to obtain minimizing orbits for all rotation vectors [20]. Bernstein and Katok [11] show that, for symplectic maps close enough to integrable, the minimizing periodic orbits satisfy some regularity properties which are sufficient to ensure the existence of a limiting orbit as the rotation vector approaches any limit. However, they cannot prove anything about the rotation vectors of the limiting orbits.

It is simple to make examples of multi-degree of freedom symplectic maps with analogues of cantori. Simply take the product of several area-preserving maps with cantori. Furthermore, if the chosen cantori are hyperbolic, then the product cantorus persists under C^1-small couplings (by structural stability of hyperbolic sets, e.g. [40]). It is somewhat exceptional, however, for a system to be close to a product of decoupled maps.

In a previous paper [15], we approached the problem from a limit complementary to integrable systems. This has now been named the *anti-integrable limit* by Aubry. We found explict analogues of cantori for a class of multidimensional generalizations of the 'sawtooth map' [2, 36, 3, 4], and used hyperbolicity to prove the existence of similar invariant sets for some smooth systems.

In this paper, we apply ideas about the anti-integrable limit, appearing in [6, 5, 41] for the case of one degree of freedom, to deduce much stronger results. We study symplectic maps with a generating function (the definition is recalled in section 2) of the form

$$h(x, x') = V(x) + U(x') + \varepsilon T(x, x') \qquad x, x' \in \mathbb{R}^d \tag{1}$$

with

$$h(x+m, x'+m) = h(x, x') \qquad \forall m \in \mathbb{Z}^d \tag{2}$$

and where $\varepsilon \neq 0$ is small. We assume, V, U and T are C^2. Without loss of generality (by the coordinate change $Y = y + DU(x)$, $X = x$, *cf.* (5)), we take U of (1) to be zero.

The anti-integrable limit of (1) is the case $\varepsilon = 0$.

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$$h(x, x') = V(x) + U(x'). \tag{3}$$

We say a symplectic map is close to the anti-integrable limit (3) if it has a generating function which is C^2-close to (3).

We establish the existence of higher-dimensional analogues of cantori of all incommensurate rotation vectors for all symplectic maps close enough to a non-degenerate anti-integrable limit. Our argument is one of continuation, and can be seen as an analogue of KAM theory (though it is much simpler than KAM theory). We call it *anti-KAM theory*. This is not to be confused with *converse KAM theory*, criteria for non-existence of invariant tori [32, 29, 28, 25] which, though related, is different.

The idea of continuing from a fully chaotic (Anosov) limit was tried by Percival, Vivaldi and others [38, 37, 12]. The idea of [6, 5, 41], continuation from a singular limit, is more fruitful.

The plan of our paper is as follows. First we recall the variational formulation of symplectic maps (section 2). Next we discuss the anti-integrable limit and the main continuation theorem (section 3). From this we prove the existence of cantori for symplectic maps near enough to a non-degenerate anti-integrable limit (section 4). In section 5, we calculate cantori explicitly for all generic multidimensional sawtooth mappings. This allows us to deduce the asymptotic form of cantori in the anti-integrable limit for general symplectic maps (section 6). We close with a short discussion (section 7).

2. Generating functions

A function $h: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is said to *generate* the symplectic map F: $(x, y) \mapsto (x', y)$ on $R^d \times R^d$, if

$$y' = D_2 h(x, x') \tag{4}$$

$$y = -D_1 h(x, x') \tag{5}$$

and (5) defines a diffeomorphism $(x, x') \mapsto (x, y)$ of $\mathbb{R}^d \times \mathbb{R}^d$. Here D_1, D_2 refer to the derivatives with respect to the first and second arguments, respectively (which are themselves vectors). There are various conditions which guarantee that h generates a map F (*e.g.* [18, 21, 28]); one such condition is that $D_{12}h$ is uniformly definite. Furthermore, (1) generates a map provided $\varepsilon \neq 0$ and the map $(x, x') \mapsto (x, -D_1 T(x, x'))$ is a diffeomorphism of $\mathbb{R}^{2d}$. If h satisfies the integer translation invariance property of (2), then F induces an exact symplectic map f on $\mathbb{T}^d \times \mathbb{R}^d$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (the real numbers modulo integer translations) is the circle.

In the integrable limit, coordinates can be chosen so that h depends only on $(x' - x)$. In the anti-integrable limit, h has no joint dependence on x and x'; this corresponds to $\varepsilon = 0$ for (1).

The orbits of F (with their time labelling) correspond to the sequences $x \in (\mathbb{R}^d)^{\mathbb{Z}}$ for which the *action*

$$W_{m,n}(x) = \sum_{t=m}^{n-1} h(x^t, x^{t+1}) \tag{6}$$

is stationary with respect to variations in x fixing the end-points x^m, x^n (for $x \in (\mathbb{R}^d)^{\mathbb{Z}}$; we use a superscript to indicate the time index in $\mathbb{Z}$ and a subscript for the dimension index in $\{1, \ldots d\}$). Orbits of f correspond to the equivalence classes of sequences under the action of $\mathbb{Z}^{d+1}$ given by

$$x \mapsto T_{\tau,m}(x),\ (\tau, m) \in \mathbb{Z} \times \mathbb{Z}^d \qquad T_{\tau,m}(x)^t = x^{t+\tau} + m. \tag{7}$$

There is a natural homeomorphism from the set of sequences to the set of orbits, when h generates a map. We endow the space of sequences, $(\mathbb{R}^d)^{\mathbb{Z}}$, with the product topology, i.e. if $y, x(k) \in (\mathbb{R}^d)^{\mathbb{Z}}$, $\forall k \in \mathbb{Z}_+$, we say $x(k) \to y$, as $k \to \infty$ if $x(k)^t \to y^t$ as $k \to \infty$, $\forall t \in \mathbb{Z}$.

The set of stationary states in $(\mathbb{R}^d)^{\mathbb{Z}}/\mathbb{Z}^d$ (with the quotient of the subset of the product topology) is denoted by $\mathcal{S}$.

Equation (5) then gives a homeomorphism g (cf [9]) from $\mathcal{S}$ to $\mathbb{T}^d \times \mathbb{R}^d$, defined by

$$g(x) = (x^0, y^0) \qquad y^0 = -D_1 h(x^0, x^1). \tag{8}$$

The action of f on $\mathbb{T}^d \times \mathbb{R}^d$ is equivaent to the shift σ on $\mathcal{S}$: $\sigma(x)^t = x^{t+1}$.

3. The anti-integrable limit

The anti-integrable limit is singular in that h does not generate a map. Nonetheless, from the variational point of view it is perfectly well behaved. For the anti-integrable system, the stationary sequences are precisely those for which x^t is a critical point of V, for all $t \in \mathbb{Z}$. We denote the set of critical points of V by E, and the set of sequences $x: \mathbb{Z} \to E$ by Σ.

Choosing a norm $|\cdot|$ on $\mathbb{R}^d$, for $x \in (\mathbb{R}^d)^{\mathbb{Z}}$ we define

$$b(x) = \sup_{t \in \mathbb{Z}} |D_2 T(x^{t-1}, x^t) + D_1 T(x^t, x^{t+1})|. \tag{9}$$

We will restrict attention to those sequences for which b is finite. We let $\Sigma_B \subset \Sigma$ denote the set of sequences in Σ with $b(x) \leqslant B$. It is invariant under the action of $\mathbb{Z}^{d+1}$.

Now let us suppose that the critical points of V are non-degenerate (i.e. $\det D^2V \neq 0$ at each of them), as is generically the case. Then each $x \in \Sigma$ is a non-degenerate critical point of the action. It follows that (cf [6, 5, 41] for the case $d = 1$):

Theorem 1. Given $B > 0$, there is an $\varepsilon_0(B) > 0$ such that all stationary states of Σ_B persist for $\varepsilon < \varepsilon_0$ and remain non-degenerate.

Proof. The equation for a stationary state is

$$D_2 h(x^{t-1}, x^t) + D_1 h(x^t, x^{t+1}) = 0 \qquad \forall t \in \mathbb{Z}. \tag{10}$$

Let $B \in \mathbb{R}_+$ and $x(0) \in \Sigma_B$. Let z be the deviation of a state x from $x(0)$:

$$x = x(0) + z. \tag{11}$$

Then (10) can be written as

$$F(z) = \varepsilon G(z) \qquad z \in (\mathbb{R}^d)^{\mathbb{Z}} \tag{12}$$

where

$$F(z)^t = DV(x^t) \tag{13}$$

$$G(z)^t = -D_2 T(x^{t-1}, x^t) - D_1 T(x^t, x^{t+1}). \tag{14}$$

Let $\mathcal{B}$ be the Banach space of bounded sequences $z \in (\mathbb{R}^d)^{\mathbb{Z}}$ with the norm

$$\|z\| = \sup_{t \in \mathbb{Z}} |z^t|. \tag{15}$$

By the hypotheses on Σ_B, F and G map a neighbourhood of 0 in $\mathcal{B}$ into $\mathcal{B}$, and they are differentiable. The operator $D^2V(x(0)^t)$ is invertible and its inverse is bounded uniformly, since non-degeneracy of the critical points of $V: \mathbb{T}^d \to \mathbb{R}$ and compactness of $\mathbb{T}^d$ implies there are only finitely many of them. Thus $DF(0)$ is invertible with bounded inverse. Hence by the implicit function theorem (e.g. [39]) it follows that for ε small enough (depending only on V, T and B), there is a unique continuous solution $z(\varepsilon)$ of (12) with $z(0) = 0$, and hence a unique continuation $x(\varepsilon)$ of $x(0)$ as a stationary state. Furthermore $x(\varepsilon)$ is non-degenerate.

We denote the mapping $x(0) \mapsto x(\varepsilon)$ by Φ.

In contrast to KAM theory, it is straightforward to obtain explicit estimates of ε_0

in theorem 1. For example, suppose that $\forall t \in \mathbb{Z}$, $|D^2V(x^t)^{-1}| \leqslant \alpha^{-1}$ on $|x^t - x(0)^t| \leqslant \delta$, so $\|DF^{-1}\| \leqslant \alpha^{-1}$ on $\|z\| \leqslant \delta$. Also suppose that $\|DG\| \leqslant \beta$ on $\|z\| \leqslant \delta$, and that $\|G(0)\| \leqslant B$. Then $x(0)$ can be continued, non-degenerately, with

$$\frac{\mathrm{d}z}{\mathrm{d}\varepsilon} = (DF - \varepsilon DG)^{-1}G(z) \tag{16}$$

as long as $(DF - \varepsilon DG)$ remains invertible. Now under the conditions

$$\|z\| \leqslant \delta \qquad \varepsilon < \alpha/\beta \tag{17}$$

we have the following two estimates:

$$\|G(z)\| \leqslant B + \beta\|z\| \tag{18}$$

$$\|(DF - \varepsilon DG)^{-1}\| \leqslant (\alpha - \varepsilon\beta)^{-1}. \tag{19}$$

So long as conditions (17) hold then

$$\frac{\mathrm{d}\|z\|}{\mathrm{d}\varepsilon} \leqslant \frac{B + \beta\|z\|}{\alpha - \varepsilon\beta}. \tag{20}$$

This is easily integrated to give

$$\|z\| \leqslant \frac{B\varepsilon}{\alpha - \varepsilon\beta}. \tag{21}$$

Hence $\|z\| \leqslant \delta$ for

$$\varepsilon \leqslant \varepsilon_0 = \frac{\alpha\delta}{B + \beta\delta} \tag{22}$$

and so $x(0)$ can be continued, and remains non-degenerate, for at least this range of ε.

Theorem 1 has many consequences. Our aim in the present paper is to find analogues of cantori. In order to do this, we need the following.

Theorem 2. The f-invariant set $C_\varepsilon \subset \mathbb{T}^d \times \mathbb{R}^d$ obtained for $\varepsilon < \varepsilon_0$ by continuation from a compact subset $C \subset \Sigma_B/\mathbb{Z}^d$ is homeomorphic to C.

Proof. From the implicit function theorem, the mappings

$$\varepsilon \mapsto x^t(\varepsilon) \qquad x(0) \in C \qquad t \in \mathbb{Z} \tag{23}$$

are uniformly continuous. Hence the mapping $\Phi: C \to \mathcal{S}$, with the product topology on both sides, is continuous. It is 1–1 because the stationary states remain non-degenerate and hence cannot collide. The inverse of a continuous bijection from a compact set is continuous. Hence Φ is a homeomorphism from C to its image. Since, when $\varepsilon \neq 0$, the map g from $\mathcal{S}$ to $\mathbb{T}^d \times \mathbb{R}^d$ is a homeomorphism, the composition $g\Phi$ is a homeomorphism of C onto its image C_ε.

As an example of application of this proposition, we deduce a significant result for maps f with generating function of the form

$$h(x, x') = V(x) + \frac{\varepsilon}{2}(x' - x)^{\mathrm{T}}T(x' - x) \qquad V(x + m) = V(x) \qquad \forall m \in \mathbb{Z}^d \tag{24}$$

where T is a non-degenerate symmetric matrix (e.g. $d=1$: the standard map [16, 19]; $d=2$: the Froeschlé map [17]). Such maps commute with the group generated by

$$(x, x') \mapsto (x+m, x'+n) \qquad \forall m, n \in \mathbb{Z}^d \tag{25}$$

and can therefore be regarded as maps on $\mathbb{T}^{2d}$. Choose B large enough so that the graph of the allowed transitions between successive pairs of critical points for sequences in Σ_B has positive entropy. Let Σ' be the set of sequences corresponding to an irreducible component of Σ_B with positive entropy. The set $\Sigma'/\mathbb{Z}^{2d}$ is then compact, and forms a Cantor set, in the quotient of the product topology.

Corollary. The invariant set obtained from Σ' for f, regarded as a map on $\mathbb{T}^{2d}$, is a Cantor set.

Remark. All compact invariant sets obtained by theorm 1 are uniformly hyperbolic. This follows from a general result of [8], namely that uniform hyperbolicity for symplectic twist maps is equivalent to boundedness of $\|D^2W^{-1}\|$.

4. Remnants of KAM tori

Theorem 2 can be used to prove existence of all sorts of orbits near the anti-integrable limit (cf [5]). Our interest here is to use it to find remnants of the KAM tori from the integrable limit.

A KAM torus for a symplectic map $f:\mathbb{T}^d\times\mathbb{R}^d\to\mathbb{T}^d\times\mathbb{R}^d$ with a generating function h corresponds to a set of stationary sequences of the form

$$x^t = X(\theta+\omega t) \qquad \theta\in\mathbb{R}^d \tag{26}$$

with y^t defined by the homeomorphism g of (8), and $X:\mathbb{R}^d\to\mathbb{R}^d$ a continuous function such that

$$X(\theta+m) = X(\theta)+m \qquad \forall m\in\mathbb{Z}^d \tag{27}$$

and $\omega\in\mathbb{R}^d$. We call the vector ω its *rotation vector* and the function X the *envelope function.*

We define a *remnant torus* for f to be any invariant set specified in the same way except that X is not required to be continuous. We include in the remnant all the sequences which are limit points of the above sequences. So there may be more than one specification of X.

We say a remnant torus is a *cantorus* if it is a Cantor set (the definition was recalled in section 1). This generalizes the notion of the cantorus for $d=1$, though it includes many sets other than the minimizing set, cf [34]. The anti-integrable limit does not correspond to a map, so the definition of a cantorus does not apply directly. But we extend it to include any set of stationary states of the form (26), (27) forming a Cantor set in $\mathcal{S}$.

Theorem 3. For B large enough, Σ_B contains cantori of all incommensurate rotation vectors.

Proof. Choose a critical point $e\in E$, and choose a fundamental domain for the action of $\mathbb{Z}^d$ such that e lies in its interior. Then simply take $X(\theta)=[\theta]$, the map from $\mathbb{R}^d$ to the copy of e in the same fundamental domain as θ. Ambiguites in what to do with points on the boundary of a fundamental domain are taken care of by the inclusion of all limit sequences in the remnant. Then $b(X(\theta))$ is bounded; take B to be an upper bound. After taking the quotient by $\mathbb{Z}^d$, the resulting set is a Cantor set: it is compact because it is closed and bounded, it is totally disconnected because only points of the lattice $e + \mathbb{Z}^d$ are used, and it has no isolated points because the rotation vector is chosen to be incommensurate.

Note that the chosen critical point e does not have to be a minimum, not even a local one. This proof can clearly be generalized to give many other cantori arising from using more than one critical point.

Theorem 4 (anti-KAM theorem). If V has a non-degenerate critical point e, then all systems near the anti-integrable limit $h(x, x') = V(x)$ possess cantori of all incommensurate rotation vectors.

Proof. From theorems 1, 2 and 3.

By the remark at the end of theorem 3, all these cantori are uniformly hyperbolic.

Note that the same method also gives interesting sets for ω commensurate, which are also Cantor sets unless ω is rational, when they are just periodic orbits.

We will derive an asymptotic form for these cantori, by finding them explictly for a piecewise linear map whose generating function agrees with that of the original system to quadratic order around the critical point.

5. Explicit cantori for sawtooth maps

We compute these cantori explicitly for a class of piecewise linear maps. These are a generalization of the $d = 1$ sawtooth mapping for which explicit cantori were found [2, 36, 3, 4]. A subclass was treated in [15].

Let $\{x\}$ denote the reduction of $x \in \mathbb{R}^d$ to some fundamental domain K for the action of $\mathbb{Z}^d$ by integer translations, such that 0 is in the interior of K. Define $[x]$ to be the mapping from x to the point of $\mathbb{Z}^d$ in the same copy of the fundamental domain. The sawtooth maps are defined by the generating function

$$h(x, x') = \frac{\varepsilon}{2}(x' - x)^{\mathrm{T}}T(x' - x) + \tfrac{1}{2}\{x\}^{\mathrm{T}}Q\{x\} \tag{28}$$

with T and Q non-degenerate symmetric matrices, where $^{\mathrm{T}}$ denotes transpose. The resulting difference equations are

$$\varepsilon T(x^{t+1} - 2x^t + x^{t-1}) = Q\{x^t\}. \tag{29}$$

Since T is non-degenerate, it is invertible, and this reduces to

$$\varepsilon(x^{t+1} - 2x^t + x^{t-1}) = A\{x^t\} \tag{30}$$

where $A = T^{-1}Q$.

If A is diagonal then the components of the above equation decouple and they can be treated as independent 2D sawtooth maps. In general, however, the equations cannot be decoupled.

Let q_i be the eigenvalues of A, which may be complex, but are never zero because Q is non-degenerate, and suppose A has diagonal Jordan normal form. Then there exists an invertible matrix B which diagonalizes A, i.e. $A = B^{-1}DB$, with $D = \mathrm{diag}[q_1, \ldots q_d]$. Let ρ_i be the root with modulus greater than 1 of

$$\rho - (2 + q_i/\varepsilon) + \rho^{-1} = 0. \tag{31}$$

Note that since $q_i \neq 0$, and the product of the roots is 1, this exists for ε small enough. Indeed

$$\rho_i \sim q_i/\varepsilon \qquad \text{as } \varepsilon \to 0. \tag{32}$$

Let

$$\alpha_i = (1 + 4\varepsilon/q_i)^{-1/2} \tag{33}$$

and for $n \in \mathbb{Z}$, let

$$b_n = \mathrm{diag}[\alpha_i \rho_i^{-|n|}]. \tag{34}$$

Choose a fundamental domain K' with 0 in the interior, not necessarily equal to K, and define functions $\{.\}'$ and $[.]'$ analogously.

Theorem 5. For ε small enough the following function defines a remnant torus of rotation vector ω:

$$X(\theta) = \theta - \sum_{n \in \mathbb{Z}} B^{-1} b_n B \{\theta + n\omega\}'. \tag{35}$$

Note that $B^{-1}b_nB$ is real even if there are complex eigenvalues, because it is equal to its complex conjugate.

Proof. We look for a solution of the form $X(\theta) = \theta + \psi(\theta)$. We first derive the solution under the assumptions that $[X(\theta)] = [\theta]'$ and the image of X has no points on the discontinuity set of $\{\}$, and then verify these assumptions.

Under the assumptions above,

$$\{X(\theta)\} = X(\theta) - [X(\theta)] = \theta + \psi(\theta) - [\theta]' = \{\theta\}' + \psi(\theta). \tag{36}$$

Therefore ψ must satisfy

$$\varepsilon(\psi(\theta + \omega) - 2\psi(\theta) + \psi(\theta - \omega)) = A(\{\theta\}' + \psi(\theta)). \tag{37}$$

Make the ansatz

$$\psi(\theta) = \sum_{n=-\infty}^{\infty} a_n \{\theta + n\omega\}'. \tag{38}$$

This is a solution if and only if

$$\varepsilon(a_{n-1} - 2a_n + a_{n+1}) = Aa_n \qquad n \neq 0 \tag{39}$$

$$\varepsilon(a_{-1} - 2a_0 + a_1) = A(I + a_0). \tag{40}$$

Define $b_n = -Ba_nB^{-1}$. Then the components of b_n satisfy independent recurrence relations. The solution which decays as $n \to \pm\infty$ is given by

$$b_n = \mathrm{diag}[\alpha_i \rho_i^{-|n|}]. \tag{41}$$

It remains to show that the assumptions are satisfied for ε small enough. Now X

is differentiable in the sense of distributions, with derivative $DX = 0$ at all points θ for which for all $n \in \mathbb{Z}$, $\theta + n\omega$ is not a point of discontinuity of $\{\}'$, and a jump of magnitude $-B^{-1}b_n Bu$ when $\theta + n\omega$ crosses from one fundamental domain to a translate by u.

So for $\theta + K'$

$$|X(\theta) - X(0)| \leq \eta = \sum_{n \in \mathbb{Z} \setminus 0} |B^{-1} b_n B| = \mathcal{O}(\varepsilon) \tag{42}$$

because $|b_n| \leq C(\varepsilon/q)^{|t|}$ for some C, q. Also

$$|X(0)| \leq \eta \sup_{\theta \in K'} \{\theta\}' = \mathcal{O}(\varepsilon) \tag{43}$$

So

$$|X(\theta)| = \mathcal{O}(\varepsilon) \qquad \theta \in K'. \tag{44}$$

Hence for ε small enough, the assumptions are satisfied.

An example of such a cantorus with $d = 2$ was shown in [15], where A had real eigenvalues. Figure 1 shows a variety of examples with complex eigenvalues.

Notes

(i) In the case $d = 1$ if one chooses the fundamental domains to be $K = K' = [-\frac{1}{2}, \frac{1}{2}]$, then the solution (35) is valid for all ε. Numerical evaluation of (35) for some cases with $d = 2$, however, shows that it is not always correct for the natural generalization $K = K' = [-\frac{1}{2}, \frac{1}{2}]^d$, $d > 1$. Nonetheless, we conjecture that given T and Q of (28), there always exist fundamental domains K, K' such that (35) is valid.

(ii) It would probably be a straightforward extension of the analysis to also derive explicit cantori for sawtooth maps for which A has non-diagonal Jordan normal form.

6. Form of cantori near the anti-integrable limit

Since the cantori of the multidimensional sawtooth maps avoid the discontinuities for ε small enough, we can smooth out the discontinuities without making any difference to the cantori, thus obtaining C^∞ examples with explicit cantori.

More importantly, given any symplectic map near a non-degenerate anti-integrable limit, we can derive an asymptotic form for these cantori, by approximating the generating function by its quadratic part at the critical point e.

Without loss of generality, the Taylor expansion of T about e begins with a quadratic form in $(x' - x)$. This is because constants make no difference, and terms in x, x', x^2, x'^2 can be absorbed into V by change of variable. Then the quadratic part of h, truncated to a fundamental domain containing e in its interior, is the generating function for a sawtooth map. The cantori of the sawtooth are contained within $\mathcal{O}(\varepsilon)$ of the critical point. Higher-order terms in T and V move points by $\mathcal{O}(\varepsilon^2)$ (implicit function theorem estimates). Hence the cantori for the full system are within a relative distance $\mathcal{O}(\varepsilon)$ of those for the corresponding sawtooth map.

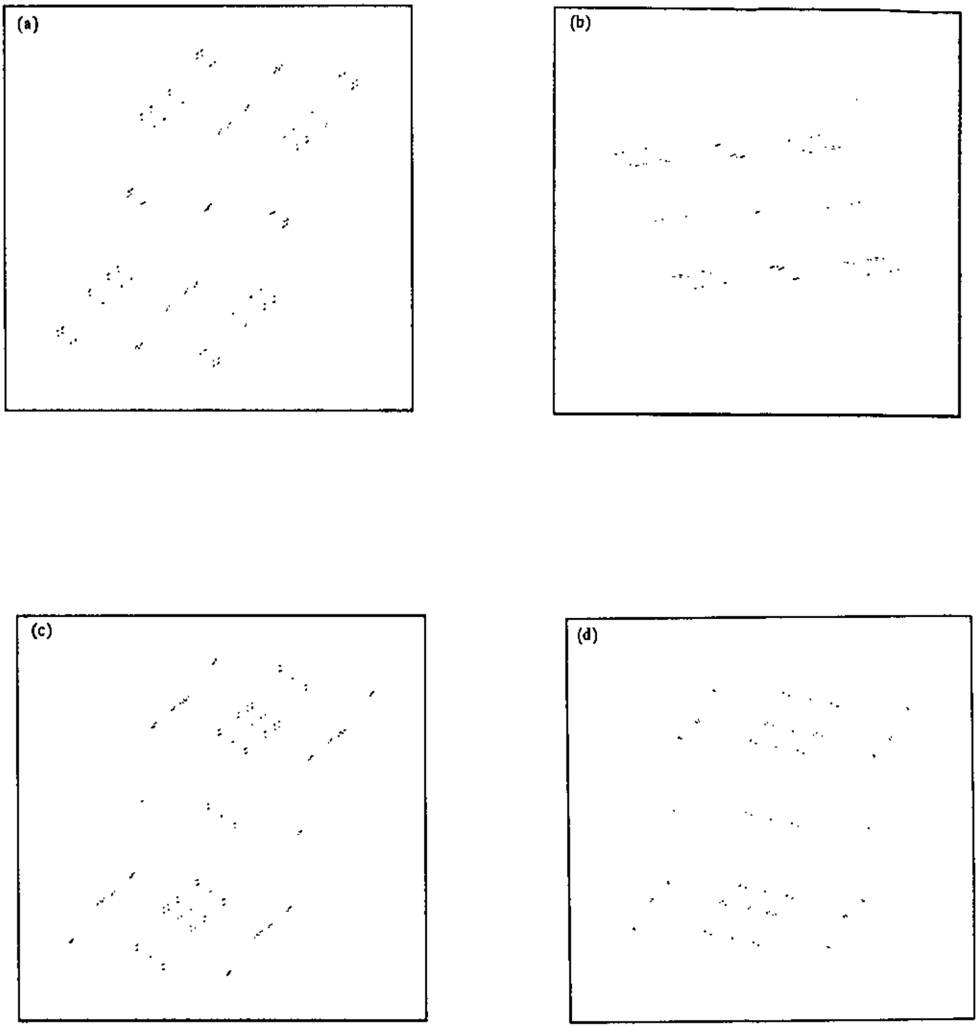

Figure 1. Configuration space projections of cantori for some four-dimensional saw-tooth maps: (a) and (b) have $\omega = ((\sqrt{5}-1)/2,\ 1/\sqrt{2})$ while (c) and (d) have $\omega = (\tau^{-1}, \tau^{-2})$, where τ is the real root of $\tau^3 - \tau - 1$. The matrices A for (a) and (c) are

$$A \approx \begin{pmatrix} 1.915 & -2.056 \\ 0.522 & 0.085 \end{pmatrix}$$

and for (b) and (d) are

$$A \approx \begin{pmatrix} 1.085 & 2.086 \\ -0.521 & 2.915 \end{pmatrix} \qquad \begin{pmatrix} 1.321 & -1.188 \\ 0.297 & 0.679 \end{pmatrix}.$$

For all four cases the eigenvalues are $q = 1 \pm \mathrm{i}/2$.

7. Discussion

We have proved that all symplectic maps close enough to a non-degenerate anti-integrable limit possess analogues of cantori for all incommensurate rotation vectors, and found their asymptotic form.

Some questions remain:

(i) Do these cantori continue to the KAM tori of the integrable limit? Are there intermediate stages in which there is a remnant torus which is a locally a Cantor set cross a manifold, or a Sierpinski gasket? Numerical results are unclear [15].

(ii) Are our cantori composed of orbits of minimal action (if we take e to be the global minimum of V, and T to be convex)?

(iii) Can we deduce something about transport (cf [26, 27]) for multi-degree of freedom symplectic maps by approximation by sawtooth maps (cf. [14])?

It will be interesting to pursue this line of research.

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