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Hamiltonian perturbation theory: periodic orbits, resonances and intermittency

P Lochak†
DMI, Ecole Normale Supérieure, 45 rue d'Ulm, F-75230 Paris Cedex 05. France

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Abstract. This paper gives an almost self-contained and semi-formal account of a new approach of perturbation theory for Hamiltonian systems (a detailed exposition can be found in a previous paper). We particularly emphasize the connection between some basic physical ideas and their mathematical counterparts.

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1. Introduction

The purpose of this paper is to present, in a non-technical way, the main ideas which underlie the theory detailed in [1, 2]. Essentially all the results we shall mention can be found in these papers; notice that some details have been solved since these papers were written, which has allowed us to improve the results to the point at which the most significant quantities have become essentially optimal. Regarding these technical improvements, we refer to [3] and [4]. Our intent here is rather to underscore the physical ideas which give rise to the mathematical developments and serve as a guide for intuition. It may be useful to notice that, although we provide no formal proof of the results we mention, the only result which does require some machinery is the one stated here as theorem 1; once this has been obtained, all further results require little more than a few lines of book-keeping type computations.

The set-up of the problem is classical; start with a near integrable system:

$$H(p,q) = h(p) + \varepsilon f(p,q) \qquad \text{with} \qquad (p,q) \in \mathbb{R}^n \times \mathbb{T}^n,\ \mathbb{T} = \mathbb{R}/\mathbb{Z} \tag{1}$$

where (p, q) are action-angle variables of the integrable Hamiltonian h. We assume that H is *analytic* over some domain $\mathcal{D} = \mathcal{D}(R, \rho, \sigma)$, $(\rho > 0, \sigma > 0)$ defined as follows: let B_R be the ball of radius R around the origin, then

$$\mathcal{D} = \mathcal{D}(R,\rho,\sigma) = \left\{(p,q) \in \mathbb{C}^{2n}, \operatorname{dist}(p, B_R) \leqslant \rho; \operatorname{Re}(q) \in \mathbb{T}^n; |\operatorname{Im}(q)| \leqslant \sigma\right\}$$

with $|\operatorname{Im}(q)| = \operatorname{Sup}_i(|\operatorname{Im}(q_i)|)$. Note that the real part of $\mathcal{D}$ is nothing but $B_{R+\rho} \times \mathbb{T}^n$.

Assume further that h is *convex*. More precisely, denote $\omega(p) = \nabla h(p) \in \mathbb{R}^n$ and $A(p) = \nabla^2 h(p) \in \mathcal{M}_n(\mathbb{R})$ the frequency vector and the Hessian matrix. We assume that

† e-mail address: lochak@dmi.ens.fr

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$A(p)$ is sign definite—say—positive. Let m be a lower bound of the spectrum of A over the real part of the domain, then

$$\forall p \in B_{R+\rho} \subset \mathbb{R}^n, \forall v \in \mathbb{R}^n : A(p)v.v \geqslant m\|v\|^2$$

$m > 0$ and $u.v$ denotes the scalar product of two vectors u, v. We will denote M the operator norm of A on $\mathcal{D}$ (the complex domain): $\|A(p)v\| \leqslant M\|v\|$.

Analyticity and convexity, which may be relaxed to quasi-convexity (see e.g. [1, 5, 8]), are the two essential conditions in what follows and are analytic and geometric in nature respectively. We refer once and for all to [5–7] for background information, and to [1–4] for detailed proofs, amplifications and comments.

In [5], Nekhoroshev proved the following fundamental result:

Theorem 0. For any initial condition $(p(0), q(0))$ with $p(0) \in B_R$, one has

$$\|p(t) - p(0)\| \leqslant R(\varepsilon) = c\varepsilon^b \qquad \text{for} \quad |t| \leqslant \mathcal{T}(\varepsilon) = \exp(c/\varepsilon^a)$$

provided ε is small enough, i.e. $|\varepsilon| \leqslant \varepsilon_0$.

This can be summarized by saying that *action variables are stable over exponentially long times if the unperturbed part of the system is analytic and convex* (this may be enlarged to the class of the so-called 'steep' functions; cf [5] and [21]). We call $R(\varepsilon)$ the *radius of confinement*, $\mathcal{T}(\varepsilon)$ the *time of stability*, ε_0 the *threshold of stability*; (a, b) are the *stability exponents*, and they satisfy $0 < a, b \leqslant 1$.

2. Global stability

In theorem 0, c stands for some constant (not the same every time) depending on the various parameters. To make this a little more precise, we put $\|f\|_{\mathcal{D}} = E$ where $\|f\|_{\mathcal{D}}$ is simply the maximum of $|f|$ over $\mathcal{D}$. We also let $\omega_0 = \omega(0)$, $\Omega = \|\omega_0\|$ and collect the parameters which describe the system as $\mathcal{P} = (R, \rho, \sigma, M, m, E, \Omega)$. In order to simplify the notation, we introduce the following symbol: if α and β are any two scalar quantities, we write $\alpha \preceq \beta$ if there is a constant $c = c(\mathcal{P})$ such that $\alpha \leqslant c\beta$; similarly, $\alpha \succeq \beta$ means $\alpha \geqslant c\beta$ and $\alpha \asymp \beta$ means $\alpha = c\beta$. In fact, the dimensional structure of the constants which appear (they are computed in [1–4]), i.e. their dependence w.r.t. the set of parameters $\mathcal{P}$, is easy to disentangle, but we shall not pursue this here.

The main novelty introduced in [1] and [2] consists in focusing attention on periodic orbits or rather periodic tori. Assuming a possible shift of the origin, restriction of the domain etc (we shall not mention these technical details any longer), suppose that ω_0 is a *rational vector* with period $T > 0$; this amounts to requiring that $T\omega_0 \in Z^n$. The torus $p = 0$ is then filled with the closed linear orbits of the unperturbed system: $q(t) = q(0) + \omega_0 t$.

In this context, the following result is the fundamental building block, to which one needs only add simple arithmetical considerations, in order to derive all the subsequent results.

Theorem 1.

$$\|p(0)\| \leqslant r \Rightarrow \|p(t)\| \preceq r \qquad \text{if} \quad |t| \preceq e^{cs} \quad (c = c(\mathcal{P}))$$

provided the following two conditions hold:

(i) $s\ r\ T \preceq 1$

(ii) $r \succeq \sqrt{\varepsilon}$.

It is essential that here, the quantities r, i.e. the radius of the 'influence zone' of the periodic torus $p = 0$, s, which governs the time of stability, and the period T of the torus are viewed are free parameters, subject only to (i) and (ii). True, there are other mild conditions which are required in order for the above result to hold; they can be written as

(iii) $r \preceq 1$

(iv) $\varepsilon \preceq 1$

(v) $s > 1$.

(iii) and (iv) take care of harmless and uninteresting things like the domains of definition of the various quantities, the radius of invertibility of the frequency map $p \to \omega(p)$ (which is indeed a local diffeomorphism) etc; if (v) does not hold, everything is empty since one is interested in estimates over *long* time intervals so that one wants $s \gg 1$. We shall not mention (iii), (iv) and (v) any more.

Perturbation theory uses, roughly speaking, analysis, geometry and arithmetics, all three at a rather elementary level. Using the periodic orbit method which we put forward reduces all the analysis and geometry that is needed to the above result. Let us thus dwell a bit longer on its content.

From the *analytic* viewpoint, this is nothing but *one* phase averaging . The simplest possible problem along this line is embodied in the following equation:

$$\dot{x} = \varepsilon f(x, t) \qquad x \in R^n \tag{2}$$

where f is analytic w.r.t. x (Lipschitz in t) and periodic, say of period one, in t. One seeks to 'eliminate' the time by a near identity change of variables $x = y + \varepsilon u(y, t, \varepsilon)$, where u is unknown and y satisfies the *autonomous* equation $\dot{y} = \varepsilon g(y, \varepsilon)$. This is always possible at a *formal* level: there are formal power series $\hat{u}$ and $\hat{g}$ in ε, whose coefficients are analytic functions in (y, t) which do just that. But these series are generically divergent; in fact the coefficients increase like $k!$ (k labelling the coefficients), as was recently proven in [10], and this is why we say they are of Gevrey order one (there is no standard way of labelling Gevrey indices). This is probably one of the simplest cases of divergence in the theory of normal forms, one which arises without the—explicit—occurrence of 'small divisors'. Notice that the whole problem has much in common with that of embedding a near identity analytic diffeomorphism into a flow. One can now use a least-term cut-off to truncate $\hat{u}$ and $\hat{g}$, and obtain functions u and g such that

$$x = y + \varepsilon u(y, t, \varepsilon) \qquad \dot{y} = \varepsilon g(y, \varepsilon) + O(e^{-c/\varepsilon}). \tag{3}$$

So, one has eliminated the time variable to within an exponentially small remainder and this is optimal in general (see also the discussion in [2], appendix 2). That (3) can be obtained was proven in [9]. There, Neistadt uses a recursion scheme, which is a much simplified version of the analytic part of [5], and not majorizing series. In fact, in perturbation theory, one often has the choice of using either an iterative procedure or a direct method, writing down the normalizing series (here $\hat{u}$ and $\hat{g}$) and controlling the growth of the coefficients. For instance, Siegel's original proof of his famous holomorphic linearization theorem used majorizing series, and it foreshadowed Kolmogorov's celebrated result in 1954, where the latter used an iterative procedure, as is almost always the case nowadays, in the framework of KAM theory.

The scheme for proving theorem 1 is almost the same as that of [9]. This is implemented in [3] without any explanation of the constants, which were given in [4]. For the sake of comparison, one should beware of the fact that the small parameter ε in (2) corresponds

to $\sqrt{\varepsilon}$ in the Hamiltonian problem. This analytic part is *not* Hamiltonian in nature, as exemplified by Neistadt's result on equation (2), although it is necessary to retain the symplectic nature of the problem for future use. Condition (i) in theorem 1 refers to this analytic part: s is the number of steps in the iterative procedure, and the remainder decreases geometrically; in fact, it decreases by a factor $c\,srT$ ($c = c(\mathcal{P})$) at each step, and this factor should be smaller than 1; hence (i).

The analogue of Neistadt's result for equation (2) follows from theorem 1 if one looks at a fixed torus so that T is fixed too (in (2), $T = 1$); then one can choose $r \asymp \sqrt{\varepsilon}$ and $s \asymp \frac{1}{rT}$. We state the resulting estimate as

Corollary 1.

$$\|p(0)\| \preceq \sqrt{\varepsilon} \Rightarrow \|p(t)\| \preceq \sqrt{\varepsilon} \qquad \text{if} \quad |t| \preceq \exp(\frac{c}{T\sqrt{\varepsilon}}) \quad (c = c(\mathcal{P}))$$

provided ε is small enough.

One thus finds stability of a $\sqrt{\varepsilon}$-neighbourhood of a T-periodic torus over a time interval on the order of $\exp(\frac{c}{T\sqrt{\varepsilon}})$. In other words, this domain of phase space admits the stability exponents $(a, b) = (\frac{1}{2}, \frac{1}{2})$. Below, we shall meet many other such 'local stability estimates', but right now we would like to obtain a *global* result, i.e. one which is valid for any point in phase space.

We notice that in [1] and [2], the analytic part was slightly less accurate so that we obtained $\varepsilon^{\frac{1}{3}}$ instead of $\varepsilon^{\frac{1}{2}}$ in the above estimate, also resulting in a slight loss for the global stability exponent $a(n)$ to be derived below. This was subsequently repaired, simply by inspecting the scheme in [9]. Apart from this (changing 3's into 2's) all the—global and local stability—results are already contained in [1] (and *a fortiori* [2]).

At this point, one should stress a feature of theorem 1, which can hardly be overemphasized: it is essentially independent of the total number n of degrees of freedom, 'essentially' referring to the fact that n in fact enters implicitly, through the Euclidean norm $\|\cdot\|$ of the n vectors. As it stands here, using Euclidean distance, it is thus not quite adapted to 'passing to the thermodynamical limit', i.e. somehow letting n tend to infinity. Again, this—near—indepedence on the total dimension (number of frequencies) is very important, from a physical viewpoint. It was really a major contribution of Italian researchers to let this property emerge and understand its significance, in a slightly different context (see [11–15] and references therein); we shall comment more on this later, notably when we discuss the stability of resonances in section 3. Now this physical property is, so to speak, encapsulated in theorem 1, and, at this level, the contribution of [1] and [2] was perhaps to recognize the period T as the important parameter, which should remain free, and with which one can subsequently play in many ways, using arithmetics or more precisely simultaneous approximation (see below).

We have already mentioned that the 'analytic' part of the proof of theorem 1, which consists in one phase averaging, is *not* linked to the symplectic nature of the problem *nor* to the convexity requirement. Again, this is shown by the treatment of equation (2) and, for instance, the significant physical results obtained in [11] and [12] also have nothing to do with convexity; there, one is essentially concerned with perturbations of harmonic oscillators. What *is* important, is the requirement of analyticity, which cannot be dispensed with, if one wants to get exponentially small remainders (or exponentially long times). We dwell a little longer on these remarks because the analytic part of theorem 1, simple as it

may be, is the only place in the whole theory where non-optimal features can creep in. The geometrical and arithmetical parts *are* esentially optimal, even from a realistic, numerical viewpoint. The analytic part having been reduced to simple one-phase averaging is itself rather transparent. In fact, we believe that theorem 1 is optimal in some sense, including, e.g., the dimensional dependence (detailed in [3] and [4]) of the implied constants w.r.t. the set of parameters $\mathcal{P}$. Of course, the numerical factors cannot be optimal, but they could be pushed to realistic values using a computer-assisted proof, for some specific Hamiltonian.

Let us now say a word about the simple *geometry* involved in the proof of theorem 1. The stability of periodic tori, as stated in theorem 1, does require convexity and energy conservation, hence the canonical character of the motion. The idea of combining energy conservation with convexity in order to simplify Nekhoroshev's original trapping mechanism appeared in [7] and was subsequently used in many papers ([1–4, 8], etc). This simple reasoning is the only place where the Hamiltonian character of the equations is really needed and it can be roughly summarized as follows.

The perturbed Hamiltonian $H(p, q)$ may be considered as the sum of a 'kinetic part' $h(p)$ and a perturbation. The convexity of h provides a quadratic potential well; indeed one is more or less reduced to the case when $h(p) = \frac{1}{2}mp^2$, considered near $p = 0$ (m is the quantity entering in $\mathcal{P}$). Total energy is conserved, and the action p may vary until the kinetic energy $\frac{1}{2}mp^2$ is balanced by the $O(\varepsilon)$-perturbation; hence p may experience $O(\sqrt{\varepsilon})$-variations. This introduces condition (ii) in theorem 1, namely $r \geq \sqrt{\varepsilon}$ in order to ensure stability, and this in turn forces the *a priori* inequality $b \leqslant \frac{1}{2}$ for the exponent of stability governing the radius of confinement. $\sqrt{\varepsilon}$ represents what physicists call the 'resonance width'; one should keep in mind the simple pendulum equation with Hamiltonian

$$H = \tfrac{1}{2}m\dot{x}^2 + \varepsilon \sin x(1 + \varepsilon\varphi(t)) \tag{4}$$

φ periodic with period 1. This example also demonstrates that variations in the action ($= m\dot{x}$) of order $\sqrt{\varepsilon}$ usually take place over short timescales ($\sim \frac{1}{\sqrt{\varepsilon}}$).

Now that we have theorem 1 available, we can play with the various parameters, adding in only arithmetics. We have already mentioned the obvious case of a fixed given torus (corollary 1). In general, one may of course constrain $s \asymp \frac{1}{rT}$, keeping r a free parameter; this gives a measure of the way in which the influence of a periodic torus decreases, as the distance increases. We state this in corollary 2.

Corollary 2.

$$\|p(0)\| \leqslant r \Rightarrow \|p(t)\| \preceq r \qquad \text{if} \quad |t| \preceq \exp\left(\frac{c}{rT}\right) \quad (c = c(\mathcal{P}))$$

provided $r \geq \sqrt{\varepsilon}$ and ε is small enough.

To obtain a global stability result, one which is valid uniformly over phase space, one needs yet another choice of the parameters. Indeed, one should force $s \asymp \varepsilon^{-a}$; at this point $a > 0$ is still undefined, and we are looking around a period T torus. Set

$$s \asymp \varepsilon^{-a} \qquad rT \asymp \varepsilon^{-a}.$$

Condition (ii) in theorem 1 is satisfied provided $T \preceq \varepsilon^{-\frac{1}{2}+a}$, which is a bound on the period. This is very sensible since, as is intuitively clear (and confirmed by condition (i) of theorem 1), averaging deteriorates when the period of the motion increases. We thus obtain the following statement:

Corollary 3.

$$\|p(0)\| \preceq \frac{\varepsilon^a}{T} \Rightarrow \|p(t)\| \preceq \frac{\varepsilon^a}{T} \qquad \text{if } |t| \preceq e^{c/\varepsilon^a} \cdot \ (c = c(\mathcal{P}))$$

provided $T \preceq \varepsilon^{-\frac{1}{2}+a} (0 < a \leqslant \frac{1}{2})$ and ε is small enough.

Although this statement may appear a little more contrived than corollaries 1 and 2, it has a simple interpretation. Let $z(t) = (p(t), q(t))$ denote the trajectory in phase space, and let $z_0(t_0, t) = (p(t_0), \omega_0(t-t_0)+q(t_0))$; $z_0(t_0, t)$ is thus the trajectory starting from $z(t_0)$ and following the unperturbed flow on the reference torus $p = 0$ (with frequency $\omega(0) = \omega_0$). The above statement implies that

$$\|z(t) - z_0(t_0, t)\| \preceq \varepsilon^a \qquad \text{if} \quad t_0 \leqslant t \leqslant t_0 + T \cdot \text{ and } \quad |t_0| \preceq e^{c/\varepsilon^a}.$$

One can rephrase this by saying that the perturbed trajectory is *almost periodic* within $O(\varepsilon^a)$, with almost period T, over an exponentially long timescale. This is somewhat more precise than just stability of the action variables and reveals some of the nature of the periodic orbit method. In effect, one tries to approximate the true motion over an exponentially long time with periodic trajectories having the shortest possible periods. This will become apparent when we add some arithmetics in order to obtain a global statement.

Before going into this, let us again spell out the key ingredients of our approach.

(i) One needs a good estimate for the stability of the neighbourhoods of periodic tori. This stability is a specific feature of convex systems, in which case it is shown by using one-phase averaging (theorem 1).

(ii) Results valid over the whole of phase space (global stability) or particular subsets thereof (see sections 3 and 4) are obtained by approximating corresponding points (initial conditions of the trajectories to be studied) by means of periodic points in action space. A periodic point is a point belonging to a periodic torus of the unperturbed system, that is it is a point p with *rational* frequency $\omega(p)$ (there is a $T > 0$ such that $T\omega(p) \in \mathbb{Z}^n$). Approximating points in action space by periodic ones corresponds to using the theory of simultaneous approximation in frequency space.

We stress that this approach is fundamentally different from the usual one (as implemented for the first time in [5] as far stability over exponentially long times is concerned); roughly speaking, the latter focuses first on very non-resonant parts of phase space and uses linear approximation in order to control the so-called small divisors (that is, quantities of the form $\omega(p) \cdot k$, $k \in \mathbb{Z}^n \backslash \{0\}$).

We illustrate now the strategy described above by working out the global stability estimate, using theorem 1 and approximation. To this end we need only the most basic result of simultaneous approximation, namely the Dirichlet theorem, which we recall:

Theorem (Dirichlet). Let $\omega \in R^n$ and $Q > 1$ be a real number. There exists an *integer* q, $1 \leqslant q < Q$, such that $\langle q\omega \rangle \leqslant Q^{-\frac{1}{n}}$.

Here we have used the following notation, for $\alpha \in R^n$

$$\langle \alpha \rangle \equiv \inf_{\zeta \in Z^n} \|\alpha - \zeta\|_\infty = \inf_{\zeta \in Z^n} \Big(\sup_{i=1,\ldots,n} |\alpha_i - \zeta_i| \Big).$$

$\zeta = (\zeta_i)$ is thus the point of the integer lattice Z^n closest to $\alpha = (\alpha_i)$, using the norm $\|\cdot\|_\infty$ of the largest component for vectors of R^n. In particular

$$\langle \alpha \rangle \leqslant \operatorname{dist}(\alpha, Z^n) \leqslant \sqrt{n} \langle \alpha \rangle$$

where 'dist' denotes the Euclidean distance.

Dirichlet's theorem (or lemma) is an immediate consequence of the 'box counting principle' and is found in the first few pages of any book dealing with approximation theory (e.g., [22]). Notice that if $\omega = (\omega_1, \omega_2)$ where $\omega_1 \in Z^m$ and $\omega_2 \in R^d$, $m + d = n$, one has $\langle q\omega \rangle = \langle q\omega_2 \rangle$; approximating ω is the same thing as approximating ω_2 and n may accordingly be changed into d in the statement of the result. This simple remark will be crucial when dealing with 'resonance surfaces' in section 3.

The reasoning leading to global stability now goes as follows. Let $p^* = p(0)$ be an arbitrary point in action space (the initial phase $q(0)$ is arbirary as well and does not enter the discussion). We wish to apply corollary 3 around some rational point p, having period T, and lying close to p^*. First rescale one of the components of $\omega^* = \omega(p^*)$, say the first, to an integer (the effect of this rescaling on the parameters $\mathcal{P}$ is obvious and detailed in [1, 2]) and apply Dirichlet's theorem, to the effect that

$$\forall Q > 1, \exists \omega \qquad T\text{-periodic such that:} \|\omega - \omega^*\| \leqslant \frac{\sqrt{n-1}}{TQ^{\frac{1}{n-1}}} \quad 1 \leqslant T \leqslant Q.$$

$\| \cdot \|$ again denotes the Euclidean norm. For $\|\omega - \omega^*\|$ small enough, there exists a corresponding point p such that $\omega = \omega(p)$ ($p \to \omega(p)$ is locally invertible). In fact

$$\|p - p^*\| \leqslant \frac{\sqrt{n-1}}{m} \frac{1}{TQ^{\frac{1}{n-1}}} \qquad 1 \leqslant T \leqslant Q. \tag{5}$$

We want to apply corollary 3 around p (so the origin of the coordinates is p, not 0 as in the statement of the corollary); $p(0) = p^*$ should be close enough to p so as to lie in what we call the 'influence zone' of the periodic torus defined by p in phase space. More precisely, we need $\|p - p^*\| \leqslant r \asymp \frac{\varepsilon^a}{T}$; by (4), it is enough to require

$$\frac{\varepsilon^a}{T} \succeq \frac{\sqrt{n-1}}{TQ^{\frac{1}{n-1}}}$$

so *define* Q through

$$\frac{\varepsilon^a}{T} \asymp \frac{\sqrt{n-1}}{TQ^{\frac{1}{n-1}}} \qquad \text{or} \quad Q^{\frac{1}{n-1}} \asymp \sqrt{n-1} \quad \varepsilon^{-a}. \tag{6}$$

The constant $\sqrt{n-1}$ is rather uninteresting in this context, but we have to comply with the definition of the symbol '$\asymp$'. In order for corollary 3 to apply, we need $T \preceq \varepsilon^{-\frac{1}{2}+a}$, and since $T \leqslant Q$, this is satisfied provided

$$Q \preceq \varepsilon^{-\frac{1}{2}+a}. \tag{7}$$

(6) and (7) together immediately yield the inequality $a \leqslant \frac{1}{2n}$. So pick $a = \frac{1}{2n}$ and since $r \asymp \frac{\varepsilon^a}{T} \preceq \varepsilon^a$, one may set $b = a$. This proves theorem 0 (Nekhoroshev's theorem) with the values $a = b = \frac{1}{2n}$. For the sake of further reference, we state this result as

Theorem 2. For any initial condition $(p(0), q(0))$ (say with $p(0) \in B_R$ so that the trajectory is defined), one has

$$\|p(t) - p(0)\| \preceq \varepsilon^b \qquad \text{for} \quad |t| \preceq \exp(c/\varepsilon^a) \quad (c = c(\mathcal{P}))$$

provided ε is small enough; one has $a = b = \frac{1}{2n}$.

The reader may wish to return to the interpretation of corollary 3 in terms of 'almost almost periodic' functions; now that T is not fixed anymore, Nekhoroshev's result can be interpreted—and made somewhat more precise—by saying that the trajectory in phase space, including angle variables, is approximated over an exponentially long time by pieces of periodic trajectories, with an algebraically long period ($T \preceq \varepsilon^{-\frac{1}{2}+a}$). This point of view is characteristic of the method we use and will be made more accurate in section 4.

The first stability exponent a, governing the time of stability, is probably the most important quantity (the value of b is discussed at length in section 4). Let us summarize the meaning of the value $a = \frac{1}{2n}$. One may figure this as $a = \frac{1}{2} \times 1 \times \frac{1}{n} : \frac{1}{2}$ simply refers to the fact that $\varepsilon^{\frac{1}{2}}$ rather than ε itself is the natural small parameter. 1 refers to the exponentially small remainder for one-phase averaging (see equations (2) and (3)); this is a Gevrey-1 result as we mentioned before, and we refer to [2] and [10] for more details. Finally, the factor $\frac{1}{n}$ is forced by Dirichlet's theorem. In our opinion, it is important that the periodic orbit methods makes this almost crystal clear and shows why this value is likely to be—generically and for $n > 2$—optimal. This is discussed further in [1] and [2]; there, a slight imperfection in the analytic scheme led essentially to the value $a = \frac{1}{2n+1}$ although it was clear that $a = \frac{1}{2n}$ was the 'right' value, as was found in [3, 4, 8]; in the latter paper, Pöschel uses the 'traditional' method (as in [7]), improving its geometrical part to reach the value $a = \frac{1}{2n}$.

In fact, Chirikov had predicted this same value a long time ago (see, in particular, [16]), starting so to speak, from the other side. Specifically, heuristic reasoning led him to predict that the 'speed of Arnold diffusion' (this quantity is somewhat difficult to define in a rigorous way) in a system with n independent phases can be of the order of $\exp(-c\varepsilon^{-\frac{1}{2n}})$. This reasoning was adapted and made more transparent in [1] and [2], where we also describe a heuristic picture of Arnold diffusion more in accordance with the periodic orbits method. Put slightly differently, Chirikov predicted that on a timescale of order $\exp(c\varepsilon^{-\frac{1}{2n}})$ the action variables can actually change by a quantity of order 1, so that stability does not prevail any longer.

All this says that in a way we have really reached the limits of canonical perturbation theory: over longer timescales, Arnold diffusion, which is poorly understood, at a rigorous or even 'semi-rigorous' level, should really be taken into account. So the main result of [1] and [2] was in fact to fill the gap between Chirikov's long-standing prediction for the maximal possible speed of Arnold diffusion and the existing results for the maximal timescale over which stability prevails; indeed, in [5–7], $a(n)$ was of the order of $1/n^2$ and there was no clue as to what the 'best' value should be like.

3. Local stability

We shall now discuss local estimates, i.e. explore the phase space and find places where the value of a and/or b can be improved. This section is concerned with the local values of a; the second exponent b is the subject of section 4. In some sense, we are moving in a circular way: indeed, we started from the periodic orbits—or tori—and we know already that in a $O(\sqrt{\varepsilon})$ of a given torus, or any set of tori with periods bounded from above, one has $a = b = \frac{1}{2}$; this is the content of corollary 1. These are the best possible values one can expect: a is bounded from above by $\frac{1}{2}$ because one-phase averaging generates Gevrey-1 series and no better generically; b is also bounded from above by $\frac{1}{2}$ because the geometric reasoning using convexity is optimal: the width of a resonance is indeed of the order of $\varepsilon^{\frac{1}{2}}$

(recall the pendulum (4)). Also, the deformation of invariant KAM tori can be of the order of $\varepsilon^{\frac{1}{2}}$.

In between the global estimate of theorem 2 and the estimate around periodic tori of corollary 1, we come across the hierarchy of resonances and the fascinating phenomenon of *stabilization via resonance*. We shall first show how it comes out easily using the periodic orbit method and then add some physical and historical comments.

Let $\mathcal{M} \in Z^n$ be a sublattice of rank m, which means that the plane spanned by the vectors of $\mathcal{M}$ has dimension m. One defines the corresponding resonance plane $P_{\mathcal{M}}$ in frequency space as

$$P_{\mathcal{M}} = \{\omega \in R^n \ \omega \cdot k = 0 \quad \forall k \in \mathcal{M}\}$$

there corresponds to it a resonance surface in action space:

$$S_{\mathcal{M}} = \{p \in R^n \ \omega(p) \cdot k = 0 \quad \forall k \in \mathcal{M}\}.$$

The resonance is said to be of multiplicity m; $P_{\mathcal{M}}$ and $S_{\mathcal{M}}$ have dimension d, $d + m = n$.

Assume that $p(0) = p^* \in S_{\mathcal{M}}$, hence $\omega^* = \omega(p^*) \in P_{\mathcal{M}}$. Then, by a *linear* symplectic change of coordinates, one may in fact assume that $\omega^* = (0, \omega')$, $0 \in R^m$, $\omega' \in R^d$. Recall now what was said when stating Dirichlet's theorem: approximating ω^* is obviously the same as approximating ω' and consequently, $\frac{1}{d}$ may be substituted for $\frac{1}{n}$ in the statement of Dirichlet's theorem. This substitution goes through, so that one finds ultimately the value $a = b = \frac{1}{2d}$ for a trajectory starting from $(p(0), q(0))$, with $p(0) = p^*$. This is in fact readily extended to initial conditions lying *close* to the resonance surface; the final result reads

Theorem 3. Let the initial condition $(p(0), q(0))$ satisfy: $\operatorname{dist}(p(0), S_{\mathcal{M}}) \preceq \sqrt{\varepsilon}$, $\operatorname{rank} \mathcal{M} = m$, $\dim S_{\mathcal{M}} = d$, $m + d = n$; then

$$\|p(t) - p(0)\| \preceq \varepsilon^{\frac{1}{2d}} \qquad \text{for} \quad |t| \preceq \exp(c_{\mathcal{M}} \varepsilon^{-\frac{1}{2d}})$$

provided ε is small enough.

The constant $c_{\mathcal{M}} = c_{\mathcal{M}}(\mathcal{P})$ now also depends on the lattice $\mathcal{M}$ through the linear transformation reducing ω^* to the standard form $(0, \omega')$. Incidentally, this allows the *order* of a resonance to be defined in an intrinsic way; details can be found in [2].

This result, which is obtained in a completely effortless way, should not be underestimated. It contains the 'global' result (theorem 2), because any point belongs to $S_{\mathcal{M}}$ with $\mathcal{M} = \emptyset$ and $d = n$. Literally, theorem 3 asserts that resonant or even near-resonant points in the phase space of an analytic near-integrable Hamiltonian system are much *more* stable than 'generic' points, provided the integrable part is convex. Let us stress that this is a qualitative distinctive feature of *convex* (or quasi-convex) systems. Otherwise, in particular for steep non-convex Hamiltonians, it simply does not hold, because theorem 1 is not valid. The result obviously stems from the fact that resonant—or near-resonant—points can be approximated by rational points at a much faster rate than generic points. In particular, neighbourhoods of periodic points (in action space) or tori (in phase space), which need not be approximated at all, correspond to $m = n - 1, d = 1$. Theorem 3 then reduces to corollary 1, with $c_{\mathcal{M}} \asymp \frac{1}{T}$.

Because $m = n - 1$, neighbourhoods of periodic tori coincide with 'resonant zones of maximal multiplicity'. The peculiarity of these zones and their exceptional stability was

somehow anticipated long ago (cf in particular [5, section 11.5]), but they appeared as an extreme in the hierarchy of resonances. In [1] and [2], the ideas contained in prior works (see below) were pushed to their logical consequences and things were, so to speak, turned inside out. So, instead of 'resonant zones of maximal multiplicity', one would speak of periodic orbits (or tori), described by *one* main parameter, namely the period T, and these are regarded as the fundamental primitive objects of the theory. This naturally leads to simultaneous instead of linear approximation, approximate recurrence times instead of resonances etc.

One important 'psychological' difference is that resonances are usually felt as a hindrance when trying to carry out some perturbative scheme, whereas periodic orbits appear as an 'opportunity'. Indeed, from this point of view, when the dimension increases, results deteriorate not because of the profusion of resonances and the corresponding 'small divisors', but rather due to the global scarcity of periodic orbits, measured—optimally—by the exponent $\frac{1}{n}$ in Dirichlet's theorem. There always remain however, domains of phase space with 'many' periodic orbits, namely resonant zones of small dimension d (large multiplicity m), over which stability prevails essentially as in dimension d, irrespective of the global dimension n of phase space. The latter may eventually then 'tend to infinity', i.e. one can explore the so-called thermodynamical limit.

In this vein, since the end of the 1970s, a group of Italian physicists investigated the mechanical foundations of statistical thermodynamics, guided, among other things, by the deep physical intuitions of Boltzman and Jeans (cf [11–13] and references therein). By studying simple models like one-dimensional chains of oscillators and rotators ([14, 15]), they understood the possibility of this phenomenon of stabilization *via* resonance. More precisely, they showed that nonlinear localization has to do with resonance and, that when only a finite number of degrees of freedom are excited (d in theorem 3), one can obtain results which are essentially independent of the total number n of frequencies in the problem, so that these results have a chance to survive at the thermodynamical limit, and thus be pertinent in the context of statistical mechanics. Of course, there is more in their work than can be gathered in this terse summary, but this was really a breakthrough and the author is much indebted to them for letting him share some of their insights.

Because we believe that long time stability of resonances is really an important new phenomenon with far-reaching physical consequences, it may be worth summarizing some historical facts and heuristic ideas. It was first recognized in celestial mechanics that resonances sometimes—not always!—seem to be 'more stable than expected'. This was particularly forcefully advocated in the work of Molchanov [17, 18] at a purely observational and arithmetical level. Molchanov essentially emphasized the existence of many 'simple' resonance relations between the planets of the solar system and inside satellite subsystems around Jupiter, Saturn and Uranus. He was immediately criticized on the grounds that these relations were not really 'astonishing' and would often occur among numbers or vectors picked 'at random'. In any case, since then, much work has been devoted to the occurrence of resonances in celestial mechanics. Many of them are explained in a satisfactory way by *non*-Hamiltonian effects, something which leads to the idea that they can *all* be ascribed to such effects (see in particular [5], section 2.1.D and references therein). This is a very important point. In order to understand that resonances can be more stable *within* the framework of Hamiltonian mechanics, it seems that there are at least two strong prejudices to overcome:

—the first one is the usual intuition and experience of physicists, engineers and applied mathematicians. *All* books stress the *de*stabilizing effect of resonance, which is in fact basically a *linear* effect. The common opinion is that stability can be restored only by

non-conservative, i.e. dissipative, forces, which indeed can lead to various mechanisms of capture into resonance (see [20, 21]). Theorem 3 precisely says that resonances can be stable in the absence of irreversibility.

—The second psychological obstacle is KAM theory, predicting the stability of 'very non-resonant' points in phase space. Theorem 3 introduces a kind of paradox: there is a set in phase space, namely the union of all surviving Kolmogorov tori for a given strength of the perturbation, which is stable *ad aeternum*, and this comprises only very *non* resonant initial conditions; but there is another set, comprising only very resonant points and, in particular, the neighbourhoods of periodic tori for the unperturbed system with not too large periods, which is stable over very long times, but—generically—unstable eventually. So there is a kind of incompatibility between long time stability and eternal stability. Things are not nearly so simple, however, and there are all sorts of pitfalls and caveats to be added. For instance, in the neighbourhood of a Kolmogorov torus, one should be able to develop a two-parameter perturbation theory (the strength of the perturbation and the distance from the torus) which describes the drift away from the invariant torus, so that the existence of the latter influences the long time stability of points in its vicinity.

To summarize, one can say that there are four important possibilities for the integrable part $h(p)$ of the near integrable Hamiltonian $H(p, q)$:

(i) h is linear: $h(p) = \omega \cdot p$.

(ii) h is genuinely nonlinear: $\det \nabla^2 h(p) \neq 0$.

(iii) h is steep; this is a kind of weak transversality condition (see [1, 5, 21]).

(iv) h is convex: $\nabla^2 h(p)$ is a positive—or negative—definite symmetric matrix (this can be relaxed into quasi-convexity; see, e.g., [1, 2, 8]).

These cases do not exhaust the possibilities; in particular, many degeneracies can arise, and they are in fact more the rule than the exception in celestial mechanics. Nor are they mutually exclusive: (iv) is a subcase of (iii) which in turn is included in (ii). (ii) is enough for KAM theory to apply; (iii) is a sufficient (and almost necesssary) condition to derive stability estimates over exponentially long times. But stability of resonances applies only if (iv) holds, emphasizing a distinctive feature of convexity. One should notice however that resonance can lead to interesting and somewhat unexpected phenomena even in the linear case (i), like preventing the interaction between some subsystems of the whole system (see [11, 12]).

4. Trapping and intermittency

We have now more or less exhausted what we wanted to emphasize about the local value of the time of stability, as reflected by the first exponent a, and we now turn to a discussion of the size of the radius of confinement, through the value of the second exponent b. Theorem 2 states that $a = b = \frac{1}{2n}$ holds uniformly in phase space, and we have explained why the trapping mechanism provided by the convexity of the unperturbed part ('kinetic energy') implies that $b \leqslant \frac{1}{2}$ (generically of course). The basic question is: *does the pair* $(a, b) = (\frac{1}{2n}, \frac{1}{2})$ *hold uniformly in phase space?*

Our guess is that this is *not* the case, and that the situation is somewhat more subtle, giving perhaps rise to a kind of intermittency phenomenon. Before we go into this matter, we should however mention a few facts.

After [1] and [2] had been completed, Pöschel (in [8]) took up the 'traditional' method and essentially, by improving the geometric part of [7], he derived the results we have decribed above; in particular, he obtained the value $a = b = \frac{1}{2n}$ for the exponents (as we

mentioned, these results were not quite as neat in [1, 2]). This is not surprising in itself: the usual approach to perturbation theory, which rests on Fourier series, linear approximation etc, allows one to derive the results that were obtained using the periodic orbits. The latter presents itself as an alternative approach, which allows one to guess, understand and explain some essential phenomena in a more transparent—and technically less cumbersome—way. We believe the duality between the two approaches to be very important in itself. It is already quite deep at the level of pure arithmetics, where it is embodied in the so-called Khinchin's transfer principles (see [2, 22]) between simultaneous and linear diophantine approximations. We refer again to [2] for a more thorough discussion.

There seem to be some differences between the results which can be obtained using one method or the other. The periodic orbits method shows how the perturbed trajectory in phase space (i.e. including the motion of the angle variables) behaves over exponentially long times as some kind of almost periodic function, with well-defined almost periods. This was stressed in connection with corollary 3, which leads directly to theorem 2, and will be developed below. This description is in itself more precise than just the stability of the action variables, which it implies. In some sense, usual—modern—perturbation theory is based on a geometric analysis of phase space, whereas, by using periodic orbits, one focuses on a kind of *time* series analysis of the trajectory, which may be closer to physical intuition (and perhaps also the analysis of numerical experiments).

The periodic orbit method does not yield the best possible control of the trajectory over shorter times; in this respect, the 'traditional' (since [5]) partitioning of action space into a hierarchy of resonance zones remains useful. In particular, it was shown in [7] and [8] that one may take exponent b independent of the dimension n, over shorter timescales. Specifically, Pöschel shows in [8] that the pair $(a, b) = (\frac{\mu}{2n}, \frac{\mu}{2n} + \frac{1}{2}(1-\mu))$ holds uniformly in phase space for any $\mu \in (0, 1)$. Such a result can probably not be obtained using periodic orbits. We stress that this is not a question of arithmetics: linear and simultaneous approximation do carry the same information, but it is encoded in very different ways. The encoding is more compact for simultaneous approximation but is is less detailed, if truncated at a finite order: the equivalence is asymptotic.

We now describe the intermittency phenomenon which should prevent $(a, b) = (\frac{1}{2n}, \frac{1}{2})$ obtaining in the whole of phase space. This discussion relies heavily on the periodic orbits and the specificity of simultaneous approximation (the results appear in [1] and in an abridged form in [2]). We have to start with some definitions and pieces of notation, referring to standard textbooks for more details (e.g., [22]; see also [2], appendix 1 and references therein). First recall the notation $\langle\alpha\rangle$, for $\alpha \in R^n$, which was used in the statement of Dirichlet's theorem. Diophantine sets of badly approximable vectors are defined as

$$\Omega_n(\tau, \gamma) = \left\{\alpha \in R^n, \forall q \in Z_+,\ \langle q\alpha\rangle \geqslant (\gamma/q)^{\frac{1}{n}(1+\tau)}\right\}.$$

For any $\alpha \in R^n$, we also define its *periods* $(q_i)_{i\geqslant 0}$ as the sequence of positive integers such that

$$q_0 = 1 \qquad \text{and} \qquad \forall q \in Z_+ \quad q < q_{i+1} \Rightarrow \langle q\alpha\rangle \geqslant \langle q_i\alpha\rangle.$$

Let $\mathbf{p}_i \in Z^n$ be such that $\langle q_i\alpha\rangle = \|q_i\alpha - \mathbf{p}_i\|_\infty$ and define $\alpha_i = \mathbf{p}_i/q_i$, which is a rational point of period q_i. The α_i's are called the best Dirichlet approximations of α. Dirichlet theorem asserts that $\langle q_i\alpha\rangle \leqslant q_{i+1}^{-\frac{1}{n}}$. On the other hand, one can show that for any α (with at least one irrational component), the sequence (q_i) increases at least geometrically; in fact

$$\liminf_{i\to 0} (q_i)^{1/i} \geqslant g_n > 0$$

with a uniform estimate $g_n \geqslant 1 + 2^{-n-1}$.

The above quantities are adequate for arithmetics, where one wants to work with *integer* qs. In particular, the sequence $(q_i)_{i>0}$ constitutes a very compact way of encoding the arithmetical properties of a vector $\alpha \in R^n$; it should be compared with the essentially $(n-1)$-dimensional set of values of the linear forms (small divisors) $\alpha \cdot k$, $(k \in Z^n)$ used in linear approximation theory.

Simultaneous approximation in dimension n corresponds to *non-homogeneous* linear approximation of the same dimension, because it is equivalent to approximate $\alpha \in R^n$ or $(1, \alpha) \in R^{n+1}$, if one uses the sequence $(\langle q\alpha \rangle)$, with $q \in Z^+$. When it comes to dynamics however, we want to use real—not necessarily integral—values for the period T and this unfortunately causes some minute notational and definitional minute problems which are harmless but may slightly obscure the situation. We briefly summarize these unpleasant details.

Let $h(p)$ be the convex unperturbed Hamiltonian we started with; the frequency map $p \to \omega(p) = \nabla h(p)$ is a local diffeomorphism between action and frequency spaces. Let again p^* be a given point in action space, $\omega^* = \omega(p^*)$ the corresponding frequency, and $h^* = h(p^*)$ the unperturbed energy. One would like to look at the quantities $\langle T\omega^* \rangle$, T real, and define a sequence (T_i) of periods and related best approximations (ω_i). There are at least three natural ways of achieving this, which are equivalent, except for rather irrelevant differences which cause notational problems:

(i) One may define the (T_i)s via the best approximation property:

$$\langle T\omega^* \rangle \geqslant \langle T_i\omega^* \rangle \qquad \text{for } T \leqslant T_{i+1}.$$

(ii) One may rescale one of the components of ω^* to unity (or any integer), i.e. put $\omega^* = w(1, \alpha)$, $w > 0, \alpha \in R^{n-1}$ and then set $T_i = q_i/w$, $q_i \in Z^+$ corresponding to α as above. This underscores the fact that working with real Ts is essentially the same as working with integers in $n-1$ dimensions.

(iii) One may restrict attention to the energy surface $h(p) = h^*$. Indeed, if $\omega = \omega(p)$, $h(p) = h^*$, the inverse image, in action space, of the line $\{\lambda\omega, \lambda \in R\}$, is locally, for λ close to 1, a curve which intersects the energy surface transversally at p. One may then restrict attention to rational frequencies such that the corresponding action lies on the surface $h(p) = h^*$. This singles out a discrete sequence of values for the periods T from which one extracts the sequence (T_i) corresponding to the best approximations in this sense. All this amounts to viewing a frequency ω as a vector in projective space PR^{n-1} and this is indeed definitely to be used when h is only assumed to be *quasi*-convex, i.e. if one assumes that the energy surface $h(p) = h^*$ is—geometrically—strictly convex in action space.

The reader will easily convince himself that (i), (ii) and (iii) are equivalent, more precisely that the corresponding quantities differ by factors close to 1 and tending to 1 as $i \to \infty$. So, one can essentially forget about all this and keep the following in mind: given p^* and $\omega^* = \omega(p^*)$, one define sequences (T_i), (ω_i), (p_i), (r_i) as follows:

(i) (T_i) is the sequence of the periods of ω^*.

(ii) (ω_i) is the corresponding sequence of best approximations; ω_i has period T_i and in fact, $\omega_i = \zeta_i/T_i$, $\zeta_i \in Z^n$, $\langle T_i\omega^* \rangle = \|T_i\omega^* - \zeta_i\|_\infty$.

(iii) p_i is defined by the identity $\omega_i = \omega(p_i)$; this is possible at least for i large enough since $p \to \omega(p)$ is a local diffeomorphism.

(iv) $r_i = \|p_i - p\|$ is the Euclidean distance from p^* to the rational point p_i.

Everything now happens as if the dimension were $n-1$, not n. In particular, by Dirichlet:

$$\|\omega_i - \omega^*\| \leqslant \frac{\sqrt{n-1}}{T_i T_{i+1}^{\frac{1}{n-1}}}.$$

One also has

$$r_i = \|p_i - p^*\| \leqslant \frac{\sqrt{n-1}}{m} \frac{1}{T_i T_{i+1}^{\frac{1}{n-1}}}.$$

This is valid in the convex case, to which we restrict ourselves here; treating the quasi-convex situation would necessitate some mild modifications.

The four sequences (T_i), (ω_i), (p_i), (r_i), are intrisincally defined modulo inessential choices; we define yet an other sequence (ε_i) as follows. Looking back at condition (ii) in theorem 1, namely $r \succeq \sqrt{\varepsilon}$, we write it more explicitly as $r \geqslant r_0\sqrt{\varepsilon}$ with $r_0 = r_0(\mathcal{P})$ some number having the dimension of a length. ε_i is defined through the equality $r_i = r_0\sqrt{\varepsilon_i}$. (ε_i) is not really intrinsic (because r_0 is not) but the ratios $(\varepsilon_i/\varepsilon_{i+1}) = (r_i/r_{i+1})^2$ are.

The five above defined sequences encode all the important properties of a given point p^*. We now return to dynamics and again consider trajectories of the perturbed system starting at $(p(0), q(0)$, $p(0) = p^*$. To make things clearer, *from now on we redefine the symbols* $\preceq$, $\asymp$ *and* $\succeq$ *so that they possibly include a dependence on the dimension n of the system*; in other words, $\alpha \preceq \beta$ now means $\alpha \leqslant c(\mathcal{P}, n)\beta$ and similarly for $\asymp$ and $\succeq$. This will take care of some simple algebraic factors involving n. For example, we shall now write

$$\|\omega_i - \omega^*\| \preceq \frac{1}{T_i T_{i+1}^{\frac{1}{n-1}}} \qquad r_i \preceq \frac{1}{T_i T_{i+1}^{\frac{1}{n-1}}}. \tag{8}$$

We assume one of the components of ω^* has been rescaled to unity, $\omega^* = (1, \alpha)$, $\alpha \in R^{n-1}$. The global stability estimate was obtained by applying corollary 3. For the radius of confinement,we have the estimate

$$\|p - p^*\| \leqslant r \asymp \frac{\varepsilon^a}{T} \qquad 1 \leqslant T < Q.$$

The basic problem is that T may lie anywhere between 1 and Q. If $T \asymp Q$, and since $Q \asymp \varepsilon^{-(n-1)a}$, we find that $\frac{\varepsilon^a}{T} \asymp \frac{\varepsilon^a}{Q} \asymp \sqrt{\varepsilon}$ $(a = \frac{1}{2n})$, and thus $b = \frac{1}{2}$. More generally, if ω^* is badly approximated, T tends to be close to Q; assume specifically that $\omega^* \in \Omega_{n-1}(\tau, \gamma)$, by which we mean in fact $\alpha \in \Omega_{n-1}(\tau, \gamma)$. Then the following holds:

$$\frac{1}{Q^{\frac{1}{n-1}}} \geqslant \langle T\omega^* \rangle \geqslant (\frac{\gamma}{T})^{\frac{1}{n}(1+\tau)}.$$

This yields

$$T \geqslant \gamma Q^{\frac{1}{1+\tau}} \succeq \gamma \varepsilon^{-a\frac{n-1}{1+\tau}}.$$

Since $a = \frac{1}{2n}$, this implies

$$\frac{\varepsilon^a}{T} \preceq \gamma^{-1} \varepsilon^{\frac{n+\tau}{2(1+\tau)}}$$

this proves the following.

Corollary 4. If $\omega^* \in \Omega_{n-1}(\tau, \gamma)$, one has

$$\|p(t) - p*\| \preceq \gamma^{-1}\varepsilon^{\frac{n+\tau}{2(1+\tau)}} \qquad \text{for } |t| \preceq \exp(c\varepsilon^{-\frac{1}{2n}}).$$

In particular, almost everywhere, the stability exponents $(a, b) = (\frac{1}{2n}, \frac{1}{2} - \eta), \forall \eta > 0$ are valid.

This is not a uniform estimate, because of the γ dependence. To restore uniformity, one can state the result on sets of large measure, e.g., of the type $\Omega_{n-1}(\tau, \gamma(\varepsilon))$ with—say—$\gamma(\varepsilon) = \gamma_0\varepsilon^{-\xi}$, for some $\xi > 0$.

Suppose now we try to apply corollary 1 instead of corollary 3. We readily get a confinement on the order of $\sqrt{\varepsilon}$, since $\|p(t) - p^*\| \leqslant r_0\sqrt{\varepsilon}$. But this applies only if

$$\frac{1}{TQ^{\frac{1}{n-1}}} \preceq \sqrt{\varepsilon}.$$

This inequality, unlike in the application of corollary 3, does not define Q, since T lies anywhere between 1 and Q. Again, $b = \frac{1}{2}$ holds if $T \asymp Q$. Let now p^*, and thus ω^*, be arbitrary, but assume that $\varepsilon = \varepsilon_i$, ($i$ large enough). By definition of the sequences (r_i) and (ε_i), one may apply corollary 1 around p_i, as p^* lies exactly on the boundary of the influence zone of this torus. The time of validity is on the order of $\exp(\frac{c}{T_i\sqrt{\varepsilon_i}})$. But

$$r_0\sqrt{\varepsilon_i} = r_i \preceq \frac{1}{T_i T_{i+1}^{\frac{1}{n-1}}} < \frac{1}{T_i^{\frac{n}{n-1}}}$$

which yields $T_i\sqrt{\varepsilon_i} \preceq \varepsilon_i^{\frac{1}{2n}}$. This we express as the following.

Corollary 5. For *any* point p^*, there exists a sequence $\varepsilon_i \to 0$ (depending on p^*) such that, for i large enough, one has

$$\|p(t) - p^*\| \preceq \sqrt{\varepsilon_i} \qquad \text{when} \quad |t| \preceq \exp(c\varepsilon_i^{\frac{1}{2n}}) \quad c = c(\mathcal{P}, n).$$

So, the pair $(a, b) = (\frac{1}{2n}, \frac{1}{2})$ prevails for any point along a subsequence, when the perturbation goes to zero. This form of the intermittency phenomenon we alluded to can be simply described as follows: let $\varepsilon_i \geqslant \varepsilon \geqslant \varepsilon_{i+1}$; then, if $\varepsilon = \varepsilon_i$, corollary 1 may be applied around p_i; but as soon as ε crosses this value, so that $\varepsilon < \varepsilon_i$, one has to apply it around p_{i+1}, and the time of validity drops discontinuously (w.r.t. ε) from $\exp(\frac{c}{T_i\sqrt{\varepsilon}})$ to $\exp(\frac{c}{T_{i+1}\sqrt{\varepsilon}})$.

The above result is somewhat artificial, because the sequence (ε_i) is not really intrinsic. This is not the case for the sequence (r_i) however, and one may wish to apply corollary 2 rather than corollary 1, producing the following.

Corollary 6. Assume $\varepsilon \leqslant \varepsilon_i$, i large enough, $p(0) = p^*$ arbitrary; then one has

$$\|p(t) - p^*\| \preceq r_i \qquad \text{when} \quad |t| \preceq \exp(cT_{i+1}^{\frac{1}{n-1}}) \qquad c = c(\mathcal{P}, n).$$

Indeed, one can apply corollary 2 around p_i and use (8) to conclude.

In order to extract from corollary 5 a statement which is valid for any small enough ε but only over a certain part of phase space, one should introduce yet other types of diophantine sets. Define

$$\Omega(\tau,\gamma)=\{\omega,\forall i\geqslant 0,\ T_{i+1}\leqslant \gamma T_i^{1+\tau}\}.$$

One thus imposes a bound on the rate of increase of the sequence of the periods. It is important to point out that the sets $\Omega(\tau,\gamma)$ do not really describe the *rate* of approximation by rationals, but rather its *regularity*. In particular, if $\omega=(0,\omega')$, $0\in R^m$, $\omega'\in R^d$, $\omega\in\Omega(\tau,\gamma)$ if and only if $\omega'\in\Omega(\tau,\gamma)$; this is why this set is essentially independent of n. In dynamical words, when a vector is resonant with multiplicity m and no more, a trajectory of the corresponding linear flow on the n torus is not dense; rather, the torus is foliated into tori of lower dimension d over which trajectories are dense. Belonging to $\Omega(\tau,\gamma)$ depends only on the motion on these lower dimensional tori. It is conceivable that this kind of arithmetical conditions may turn out to be the most useful and natural under various circumstances. Notice the inclusion $\Omega_n(\tau,\gamma)\subset\Omega(\tau,\gamma^{-(1+\tau)})$, which comes from the inequalities:

$$\left(\frac{\gamma}{T_i}\right)^{\frac{1}{n}(1+\tau)}\leqslant\langle T_i\omega\rangle\leqslant T_{i+1}^{\frac{1}{n}}$$

valid when $\omega\in\Omega_n(\tau,\gamma)$; here we use integer T_is and the right-hand side inequality is equivalent to Dirichlet's theorem.

Let us now return to the setting of corollary 5; assume that $\varepsilon_{i-1}>\varepsilon\geqslant\varepsilon_i$ and that $\omega^*\in\Omega(\tau,\gamma)$. Apply again corollary 1 around p_i; this yields a confinement on the order of $\sqrt{\varepsilon}$, over a time interval on the order of $\exp(\frac{c}{T_i\sqrt{\varepsilon}})$. T_i can now be compared with T_{i-1} thanks to the definition of $\Omega(\tau,\gamma)$ and Dirichlet theorem produces

$$r_0\sqrt{\varepsilon}\leqslant r_{i-1}\preceq\frac{1}{T_{i-1}T_i^{\frac{1}{n-1}}}<\frac{1}{T_{i-1}^{\frac{n}{n-1}}}$$

which implies $T_{i-1}\preceq\varepsilon^{-\frac{n-1}{2n}}$; hence

$$T_i\sqrt{\varepsilon}\leqslant\gamma T_{i-1}^{1+\tau}\sqrt{\varepsilon}\preceq\gamma\varepsilon^{\frac{1}{2n}(1-(n-1)\tau)}$$

yielding the following.

Corollary 7. If $\omega(p^*)\in\Omega(\tau,\gamma)$, $\tau<\frac{1}{n-1}$, local stability at p^* holds with exponents $(a,b)=(\frac{1}{2n}(1-(n-1)\tau),\frac{1}{2})$.

In particular, almost everywhere, the stability exponents $(\frac{1}{2n}-\eta,\frac{1}{2})$ are valid $\forall\eta>0$.

To conclude, we have seen that the stability exponents $(a,b)=(\frac{1}{2n},\frac{1}{2n})$ hold uniformly in phase space, or indeed any pair $(\frac{\mu}{2n},\frac{(1-\mu)}{2}+\frac{\mu}{2n})$, $\mu\in(0,1)$ (cf [8]). Corollaries 4 to 7 give a detailed description of some possible local refinements. They also serve to point out what the important quantities are, and that ε, r and T should somehow be treated on an equal footing.

There are many other possibilities, if one combines the above statements with considerations on resonances as in section 3. For instance, one may replace n by d in

corollary 7 on a resonance surface $S_{\mathcal{M}}$, 'almost everywhere' referring to the superficial measure over $S_{\mathcal{M}}$ (this comes from the definition of $\Omega(\tau, \gamma)$). Or one finds that over a 'slice' defined by $\mathrm{dist}(S_{\mathcal{M}}, p^*) \preceq \sqrt{\varepsilon}$, the exponents $(\frac{1}{2d} - \eta, \frac{1}{2})$ are valid almost everywhere (w.r.t. to n-dimensional Lebesgue measure) $\forall \eta > 0$. We leave it to the reader to enrich the combinatorics of these results, which are proved in a straightforward way.

This emphasizes the significance of the question which was already raised at the beginning of this section: is the pair $(\frac{1}{2n}, \frac{1}{2})$ in fact valid *everywhere*—and uniformly so—in phase space?

Whatever the answer, it would be interesting to know more about the geometry of the arithmetical sets $\Omega_n(\tau, \gamma)$ and $\Omega(\tau, \gamma)$, as well perhaps as 'truncated' versions of these. Indeed, as a final remark, we notice that it is slightly paradoxical that one should introduce diophantine conditions, while studying the behaviour of a system over *finite* times, because arithmetical conditions are of essentially asymptotic nature. In fact, there are two additional flexibilities which we have not fully used:

(i) We are interested in phenomena which occur over exponentially long times; on the other hand, the sequence of the periods (T_i) of *any* vector increases at least geometrically. We could therefore restrict attention to indices i such that $i = \mathrm{O}(\varepsilon^{-c})$, for some $c > 0$; this is the 'simultaneous' analogue of the usual ultraviolet cut-off.

(ii) There is an additional freedom related to the initial condition. Given a point p^*, it is only necessary to find a point whose frequency has nice arithmetical properties and which lies close to p^*. This was in fact used when extending local estimates from initial conditions lying *on* a resonance surface to points which may be $\mathrm{O}(\sqrt{\varepsilon})$ from the surface.

Combining these two possibilities could lead to an improvement in the results of this section.

5. Prospects and problems

In [1] and [2], rather detailed comments were made about how the periodic orbit method goes along with a somewhat new viewpoint on perturbation theory. We shall not repeat them here, beyond the few hints which are scattered in the previous sections, but would rather like to indicate some specific problems which, it seems to us, deserve to be investigated. Also, in this section, we shall refrain from giving detailed references, because they would be too numerous and would only obscure the matter. Other directions and possibilities are mentioned in [2].

—The first direction has already been alluded to. Having broken the problem down into small parts, 'analysis' was essentially reduced to one-phase averaging, namely the investigation of equation (2). Notice that although our original problem is multi-dimensional, we never met with small divisors, Linstedt–Poincaré series or the like. Now (2) embodies, as we mentioned, a simple case of divergence 'without small divisors'. There are some natural questions which come to mind suggested by the results which appear in [9] and [10]. Write the series $\widehat{u}$ and $\widehat{g}$ as

$$\widehat{u} = \sum_{k \geqslant 0} u_k(y, t)\varepsilon^k \qquad \widehat{g} = \sum_{k \geqslant 0} g_k(y)\varepsilon^k$$

where the u_ks and g_ks are analytic functions. The Borel transforms w.r.t. ε of these series read as

$$\mathcal{B}u = \sum_{k \geqslant 0} u_k(y, t)\frac{\zeta^k}{k!} \qquad \mathcal{B}g = \sum_{k \geqslant 0} g_k(y)\frac{\zeta^k}{k!}$$

where ζ is the variable dual to ε. These new series are now analytic for ζ small enough; what do their singularities look like in the ζ plane? The best one can hope for is that there appear essentially *isolated* singularities, which allow for continuation to an—infinitely many sheeted—Riemann surface. Ecalle's theory of 'resurgent' functions could then perhaps be applicable. This hope is fed by the fact that the time-1 map of (2) is ε-close to identity and that germs of holomorphic maps from $(C^n, 0)$ to itself which are tangential to identity can indeed be analysed through resurgence theory. This could lead to the following kind of dichotomy: either $\widehat{u}$ and $\widehat{g}$ are in fact convergent for ε small enough (this is the 'integrable' case) or they are of Gevrey order 1 and no less. In other words, they diverge in some sense *at least* as $\sum_{k\geqslant 0} k!\varepsilon^k$. Such assertion about a minimal order of divergence would also perhaps have a bearing on the estimates from *below* of some exponentially small quantities, as the splitting of the separatrices of the perturbed pendulum (4).

—A second class of problems has to do with using simultaneous approximation in dynamical system theory. We shall be very sketchy and refer again to [2] for some more information. An obvious question is: can one use simultaneous approximation in KAM theory? The answer is essentially: yes, one can indeed... (Rüssman, private communication). Any frequency vector ω being described by the sequence $(T_i)_{i\geqslant 0}$ of its periods, what are the natural diophantine conditions that ensure the conservation of a torus with frequency ω? How, for instance, does Brjuno's condition (multi-dimensional case) read in this language? Notice that a condition of the type $\omega \in \Omega(\tau, \gamma)$ does not entail that ω is non-resonant. So under some extra conditions, there may be tori attached to such frequencies, but not necessarily of maximal dimension.

—Returning to long time perturbation theory, a natural task is to find the optimal exponents in Nekhoroshev's theorem for steep non-convex Hamiltonians. There, the periodic orbit method does not apply and one has to revert to Nekhoroshev's original method, as improved in [8], and rewrite some geometric lemmas (on 'almost plane curves') described in [5]. It would be interesting—and probably difficult—to link the steepness indices and stability exponents that appear, with singularity theory, via the algebraic description of steep analytic functions given by Illyashenko (see references in [2]). The latter provides a sufficient and almost necessary condition on the jet of an analytic function for it to be steep. It thus somehow parallels Rüssman's non-degeneracy condition ensuring the applicability of KAM theory.

—The periodic orbit methods is particularly well suited for the investigation of systems with (possibly infinitely) many degrees of freedom. The discrete case of chains, crystals etc, has been under investigation for some years and we shall not report on this here. What about PDEs? Can one use a suitable modification of the method to attack some specific problems, as for instance particular cases of homogeneization phenomena?

—In a different vein, notice that the principal symbols of (pseudo)differential operators are usually *not* of the form studied in classical mechanics. So a large part of the theory would have to be rewritten *ab initio* before it can be applied to study the charateristic flows of *linear* problems. One may think of media with slowly varying structure coefficients, say an optical medium with slowly varying refraction index. The corresponding classical system is then integrable in the adiabatic approximation, i.e. if one freezes the coefficients. Under suitable conditions, a form of Nekhoroshev's theory should predict that light rays ('characteristic strips') look almost like straight lines, over a distance which is exponential w.r.t. the typical length of variation of the refraction index. What does that say about the waves?

—Finally and, somewhat connected to this type of questions, is the problem of the relevance of stability theory over exponential times to semi-classical mechanics. The semi-

classical theory of near integrable systems is a two-parameter theory (ε and $\hbar$) with highly non-commuting limits. Now *geometric* objects allow for the construction of quasimodes, as has been done for KAM tori (of maximal dimension) and stable periodic orbits. But finite-time results are obviously difficult to translate into spectral information. One of the only known ways of achieving this is by analytic continuation in the spectral plane, uncovering 'resonances', in the sense of quantum theory, but this is restricted to a very special class of problems (Schrödinger equation with dilation analytic potentials in particular). 'Classical' perturbation theory, of which results over exponentially long times appear as a crowning achievement, is thus really difficult to use semi-classically, and in fact, to the author's knowledge, only very simple results (essentially Birkhoff's normal form theory to a finite order) have been applied to date, at least in a rigorous mathematical setting. Perhaps the periodic orbit methods can be useful in that respect, although one should remember that we used *unperturbed* periodic orbits. Drawing a parallel with the chaotic (hyperbolic) case, in which the true periodic orbits seem to play a decisive role in analysing the spectrum (cf the Gutzwiller–Guillemin–Duistermaat trace formula) is tantalizing, but could be misleading.

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