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BIFURCATIONS OF RELATIVE EQUILIBRIA*

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Abstract. This paper discusses the dynamics and bifurcation theory of equivariant dynamical systems near *relative equilibria*, that is, group orbits invariant under the flow of an equivariant vector field. The theory developed here applies, in particular, to secondary steady-state bifurcations from invariant equilibria. Let Γ be a compact group of symmetries of R^n and let x_0 be in R^n. Suppose that f is a smooth Γ-equivariant vector field and Σ the isotropy group of x_0. It is shown that there exists a Σ-equivariant vector field f_N, defined on the space normal to X at x_0, and that the local asymptotic dynamics of f are closely related to the local asymptotic dynamics of f_N. Next those bifurcations of X are studied which occur when an eigenvalue of $(df_N)_x$ crosses the imaginary axis. Properties of the vector field f_N imply that branches of equilibria and periodic orbits of f_N correspond to trajectories of f which are dense in tori. Field [*Equivariant dynamical systems*, Trans. Amer. Math. Soc., 259 (1980), pp. 185-205] found bounds on the dimensions of these tori. Some of his results are extended. This theory is applied to the following specific problems:

(1) Bifurcations of systems with $O(2)$ symmetry.
(2) Bifurcations of steady-state solutions of the Kuramoto-Sivashinsky equation.
(3) Secondary bifurcations in the planar Bénard problem.

Key words. bifurcation, symmetry, relative equilibria

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Introduction. In this work we discuss the dynamics and bifurcation theory of equivariant dynamical systems near group orbits invariant under the action of the flow. Such group orbits are called *relative equilibria.* The simplest example of a relative equilibrium is a group orbit of equilibria. A group orbit of equilibria can be characterized as a relative equilibrium on which the flow is trivial. The symmetry groups we consider are compact and have positive dimension, so, in particular, they must contain a subgroup isomorphic to $SO(2)$. For such groups the flow trajectories on the relative equilibria can be nontrivial. A well-known example of such nontrivial trajectories on relative equilibria are rotating waves, that is, solutions given by $x(t)=\theta(t)x_0$, where $\theta(t)$ parametrizes $SO(2)$.

A special case of a relative equilibrium is an invariant equilibrium, that is, an equilibrium invariant under all the symmetries of the system. Such equilibria often arise in applications, and their bifurcations have been extensively studied. Often bifurcations of invariant equilibria are characterized by symmetry breaking; that is, the invariant equilibrium bifurcates to branches of equilibria no longer invariant under the action of the symmetry group. In other words, nontrivial relative equilibria often occur as a result of bifurcations of invariant equilibria. In this context bifurcations of relative equilibria correspond to secondary bifurcations from an invariant equilibrium.

Let X be a group orbit of equilibria of an equivariant vector field f. If X has positive dimensions, then the conditions determining when f can undergo a bifurcation near X are quite different than in the case of an invariant equilibrium. In particular, no element of X can be hyperbolic, since the directions along the group orbit must be neutrally stable. More precisely, for any $x\in X$ the tangent space T_xX is contained in the kernel of the derivative $(df)_x$. It follows that X will be normally hyperbolic if

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$(df)_x$ has no purely imaginary eigenvalues and the algebraic multiplicity of zero as an eigenvalue of $(df)_x$ equals the dimension of T_xX. A bifurcation of X will occur if $(df)_x$ has an eigenvalue on the imaginary axis whose (generalized) eigenvector is not contained in the tangent space of X.

An interesting phenomenon has been observed in examples of bifurcations of relative equilibria: an orbit of equilibria can lose stability by having an eigenvalue passing through zero and bifurcate to a group orbit consisting of nontrivial flow trajectories. The flow on the new relative equilibria is a slow drift given by the action of a curve of group elements. Several authors, who studied bifurcations of relative equilibria, found that the resulting dynamics could be described in terms of dynamics related to standard bifurcations modulated by a drift along the group orbit. The following articles focused on bifurcations of relative equilibria where this feature has been observed. Chossat [1986] has shown that the bifurcation of standing waves in the problem of degenerate Hopf bifurcation with $O(2)$ symmetry leads to quasi-periodic motion on a group invariant two-dimensional torus. Iooss [1986] has shown that a Hopf-Hopf mode interaction in the Taylor-Couette problem ($O(2)\times SO(2)$ symmetry) leads to a three-frequency flow. Danglemayr [1986] has found a rotating wave in the problem of steady-state mode interaction with $O(2)$ symmetry. Chossat and Golubitsky [1988] have studied a related problem of Hopf bifurcation of a group orbit of standing waves and have discovered that this bifurcation leads to a three-frequency motion, with one of the frequencies given by the drift along the orbit. In their paper Chossat and Golubitsky have formulated the following theorem: the flow near a relative equilibrium can be decomposed into the flow in the direction along the orbit and the flow in the direction normal to the orbit. The precise statement and the proof of this theorem is the starting point of this work.

Section 1 of the paper contains some background information on Lie group theory. The remaining part of the paper is divided into two parts. The first part, §§ 2–5, is devoted to the theoretical aspects of the problem. The second part, §§ 6–8, focuses on specific group actions and specific dynamical systems and is designed to show the application of the ideas developed in the first part. The reader more interested in the second part of the paper will only need to know the definitions and the statements of theorems contained in the first part. The following is a brief description of the topics discussed in each of the sections.

In § 2 we give a precise description of how the previously mentioned decomposition of the vector field can be accomplished. We show that near relative equilibria the vector field can be written as a sum of equivariant components: one tangent to the group orbits and the other normal to the original orbit X (Theorem 2.1). As a consequence of this decomposition each bounded solution near a relative equilibrium is contained in the group orbit of a solution of the normal vector field f_N (Theorem 2.2). In the remainder of § 2 we show that the asymptotic dynamics of f can be determined by the asymptotic dynamics of the normal vector field modulo drifts along the orbit. Some results of § 2, including an alternative proof of Theorem 2.1, can be found in Vanderbauwhede, Krupa, and Golubitsky [1989].

The results of § 2 imply that bifurcations of f can be analyzed in two steps. The first step is to describe bifurcations of the normal vector field f_N and the second step is to find the corresponding drifts along group orbits. Let x be in X. In § 3 we argue that generic bifurcations of f_N can be described as bifurcations of a generic Σ_x-equivariant vector field, where Σ_x is the isotropy subgroup of x.

Suppose that f describes a family of vector fields, rather than a single vector field. In § 4 we study bifurcations of relative equilibria occurring when an eigenvalue of

$(df_N)_x$ passes through zero. We analyze the case when a relative equilibrium X bifurcates to another relative equilibrium Y. Let $y \in Y$, let Σ be the isotropy subgroup of y and $N(\Sigma)$ the normalizer of Σ. Field [1980] proves a theorem stating that the flow on Y is given by a linear flow on a torus whose dimension is bounded by, and generically equal to, rank $(N(\Sigma)/\Sigma)$ (rank $(N(\Sigma)/\Sigma)$ equals the dimension of a maximal torus in $N(\Sigma)/\Sigma$). The main theorem of § 4 (Theorem 4.1′) states that there exists a generic set of perturbations of f whose elements have the following property: for all except countably many values of the parameter the dimension of the flow on Y is maximal.

In § 5 we study Hopf bifurcations of relative equilibria, that is, bifurcations occurring when an eigenvalue of $(df_N)_x$ passes through a nonzero point on the imaginary axis. We apply the standard Hopf bifurcation theorems to find periodic solutions of the normal component f_N. Let Y be a periodic orbit of f_N and let Σ be the isotropy subgroup of the elements of Y. Field [1980] shows that the corresponding trajectories of f are dense in tori whose dimension is bounded by rank $(N(\Sigma)/\Sigma)+1$. Let $\tilde{\Sigma}$ be the group consisting of all the symmetries that leave Y invariant. In Theorem 5.1 we derive a new bound, given by rank $(N(\tilde{\Sigma})/\tilde{\Sigma})+1$ and show that this bound is attained for a generic vector field. Next, in Theorem 5.2, we consider a family of vector fields f, such that f_N has a Hopf bifurcation and show that there exists a generic set of perturbations of f whose elements are such that for all except countably many values of the parameter the dimension of the flow on the manifolds ΓY is maximal.

In § 6 we present a classification of generic secondary steady-state and Hopf bifurcations with symmetry group $O(2)$. In this context bifurcations of the normal vector field correspond to steady-state and Hopf bifurcations with D_k symmetry. Using the results of §§ 3 and 4 we determine for which bifurcations of the vector field f_N the bifurcating solutions of the full vector field f generically have nontrivial drift along group orbits.

In § 7 we analyze bifurcations of the zero solution of the Kuramoto–Shivashinsky equation, which has $O(2)$ symmetry. We summarize the results of a computer-assisted study done by Kevrekedis, Nicolaenco, and Scovel [1988] and compare their numerical results with the predictions of $O(2)$ bifurcations, as described in § 6.

In § 8 we classify the possible generic steady-state bifurcations in the planar Bénard problem. The generic primary bifurcations in the Bénard problem are to two types of equilibria: hexagons (with symmetry D_6) and rolls (with symmetry $O(2)\oplus Z_2$). We consider secondary steady-state bifurcations of hexagons and rolls and show that the resulting trajectories are either equilibria or rotating waves.

1. Preliminaries. Let Γ be a compact Lie group. We consider a smooth linear action of Γ on R^n. With no loss of generality we can assume that this action is orthogonal and hence identify Γ with a subgroup of $O(n)$ (see Bredon [1972, I, 3.5]). Let X be a compact and Γ-invariant submanifold of R^n. For $x \in X$ let N_x be the set of vectors normal to X. Note that N_x is a vector subspace of R^n, since it passes through zero. Let $N(X)$ be the bundle with base space X and fibers N_x; $N(X)$ is called the *normal bundle* of X. The bundle $N(X)$ is smooth (see Guillemin and Pollack [1974, p. 71]). The action of Γ on $N(X)$ is defined by the formula $\gamma(x, v) = (\gamma x, \gamma v)$. To see that this action is well defined observe that the orthogonality of the action of Γ implies that $\gamma N_x = N_{\gamma x}$. Let $\beta : N(X) \to R^n$ be defined as $\beta((x, u)) = x + u$. It is easy to see that the map β is Γ-equivariant and a local diffeomorphism. It follows that an invariant neighborhood of X in R^n can be identified, via the map β, with a neighborhood of the zero section in $N(X)$. For $x \in X$ let $\hat{N}_x = \{(x, v) : v \in N_x\}$. Note that $\hat{N}_x \subset N(X)$

and the image of $\hat{N}_x$ under β is in N_x. Given a Γ-equivariant vector field $f: R^n \to R^n$ let β^*f be defined by $\beta^*f(y) = [d\beta_{\beta(y)}]^{-1}f(\beta(y))$, $y \in N(X)$. The map β^*f is called the *pullback* of f to $N(X)$. Let $x \in R^n$ and $X = \Gamma x$. It follows that studying local dynamics of f near x is equivalent to studying the dynamics of its pullback to $N(X)$.

For $x \in R^n$ (or $N(X)$) let

$$\Sigma_x = \{\sigma \in \Gamma: \sigma x = x\}$$

be the *isotropy subgroup* of x.

The homogeneous space Γ/Σ_x is not, in general, a group, but it has the structure of a smooth manifold and the quotient map $\pi: \gamma \nrightarrow \gamma\Sigma_x$ is a surjection (see Bredon [1972, p. 302]). The map $\gamma\Sigma_x \nrightarrow \gamma x$ is a diffeomorphism between Γ/Σ_x and the orbit Γx of x (see Bredon [1972; VI, 1.2]). Fix $x \in R^n$ and let $\Sigma = \Sigma_x$. Suppose that there exists a neighborhood U of $e\Sigma$ in Γ/Σ and a map $\sigma: U \to \Gamma$ such that $\pi\sigma(u) = u$ for all $u \in U$. The map σ is called a *local cross section* of π (for a more general definition see Bredon [1972, p. 39]). We now construct a local cross section of π. Let Δ be a submanifold of Γ transverse to Σ at e with $e \in \Sigma$ and $\dim \Sigma + \dim \Delta = \dim \Gamma$. Note that a neighborhood $\hat{U}$ of e in Δ is diffeomorphic to a neighborhood of $e\Sigma$ in Γ/Σ and this diffeomorphism is given by $\pi|\hat{U}$. Let $\sigma = (\pi|\hat{U})^{-1}$. Clearly, the map σ is a local cross section of π.

Let σ be a cross section of π defined on a neighborhood U of $e\Sigma$ in Γ/Σ. A simple argument shows that the map $\phi: \sigma(U) \times N_x \to R^n$ defined as $\phi(u, y) = \sigma(u)y$ is a local diffeomorphism. Let N_x^ε be a disc of radius ε around x and let ε be chosen so that $\phi: \sigma(U) \times N_x^\varepsilon$ is a diffeomorphism. Let $V^\varepsilon = \Gamma N_x^\varepsilon$. Orthogonality of the action implies that N_x and N_x^ε are Σ_x-invariant. Let $X = \Gamma x$. We chose ε so that the set V^ε is equivariantly diffeomorphic to a neighborhood of the zero section in $N(X)$. Observe that if $y \in \hat{N}_x$ then it is clear that $\Sigma_y \subset \Sigma_x$. Hence if $y \in N_x^\varepsilon$ then $\Sigma_y \subset \Sigma_x$. We have the following proposition.

PROPOSITION 1.1. *Every smooth and Σ_x-equivariant vector field $g: N_x^\varepsilon \to R^n$ has a unique smooth and Γ-equivariant extension $f: V^\varepsilon \to R^n$.*

Proof. We define f by requiring that $f(\gamma y) = \gamma g(y)$ for $\gamma \in \Gamma$, $y \in N_x^\varepsilon$. To see that f is well defined let $\gamma_1 y = \gamma_2 y$, $y \in N_x^\varepsilon$, $\gamma_1, \gamma_2 \in \Gamma$. Then $\gamma_1^{-1}\gamma_2 y = y$, so $\gamma_1^{-1}\gamma_2 \in \Sigma_y \subset \Sigma_x$ and $g(\gamma_1^{-1}\gamma_2 y) = g(y)$, implying $\gamma_1 g(y) = \gamma_2 g(y)$. Hence f is well defined.

Let U, σ, and ϕ be as defined prior to the statement of Proposition 1.1. Let $\hat{U} = \phi(U)$. Then $f|\hat{U} = \phi \circ id \times h \circ \phi^{-1}$. It follows that f is smooth on $\hat{U}$. Smoothness of f on V^ε follows from equivariance and smoothness of the action.

2. Dynamics near relative equilibria. Let Γ be a Lie subgroup of $O(n)$ acting orthogonally on R^n and let $f: R^n \to R^n$ be a C^∞ smooth Γ-equivariant vector field. Fix x_0 in R^n and let X denote the group orbit of x_0. We say that the set X is a *relative equilibrium* of f if X is invariant under the flow of f. The subject of this work is to study bifurcations of relative equilibria. In this section we develop a systematic way of analyzing dynamics near a relative equilibrium X. We first describe our results, deferring the proofs to the end of the section.

We begin by defining the concepts of a tangent vector field and a normal vector field. Let $g: R^n \to R^n$. We say that g is a *tangent vector field* if $g(u)$ is tangent to the group orbit of x for all x in R^n. For x in X let N_x be the space of vectors normal to X at x. We say that g is a *normal vector field* if for every x in X the space N_x is invariant under the flow of g. Note that a normal vector field does not have to be normal to group orbits other than X.

This section contains two main theorems. The first theorem states that near the group orbit X the vector field f can be written as a sum of a smooth Γ-equivariant normal vector field f_N and a smooth Γ-equivariant tangent vector field f_T. We refer to this theorem as the decomposition theorem. The second theorem states that near X the dynamics of f can be described as dynamics of f_N modulated by drifts along group orbits.

The decomposition theorem is a consequence of a technical lemma. The lemma states that near X there exists a smooth, Γ-invariant bundle K whose fibers are tangent to group orbits and whose restriction to X is the tangent bundle of X. Before stating the lemma we discuss the concept of the normal bundle of X. We observe that a Γ-invariant neighborhood of X can be identified with a neighborhood of the zero section in the normal bundle of X. We conclude that the dynamics of f can be understood in terms of the dynamics of the pullback of f to the normal bundle. In the proof of the decomposition theorem we identify f with its pullback to the normal bundle.

Following the proof of the decomposition theorem we discuss the most important implications of the two main theorems. We remark that Γ-equivariance of f_N implies that the dynamics of f_N is completely determined by its dynamics on the invariant set N_{x_0}. We also describe a way of finding a global center manifold near a relative equilibrium X. When X is an orbit of equilibria we show that the global center manifold is the union of local center manifolds constructed for each normal space N_x.

Next we discuss a method of explicit computation of f_N. We present the general form of "coordinates along group orbits" and "in the direction normal to the orbit." Such coordinates have been used to study specific examples of bifurcations of relative equilibria.

We begin by assuming that $X = \Gamma x_0$ is the group orbit of x_0, but not necessarily a relative equilibrium of f. The following theorem is the first of the two main results of this section.

THEOREM 2.1. *There exists a Γ-invariant neighborhood U of X in R^n, a smooth and Γ-equivariant normal vector field f_N, and a smooth and Γ-equivariant tangent vector field f_T such that*

$$f(u) = f_T(u) + f_N(u)$$

for all u in U.

Let g denote the restriction of f_N to the space N_{x_0}. Let U be the neighborhood defined in Theorem 2.1 and suppose that $u(t)$ is a trajectory of f contained in U for all $t \geqq 0$. We now state the second of the two main theorems of this section.

THEOREM 2.2. *There exists a smooth curve of group elements $\gamma(t)$ and a trajectory $y(t)$ of the vector field g such that $\gamma(t)y(t) = u(t)$ for all $t \geqq 0$.*

Let $\Pi: N(X) \to X$ be the bundle projection, that is, $\Pi((x, v)) = x$. The following lemma is the main technical result necessary to prove Theorem 2.1.

LEMMA 2.3. *There exists a smooth Γ-invariant subbundle K of $TN(X)$ such that for all $y \in N(X)$*

(i) $K_y \subset T_y \Gamma y$

(ii) $K_y \oplus N_{\Pi(y)} = R^n$.

Note that we cannot define K_y as $T_y \Gamma y$ since the dimension of group orbits may increase near x_0 (the fact that it cannot decrease is a consequence of the inclusion $\Sigma_y \subset \Sigma_{\Pi(y)}$ for $y \in \hat{N}_{\Pi(y)}$). In fact, proving Lemma 2.3 is the main technical difficulty of this section. We defer the proof to the end of the section. The proof of Theorem 2.2 is also deferred, since it relies on the proof of Lemma 2.3. The proof of Theorem 2.1 is a simple consequence of Lemma 2.3. In the proof we assume that f is a vector field on $N(X)$; that is, $f: N(X) \to TN(X)$.

Proof of Theorem 2.1. Suppose $y \in N(X)$. Let $P: TN(X) \to TN(X)$ be defined by $P(y, v) = P_y v$, where P_y is a projection with $\ker P_y = K_y$ and $\operatorname{Im} P_y = N_{\Pi(y)}$. The map P is smooth since the spaces $\ker P_y$ and $\operatorname{Im} P_y$ vary smoothly with y. Equivariance of P follows from invariance of K and equivariance of Π. Let $f_N(y) = P(y, f(y))$, $f_T(y) = f(y) - f_N(y)$. As required f_N is a normal vector field, f_T is a tangent vector field, and they are both smooth and Γ-equivariant.

We now discuss some implications of the two main theorems. Recall that orthogonality of the action implies that N_{x_0} is Σ_{x_0}-invariant. Let g be the vector field defined following the statement of Theorem 2.1; that is, g is the restriction of f_N to the space N_{x_0}. Since f_N is Γ-equivariant it follows that g is Σ_{x_0}-equivariant. Let $k = \operatorname{codim} X$. Note that $\dim N_{x_0} = k$. It follows that in order to understand the dynamics of f near X we need to carry out two steps:

(a) Analyze the dynamics of the k-dimensional Σ_{x_0}-equivariant vector field g.

(b) Find the drift along group orbits $\gamma(t)$.

Suppose that X is a relative equilibrium; in this case every x in X is an equilibrium of f_N. In particular, the point x_0 is an equilibrium of g. Let m be a positive integer. The equivariant center manifold theorem (cf. Ruelle [1973, Thm. 1.2]) implies that near x_0 the vector field g has a C^m smooth Σ_{x_0}-invariant center manifold. Let M_{x_0} denote such a center manifold. Let $M = \Gamma M_{x_0}$. The smoothness of the action and Σ_{x_0}-invariance of M_{x_0} together imply that M is C^m smooth. Theorem 2.2 also implies that all trajectories of f contained in a sufficiently small neighborhood of X approach M as time goes to infinity. We say that M is the *center manifold* of the relative equilibrium X for the vector field f.

When X consists of equilibria it is natural to ask whether the global center manifold M is a local center manifold for every element of X. We answer this question in the affirmative by verifying that for all x in X the tangent space to M at x equals the center subspace of $(df)_x$. Let E^c be the restriction of the tangent bundle of M to the relative equilibrium X. The bundle E^c is called the *center bundle* of X. We have the following proposition.

PROPOSITION 2.4. *Let x be in X. The fiber of the center bundle E^c at x is the center subspace of $(df)_x$.*

Proof. We first prove that $T_x X$ is contained in $\ker (df)_x$. Any vector $u \in T_x X$ can be written as $(d/ds)\gamma(s)x|_{s=0}$. We use the chain rule and the fact that $f(\gamma(s)x) = 0$ to obtain

$$(df)_x u = (df)_x \left(\frac{d}{ds} \gamma(s)x|_{s=0} \right) = \frac{d}{ds} f(\gamma(s)x)|_{s=0} = 0.$$

Hence zero is an eigenvalue of $(df)_x$ with multiplicity greater than or equal $\dim X$. Let $v \in N_x$. Theorem 2.1 implies that $(df)_x v = (df_N)_x v + (df_T)_x v$. We show that $(df_N)_x v \in N_x$ and $(df_T)_x v \in T_x X$. This implies that $(df)_x$ can be written in the form:

$$(df)_x = \begin{pmatrix} 0 & (df_T)_x \\ 0 & (df_N)_x \end{pmatrix}. \tag{2.1}$$

The proposition follows from equation (2.1), since (2.1) implies that all nonzero eigenvalues of $(df)_x$ are also eigenvalues of $(df_N)_x$.

We now prove that (2.1) is valid. The vector field f_N is Γ-equivariant and $f_N(x) = 0$. The argument presented at the beginning of this proof implies that $T_x X \subset \ker (df_N)_x$. Since $f = f_N + f_T$ it follows that $T_x X \subset \ker (df_T)_x$. Recall from the proof of Theorem 2.1 that $f_N(y) = P_y f(y)$, where P_y is a projection with $\ker (P_y) = T_y \Gamma y$ and

$\operatorname{Im}(P_y) = N_{\Pi(y)}$. Hence by the chain rule

(2.2) $$(df_N)_x = (d_y P_y f(x))_x + P_x(df)_x.$$

But $f(x) = 0$, so

(2.3) $$(df_N)_x = P_x(df)_x.$$

It follows that $(df_T)_x(N_x) \subset T_x X$ and $(df_N)_x(N_x) \subset N_x$. Equation (2.1) now follows.

In many applications a bifurcation of a group orbit of equilibria X occurs when $(df)_{x_0}$ maps a vector $v \in N_{x_0}$ to $T_{x_0}X$. The vector v then becomes a generalized nullvector of $(df)_{x_0}$. We have the following proposition.

PROPOSITION 2.5. *The vector v is a null vector of $(df_N)_{x_0}$.*

Proof. Proposition 2.5 follows from identity (2.3).

In applications we need to explicitly compute the vector field f_N. This can be done by changing variables to coordinates in the normal space N_{x_0} and a complementary set of coordinates "along group orbits." Such coordinates have been used by Chossat [1986], Iooss [1986], Danglemayr [1986], and others to study bifurcations of relative equilibria. In the form presented here they were suggested by Chossat and can be found in Moutrane [1988]. In our presentation we assume that f is a vector field on the normal bundle $N(X)$; that is, $f: N(X) \to TN(X)$. For a Lie group Δ, let $\mathscr{L}(\Delta)$ denote the Lie algebra of Δ. Let $\exp: \mathscr{L}(\Gamma) \to \Gamma$ be the exponential mapping. Let $V \subset \mathscr{L}(\Gamma)$ be the orthogonal complement of $\mathscr{L}(\Sigma_{x_0})$ in $\mathscr{L}(\Gamma)$ (the space V will be defined more precisely in the proof of Lemma 2.3). Let $\theta: V \times \hat{N}_{x_0} \to N(X)$ be given by $\theta(\xi, y) = (\exp \xi)(y)$. The linear map $(d\theta)_{(0,x_0)}$ is an isomorphism, and hence θ is a local diffeomorphism. Let $h = \theta^* f$. Note that for every $y \in N(X)$ the fiber of the tangent bundle $T_y N(X)$ can be written as $T_y N(X) = V \oplus N_{\Pi(y)}$. The vector field h is defined on $V \times \hat{N}_{x_0}$ and has the following property.

PROPOSITION 2.6. *If h is written in the form $h = (h_1, h_2)$, with $h_1 \in V$ and $h_2 \in N_{x_0}$, then $h_2(0, y) = f_N(y)$ for all $y \in \hat{N}_{x_0}$.*

The proof of Proposition 2.6 relies on the proof of Lemma 2.3 and therefore will be given at the end of the section.

Proof of Lemma 2.3. Let $\mathscr{L}(\Gamma)$ denote the Lie algebra of Γ and let $\exp: \mathscr{L}(\Gamma) \to \Gamma$ be the exponential mapping. We begin by recalling two concepts related to the Lie algebra $\mathscr{L}(\Gamma)$. The *action* of $\mathscr{L}(\Gamma)$ on $N(X)$ is defined by

$$\xi y = \frac{d}{dt}(\exp t\xi)y|_{t=0} \quad \text{for } \xi \in \mathscr{L}(\Gamma), \quad y \in N(X).$$

The *adjoint action* of Γ on $\mathscr{L}(\Gamma)$ is defined by

$$\operatorname{Ad}_\gamma \xi = \frac{d}{dt}\gamma(\exp t\xi)\gamma^{-1}|_{t=0} \quad \text{for } \gamma \in \Gamma, \quad \xi \in \mathscr{L}(\Gamma).$$

Note that

(2.4) $$\gamma \xi y = \operatorname{Ad}_\gamma \xi \gamma y \quad \text{for } \gamma \in \Gamma, \quad \xi \in \mathscr{L}(\Gamma), \quad y \in N(X).$$

Recall that $k = \operatorname{codim} X$. We prove that finding the bundle K is equivalent to finding a bundle E over $N(X)$ whose fibers are k-dimensional subspaces of $L(\Gamma)$ having the following property:

(2.5) $$E_{\gamma y} = \operatorname{Ad}_\gamma E_y \quad \text{for } \gamma \in \Gamma.$$

Suppose that the bundle E has been found. Then we define the fiber K_y of the bundle K as the set of all images of y under the action of elements of E_y; that is,

$$K_y = \{\xi y : \xi \in E_y\}.$$

If $v \in K_y$ then (2.4) implies that $\gamma v \in K_{\gamma y}$. Hence K is Γ-invariant. We now prove that K is a smooth bundle. For $y \in N(X)$ let $\Phi_y : \Gamma \to N(X)$ be defined by $\Phi_y(\gamma) = \gamma y$, $\gamma \in \Gamma$. For $\xi \in \mathscr{L}(\Gamma)$ we have

$$\xi y = d\Phi_y(e)\xi.$$

It follows that $K_y = d\Phi_y(e)E_y$. Smoothness of K now follows from smoothness of E and from smooth dependence of Φ_y on y. The definition of the action of $\mathscr{L}(\Gamma)$ implies that the fibers K_y are tangent to group orbits.

It remains to prove the existence of the bundle E. Suppose $\langle\!\langle\, , \rangle\!\rangle$ is an inner product on $\mathscr{L}(\Gamma)$ invariant with respect to the adjoint action. Such an inner product always exists for a finite-dimensional action of a compact Lie group (see, for example, Golubitsky, Stewart, and Schaeffer [1988, Prop. XI, 1.3]). Let V be the orthogonal complement of $\mathscr{L}(\Sigma_{x_0})$ taken with respect to $\langle\!\langle\, , \rangle\!\rangle$; that is, $V = \mathscr{L}(\Sigma_{x_0})^\perp$. Let E be the bundle over $N(X)$ with $E_y = \mathrm{Ad}_\gamma V$, $y \in \gamma \hat{N}_{x_0}$. To see that E is well defined suppose that $\gamma_1 y = \gamma_2 y$, $y \in \hat{N}_{x_0}$. From the properties of the normal bundle it follows that $\gamma_1\gamma_2^{-1} \in \Sigma_{x_0}$. It follows that $\mathrm{Ad}_{\gamma_1\gamma_2^{-1}} V = V$ or $\mathrm{Ad}_{\gamma_1} V = \mathrm{Ad}_{\gamma_2} V$, implying that $E_{\gamma_1 y} = E_{\gamma_2 y}$. Also E_y is defined for all $y \in N(X)$ since $N(X) = \bigcup_{\gamma \in \Gamma} \gamma \hat{N}_{x_0}$. Equation (2.4) is automatically satisfied for the fibers of E.

For smoothness of E let U be a neighborhood of $e\Sigma_{x_0}$ in Γ/Σ_{x_0} and let $\sigma : U \to \Gamma$ be a local cross section of π. Let $\phi : U \times N_{x_0} \to N(X)$ be given as $\phi(u, y) = \sigma(u)(x_0, y)$. The map ϕ is a local diffeomorphism near $(e\Sigma_{x_0}, 0)$. It follows that the map $\Psi : U \times N_{x_0} \times V \to E$ given by $\Psi(u, y, \xi) = (\sigma(u)y, \mathrm{Ad}_{\sigma(u)}\, \xi)$ is a local bundle diffeomorphism. This shows smoothness of E near $(x_0, 0)$. To show smoothness near $(\gamma x_0, 0)$ we use the map $\gamma\Psi$ and the relation (2.4).

Proof of Theorem 2.2. Suppose $u(0) = u_0$. Let γ_0 be the element of Γ such that $u_0 \in \gamma_0 N_{x_0}$. Let $y_0 = \gamma_0^{-1} u_0$ and let $y(t)$ be the integral curve of f_N with $y(0) = y_0$. Let $\cdot$ denote differentiation with respect to t. To prove the theorem we need to find a curve $\gamma(t)$ with $\gamma(0) = \gamma_0$ and such that

$$(\gamma(t)y(t))^\cdot = f(\gamma(t)y(t)). \tag{2.6}$$

The idea of the proof is to reduce (2.6) to an initial value problem on Γ. We observe that the left-hand side of (2.6) can be written as

$$\frac{d}{ds}\gamma(t+s)y(t)|_{s=0} + \gamma(t)\dot{y}(t).$$

By assumption $\dot{y}(t) = f_N(y(t))$. It follows that (2.6) can be rewritten as

$$\frac{d}{ds}\gamma(t+s)y(t)|_{s=0} = \gamma(t)f_T(y(t)). \tag{2.7}$$

Let V be the subspace of $\mathscr{L}(\Gamma)$ defined in the proof of Lemma 2.3. It follows from the proof of Lemma 2.3 and from the construction of the vector field f_T that there exists a curve $\xi(t)$ of elements of V such that $f_T(y(t)) = \xi(t)y(t)$. Equation (2.7) can now be rewritten as

$$\frac{d}{ds}\gamma(t+s)y(t)|_{s=0} = \gamma(t)\xi(t)y(t). \tag{2.8}$$

Consider the initial value problem:

$$\dot{\gamma} = \gamma(t)\xi(t), \qquad \gamma(0) = \gamma_0. \tag{2.9}$$

By standard theory of ordinary differential equations, (2.9) has a unique solution $\gamma(t)$. It is clear that if $\gamma(t)$ is a solution of (2.9) then $\gamma(t)y(t)$ satisfies (2.8). The theorem now follows.

Proof of Proposition 2.6. Let $y \in \hat{N}_{x_0}$. The proof of Lemma 2.3 and the construction of f_T imply that there exists a unique $\xi_0 \in V$ such that $f_T(y) = d/dt\,(\exp t\xi_0)y|_{t=0}$. Note that $d/dt\,(\exp t\xi_0)y|_{t=0} = (d\theta)_{(0,y)}(\xi_0, 0)$ implying

$$(d\theta)_{(0,y)}(\xi_0, 0) = f_T(y). \tag{2.10a}$$

Also $\theta|\hat{N}_{x_0} = id$. Hence

$$(d\theta)_{(0,y)}(0, f_N(y)) = f_N(y). \tag{2.10b}$$

Combining (2.10a) and (2.10b) we obtain $(d\theta)_{(0,y)}(\xi_0, f_N(y)) = f_N(y) + f_T(y) = f(y)$. Also, since $y \in \hat{N}_{x_0}$ we have $\theta^* f(y) = (d\theta)^{-1}_{(0,y)} f(y)$. Hence $h_2(0, y) = f_N(y)$.

3. Bifurcations of the normal vector field. Let $F: R^n \times R \to R^n$ be a family of vector fields and assume that $F(x_0) \in T_{x_0}X$; that is, X is a relative equilibrium. The results of § 2 imply that the dynamics of F can be described as follows: the trajectory of F with initial condition y_0 is contained in the group orbit of the trajectory of F_N with the same initial condition. We will utilize this property of the dynamics and divide the bifurcation analysis into two steps. The first step will be to analyze bifurcations of the normal vector field. Then, given a bifurcating solution of the normal vector field, say $y(t)$, we will study the dynamics of F on the set $Y = \{\gamma y(t): \gamma \in \Gamma, t \in R\}$. Note that Y is Γ-invariant and, by Theorem 2.2, it is invariant under the flow of F. This program will be carried out for two kinds of trajectories of F_N—equilibria and periodic orbits.

In this section we discuss the first part of the bifurcation analysis, that is, bifurcations of the normal vector field. Suppose that $\dim N_{x_0} = k$. We prove that generic bifurcations of F_N can be described in terms of generic bifurcations of Σ_{x_0}-equivariant vector fields on R^k. More specifically, we show that a property generic in the class of smooth, Σ_{x_0}-equivariant vector fields on R^k is also generic in the class of normal vector fields on N_{x_0}. Let $G(\cdot, \lambda)$ be the restriction of the vector field $F_N(\cdot, \lambda)$ to N_{x_0}. Let $g = G(\cdot, 0)$. Suppose that $(dg)_{x_0}$ has an eigenvalue on the imaginary axis and let E be the center subspace of $(dg)_{x_0}$. Suppose that G has a steady-state bifurcation; that is, $(dg)_{x_0}$ has a zero eigenvalue. Then we have the following proposition.

PROPOSITION 3.1. *Generically the space E equals the nullspace of $(dg)_{x_0}$ and the action of Σ_{x_0} on E is irreducible.*

Proposition 3.1 follows from Proposition 1.1 and standard results in equivariant bifurcation theory (see, for example, Golubitsky, Stewart, and Schaeffer [1988, Prop. XII, 3.4]).

Suppose that W is a subspace of R^k. We say that the action of Σ_{x_0} on W is Γ-*simple* if it is irreducible but not absolutely irreducible or if there exists a space V such that $W = V \oplus V$ and the action of Σ_{x_0} on V is absolutely irreducible.

Suppose now that G has a Hopf bifurcation; that is, $(dg)_{x_0}$ has a purely imaginary eigenvalue $i\omega$. The following proposition gives a characterization of the space E.

PROPOSITION 3.2. *Generically the space E is the generalized eigenspace of $i\omega$ for $(dg)_{x_0}$ and the action of Σ_{x_0} on E is Γ-simple.*

Proposition 3.2 follows from Proposition 1.1 and standard results in bifurcation theory (see, for example, Golubitsky, Stewart, and Schaeffer [1988, Prop. XVI, 1.4]).

A center manifold reduction coupled with a change of coordinates allows us to reduce the original bifurcation problem for G to a bifurcation problem posed on $E \times R$. We divide the analysis into two cases:

(i) The action of Σ_{x_0} on E is trivial.

(ii) The action of Σ_{x_0} on E is nontrivial.

Case (i) is much simpler and can be analyzed simultaneously for all groups Γ. In particular no symmetry breaking takes place. The following proposition summarizes the bifurcation analysis for this case.

PROPOSITION 3.3. *Suppose that the action of Σ_{x_0} on E is trivial. If $(dg)_{x_0}$ has a zero eigenvalue, then generically G has a limit point bifurcation. If $(dg)_{x_0}$ has a purely imaginary eigenvalue $i\omega$, then generically $i\omega$ is a simple eigenvalue of $(dg)_{x_0}$ and G has a Hopf bifurcation to a unique periodic solution.*

The proof of Proposition 3.3 follows from standard results in bifurcation theory. In the remainder of this work, unless otherwise stated, we will assume that the action of Σ_{x_0} on E is nontrivial.

Let $y=y(\lambda)$ be a branch of equilibria of G and $Y=Y(\lambda)$ a branch of periodic orbits of G. Let Σ be the isotropy subgroup of y and Σ_Y the group of symmetries mapping Y into itself. In §§ 4 and 5 we show that the trajectories of F on Γy are dense in tori whose dimension is bounded by rank $(N(\Sigma)/\Sigma)$ (the dimension of a maximal torus in $N(\Sigma)/\Sigma$) and the trajectories on ΓY are dense in tori whose dimension is bounded by rank $(N(\Sigma_Y)/\Sigma_Y)+1$. Generically these tori are of maximal dimension. In the context of Proposition 3.3, this maximal dimension equals rank $(N(\Sigma_{x_0})/\Sigma_{x_0})$ for trajectories on Γy and rank $(N(\Sigma_{x_0})/\Sigma_{x_0})+1$ for trajectories on ΓY.

4. Steady-state bifurcations. Let x_0 be in R^n and let $X=\Gamma x_0$. Suppose that $F: R^n \times R \to R^n$ is a smooth family of equivariant vector fields and X is a relative equilibrium of F for all values of λ. Theorem 2.1 guarantees that F can be decomposed as $F=F_N+F_T$, where F_N is a family of normal vector fields and F_T is a family of tangent vector fields. Let $G(\cdot,\lambda)$ denote the restriction of $F_N(\cdot,\lambda)$ to the normal space N_{x_0}. Note that x_0 is an equilibrium of G for all values of λ. We call x_0 the *trivial equilibrium* of G. We say that the family F has a *steady-state bifurcation* near X, if there exists a branch of nontrivial equilibria of G emanating from x_0. Note that such a bifurcation will generically occur if $(dG)_{(x_0,0)}$ has a zero eigenvalue and the action of the isotropy subgroup Σ_{x_0} on the center subspace of $(dG)_{(x_0,0)}$ is nontrivial.

Suppose that F has a steady-state bifurcation. Let $y(\lambda)$, $0 \leqq \lambda < \lambda_0$, be a bifurcating branch of nontrivial equilibria of G. We assume that all the equilibria $y(\lambda)$ have the same isotropy subgroup Σ. We also assume that the map $\lambda \twoheadrightarrow y(\lambda)$ is smooth on the open interval $(0, \lambda_0)$. Let $Y(\lambda)$ denote the group orbits of the equilibria $y(\lambda)$. Theorem 2.2 guarantees that the sets $Y(\lambda)$ are invariant under the flow of F. The goal of this section is to analyze that flow of F on the sets $Y(\lambda)$.

Let $z(\varepsilon, t)$ be the trajectory of F with initial condition $y(\lambda)$. Equivariance of F implies that each trajectory on $Y(\varepsilon)$ is given as $\gamma z(\lambda, t)$, for some $\gamma \in \Gamma$. Hence, to understand the dynamics on Y it suffices to analyze the structure of $z(\lambda, t)$. Let $N(\Sigma)$ denote the normalizer of Σ. Our analysis is based on the following observations:

(a) The trajectory $z(\lambda, t)$ is contained in $N(\Sigma)y(\lambda)$.

(b) There exists an integer $k \geqq 0$ such that $z(\lambda, t)$ can be described as k-frequency drift along the group orbit Y. More precisely, there exists a k-torus $\mathbb{T} \subset \Gamma$ such that $z(\varepsilon, t)$ is dense in $\mathbb{T}y$.

Field [1980, Prop. B1] has proved that the number of independent frequencies of the drift is bounded by the dimension of a maximal torus in $N(\Sigma)/\Sigma$. The result of Field can be easily deduced from properties (a) and (b).

We now state the main result of this section.

THEOREM 4.1. *For a generic family F the dimension of the drift along the orbit $Y(\lambda)$ equals the dimension of a maximal torus in $N(\Sigma)/\Sigma$ for all except countably many values of λ.*

Theorem 4.1 is an extension of Proposition B1 in Field [1980]. Dancer [1980] also obtained results relevant to the problem discussed in this section. Suppose that f is a smooth Γ-equivariant vector field and y is an equilibrium of f. Let Σ be the isotropy subgroup of y. In the proposition on p. 88 Dancer proved that if $\dim N(\Sigma) > \dim \Sigma$ then generically all equilibria of f which are sufficiently near y lie in the group orbit of y. Property (b) is a generalization of this result.

In the latter part of this section we state a more precise version of Theorem 4.1. In order to do this we need to review some concepts and results from Lie group theory.

Before we can prove Theorem 4.1 we need to analyze the flow on a relative equilibrium of a single vector field f. Let Y be a relative equilibrium of f and suppose that Σ is the isotropy subgroup of some $y \in Y$. We prove that a trajectory on Y is dense in a k-dimensional torus and that generically k equals the dimension of a maximal torus in $N(\Sigma)/\Sigma$.

In Proposition 4.6 we prove an important technical result stating that the set of all images of a point y in R^n under a smooth Γ-equivariant map is Fix (Σ_y). This result is stated without proof in Lemma A of Field [1980].

Proposition 4.10, which is stated following the proof of Theorem 4.1′ describes what happens when the drift fails to be of maximal dimension. The proposition asserts that for a generic family the dimension of the drift can only decrease by 1. Field [1988] proves that if the dimension of a maximal torus in $N(\Sigma)/\Sigma$ equals 1, then for a generic family F the set $Y(\lambda)$ contains no equilibria. This result does not follow from Theorem 4.1.

In order to state a more precise version of Theorem 4.1 we need to review the concepts of maximal tori and rank of a Lie group. Let Δ be a Lie group. We say that a Lie subgroup $\mathbb{T}$ of Δ is a *torus* if $\mathbb{T}$ is compact, Abelian, and connected. A torus is called *maximal* if it is not properly contained in any other torus. The following is the main result on maximal tori.

THEOREM 4.2. *In a Lie group Δ any two maximal tori are conjugate, and every element of Δ is contained in a maximal torus.*

The proof of Theorem 4.2 can be found in Bröcker and tom Dieck [1985, Thm. (1.6), p. 159].

Theorem 4.2 implies that all maximal tori are of the same dimension. The dimension of maximal tori in Δ is called the *rank* of Δ.

Let $l = \operatorname{rank} \Delta$ and let $\xi \in \mathscr{L}(\Delta)$. We say that ξ *generates a maximal torus* in Δ if the set $\{\exp t\xi : t \in R\}$ is dense in a torus of dimension l. We have the following proposition.

PROPOSITION 4.3. *The set of $\xi \in \mathscr{L}(\Delta)$ which generates a maximal torus is residual (an intersection of open and dense sets).*

Proof. Let $\mathbb{T}$ be a maximal torus in Δ. We identify $\mathbb{T}$ with R^l/Z^l and $\mathscr{L}(\mathbb{T})$ with R^l (see Bröcker and tom Dieck [1985, Cor. I, eq. (3.7)]). Let

$$P^m = \{\xi \in \mathscr{L}(\mathbb{T}) : \xi = (\xi_1, \xi_2, \cdots, \xi_l) \text{ and } \textstyle\sum m_j \xi_j = 0\}$$

and let

$$E^m = \bigcup_{\sigma \in \Delta} \operatorname{Ad}_\sigma P^m.$$

Since the group Δ is compact, it follows that the image of $\mathscr{L}(\Delta)$ under the exponential mapping is the connected component of the identity in Δ (see Bröcker and tom Dieck [1985, Thm. IV, eq. (2.2)]). Hence, by Theorem 4.2, each $\zeta \in \mathscr{L}(\Delta)$ has the form $\operatorname{Ad}_\sigma \xi$,

$\xi \in \mathscr{L}(\mathbb{T})$. Since $\exp \mathrm{Ad}_\sigma \xi = \sigma \exp \xi \sigma^{-1}$ it follows that ζ generates a maximal torus if and only if ξ does. Also ξ generates a maximal torus if it is in the complement of the sets E^m for all $m \in Z^l$. The sets E^m are nowhere dense (see the proof of Theorem 4.1′), so the complement of their union is a residual set.

Throughout we assume the following conditions on the family of vector fields F:

(S1) The orbit X is a trivial relative equilibrium of F. In other words, $F_N(\lambda, x_0) = 0$ for all values of λ.

(S2) There exists $\lambda_0 > 0$ and a branch of relative equilibria of F_N, parametrized as $y(\lambda)$, $0 < \lambda < \lambda_0$. The mapping $\lambda \nrightarrow y(\lambda)$ is smooth on $(0, \lambda_0)$. The points $y(\lambda)$ have isotropy Σ.

Let $C^\infty_T(R^n \times R, R^n)$ denote the space of smooth families of equivariant tangent vector fields on R^n. For a family F satisfying (S1) and (S2) let $Y(\lambda) = \Gamma y(\lambda)$. We now state Theorem 4.1 more precisely.

THEOREM 4.1′. *Suppose that a family of vector fields F satisfies* (S1) *and* (S2). *Then*

(i) *Trajectories on the manifolds $Y(\lambda)$ are dense in tori of dimension bounded by* rank $(N(\Sigma)/\Sigma)$.

(ii) *There exists a residual set $\mathscr{B} \subset C^\infty_T(R^n \times R, R^n)$ such that for every $H \in \mathscr{B}$ there exists a countable set $I_0 \subset (0, \lambda_0)$ such that for every $\lambda \in (0, \lambda_0) \backslash I_0$ trajectories of $F + H$ on the manifolds $Y(\lambda)$ are dense in tori of dimension equal to* rank $(N(\Sigma)/\Sigma)$.

Note that Theorem 4.1′ is more general than Theorem 4.1: we assume that $y(\lambda)$ is a branch of relative equilibria of F_N rather than a branch of equilibria of F_N. This assumption does not increase the complexity of the proof.

Before proving Theorem 4.1′ we analyze the following simpler situation. Suppose that $g: R^n \to R^n$ is a Γ-equivariant vector field with the following properties:

(VS1) The orbit X is a relative equilibrium of g.

(VS2) There exists $y_0 \in R^n$, $y_0 \notin X$ such that $Y = \Gamma y_0$ is a relative equilibrium of g.

The problem of finding the dynamics on Y has been solved by Field [1980]. Here we briefly present his results. We start with the following proposition.

PROPOSITION 4.4. *Suppose that $g(y_0) = v$. Let $\xi \in \mathscr{L}(\Gamma)$ be such that $\xi y_0 = y$. Then $y(t) = \exp(t\xi) y_0$ is the integral curve of g with $y(0) = y_0$.*

Proof.

$$\dot{y}(\tau) = \frac{d}{dt} \exp(t\xi) y_0 \big|_{t=\tau}$$

$$= \exp(\tau\xi) \frac{d}{dt} \exp((t-\tau)\xi) y_0 \big|_{t=\tau}$$

$$= \exp(\tau\xi) \frac{d}{ds} \exp(s\xi) y_0 \big|_{s=0}.$$

By definition $(d/ds) \exp(s\xi) y_0|_{s=0} = \xi y_0$. Hence

$$\dot{y}(\tau) = \exp(\tau\xi) \xi y_0 = \exp(\tau\xi) g(y_0)$$

$$= g(\exp(\tau\xi) y_0) = g(y(\tau)).$$

For $h: R^n \to R^n$ let $\|h\| = \sup_{x \in R^n} |h(x)|$. The following theorem gives a complete description of dynamics on relative equilibria of a vector field g.

THEOREM 4.5 (Field [1980, Prop. B1]). *Suppose that $g: R^n \to R^n$ is an equivariant vector field satisfying* (VS1) *and* (VS2) *and let Σ be the isotropy subgroup of y_0. Then*

(i) *Every flow trajectory contained in Y has the form* $z(t) = \exp(t\xi)y$, *for some* $\xi \in \mathscr{L}(N(\Sigma))$.

(ii) *The dimension of the torus* $\mathbb{T} = \text{cl}\{\exp(t\xi)y_0: t \in R\}$ *is less than or equal to* $\text{rank}(N(\Sigma)/\Sigma)$.

(iii) *For every* $\varepsilon > 0$ *there exists a vector field h, such that* $\|h\| < \varepsilon$, *Y is a relative equilibrium of h, and the dimension of the closure of trajectories of* $g + h$ *on Y equals* $\text{rank}(N(\Sigma)/\Sigma)$.

To prove Theorem 4.5 we need to answer the following question. What are the possible images of the vector y_0 under Γ-equivariant vector fields? Suppose that V is the space of all possible images of y under Γ-equivariant vector fields; that is,

$$V = \{h(y), h: R^n \to R^n \text{ is a } \Gamma\text{-equivariant vector field}\}.$$

If h is a Γ-equivariant vector field, then it follows that $h(y_0)$ is fixed by all elements in Σ_{y_0}. Hence $V \subset \text{Fix}(\Sigma_{y_0})$. The following proposition shows that the other containment also occurs.

PROPOSITION 4.6. *The space V is equal to the fixed-point space of* Σ_{y_0}; *that is,*

$$V = \text{Fix}(\Sigma_{y_0}).$$

Proof. Let $Y = \Gamma y_0$ and let $\Sigma = \Sigma_{y_0}$. Suppose that $v \in \text{Fix}(\Sigma)$. We first show that there exists a smooth and Γ-equivariant vector field $g: R^n \to R^n$ such that $g(y_0) = v$. Let $N(Y)$ be the normal bundle of Y. Recall that $N(Y)$ can be identified with an invariant neighborhood U of Y in R^n. It follows that $TN(Y)$ can be identified with R^n. We define a vector field $h: N_{y_0} \to TN(Y)$ by $h(z) = v$. Clearly, h is smooth and Σ-equivariant. By Proposition 1.1 we can extend h to a smooth, Γ-equivariant vector field g_1 on $N(Y)$. The properties of the normal bundle $N(Y)$ imply that the vector field g_1 can be identified with a vector field g_2 defined on U. Let $\alpha: R^n \to R$ be a smooth, invariant function such that $\alpha(y_0) = 1$ and $\alpha(x) = 0$ for all $x \notin U$. Let g be defined as follows:

$$g(x) = \begin{cases} \alpha(x) g_2(x) & \text{if } x \in U, \\ 0 \quad \text{if } x \notin U. \end{cases}$$

Clearly, g is smooth, Γ-equivariant, and $g(y_0) = v$.

Remark 4.7. The vector field h described in Theorem 4.5(iii) can be chosen so that $g + h$ is a polynomial vector field. The existence of such h can be shown using the equivariant version of the Stone–Weierstrass approximation theorem (see Poenaru [1976, proof of Prop. 1, p. 20]).

The final ingredient necessary to prove Theorem 4.5 is given by the following elementary lemma.

LEMMA 4.8. *The following equality holds for any* $y \in R^n$:

$$T_y Y \cap \text{Fix}(\Sigma_y) = \{\xi y: \xi \in \mathscr{L}(N(\Sigma_y))\}.$$

Proof. Let $\xi \in \mathscr{L}(\Gamma)$, $y \in R^n$. Then $\xi y \in \text{Fix}(\Sigma_y)$ if and only if $\exp \xi y \in \text{Fix}(\Sigma_y)$. This implies that $\sigma \exp \xi y = \exp \xi y$ for all $\sigma \in \Sigma_y$. It follows that there exists $\eta \in \mathscr{L}(N(\Sigma))$ such that $\eta y = \xi y$. The lemma now follows.

Proof of Theorem 4.5. The theorem is an easy consequence of Proposition 4.4, Proposition 4.6, and Lemma 4.8.

In the remainder of this section we prove Theorem 4.1′. Let F be a family of vector fields satisfying (S1) and (S2). We begin by defining a map which assigns to each $H \in C^\infty_T(R^n \times R, R^n)$ a curve ξ in $\mathscr{L}(N(\Sigma)/\Sigma)$ such that $(F+H)(y(\lambda), \lambda) = \xi(\lambda) y(\lambda)$ for each λ in some interval I. Recall that $T_y Y \cap \text{Fix}\,\Sigma = \{\xi y: \xi \in \mathscr{L}(N(\Sigma))\}$.

Let V_Σ be the set of $y \in R^n$ with isotropy group Σ. For $y \in V_\Sigma$ the space $\{\xi y: \xi \in \mathscr{L}(N(\Sigma))\}$ is isomorphic to $\mathscr{L}(N(\Sigma)/\Sigma)$. Let $\Xi: T_yY \cap \operatorname{Fix}\Sigma \to \mathscr{L}(N(\Sigma)/\Sigma)$ denote this isomorphism. It is clear that Ξ changes smoothly as y is being varied in V_Σ. Let I be a subinterval of $(0, \lambda_0)$. The map $\Theta: C^\infty_T(R^n \times R, R^n) \to C^\infty(I, \mathscr{L}(N(\Sigma)/\Sigma))$ is defined as

$$\Theta(H)(\lambda) = \Xi((F+H)(y(\lambda), \lambda)).$$

We prove Theorem 4.1′ by showing that for a residual set of $H \in C^\infty_T(R^n \times R, R^n)$ the curve $\Theta(H)$ is transverse to all the sets E^m (see Proposition 4.3). Before presenting the proof we review some concepts related to the Whitney C^∞ topology. For a more complete treatment of this topic see Golubitsky and Guillemin [1974]. Let Z, W be smooth manifolds. For a positive integer q let $J^q(Z, W)$ denote the space of q-jets of smooth maps from Z to W. We describe a neighborhood basis of a map f in the Whitney C^∞ topology on $C^\infty(Z, W)$. Let q be a positive integer and let d_q be a metric on $J^q(Z, W)$ compatible with its topology (such a metric exists by (I, 5.9) in Golubitsky and Guillemin [1974]). Let $\delta: Z \to R^+$ be a continuous function. Let

$$U_{q,\delta} = \{g \in C^\infty(Z, W): d_q(j^q f(x), j^q g(x)) < \delta(x) \text{ for all } x \in Z\}.$$

The collection of the sets $U_{q,\delta}$ for all choices of q and δ forms a neighborhood basis of f in the Whitney C^∞ topology.

Suppose that Z is an open subset of R^p for some p, and W is a vector space. Then the above-mentioned metrics d_q can be chosen as follows. Suppose $s = \dim W$. We identify W with R^s. For a positive integer q and $g \in J^q(Z, W)$ let

$$\|g\|_q(x) + |x| + |g(x)| + \sum_{1 \leq |\alpha| \leq q} \left| \frac{\partial^{|\alpha|} g(x)}{\partial x^\alpha} \right|.$$

Here α denotes a p-vector of nonnegative integers. We define d_q on $J^q(Z, W)$ as

$$d_q(\sigma_1, \sigma_2) = \|g_1 - g_2\|_q(x),$$

where $\sigma_1, \sigma_2 \in J^q(Z, W)$ and g_1, g_2 are such that $\sigma_1 = j^q g_1(x)$ and $\sigma_2 = j^q g_2(x)$. It is easy to see that d_q agrees with the topology on $J^q(Z, W)$.

Let I be the interval used in the definition of the map Θ. Then we have Lemma 4.9.

LEMMA 4.9. *If $\bar{I} \subset (0, \lambda_0)$ then Θ is continuous in the Whitney C^∞ topology.*

Proof. Let $C(I) = \{(y(\lambda), \lambda): \lambda \in I\}$. In this proof we use the metrics d_q described prior to the statement of Lemma 4.9. The map Θ can be written as

$$\Theta(H) = \Xi \circ H | C(I).$$

Hence Θ is a composition of two maps: a map Θ_1 given as

$$\Theta(H) = H | C(I)$$

and a map Θ_2 defined as

$$\hat{H} \to \Xi \circ \hat{H}.$$

The map Θ_1 does not, in general, have to be continuous, but it is continuous if $\bar{I} \subset (0, \lambda_0)$. This follows, since $\bar{I} \subset (0, \lambda_0)$ implies that for any given q all the partial derivatives of the function $\lambda \nrightarrow y(\lambda)$ are bounded on I. Hence continuity of Θ_1 can be established through repeated application of the chain rule. The map Θ_2 is continuous by (II, 3.5) in Golubitsky and Guillemin [1974].

Proof of Theorem 4.1′. Part (i) of the theorem follows from Theorem 4.5.

We now prove part (ii). Suppose that l is the rank of $N(\Sigma)/\Sigma$. Let $\mathbb{T}$ be a maximal torus in $N(\Sigma)/\Sigma$. As in the proof of Proposition 4.3 we identify $\mathbb{T}$ with R^l/Z^l and $\mathscr{L}(\mathbb{T})$ with R^l. Recall the definitions of the sets E^m and P^m (see the proof of Proposition 4.3). In the proof of Proposition 4.3 we show that the sets E^m have the following property: a vector $\xi \in \mathscr{L}(N(\Sigma)/\Sigma)$ generates a maximal torus if and only if ξ is in the complement of E^m for all m in Z^l.

The sets E^m may not be manifolds, but we show that each E^m is a finite union of manifolds. Let $\xi \in P^m$ and let Δ be the isotropy subgroup of ξ with respect to the adjoint action of $N(\Sigma)/\Sigma$ on $\mathscr{L}(N(\Sigma)/\Sigma)$. Let $[\cdot, \cdot]$ denote the bracket in $\mathscr{L}(N(\Sigma)/\Sigma)$. It is known (see Bröcker and tom Dieck [1985, I, eq. (2.12)]) that

$$[\eta, \zeta] = \frac{d}{dt} \mathrm{Ad}_{\exp t\eta} \zeta|_{t=0} \quad \text{for all } \eta, \zeta \in \mathscr{L}(N(\Sigma)/\Sigma). \tag{4.1}$$

The containment $P^m \subset \mathscr{L}(\mathbb{T})$ implies that $[\xi, \eta] = 0$ for all $\eta \in P^m$. Let $O(\xi) = \bigcup_{\sigma \in N(\Sigma)/\Sigma} \mathrm{Ad}_\sigma \xi$ be the orbit of ξ. Let U be a small neighborhood of $e\Delta$ in $(N(\Sigma)/\Sigma)/\Delta$ and let $\sigma: U \to N(\Sigma)/\Sigma$ be a local cross section. Equation (4.1) implies that $T_\xi O(\xi) \cap P^m = \{0\}$. It follows that the map $\Psi: U \times P^m \to \mathscr{L}(N(\Sigma)/\Sigma)$, given by $\Psi(u, \eta) = \mathrm{Ad}_{\sigma(u)} \eta$, is a local diffeomorphism near (e, ξ).

Let $E^m(\Delta)$ be the set of all elements of E^m whose isotropy subgroup (with respect to the adjoint action) is conjugate to Δ, and let $P^m(\Delta) = P^m \cap E^m(\Delta)$. Note that $P^m(\Delta)$ is an open subset of $P^m \cap \mathrm{Fix}(\Delta)$. For every $\xi \in P^m$ the corresponding map Ψ is a local diffeomorphism near (e, ξ). It follows that $E^m(\Delta)$ is a smooth manifold.

Note that the number of the sets $E^m(\Delta)$ is finite. This follows from the fact that $N(\Sigma)/\Sigma$, being a compact group, has a finite number of conjugacy classes of isotropy subgroups. Clearly, $E^m = \bigcup E^m(\Delta)$.

The theorem follows from the following assertion:

(*) For every $m \in Z^l$, $\mu \in (0, \lambda_0)$, there exists an interval I containing μ and a set $\mathscr{B}^m(I) \subset C^\infty_T(R^n \times R, R^n)$ with the following properties:
(1) $\mathscr{B}^m(I)$ is residual in the C^∞ Whitney topology.
(2) $\Theta(H)$ is transverse to all the sets $E^m(\Delta)$ at each $\lambda \in I$.

We first show that the theorem follows from (*). To see this let $I_1^m, I_2^m, \cdots$ be a sequence of intervals such that $\bigcup_{i=1}^\infty I_i^m = (0, \lambda_0)$ and $\mathscr{B}^m(I_i^m)$ satisfies the properties (1) and (2). Let $\mathscr{B} = \bigcap_{i=1, j=1}^\infty \mathscr{B}^j(I_i^j)$. It follows that for every $H \in \mathscr{B}$ the curve $\Theta(H)$ is transverse to all the sets E^m at every $\lambda \in (0, \lambda_0)$. It is clear that $\mathscr{B}$ satisfies the property required in the statement of Theorem 4.1′.

We now prove (*). Fix $m \in Z^l$, a subgroup $\Delta \subset N(\Sigma)/\Sigma$ and $\mu \in (0, \lambda_0)$. Let $I \subset I_0 \subset (0, \lambda_0)$ be intervals with $\mu \in I$ and $\bar{I}_0 \subset (0, \lambda_0)$. Let $\mathscr{A}_0$ be the set of $\xi \in C^\infty(I_0, \mathscr{L}(N(\Sigma)/\Sigma))$ such that ξ is transverse to $E^m(\Delta)$ at each $\lambda \in I$. By standard transversality arguments (see Golubitsky and Guillemin [1974, (II, 4.5)]) the set $\mathscr{A}_0$ is open and dense in the Whitney C^∞ topology. We assume that I_0 is the interval used in the definition of the map Θ. Let $\mathscr{A} = \Theta^{-1}\mathscr{A}_0$. It follows from Lemma 4.9 that $\mathscr{A}$ is an intersection of open sets. We now show that $\mathscr{A}$ is dense. Fix $H \in C^\infty_T(R^n \times R, R^n)$. We construct a sequence of families $\{H_i\}$ converging to H and such that each $H_i \in \mathscr{A}$. Let $\{\xi_i\}$ be a sequence of elements of $C^\infty(I_0, \mathscr{L}(N(\Sigma)/\Sigma))$ such that each curve $\Theta(H) + \xi_i$ is in $\mathscr{A}_0$ and the curves ξ_i converge to the zero curve as $i \to \infty$. Such a sequence exists, since $\mathscr{A}_0$ is dense in $C^\infty(I_0, \mathscr{L}(N(\Sigma)/\Sigma))$. To show the existence of the sequence $\{H_i\}$ it suffices to prove that for every $\eta \in C^\infty(I_0, \mathscr{L}(N(\Sigma)/\Sigma))$ there exists a family H_η such that $\Theta(H_\eta)(\lambda) = \eta(\lambda)$ for all $\lambda \in I$ and such that for every positive integer q the size of partial derivatives of H_η of order less than or equal to q can be estimated by the size of partial derivatives of η of order less than or equal to q. We now give a more precise description of this estimate. Let $\hat{\eta}$ be a smooth curve of elements of $\mathscr{L}(N(\Sigma))$ which projects to η in $\mathscr{L}(N(\Sigma)/\Sigma)$. Suppose that α is an n-vector of positive integers and β a positive integer. Then there exists a constant C, depending only on m, and such that

$$\left|\frac{\partial^{|\alpha|+\beta} H_\eta(z,\lambda)}{\partial z^\alpha\,\partial\lambda^\beta}\right| \leqq C\left|\frac{\partial^\beta \hat{\eta}(\lambda)}{\partial\lambda^\beta}\right| \tag{4.2}$$

for all $(z,\lambda)\in V\times I_0$. Moreover, $H_\eta(z,\lambda)=0$ for $(z,\lambda)\notin V\times I_0$.

We now construct H_η. Let $y_0=y(\mu)$ and $Y=\Gamma y_0$. Recall that $N(Y)$ is equivariantly diffeomorphic to an invariant neighborhood of Y in R^n. Let V be such a neighborhood. In the sequel we identify V with $N(Y)$. By shrinking the interval I we can assume that $y(\lambda)\in V$ for all $\lambda\in I$. We can assume that $y(\lambda)\in N_{y_0}$ for all $\lambda\in I$. Otherwise, we could replace the curve $y(\lambda)$ by a curve $\hat{y}(\lambda)=\gamma(\lambda)y(\lambda)$ with $\gamma(\lambda)\in N(\Sigma)/\Sigma$. We can now define $\hat{H}(z,\lambda)$ as $\hat{\eta}(\lambda)z$ is $z\in N_{y_0}$ and extend this definition by equivariance (see the proof of Proposition 4.6). Let U_0 be a small neighborhood of $e\Sigma$ in Γ/Σ and let σ be a local cross section of π (see § 1). Let $\phi: U_0\times N^\varepsilon_{y_0}$ be defined as $\phi(u,y)=\sigma(u)y$ (here $N^\varepsilon_{y_0}$ denotes a disc of radius ε around y_0 in N_{y_0}). Recall that for ε small enough ϕ is a diffeomorphism. Let $U=\phi(U_0\times N^\varepsilon_{y_0})$. We can express every point $z\in U_0$ in local coordinates as $\sigma(u)y$, $y\in N_{y_0}$, $u\in U$. Then, for every $z\in U$, $\hat{H}(z,\lambda)=\sigma(u)\hat{\eta}(\lambda)z$. It is clear from this expression and from the smoothness of the action of Γ that (4.2) holds for all $z\in U$. From compactness of Y it follows that the bound (4.2) holds on a neighborhood V_1 of Y in R^n with possibly a different constant C. With no loss of generality we can assume that $V=V_1$. Let W be an invariant neighborhood of y_0 such that $\bar{W}\subset V$ and suppose that I is chosen so that $y(\lambda)\in W$ for all $\lambda\in I$. Let $h: R^n\times R\to R$ be a smooth Γ-invariant cutoff function vanishing on the complement of $V\times I_0$ and equal to 1 on $U\times I$. Let $H_\eta(z,\lambda)=h(z,\lambda)\hat{H}(z,\lambda)$. It is clear that H_η is globally defined and satisfies (4.2).

We complete the proof of $(*)$ by defining $\mathscr{B}^m(I)$ as the intersection of the sets $\mathscr{A}$ for all choices of Δ.

For $m_1, m_2\in Z^l$ let $E^{m_1,m_2}=E^{m_1}\cap E^{m_2}$. Note that if m_1 and m_2 are not collinear then the sets E^{m_1,m_2} have codimension 2 in $\mathscr{L}(N(\Sigma)/\Sigma)$. The union of these sets consists of the elements $\zeta\in\mathscr{L}(N(\Sigma)/\Sigma)$ which generate a torus of dimension no less than rank $(N(\Sigma)/\Sigma)-1$. In the proof of Theorem 4.1′ we could, instead of the sets E^m, use the sets E^{m_1,m_2}. Then, for every $H\in\mathscr{B}$, the curve $\Theta(H)$ would be transverse to all E^{m_1,m_2} at each $\lambda\in(0,\lambda_0)$. This would imply that if m_1 and m_2 were not collinear then E^{m_1,m_2} and $\Theta(H)$ would not intersect. This property implies the following proposition.

PROPOSITION 4.10. *Suppose that F satisfies* (S1) *and* (S2). *Then there exists a residual set $B\subset C^\infty_T(R^n\times R, R^n)$ such that if $H\in B$ then the dimension of the trajectories of $F+H$ on the sets Y is greater than or equal to* rank $(N(\Sigma)/\Sigma)-1$.

5. Hopf bifurcations. Let x_0 be in R^n and let $X=\Gamma x_0$. Suppose that $F: R^n\times R\to R^n$ is a smooth family of equivariant vector fields and X is a relative equilibrium of F for all values of λ. By Theorem 2.1 $F=F_N+F_T$, where F_N is a family of normal vector fields and F_T is a family of tangent vector fields. Let G be the family defined at the beginning of § 4; that is, $G(\cdot,\lambda)$ is the restriction of $F_N(\cdot,\lambda)$ to the normal space N_{x_0}. Recall that x_0 is the trivial equilibrium of G. We say that the family F has a *Hopf bifurcation* near X, if there exists a branch of nontrivial periodic orbits of G emanating from x_0. Note that such a bifurcation will generically occur if $(dG)_{(x_0,0)}$ has a purely imaginary eigenvalue.

Suppose that F has a Hopf bifurcation. Let $Y(\lambda)$, $\lambda_0 > \lambda > 0$, be a branch of periodic orbits of G and let $y(\lambda)$ denote the initial conditions for the trajectories $Y(\lambda)$. We assume that all the points $y(\lambda)$ have the same isotropy subgroup Σ. Let $\Sigma_{Y(\lambda)}$ be the *group of symmetries* of the set $Y(\lambda)$; that is,

$$\Sigma_{Y(\lambda)} = \{\sigma \in \Sigma_{x_0}: \sigma Y(\lambda) = Y(\lambda)\}.$$

We assume that all the sets $Y(\lambda)$ have the same group of symmetries Σ_Y.

Let $Z(\lambda)$ denote the group orbits of the sets $Y(\lambda)$. Theorem 2.2 guarantees that the sets $Z(\lambda)$ are invariant under the flow of F. The goal of this section is to analyze that flow of F on the sets $Z(\lambda)$. Our analysis is based on the following result: every trajectory of F on Z is dense in a $(k+1)$-dimensional torus, with k frequencies given by the drift along group orbits and the additional frequency corresponding to the motion along the periodic orbit Y. This result was obtained by Field [1980, Prop. B2]. Field also showed that the number of the drift frequencies is bounded by rank $(N(\Sigma)/\Sigma)$.

This section contains two main results. The first of these results is a modification of the theorem of Field. We assume that f is a smooth Γ-equivariant vector field, X is a relative equilibrium of f, and Y is a periodic orbit of f_N. Let $Z = \Gamma Y$. The theorem states that the trajectories on Z are dense in tori of dimension $k+1$, with $k \leqq$ rank $N(\Sigma_Y)/\Sigma_Y$. For some choices of f rank $(N(\Sigma_Y)/\Sigma_Y) <$ rank $(N(\Sigma)/\Sigma)$. This is illustrated in Example 5.3.

Our second main result (Theorem 5.2) deals with the dynamics of the family of vector fields F on the sets $Z(\lambda)$. The theorem states that, given a generic family of vector fields F, there exists a countable set $L_0 \subset (0, \lambda_0)$ such that if $\lambda \notin L_0$ then the trajectories on $Z(\lambda)$ are dense in tori of maximal dimension. In Proposition 5.7 we strengthen this result by showing that generically the dimension of trajectories on $Z(\lambda)$ drops only by 1.

We now state the first of the two main theorems. Let $f: R^n \to R^n$ be a smooth, Γ-equivariant vector field with the following properties:

(VH1) The orbit X is a relative equilibrium of f.

(VH2) The vector field f_N has a periodic orbit $Y = \{y(t): t \in [0, T]\}$, where T is the period of Y.

Let Σ_Y denote the group of symmetries of Y and let $Z = \Gamma Y$. We have Theorem 5.1.

THEOREM 5.1. *All trajectories on the set Z are dense in $(k+1)$-dimensional tori, where $k \leqq$ rank $N(\Sigma_Y)/\Sigma_Y$. For every $\varepsilon > 0$ there exists a smooth and Γ-equivariant vector field h such that $\|h\| \leqq \varepsilon$, h satisfies* (VH1) *and* (VH2), *and such that the trajectories of $f+h$ on the set Z are dense in tori of dimension equal to* rank $N(\Sigma_Y)/\Sigma_Y + 1$.

We now state the second main theorem. Let $F: R^n \to R^n$ be a smooth family of Γ-equivariant vector fields with the following properties:

(H1) The orbit X is a relative equilibrium of F for all values of $\lambda \in R$.

(H2) There exists $\lambda_0 > 0$ and a branch of periodic orbits of F_N, parametrized as $(\lambda, Y(\lambda))$, $0 < \lambda < \lambda_0$, with initial conditions y_λ. All the elements y_λ have the same isotropy subgroups Σ and all the sets $Y(\lambda)$ have the same group of symmetries Σ_Y. The map $\lambda \nrightarrow y_\lambda$ is smooth on the interval $(0, \lambda_0)$.

For a family of vector fields satisfying (H1) and (H2), let $Z(\lambda) = \Gamma Y(\lambda)$. We have the following theorem.

THEOREM 5.2. *If F is a family of vector fields satisfying* (H1) *and* (H2) *then there exists a set* $\mathcal{B} \subset C^\infty_T(R^n \times R, R^n)$ *with the following properties*:

(i) *For every $H \subset \mathcal{B}$ there exists a countable set $I_0 \subset (0, \lambda_0)$ such that for every $\lambda \in (0, \lambda_0) \backslash I_0$ trajectories of $F+H$ on the manifolds $Z(\lambda)$ are dense in tori of maximal dimension.*

(ii) *The set $\mathcal{B}$ is residual in the Whitney C^∞ topology.*

We now present an example of a vector field f for which $\text{rank}(N(\Sigma)/\Sigma) > \text{rank } N(\Sigma_Y)/\Sigma_Y$.

Example 5.3. Let $F: R^n \times R \to R^n$ be a smooth family of vector fields equivariant under the action of $O(2)$. Let $\kappa \in O(2) \backslash SO(2)$. Suppose that the invariant equilibrium $x=0$ bifurcates to a branch of equilibria $x(\lambda)$ with $\Sigma_{x(\lambda)} = \{e, \kappa\}$. Consider a secondary Hopf bifurcation occurring along the branch $x(\lambda)$. In other words, suppose that $(dF_N)_{(x(\lambda_0), \lambda_0)}$ has, for some λ_0, a pair of purely imaginary eigenvalues $i\omega$. Let V be the real eigenspace of $i\omega$. We assume that V is two-dimensional and that the action of κ on V is nontrivial. Then, by the standard Hopf bifurcation theorem, F_N has a branch of periodic solutions with period T and such that $y_\lambda(t+(T/2)) = \kappa y_\lambda(t)$. By Theorem 2.2 the trajectory of F corresponding to $y_\lambda(t)$ is $z_\lambda(t) = \gamma(t) y_\lambda(t)$, where $\gamma(t) \in SO(2)$ and $\gamma(0) = e$. Let Φ_t denote the flow of F. Then

$$x_\lambda(t) = \Phi_T y_\lambda(0) = \Phi_{T/2}(\Phi_{T/2} y_\lambda(0)) = \Phi_{T/2}(\gamma(T/2)\kappa y_\lambda(0)).$$

Equivariance of $\Phi_{T/2}$ implies that

$$\Phi_{T/2}(\gamma(T/2)\kappa y_\lambda(0)) = (\gamma(T/2)\kappa)^2 y_\lambda(0) = y_\lambda(0) = x_\lambda(0).$$

It follows that $x_\lambda(t)$ must be a periodic solution. Note that $\Sigma = \{e\}$, so $N(\Sigma)/\Sigma = O(2)$, but $\Sigma_Y = \{e, \kappa\}$, so $N(\Sigma_Y)/\Sigma_Y$ is discrete. Hence $\text{rank}(N(\Sigma)/\Sigma) > \text{rank}(N(\Sigma_Y)/\Sigma_Y)$.

In the remainder of the section we prove Theorems 5.1 and 5.2. In the proof of Theorem 5.1 we will use two lemmas and some background information from Lie group theory. We begin by stating and proving the first of the lemmas. Note that $\Sigma \subset \Sigma_Y \subset N(\Sigma)$. Let $\Delta = \Sigma_Y/\Sigma$. We have the following lemma.

LEMMA 5.3. *The group Δ is finite and cyclic or Δ is isomorphic to S^1.*

Proof. Let y_0 be the initial condition of the periodic orbit Y. We assume, with no loss of generality, that the period of the solution $y(t)$ is 1. We identify S^1 with R/Z. Note that Y is diffeomorphic to S^1 via the map $t \to y(t)$. Let $\rho: \Delta \to S^1$ be defined by the identity

$$y(\rho(\delta)) = \delta y_0, \qquad \delta \in \Delta.$$

Clearly, ρ is smooth and well defined. Note that the action of Δ on Y is free, so ρ must be injective. Equivariance of f implies that ρ is a Lie group homomorphism. It follows that $\rho(\Delta)$ (which is isomorphic to Δ) is a Lie subgroup of S^1 and therefore must be isomorphic to either S^1 or Z_l for some l.

We now review some of the concepts from Lie group theory, which will be used in the proof of Theorem 5.1. For a Lie group H let H_0 denote the connected component of the identity in H. We say that a subgroup K of a compact Lie group H is *topologically cyclic* if there exists $\delta \in K$ such that $K = \text{cl}\{\delta^n: n \text{ is an integer}\}$. The element δ is called the *generator* of K. We say that K is a *Cartan subgroup* of H if K is topologically cyclic and $N(K)/K$ is discrete. In the proof of Theorem 5.1 we will use the following two propositions.

PROPOSITION 5.4. *Each element $h \in H$ is contained in a Cartan subgroup K of H such that K/K_0 is generated by hK_0.*

PROPOSITION 5.5. *If K is a Cartan subgroup of H generated by z, then any $h \in H_0 z$ is conjugate to an element of $K_0 z$ via conjugation by an element of H_0.*

The statements and the proofs of Propositions 5.4 and 5.5 can be found in Bröcker and tom Dieck [1985, IV, eqs. (4.2) and (4.3)].

If K is a Cartan subgroup and K_1 a topologically cyclic group, then K/K_1 must be discrete. Proposition 5.4 implies that each topologically cyclic group is contained in a Cartan subgroup. It follows that a Cartan subgroup can be defined as a topologically cyclic group of maximal dimension.

Let K be a topologically cyclic subgroup of H. Then K_0 must be a torus in H_0 and K/K_0 must be finite and cyclic. It follows that K is isomorphic to $K_0 \times Z_l$, for some l (see Bröcker and tom Dieck [1985, I, eq. (4.14)]).

Let f be a smooth Γ-equivariant vector field satisfying (VH1) and (VH2). Theorem 2.2 guarantees that the trajectory of f with initial condition $y(0)$ has the form $\gamma(t)y(t)$, where $\gamma(t) \in (N(\Sigma)/\Sigma)_0$. We assume that Δ is finite and cyclic with generator δ_0 (see Lemma 5.3) and let T_0 be defined by the identity $y(T_0) = \delta_0 y_0$. To prove Theorem 5.1 we need to prove the following lemma.

LEMMA 5.6. *Suppose that $\gamma(T_0) = \gamma_0$ and that $\gamma_1 \in (N(\Sigma)/\Sigma)_0$. Then there exist a smooth, Γ-equivariant tangent vector field h and a curve $\gamma^1(t)$ such that*:

(i) $\gamma^1(0) = e$.

(ii) $\gamma^1(T_0) = \gamma_1$.

(iii) *$\gamma^1(t)y(t)$ is the trajectory of $f+h$.*

Proof. Let $\xi(t) = \gamma(t)^{-1}\dot{\gamma}(t)$. Note that $f_T(y(t)) = \xi(t)y(t)$. Let $\sigma = \gamma_1\gamma_0^{-1}$ and let $\zeta = \exp^{-1}\sigma$. Let $\eta(t)$ be a smooth function such that $\eta(0) = 0$, $\eta(T_0) = 1$, and

$$\frac{d^j}{dt^j}\eta(0) = \frac{d^j}{dt^j}\eta(T_0) = 0, \qquad j = 1, 2, \cdots.$$

Let $\gamma^1(t) = \exp(\eta(t)\zeta)\gamma(t)$. We define a curve $\hat{\xi}(t) = \xi(t) + \dot{\eta}(t)\gamma(t)^{-1}\zeta\gamma(t)$. Let $\xi^1(t) = \hat{\xi}(t) - \xi(t)$. Note that

$$\frac{d^j}{dt^j}\xi^1(0) = \frac{d^j}{dt^j}\xi^1(T_0) = 0, \qquad j = 1, 2, \cdots. \tag{5.1}$$

We extend the definition of ξ^1 to all of R by requiring that $\xi^1(t+T_0) = \mathrm{Ad}_\delta\, \xi^1(t)$, where δ is as defined in Lemma 5.3. Equation (5.1) implies that this extension is smooth. Let $h(y(t)) = \xi^1(t)y(t)$. Let $Z = \Gamma Y$. We extend h to Z by equivariance. Let $N(Z)$ be the normal bundle of Z. Recall that $N(Z)$ can be identified with a Γ-invariant neighborhood of Z. We extend h to $N(Z)$ by letting $h(w) = \tilde{h}(z)$ for $w \in N_z$. We use an invariant cutoff function to extend the definition of h to all of R^n. A simple computation shows that $\gamma^1(t)y(t)$ is a trajectory of $f+h$. It is also clear that $\|h\| \to 0$ as σ approaches e.

Proof of Theorem 5.1. Let Δ be as defined prior to the statement of Lemma 5.3; that is, $\Delta = \Sigma_Y/\Sigma$. Lemma 5.3 implies that Δ is either finite and cyclic or isomorphic to S^1. We divide the analysis in two cases:

(1) Δ is isomorphic to S^1.

(2) Δ is finite and cyclic.

Case (1). It follows from the proof of Lemma 5.3 that Z is a relative equilibrium of f. Therefore the dynamics on Z is described by Theorem 4.5. Hence the trajectories on Z are dense in tori of maximal dimension equal to rank $N(\Sigma)/\Sigma$. We now show that rank $N(\Sigma)/\Sigma = \text{rank } N(\Sigma_Y)/\Sigma_Y + 1$. Note that Δ is a torus of dimension 1 contained in $N(\Sigma)/\Sigma$. Hence there is a maximal torus $\mathbb{T}$ in $N(\Sigma)/\Sigma$ such that $\Delta \subset \mathbb{T}$. Clearly, $\mathbb{T} \subset N(\Sigma_Y)/\Sigma$. It follows that $\mathbb{T}/\Delta$ is a maximal torus in $N(\Sigma_Y)/\Sigma_Y$, which proves the required equality.

Case (2). Let $z(t)$ be a trajectory on Z. By Theorem 2.2 $z(t)=\gamma(t)y(t)$, $\gamma(t)\in\Gamma$. Since $\Sigma_{y(t)}=\Sigma$ we can assume that $\gamma(t)\in N(\Sigma)/\Sigma$. Let T_0 be the number defined prior to the statement of Lemma 5.6 and let $\gamma_0=\gamma(T_0)\delta_0$. Clearly, $z(T_0)=\gamma_0 y_0$. Let $\mathbb{T}=\mathrm{cl}\,\{z(t)\colon t\in R\}$ and let

$$S=\mathrm{cl}\,\{\gamma_0^k\colon k \text{ is an integer}\}.$$

Observe that $Sy_0\subset\mathbb{T}$. Let Φ_t be the flow of f. We have

$$\Phi_t\gamma_0 y_0=\Phi_t\Phi_{T_0}y_0=\Phi_{t+T_0}y_0=z(t+T_0).$$

It follows that $\Phi_t Sy_0\subset Z$ for all $t\in R$. Let us define the action of $\Gamma\times R$ on R^n as

$$(t,\gamma)w=\Phi_t\gamma w \quad \text{where } (t,\gamma)\in\Gamma\times R, \quad w\in R^n. \tag{5.2}$$

It follows that $\mathbb{T}=(S\times R)y_0$. Let Δ_0 be the isotropy subgroup of y_0 with respect to the action defined by (5.2). It is clear that $(S\times R)/\Delta_0$ is compact, connected, and Abelian. Hence $(S\times R)/\Delta_0$ is isomorphic to a torus. This implies that $\mathbb{T}$ is diffeomorphic to a torus. Note that S is topologically cyclic and γ_0 is its generator. Hence the dimension of S is less than or equal to the dimension of a Cartan subgroup containing γ_0. Note that γ_0 and δ_0 lie in the same component of the identity in $N(\Sigma)/\Sigma$, which we denote by $S(\delta_0)$. Let K be a Cartan subgroup generated by an element $\gamma_1\in S(\delta_0)$. Proposition 5.5 implies that $\dim S\leqq\dim K$ and $\dim\mathbb{T}\leqq\dim K+1$.

We now prove that for a small perturbation of f the dimension of the closure of a trajectory on Z equals $\dim K+1$. Suppose that $\gamma_1\in S(\delta_0)$ generates a Cartan subgroup. By Lemma 5.6 there exists a vector field h satisfying (VH1) and (VH2) and such that if $\tilde{z}(t)=\tilde{\gamma}(t)y(t)$ is the trajectory of $f+h$ with initial condition y_0 then $\tilde{\gamma}(T_0)\delta_0=\gamma_1$. Clearly, the dimension of the closure of $\tilde{z}(t)$ equals $\dim K+1$.

To conclude the proof we need to show that $\dim K=\operatorname{rank} N(\Sigma_Y)/\Sigma_Y$. By Proposition 5.4 we can choose K so that $\Delta\subset K$. The definition of Δ and the fact that Δ is discrete imply that $\operatorname{rank}(N(\Sigma_Y)/\Sigma)=\operatorname{rank}(N(\Sigma_Y)/\Sigma_Y)$. Let $N(\Delta)$ denote the normalizer of Δ in $N(\Sigma)/\Sigma$. Note that $N(\Delta)=N(\Sigma_Y)/\Sigma$. Clearly, $K\subset N(\Delta)$. We show that $\operatorname{rank} N(\Delta)=\dim K$. Suppose that $\phi\in N(\Delta)_0$, and let $\mathbb{T}_0$ be the torus generated by ϕ. Let $\phi_0=\phi\delta_0$ and let $\hat{K}$ be the topologically cyclic subgroup generated by ϕ_0. Since $\phi\in N(\Delta)$ we have $\phi\Delta\phi^{-1}=\Delta$, which implies that

$$\phi\delta_0=\delta_0^m\phi \quad \text{for some } m. \tag{5.3}$$

Note that continuity implies that m is independent of ϕ. It follows that for any positive integer j we must have $\phi_0^j=\delta_0^s\phi^j$ for some s (depending on j but independent of ϕ). Since ϕ_0 is the generator of $\hat{K}$ it follows that for some l we must have $\phi_0^l\in\hat{K}_0\subset N(\Delta)_0$. By continuity we must have $\phi_0^l\in N(\Delta)_0$ for all $\phi\in N(\Delta)_0$. Now (5.3) implies that $\delta_0^s\phi^l\in N(\Delta)_0$ for some s (independent of $\phi\in N(\Delta)_0$). It follows that for some ϕ the torus generated by $\delta_0^s\phi^l$ is a maximal torus in $N(\Delta)$. It follows that $\dim\mathbb{T}_0\leqq\dim\hat{K}\leqq\dim K$. The inequality $\dim K\geqq\operatorname{rank} N(\Delta)$ follows from the fact that K_0 is a connected Abelian subgroup of a compact Lie group; hence it is contained in a maximal torus. It follows that $\operatorname{rank} N(\Delta)\geqq\dim K$.

Let F be a family of vector fields satisfying (H1) and (H2). Suppose that Δ is finite and cyclic and let T_0 be as defined prior to the statement of Lemma 5.6. Let $x_\lambda(t)$ be the trajectory of $F(\cdot,\lambda)$ with initial condition y_λ. By Theorem 2.2 $x_\lambda(t)=\gamma_\lambda(t)y_\lambda(t)$ ($y_\lambda(t)$ is the periodic orbit of $F_N(\cdot,\lambda)$). Let $\gamma(\lambda)=\gamma_\lambda(T_0)$. The proof of Theorem 5.2 is based on the following assertion: a generic F_T gives rise to a generic curve γ. Given H satisfying (H1) and (H2) let $\gamma_\lambda^H(t)y_\lambda(t)$ be the trajectory of $H(\cdot,\lambda)$.

We now define a map Θ which assigns to every family of vector fields satisfying (H1) and (H2) the corresponding curve $\gamma(\lambda)$. Let $I \subset (0, \lambda_0)$ be an open interval and let

$$\Theta(H)(\lambda) = \gamma_\lambda^H(T_0), \qquad \lambda \in I.$$

Let $S(\delta_0)$ be the connected component of $N(\Sigma)/\Sigma$ containing δ_0. Note that $\Theta(H)$ is an element of $C^\infty(I, S(\delta_0))$.

Proof of Theorem 5.2. The proof is analogous to the proof of Theorem 4.1′(ii). We will therefore only outline the proof and omit the technical details. If Δ is isomorphic to S^1 then F satisfies the assumptions of Theorem 4.1′ with $y(\lambda) = y_\lambda$ being the curve of relative equilibria. Hence the theorem follows from Theorem 4.1′ and part (2) of the proof of Theorem 5.1.

Suppose that Δ is finite and cyclic. Let $Q \subset S(\delta_0)$ be the set of elements which does not generate a Cartan subgroup. We show that Q is a countable union of submanifolds of $S(\delta_0)$, each of codimension greater than or equal to 1. Let K be a Cartan subgroup containing δ_0 and generated by an element of $S(\delta_0)$. The existence of K follows from Proposition 5.4. Recall that K is isomorphic to $K_0 \times Z_l$ with $K_0 \times \{1\}$ corresponding to $\delta_0 K_0$ (we identify Z_l with $\{0, 1, 2, \cdots, l-1\}$). In K_0 we define the sets P^m (see the proof of Proposition 4.3). It follows that $\delta_0 K_0 \cap Q$ is the union of the sets $P^m \times \{1\}$. Let E^m be the union of all conjugacy classes of $P^m \times \{1\}$ by the elements of $(N(\Sigma)/\Sigma)_0$. Proposition 5.5 implies that Q is the union of the sets E^m. Consider the action of $(N(\Sigma)/\Sigma)_0$ on $N(\Sigma)/\Sigma$ defined as conjugation by a group element. As in the proof of Theorem 4.1′ we can partition E^m into manifolds $E^m(\Delta)$, consisting of all elements of E^m whose isotropy with respect to this action is conjugate to Δ.

The remaining part of the proof is analogous to the proof of Theorem 4.1′. The main objective is to show that there exists a residual subset $\mathscr{B} \subset C_T^\infty(R^n \times R, R^n)$ such that if $H \in \mathscr{B}$ then for all choices of m and Δ the curve $\Theta(H)$ is transverse to $E^m(\Delta)$ at every $\lambda \in (0, \lambda_0)$. This is done by showing that for some fixed m and Δ there exists a residual set $\mathscr{B}^m(\Delta)$, whose elements are transverse to $E^m(\Delta)$, and then taking the intersection of the sets $\mathscr{B}^m(\Delta)$.

We now fix m and Δ and regard Θ as a mapping from $C_T^\infty(R^n \times R, R^n)$ to $C^\infty(I, S(\delta_0))$. To show existence of $\mathscr{B}^m(\Delta)$ we need to prove the following properties of Θ:

(1) If the interval I is such that $\bar{I} \subset (0, \lambda_0)$, then Θ is continuous in the Whitney C^∞ topology.

(2) For each $\mu \in (0, \lambda_0)$ there exists an interval I such that $\mu \in I$ and $I \subset (0, \lambda_0)$ and a residual set $\mathscr{A} \subset C_T^\infty(R^n \times R, R^n)$ such that if $H \in \mathscr{A}$ then $\Theta(H)$ is transverse to $E^m(\Delta)$ at all $\lambda \in I$.

Property (1) follows from standard theorems on smooth dependence of solutions of ordinary differential equations on parameters (also see the proof of Theorem 2.2).

We now indicate how to prove property (2). We choose an interval I_0 such that $\mu \in I_0$ and $\bar{I}_0 \subset (0, \lambda_0)$. Given some interval $I \subset I_0$ we define $\mathscr{A}_0$ as the set of elements of $C^\infty(I_0, S(\delta_0))$ which are transverse to $E^m(\Delta)$ at all $\lambda \in I$. Standard transversality theory implies that $\mathscr{A}_0$ is residual in the Whitney C^∞ topology on $C^\infty(I_0, S(\delta_0))$. Let $\mathscr{A} = \Theta^{-1}(\mathscr{A}_0)$. The property (1) implies that $\mathscr{A}$ is an intersection of open sets. If I is small enough then $\mathscr{A}$ is dense in the Whitney C^∞ topology on $C_T^\infty(R^n \times R, R^n)$. To see this, suppose that $\gamma \in C^\infty(I_0, S(\delta_0))$ is a small perturbation of the curve $\Theta(F)$. Then there exists a small perturbation H of the family F such that $\Theta(F+H) = \gamma$. The proof of the existence of H is a straightforward generalization of Lemma 5.6.

The methods used in the proof of Theorem 5.2 can be easily generalized to prove the following proposition.

PROPOSITION 5.7. Suppose that F satisfies (H1) and (H2). Then there exists a residual set $\mathscr{B} \subset C_T^\infty(R^n \times R, R^n)$ such that if $H \in \mathscr{B}$ then the dimension of the trajectories of $F+H$ on the sets Y is greater than or equal to rank $(N(\Sigma_Y)/\Sigma_Y)$.

6. Bifurcations of relative equilibria with $O(2)$ symmetry. In this section we discuss bifurcation problems with symmetry group $O(2)$. Let $\dot{+}$ denote semidirect product. Recall that $O(2) = SO(2) \dot{+} Z_2(\kappa)$, where $Z_2(\kappa) = \{1, \kappa\}$, κ is an orientation reversing element of $O(2)$, and $SO(2)$ is the subgroup of $O(2)$ consisting of orientation preserving rotations. We assume that $O(2)$ acts on R^n and that $F: R^n \times R \to R^n$ is a smooth and equivariant family of vector fields. We restrict our attention to bifurcations of group orbits of equilibria whose isotropy subgroups are either $Z_2(\kappa)$ or D_k, $k \geqq 2$. Here D_k denotes the group of symmetries of a regular k-gon. The groups $Z_2(\kappa)$ or D_k occur as maximal isotropy subgroups for the various irreducible representations of $O(2)$; hence the bifurcations we study can occur as secondary bifurcations from an invariant equilibrium.

Let R_θ denote the rotation of the plane by the angle θ. The groups D_k are generated by the reflection κ and the rotation $R_{2\pi/k}$. Let Z_k denote the cyclic group generated by $R_{2\pi/k}$. We have $D_k = Z_k \dot{+} Z_2(\kappa)$. In the sequel we use D_1 to denote $Z_2(\kappa)$ and Z_1 to denote the trivial subgroup **1**.

Fix $x_0 \in R^n$, let $X = O(2)x_0$, and let Σ_{x_0} be the isotropy subgroup of x_0. We assume that $\Sigma_{x_0} = D_k$, $k \geqq 1$, and that X is a relative equilibrium of F. Let G be as defined in § 3; that is, G is the restriction of F_N to the normal space N_{x_0}. Let $g = G(\cdot, 0)$; and let E be the center subspace of $(dg)_{x_0}$. In this section we analyze steady-state and Hopf bifurcations of F near X. More precisely, we consider the following situations:

(a) $(dg)_{x_0}$ has a zero eigenvalue.

(b) $(dg)_{x_0}$ has a purely imaginary eigenvalue $i\omega$.

Let $\mathscr{K}(\Sigma_{x_0}) = \{\sigma \in \Sigma_{x_0}: \sigma v = v \text{ for all } v \in E\}$ be the *kernel* of the action of Σ_{x_0} on E. We assume that the action of Σ_{x_0} on E is nontrivial; that is, $\mathscr{K}(\Sigma_{x_0})$ is properly contained in Σ_{x_0}. We now state the two main results of this section: a steady-state bifurcation theorem and a Hopf bifurcation theorem. We begin with the steady-state bifurcation theorem. Suppose that $(dg)_{x_0}$ has a zero eigenvalue. We make a generic assumption that E is the nullspace of $(dg)_{x_0}$ and that the action of Σ_{x_0} on E is absolutely irreducible. The following theorem describes all the generic types of bifurcating solutions and gives the number of distinct nonconjugate branches.

THEOREM 6.1. *All the generic types of bifurcating solutions of F are listed in Table* 6.1.

We now state the Hopf bifurcation theorem. Suppose that $(dg)_{x_0}$ has a purely imaginary eigenvalue $i\omega$. We make a generic assumption that E is the eigenspace of $i\omega$ and that the action of Σ_{x_0} on E is Γ-simple. The following theorem describes all the generic types of bifurcating solutions and gives the number of distinct nonconjugate branches.

THEOREM 6.2. *All the generic types of bifurcating solutions of F are listed in Table* 6.2.

TABLE 6.1

Kernel of isotropy	Type of solution	Number of branches
Z_k	rotating wave	1
D_m, $k=2m$	steady state	1
Z_l, l/k, $l<k$	steady state	2

TABLE 6.2

Kernel of isotropy	Type of solution	Number of branches
Z_k	periodic orbit	1
D_m, $k=2m$	periodic orbit	1
Z_l, l/k, $l<k$	periodic orbit	2
	two-torus	1

In the remainder of this section we prove Theorems 6.1 and 6.2. We begin by classifying all the possible kernels of the action of Σ_{x_0} on E. We prove the following lemma.

LEMMA 6.3. *One of the following statements must hold*:

(i) $\mathcal{K}(\Sigma_{x_0})=Z_l$, $l\leqq k$, *l divides k*, $l\neq k/2$.

(ii) $k=2m$ *and* $\mathcal{K}(\Sigma_{x_0})$ *is isomorphic to* D_m.

Proof. Note that $\mathcal{K}(\Sigma_{x_0})$ is normal in Σ_{x_0}. Hence finding all the possible groups $\mathcal{K}(\Sigma_{x_0})$ is equivalent to classifying the normal subgroups of Σ_{x_0}.

We consider two cases: $\mathcal{K}(\Sigma_{x_0})\subset SO(2)$ and $\mathcal{K}(\Sigma_{x_0})\not\subset SO(2)$. Suppose that $\mathcal{K}(\Sigma_{x_0})\subset SO(2)$. Then $\mathcal{K}(\Sigma_{x_0})$ is a subgroup of Z_k. Hence $\mathcal{K}(\Sigma_{x_0})=Z_l$, $l\leqq k$, l divides k.

Suppose now that $\mathcal{K}(\Sigma_{x_0})\not\subset SO(2)$. Let $\zeta\in\mathcal{K}(\Sigma_{x_0})$, $\zeta\notin SO(2)$. Since $\mathcal{K}(\Sigma_{x_0})$ is normal in Σ_{x_0} we have $R_{4\pi/k}\zeta=R_{2\pi/k}\zeta R_{-2\pi/k}\in\mathcal{K}(\Sigma_{x_0})$. Hence $R_{4\pi/k}\in\mathcal{K}(\Sigma_{x_0})$. If $k=2m$ then it follows that $\mathcal{K}(\Sigma_{x_0})$ is generated by $R_{4\pi/k}$ and ζ and therefore is isomorphic to D_m. Otherwise $k+1$ is divisible by 2 and $(R_{4\pi/k})^{(k+1)/2}=R_{2\pi/k}$. This implies that $\mathcal{K}(\Sigma_{x_0})=D_k=\Sigma_{x_0}$, which is a contradiction.

The case $\mathcal{K}(\Sigma_{x_0})=Z_l$, $l\neq k/2$ cannot occur, since then $\Sigma_{x_0}/\mathcal{K}(\Sigma_{x_0})$ would be isomorphic to D_2 and every irreducible action of D_2 has a nontrivial kernel.

Proof of Theorem 6.1. We begin by describing the bifurcation problem for the family G. We assume that the center manifold reduction has been carried out; that is, G is a family of Σ_{x_0}-equivariant vector fields on E. Recall that $\mathcal{K}(\Sigma_{x_0})$ is normal in Σ_{x_0}. Let $\tau:\Sigma_{x_0}\to\Sigma_{x_0}/\mathcal{K}(\Sigma_{x_0})$ be the natural projection. We define the action of $\tau(\Sigma_{x_0})$ on E by $\tau(\sigma)v=\sigma v$, $v\in E$, $\sigma\in\Sigma_{x_0}$. Note that this action is well defined, since $\ker\tau=\mathcal{K}(\Sigma_{x_0})$. It follows that G is Σ_{x_0}-equivariant if and only if it is $\tau(\Sigma_{x_0})$-equivariant. We replace the action of Σ_{x_0} by the action of $\tau(\Sigma_{x_0})$.

As indicated in Table 6.1 we divide the analysis into three cases:

(1) $\mathcal{K}(\Sigma_{x_0})=Z_l$, $l<k$, l divides k.

(2) $\mathcal{K}(\Sigma_{x_0})=Z_k$.

(3) $k=2m$ and $\mathcal{K}(\Sigma_{x_0})$ is isomorphic to D_m.

Case (1). Let $m=k/l$. Since $l<k$ it follows that $m\geqq 2$. Clearly, $\tau(\Sigma_{x_0})$ is isomorphic to D_m. Since the action of $\tau(\Sigma_{x_0})$ is faithful it follows that $\dim E>1$. This implies that $\dim E=2$, since all the irreducible representations of D_m are one- or two-dimensional. Also $m\geqq 3$, since any irreducible representation of D_2 has a nontrivial kernel. It follows that the action of $\tau(\Sigma_{x_0})$ is isomorphic to the standard action of D_m on C. According to Table XIII, 5.2 in Golubitsky, Stewart, and Schaeffer [1988] a generic family G has two branches of steady-state solutions $y_1(\lambda)$ and $y_2(\lambda)$. Let $Y_1=O(2)y_1$ and $Y_2=O(2)y_2$. We now show that the sets Y_1 and Y_2 consist of equilibria of F. The results of Golubitsky, Stewart, and Schaeffer also imply that the isotropy subgroups of y_1 and y_2 with respect to the action of D_m are two-element groups, each generated by an element not contained in Z_m. Let Σ_{y_1} and Σ_{y_2} denote the isotropy subgroups of y_1 and y_2 in Σ_{x_0}. It follows that $\Sigma_{y_1}\not\subset SO(2)$ and $\Sigma_{y_2}\not\subset SO(2)$. Hence the normalizers of Σ_{y_1} and Σ_{y_2} are discrete. Theorem 4.1 implies that the group orbits Y_1 and Y_2 consist of equilibria of F.

Case (2). Note that $\tau(\Sigma_{x_0})$ is isomorphic to Z_2. Hence dim $E=1$. Since the action of Z_2 is nontrivial it follows that Z_2 acts on E as minus identity. Hence generically G undergoes a pitchfork bifurcation; that is, G has a unique (up to conjugacy) branch of equilibria $y(\lambda)$ with trivial isotropy. For more information on Z_2-equivariant bifurcation, see Golubitsky and Schaeffer [1985, Chap. XVI].

Let $Y=O(2)y$. We show that generically the trajectories of F on Y are rotating waves. Let Σ_y be the isotropy subgroup of y in Σ_{x_0}. It follows that $\Sigma_y=\mathcal{K}(\Sigma_{x_0})=Z_k$. We conclude that $N(\Sigma_y)=O(2)$. By Theorem 4.1 generically the trajectories of $F(\cdot,\lambda)$ on $Y(\lambda)$ are given by drift along circles.

Case (3). As in Case (2) $\tau(\Sigma_{x_0})$ is isomorphic to Z_2. Hence G has a branch of equilibria $y(\lambda)$, whose isotropy subgroup in Z_2 is trivial. Let Σ_y be the isotropy subgroup of y in Σ_{x_0}. Since $\Sigma_y=\mathcal{K}(\Sigma_{x_0})$ and $\mathcal{K}(\Sigma_{x_0})$ is isomorphic to D_m, it follows that $\Sigma_y\not\subset SO(2)$ and that $N(\Sigma_y)$ is discrete. By Theorem 4.1 the orbit $Y=O(2)y$ consists of equilibria of F.

Before proving Theorem 6.2 we give some background on Hopf bifurcation from an invariant equilibrium. The results we review will be used in the analysis of bifurcations of the family G. Let $\Gamma\subset O(n)$ be a Lie group acting on R^n and suppose that this action is nontrivial. Let $H:R^n\times R\to R^n$ be a family of smooth, Γ-equivariant vector fields. Let $h=H(\cdot,0)$ and suppose that $(dh)_0$ has a purely imaginary eigenvalue ωi. Suppose that the center manifold reduction has been carried out; that is, R^n is the real part of the sum of the eigenspaces of $\pm\omega i$. We make a generic assumption that the action of Γ on R^n is Γ-simple. Then the group $\{\exp(Lt)\colon t\in R\}$ is isomorphic to S^1. We define the action of S^1 on R^n as

$$(\gamma,\theta)x=\gamma\exp(L\theta)x\quad\text{where }(\gamma,\theta)\in\Gamma\times S^1\text{ and }x\in R^n.$$

The following theorem is the equivariant Hopf bifurcation theorem (see Golubitsky and Stewart [1985, Thm. 5.1] or Golubitsky, Stewart, and Schaeffer [1988, Thm. XVI, 4.1]):

THEOREM 6.4. *Suppose that Δ is a maximal isotropy subgroup of $\Gamma\times S^1$ and* dim Fix $(\Delta)=2$. *Then H has a branch of small amplitude periodic solutions $x_\lambda(t)$ such that $\sigma x_\lambda(t+\theta)=x_\lambda(t)$ for every pair $(\sigma,\theta)\in\Delta$.*

Suppose that $x_\lambda(t)$ is a branch of periodic solutions described in the statement of Theorem 6.4. Let $X(\lambda)=\{x_\lambda(t)\colon t\in R\}$. Recall the definition of the group of symmetries of the set X, denoted by Σ_X, as the set of all $\sigma\in\Gamma$ such that $\sigma x=x$ for all $x\in R^n$. Clearly, Σ_X is obtained by projecting Δ onto the first component of $\Gamma\times S^1$; that is,

$$\Sigma_X=\{\sigma\in\Gamma\colon(\sigma,\theta)\in\Delta\text{ for some }\theta\in S^1\}.$$

We refer to the group Δ as the group of spatial-temporal symmetries of the periodic orbit X.

We will now describe generic Hopf bifurcations in two cases: $\Gamma=Z_2$ and $\Gamma=D_k$, $k\geqq 3$. Assume that $\Gamma=Z_2$. We have the following proposition.

PROPOSITION 6.5. *Generically, the family H has a branch of periodic orbits $Y(\lambda)$ with $\Sigma_{Y(\lambda)}=Z_2$.*

Proof. The irreducible representations of Z_2 are absolutely irreducible and one-dimensional. Hence a Γ-simple representation of Z_2 will be two-dimensional and will have the form $R\oplus R$. The action on each of the copies of R will be given as reflection with respect to the origin. Let ζ be the nontrivial element in Z_2. Then, for $(x,y)\in R^2$

$\zeta(x, y) = (-x, -y)$. Also

$$L = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

and $\exp(Lt)$ is a rotation by angle t. It is easy to see that $Z_2(\zeta, \pi) = \{(0,0), (\zeta, \pi)\}$ is a maximal isotropy subgroup of $Z_2 \times S^1$. It follows from Theorem 6.4 that H has a branch of periodic orbits $Y(\lambda)$ with $\Sigma_{Y(\lambda)} = Z_2$. Using normal form theory we can show that generically this branch is unique.

We now discuss the case where $\Gamma = D_k$, $k \geqq 3$. We assume that the action of D_k on R^n is faithful. Let $\xi = 2\pi/k$. We define the following subgroups of $D_k \times S^1$:

$$\tilde{Z}_k = \left\{ \left(\frac{2\pi m}{k}, \frac{2\pi m}{k} \right) : m = 0, 1, \cdots, k \right\},$$

$$Z_2(\kappa) = \{(0,0), (\kappa, 0)\},$$

$$Z_2(\kappa, \pi) = \{(0,0), (\kappa, \pi)\},$$

$$Z_2(\kappa, \xi) = \{(0,0), (\kappa, \xi)\},$$

and when k is even

$$Z_2^c = \{(0,0), (\pi, \pi)\}.$$

We have the following theorem.

THEOREM 6.6. *Generically, the family H has three branches of periodic orbits*: $Y_1(\lambda)$, $Y_2(\lambda)$, *and* $Y_3(\lambda)$. *The groups of spatial-temporal symmetries of* $Y_1(\lambda)$, $Y_2(\lambda)$, *and* $Y_3(\lambda)$ *are given respectively by*:

(a) $\tilde{Z}_k$, $Z_2(\kappa)$, *and* $Z_2(\kappa, \pi)$ *if k is odd.*

(b) $\tilde{Z}_k$, $Z_2(\kappa) \oplus Z_2^c$, *and* $Z_2(\kappa, \pi) \oplus Z_2$ *if* $k \equiv 2 \pmod 4$.

(c) $\tilde{Z}_k$, $Z_2(\kappa) \oplus Z_2^c$, *and* $Z_2(\kappa, \xi) \oplus Z_2$ *if* $k \equiv 0 \pmod 4$.

Theorem 6.6 is a consequence of Theorem XVIII, 3.1 in Golubitsky, Stewart, and Schaeffer [1988].

We now present the proof of Theorem 6.2.

Proof of Theorem 6.2. We begin by describing the bifurcation problem for the family G. We assume that the center manifold reduction has been carried out; that is, G is a family of Σ_{x_0}-equivariant vector fields on E. Recall that τ is the natural projection from Σ_{x_0} onto $\Sigma_{x_0}/\mathscr{K}(\Sigma_{x_0})$. As in the proof of Theorem 6.1 we replace the action of Σ_{x_0} by the action of $\tau(\Sigma_{x_0})$. We consider three cases:

(1) $\mathscr{K}(\Sigma_{x_0}) = Z_l$, $l < k$, l divides k.

(2) $\mathscr{K}(\Sigma_{x_0}) = Z_k$.

(3) $k = 2m$ and $\mathscr{K}(\Sigma_{x_0})$ is isomorphic to D_m.

Case (1). Let $m = l/k$. We have $\tau(\Sigma_{x_0}) = D_m$. Since the action of $\tau(\Sigma_{x_0})$ on E is faithful it follows that $m \geqq 3$. The action of $\tau(\Sigma_{x_0})$ on E is Γ-simple, so we are in position to apply Theorem 6.6. Hence G has three branches of solutions $Y_1(\lambda)$, $Y_2(\lambda)$, and $Y_3(\lambda)$ whose groups of spatial-temporal symmetries in D_m are as indicated in Theorem 6.6. Let Σ_{Y_1}, Σ_{Y_2}, and Σ_{Y_3} be the groups of symmetries of these trajectories inside of Σ_{x_0}. Observe that $\Sigma_{Y_1} = Z_k$ and the groups Σ_{Y_2} and Σ_{Y_3} are not contained in $SO(2)$. It follows that $N(\Sigma_{Y_1}) = O(2)$ and the normalizers of the groups Σ_{Y_2} and Σ_{Y_3} are discrete. Let $Z_1(\lambda) = O(2)Y_1(\lambda)$, $Z_2(\lambda) = O(2)Y_2(\lambda)$, and $Z_3(\lambda) = O(2)Y_3(\lambda)$. Theorem 5.2 implies that generically the trajectories of F on Z_1 are dense in two-dimensional tori and the trajectories of F on Z_2 and Z_3 are periodic orbits.

Case (2). We have $\tau(\Sigma_{x_0})=Z_2$. By Proposition 6.5 G has a unique branch of periodic orbits $Y(\lambda)$. Let ζ be the nontrivial element in Z_2. It follows from the proof of Proposition 6.5 that (ζ, π) is a spatial-temporal symmetry of Y. Let Σ_Y denote the group of symmetries of Y inside of Σ_{x_0}. We have $\tau(\kappa)=\zeta$, so $\Sigma_Y \not\subset SO(2)$ and $N(\Sigma_Y)$ is discrete. Let $Z=O(2)Y$. It follows that the flow of F on Z consists of periodic orbits.

Case (3). We have $\tau(\Sigma_{x_0})=Z_2$. By Proposition 6.5 G has a unique branch of periodic orbits $Y(\lambda)$. Since $\mathscr{K}(\Sigma_{x_0})\not\subset SO(2)$ it follows that $\Sigma_Y\not\subset SO(2)$ and $N(\Sigma_Y)$ is discrete. Let $Z=O(2)Y$. It follows that the flow of F on Z consists of periodic orbits.

7. The Kuramoto–Sivashinsky equation. The Kuramoto–Sivashinsky equation is used to model several physical and chemical phenomena, for example, flame propagation and some aspects of the dynamics of the Belousov–Zhabotynski reaction. The following is the Kuramoto–Shivashinsky equation in one space variable:

$$u_t+4u_{xxxx}+\alpha(u_{xx}+\tfrac{1}{2}u_x^2)=0. \tag{7.1}$$

In this section we study a bifurcation problem derived from (7.1). Equation (7.1) is equivariant with respect to translations and reflections in the space variable. An approach often used in such situations is to impose periodic boundary conditions with period $L>0$. Then L becomes an additional parameter in the problem. The space variable x can be rescaled so that the boundary conditions become 2π periodic. As a result we obtain the following boundary value problem:

$$v_t+4v_{xxxx}+\alpha(v_{xx}+\tfrac{1}{2}v_x^2)=0, \qquad v(x+2\pi, t)=v(x,t) \tag{7.2}$$

where the period L has been absorbed into the parameter α. The boundary value problem (7.2) is $O(2)$-equivariant. Hence the theory developed in § 6 will apply to bifurcations of relative equilibria of (7.2).

An interesting aspect of the bifurcation analysis of the Kuramoto–Sivashinsky equation is that we can easily find primary branches of solutions with isotropy D_k, for all $k\geqq 2$. This is a consequence of the following observation. Suppose that u is a steady-state solution of (7.2). If we extend u by periodicity to the interval $[0, 2k\pi]$ and rescale the space variable by k, then the so-obtained function is an equilibrium solution of (7.2) for a different value of the parameter α. The new equilibrium is called a *replicated solution.* Note that this solution is $2\pi/k$ periodic, which implies that its isotropy subgroup contains Z_k. It is easy to see that $u=0$ is an equilibrium of (7.2). This equilibrium is stable for α near zero. As α is increased the solution $u=0$ loses stability and bifurcates to a branch of solutions with isotropy group $Z_2(\kappa)$. Hence for each $k\geqq 2$ there exists a branch of replicated equilibria with symmetry D_k.

We might expect that the secondary branches of solutions bifurcating along the replicated branches would be replications of the secondary branches bifurcating from the primary branch. According to the analysis of § 6, however, secondary bifurcations from the replicated branches can be different from secondary bifurcations from the branch with symmetry $Z_2(\kappa)$. In particular, we expect the branch with isotropy $Z_2(\kappa)$ to bifurcate to a rotating wave, and the branches with isotropy D_k to bifurcate to group orbits of equilibria. Kevrekedis, Nicolaenco, and Scovel [1988] carried out a computer-assisted study of secondary and tertiary bifurcations from the branches with isotropy groups $Z_2(\kappa)$, D_2, and D_3. Their results fit the predictions of Theorems 6.1 and 6.2; in particular, the first bifurcation along the branch of the equilibria with isotropy group $Z_2(\kappa)$ is to a rotating wave, and the first bifurcations along the branches of equilibria with isotropy groups D_2 and D_3 are to orbits of equilibria. In this section we discuss the results of Kevrekedis, Nicolaenco, and Scovel and compare them with the predictions of Theorems 6.1 and 6.2.

The numerical results of Kevrekedis, Nicolaenco, and Scovel also indicate existence of quasi-periodic solutions and dynamics related to homoclinic and heteroclinic connections. None of these arise as a result of the bifurcations discussed in § 6. Armbruster, Guckenheimer, and Holmes [1987] analyzed the $O(2)$ equivariant problem of interaction of two steady-state modes, one with isotropy group $Z_2(\kappa)$ and the other with isotropy group D_2. The dynamics they found was much like the dynamics found by Kevrekedis, Nicolaenco, and Scovel near the $Z_2(\kappa)$ and D_2 branches.

Let us now give a more detailed description of the bifurcation problem derived from the Kuramoto-Sivashinsky equation. We start by modifying the coordinates in (7.2) so that the solutions are bounded (see Kevrekedis et al. [1988]):

$$m(t)=\int_0^{2\pi} v(x,t)\,dx.$$

We use (7.2) and the fact that the integrals $\int_0^{2\pi} v_{xx}\,dx$ and $\int_0^{2\pi} v_{xxxx}\,dx$ vanish to show that

$$\dot m(t)=-\frac{\alpha}{4\pi}\int_0^{2\pi} v_x^2\,dx.$$

We now modify the coordinates by letting $u(x,t)=v(x,t)-m(t)$. We obtain

$$\begin{aligned} &u_t+4u_{xxxx}+\alpha(u_{xx}+\tfrac{1}{2}u_x^2)+m(t)=0,\\ &u(x,t)=u(x+2\pi,t). \end{aligned} \tag{7.3}$$

Let us now describe the symmetries of (7.2) and (7.3). Let X be a space of four times differentiable functions $u(x,t)$, 2π periodic in the space variable x. The $O(2)$ action on X is generated by

$$\begin{aligned} &\theta u(x,t)=u(x+\theta,t), \qquad \theta\in SO(2),\\ &\kappa u(x,t)=u(-x,t). \end{aligned}$$

It is easy to see that (7.2) and (7.3) are equivariant with respect to this action.

Let us now explain in more detail how we obtain the replicated steady-state solutions. The ideas we present can be found in Kevrekedis, Nicolaenco, and Scovel. Consider the steady-state problem corresponding to (7.3):

$$4u_{xxxx}+\alpha\left(u_{xx}+\frac{1}{2}u_x^2\right)+\int_0^{2\pi} u_x^2\,dx=0. \tag{7.4}$$

We assert the following. Suppose $u(x)$ is a steady-state solution of (7.3) with $\alpha=\alpha_0$ and let k be a positive integer. Then $w(x)=u(kx)$ is a solution of (7.3) with $\alpha=4k^2\alpha_0$. To prove the assertion we apply the left-hand side of (7.3) to w and use the fact that u is a solution.

Note that $u=0$ is a trivial solution of (7.2). To determine the stability of zero we write (7.2) as

$$u_t=F(u)$$

where

$$F(u)=-4u_{xxxx}+\alpha(u_{xx}+\tfrac{1}{2}u_x^2)+\dot m(t).$$

Then

$$dF|_{u=0}h=-4h_{xxxx}+\alpha h_{xx}.$$

It is easy to see that the functions $e^{2\pi ikx}$, where k is an integer, form a complete set of eigenvectors of $(dF)_{u=0}$ and the corresponding eigenvalues are $(2\pi)^2k^2(4k^2-\alpha)$. The first instability occurs at $k=1$ and $\alpha=4$. As α crosses 4, a branch of equilibria with isotropy group $Z_2(\kappa)$ bifurcates from the trivial solution. We will refer to it as the unimodal branch. This branch is replicated for $\alpha=4k^2$. These replicated branches will be referred to as the k-modal branches. Kevrekedis, Nicolaenco, and Scovel describe some secondary and tertiary bifurcations discovered in their numerical studies of the Kuramoto-Sivashinsky equation. If we believe that those bifurcations are generic in the sense discussed in § 6, then each one of them must match one of the cases described in Theorems 6.1 and 6.2. In what follows we summarize the findings of Kevrekedis, Nicolaenco, and Scovel and relate them to the results of Theorems 6.1 and 6.2. The bifurcation diagram based on the results of Kevrekedis, Nicolaenco, and Scovel is given in Fig. 7.1. The solid lines represent branches of asymptotically stable solutions and the dotted ones represent branches of unstable solutions.

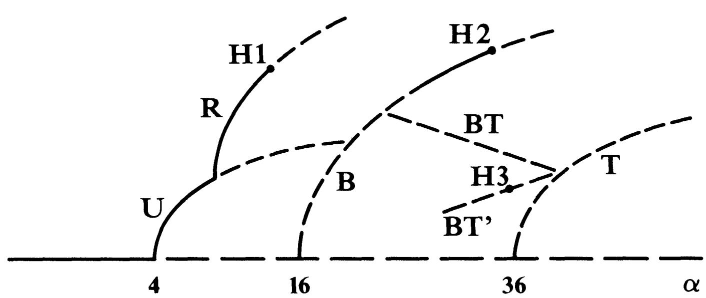

FIG. 7.1. *Secondary and tertiary bifurcation of the Kuramoto-Shivashinsky equation.*

Let U be the branch of equilibria with isotropy $Z_2(\kappa)$, B the branch of equilibria with isotropy D_2, and T the branch of equilibria with isotropy D_3. We discuss the secondary bifurcations found by Kevrekedis, Nicolaenco, and Scovel along each of these branches. We first consider steady-state bifurcations.

(1) *Steady-state bifurcations from the branch U.* At $\alpha=13.005$ the computations of Kevrekedis, Nicolaenco, and Scovel reveal that a real eigenvalue passes through zero. In this case Theorem 6.1 predicts a bifurcation to a rotating wave. The numerical experiments confirm the existence of a rotating wave. Moreover, Kevrekedis, Nicolaenco, and Scovel give an analytical proof of the existence of the rotating wave, based on the ideas of Iooss [1986].

(2) *Steady-state bifurcations from the branch B.*

(a) The first bifurcation on the branch B occurs at $\alpha=16.1399$. Theorem 6.1 predicts a bifurcation to a unique branch of orbits of equilibria with isotropy group isomorphic to Z_2 (provided that the kernel of the action of D_2 on the nullspace is not contained in $SO(2)$). This is in agreement with the computations, which show that the branch U merges with the branch B.

(b) An analogous bifurcation is observed for $\alpha=22.559$. The isotropy group bifurcating branch is $Z_2(\kappa)$. We label this branch BT since it later joins with the trimodal branch.

(3) *Steady-state bifurcations from the branch T.* At $\alpha=36.235$ two real eigenvalues of the branch T pass through zero. According to Theorem 6.1 there are two branches of equilibria bifurcating of the branch T. The isotropy of these equilibria is $Z_2(\kappa)$. The

numerical results are in complete agreement with this prediction. One of the bifurcating branches is the branch BT. Kevrekedis, Nicolaenco, and Scovel refer to the other branch as a continuation of BT and also label it BT. For this reason we label this branch as BT'.

(4) *Steady-state bifurcations from the branch BT.* Kevrekedis, Nicolaenco, and Scovel also find a bifurcation point related to a zero eigenvalue on the branch BT. They conjecture that the corresponding bifurcation is to a rotating wave. Theorem 6.1 also predicts a bifurcation to a rotating wave and hence supports the conjecture.

Kevrekedis, Nicolaenco, and Scovel discuss three Hopf bifurcation points, marked in Fig. 7.1 as H1, H2, and H3. The following are the predictions of the nature of these bifurcations derived from Theorem 6.2.

The Hopf bifurcation point 1. Point H1 corresponds to a Hopf bifurcation from the branch of rotating waves R. The group of symmetries of the branch of rotating waves is $SO(2)$. Theorem 6.2 implies that generically the bifurcating trajectories are dense in two-tori. This agrees with the predictions of Kevrekedis, Nicolaenco, and Scovel [1988, closing remarks of § 5a], who conclude from the structure of the dynamics that a doubly periodic solution is likely to exist.

The Hopf bifurcation point 2. Point H2 corresponds to a Hopf bifurcation occurring along the branch B. The isotropy group of the equilibria on the branch B is D_2. Theorem 6.2 implies that this Hopf bifurcation leads to a periodic flow. The numerical results of Kevrekedis, Nicolaenco, and Scovel indicate that the bifurcating solutions are periodic.

The Hopf bifurcation point 3. Point H3 corresponds to Hopf bifurcation occurring along the branch BT. Let Σ be the isotropy group of the equilibria on that branch. We have previously remarked that $\Sigma = Z_2(\kappa)$ or $\Sigma = Z_2(\kappa, \pi)$. It follows from Theorem 6.2 that the bifurcating solutions must be periodic orbits. Kevrekedis, Nicolaenco, and Scovel do not comment on the dynamics related to this bifurcation.

8. The Bénard problem. In this section we analyze secondary steady-state bifurcations of a dynamical system equivariant with respect to the group $\Gamma = D_6 \dot{+} \mathbb{T}^2$, where $\mathbb{T}^2$ is a two-dimensional torus and D_6 is the group of symmetries of a regular hexagon. A bifurcation problem with this symmetry arises in the analysis of the mathematical model of convection between two infinite planes. This problem is called the planar Bénard problem. We now briefly describe the symmetries of the model and the derivation of the bifurcation problem with symmetry group Γ. Detailed information on this topic and the analysis of primary bifurcations can be found in Buzano and Golubitsky [1983] or in Golubitsky, Stewart, and Schaeffer [1988, Case Study 4].

Let x, y be the coordinates in a horizontal plane and z the coordinate in the vertical direction. The model of convection is equivariant with respect to translations, reflections, and rotations in the xy plane. The group generated by these transformations is called the group of Euclidian motions in the plane and is denoted by E_2. Let w be in R^2 and let T_w denote translation by w. The group of translations in the plane is isomorphic to R^2 with the isomorphism defined by the assignment $w \nrightarrow T_w$. The group E_2 can be thought of as the semidirect product $O(2) \dot{+} R^2$ with multiplication defined by

$$(8.1) \qquad (\sigma_1, T_{w_1})(\sigma_2, T_{w_2}) = (\sigma_1\sigma_2, T_{w_1+\sigma_1 w_2}), \qquad \sigma_1, \sigma_2 \in O(2), \quad w_1, w_2 \in R^2.$$

Let $e \in R^2$ be an arbitrary vector and let f be obtained by rotating e by $\pi/3$. The hexagonal lattice H_6 is given as

$$H_6 = \{ne + mf\text{: for all pairs of integers } n \text{ and } m\}.$$

Note that H_6 is a subgroup of R^2. The solutions of the convection problem have the form $u(t, x, y, z)$. We restrict our attention to those that are periodic in both directions of the lattice. Let

$$\mathscr{X} = \{u(t, x, y, z): u(t, (x, y, z)+(e, 0)) = u(t, (x, y, z)+(f, 0)) = u(t, x, y, z)\}.$$

Clearly, the only elements of $O(2)$ that leave $\mathscr{X}$ invariant are the elements of D_6. Also the action of H_6 on $\mathscr{X}$ is trivial. Hence the group of symmetries of the convection problem restricted to $\mathscr{X}$ is given by Γ with $\mathbb{T}^2 = R^2/H_6$.

Multiplication in Γ is induced by multiplication in E_2. Let D_6 act on R^2 by the standard action. Let 1 denote the identity in D_6, zero the identity in $\mathbb{T}^2$, and e the identity in Γ. For $p \in R^2$ let p' denote the image of p under the natural projection $R^2 \to \mathbb{T}^2$. For $\sigma \in D_6$ we define $\sigma \cdot p' = (\sigma p)'$. Multiplication in Γ is given as follows:

$$(8.2) \qquad (\sigma_1, p_1')(\sigma_2, p_2') = (\sigma_1\sigma_2, p_1' + \sigma_1 \cdot p_2').$$

Here $\sigma_1\sigma_2$ is the product of σ_1 and σ_2 in D_6 and $p_1' + \sigma_1 \cdot p_2'$ is the sum of vectors in $\mathbb{T}^2$.

The Bénard problem has an invariant equilibrium (the pure conduction state). There exists a region in the parameter space where this equilibrium is stable. The known primary bifurcations are to two types of equilibria with maximal isotropy subgroups. These subgroups are D_6 and $D_2 \dot{+} S^1$ $(= Z_2 \oplus O(2))$. The equilibria with isotropy D_6 are called hexagons, and the equilibria with isotropy $Z_2 \oplus O(2)$ are called rolls. In what follows we describe the kinds of steady-state bifurcations each one of these solutions can undergo.

From now on we consider an abstract Γ-equivariant bifurcation problem. We assume that $F: R^n \to R^n \times R$ (for some n) is a smooth Γ-equivariant family of vector fields and that F has a branch of equilibria with isotropy group D_6 (which we refer to as hexagons) and a branch of solutions with isotropy group $Z_2 \oplus O(2)$ (which we refer to as rolls). We analyze the generic bifurcations of these solutions.

(A) Bifurcations of hexagons. Suppose $X = \Gamma x_0$ is a group orbit of hexagons. Let G be the restriction of F_N to the normal space $x_0 + N_{x_0}$ and let $g = G(\cdot, 0)$. We assume that $(dg)_{x_0}$ has a zero eigenvalue. Let E be the center subspace of $(dg)_{x_0}$. We make a generic assumption that E is the nullspace of $(dg)_{x_0}$ and that the action of D_6 on E is absolutely irreducible. Our bifurcation analysis will depend on the action of D_6 on E. Let $\mathscr{K}(D_6)$ be the kernel of the action of D_6 on E. We assume that the bifurcation is symmetry breaking, that is, $\mathscr{K}(D_6)$ is a proper subgroup of D_6. According to Lemma 6.3, either $\mathscr{K}(D_6) = Z_m$, $m = 1, 2, 6$, or $\mathscr{K}(D_6)$ is isomorphic to D_3. The following proposition gives a classification of generic bifurcations of hexagons.

PROPOSITION 8.1. *All generic types of bifurcating solutions of F are listed in Table* 8.1.

We now state and prove a lemma necessary to prove Proposition 8.1. Suppose that Σ is a subgroup of D_6 and let $N(\Sigma)$ denote the normalizer of Σ in Γ. Let Fix (Σ)

TABLE 8.1

Kernel of isotropy	Type of solution	Number of half branches
Z_6 or D_3	steady state	1
Z_2	steady state	2
trivial	periodic orbit	2

denote the fixed-point space of Σ taken with respect to the standard action of D_6 on R^2. We have the following lemma.

LEMMA 8.2. $\dim N(\Sigma) = \dim \mathrm{Fix}(\Sigma)$.

Proof. Let Γ_0 denote the connected component of the identity in Γ. Since Γ is a compact group it suffices to show that the normalizer of Σ in Γ_0 has the same dimension as $\mathrm{Fix}(\Sigma)$. The group Γ_0 consists of elements of form $(1, p')$, $p' \in \mathbb{T}^2$. Suppose that $(\sigma, 0)$ is in Σ. The element $(1, p')$ is in $N(\Sigma)$ if $(1, -p')(\sigma, 0)(1, p')$ is in Σ. We have

$$(1, -p')(\sigma, 0)(1, p') = (\sigma, \sigma \cdot p' - p'). \tag{8.3}$$

If (8.1) holds then we must have $\sigma \cdot p' - p' = 0$, which is equivalent to $(\sigma p - p)' = 0$. The proof now follows.

Proof of Proposition 8.1. Let $\tau: D_6 \to D_6/\mathcal{K}(D_6)$ be the natural projection. In the analysis of bifurcations of the family G we replace the action of D_6 by the action of $\tau(D_6)$. As indicated in Table 8.1 we divide the analysis into three cases:

(1) $\mathcal{K}(D_6) = Z_6$ or $\mathcal{K}(D_6)$ is isomorphic to D_3.

(2) $\mathcal{K}(D_6) = Z_2$.

(3) $\mathcal{K}(D_6)$ is trivial.

Case (1). Observe that $\tau(D_6)$ is isomorphic to Z_2. In this case a generic family G has a unique branch of equilibria $y(\lambda)$, whose isotropy subgroup in D_6 is trivial (see the proof of Theorem 6.1). The isotropy group of $y(\lambda)$ in $\tau(D_6)$ is trivial. Let Σ_y be the isotropy group of $y(\lambda)$ in D_6. It follows that Σ_y is $\mathcal{K}(D_6)$. The group $\mathcal{K}(D_6)$ contains a nontrivial rotation. It follows that the fixed-point space of $\mathcal{K}(D_6)$ with respect to the standard action of D_6 on R^2 is trivial. By Lemma 8.2 $\dim N(\Sigma) = 0$. Let $Y(\lambda) = \Gamma y(\lambda)$. By Theorem 4.1 the set $Y(\lambda)$ consists of equilibria of F.

Case (2). Observe that $\tau(D_6)$ is isomorphic to D_3. It follows from Table XIII, 5.1 in Golubitsky, Stewart, and Schaeffer [1988] that a generic family G has two half branches of equilibria $y_1(\lambda)$, $y_2(\lambda)$. Let Σ_{y_1} and Σ_{y_2} denote the isotropy subgroups of y_1 and y_2 in D_6. Both of these groups must contain $\mathcal{K}(D_6)$. Since $\mathcal{K}(D_6)$ contains the rotation by π it follows that the fixed-point spaces of Σ_{y_1} and Σ_{y_2} are trivial. Lemma 8.2 implies that the normalizers of these groups are discrete. Let $Y_1(\lambda) = \Gamma y_1(\lambda)$, $Y_2(\lambda) = \Gamma y_2(\lambda)$. Theorem 4.1 implies that the sets $Y_1(\lambda)$ and $Y_2(\lambda)$ consist of equilibria of F.

Case (3). According to Table XIII, 5.2 in Golubitsky, Stewart, and Schaeffer [1988], a generic family G has two half branches of equilibria $y_1(\lambda)$, $y_2(\lambda)$, with $\Sigma_{y_1} = \{1, \kappa\}$ and $\Sigma_{y_2} = \{1, \kappa\pi\}$. Here κ denotes the reflection through the x-axis and $\kappa\pi$ denotes the reflection through the y-axis. Clearly, these groups have one-dimensional fixed-point spaces, hence, by Lemma 8.2, their normalizers are one-dimensional. Let $Y_1(\lambda) = \Gamma y_1(\lambda)$, $Y_2(\lambda) = \Gamma y_2(\lambda)$. Theorem 4.1 implies that generically the trajectories of the flow of F on the sets $Y_1(\lambda)$ and $Y_2(\lambda)$ are rotating waves.

(B) Bifurcations of rolls. Suppose $X = \Gamma x_0$ is a group orbit of rolls. Let G be the restriction of F_N to the normal space N_{x_0} and let $g = G(\cdot, 0)$. We assume that $(dg)_{x_0}$ has an eigenvalue on the imaginary axis. Let Σ_r denote the isotropy subgroup of rolls. Let E be the center subspace of $(dg)_{x_0}$. We make a generic assumption that the action of Σ_r on E is irreducible. We will show that this implies that the action of Σ_r on E is absolutely irreducible. This implies that $(dg)_{x_0}$ must have a zero eigenvalue and hence the bifurcation we consider is a steady-state bifurcation.

The group Σ_r is generated by translations along the y-axis, reflection through the y-axis and rotation by π (see Golubitsky, Stewart, and Schaeffer [1988, p. 154]). The projection of the y-axis into the torus $\mathbb{T}^2$ is a circle which we denote by S^1. Let ξ correspond to the element of D_6 which acts on R^2 as rotation by π, and let ζ denote the element of D_6 which acts as reflection through the y-axis. The element ζ commutes with the elements of S^1 and the element ξ anticommutes with the elements of S^1. It follows that Σ_r is isomorphic to $Z_2 \oplus O(2)$, with S^1 corresponding to $SO(2)$, ξ corresponding to a reflection in $O(2)$, and ζ corresponding to the nontrivial element of Z_2. The irreducible representations of $Z_2 \oplus O(2)$ are given by the irreducible representations of $O(2)$, with ζ acting as identity or as minus identity. Since the irreducible representations of $O(2)$ are absolutely irreducible it follows that $(dg)_{x_0}|E \equiv 0$.

We now divide that analysis into two cases:

(1) S^1 acts trivially on E.

(2) S^1 acts nontrivially on E.

Case (1). In the analysis of bifurcations of G we can replace the action of Σ_r on E by the action of $D_2 = \{1, \xi, \zeta, \xi\zeta\}$. The space E must be one-dimensional and the kernel of the action of D_2 must be one of the groups:

$$(8.4) \qquad Z_2(\zeta) = \{1, \zeta\}, \qquad Z_2(\xi) = \{1, \xi\}, \qquad Z_2(\xi\zeta) = \{1, \xi\zeta\}.$$

It follows that in the analysis of bifurcations of G the action of D_2 can be replaced by the action of a group isomorphic to Z_2. Hence a generic family G has a unique branch of equilibria $y(\lambda)$ and the isotropy group of $y(\lambda)$ in D_2 is equal to one of the groups listed in (8.4). Let Σ_y be the isotropy subgroup of $y(\lambda)$ in Σ_r. We have $\Sigma_y = \langle S^1, \zeta\rangle$, or $\Sigma_y = \langle S^1, \xi\rangle$ or $\Sigma_y = \langle S^1, \xi\zeta\rangle$. In other words, Σ_y is generated by the elements of S^1 and ξ, ζ, or $\xi\zeta$. Let $Y(\lambda) = \Gamma y(\lambda)$. We have the following proposition.

PROPOSITION 8.3. *If $\Sigma_y = \langle S^1, \xi\zeta\rangle$ then the trajectories of flow of F on $Y(\lambda)$ are rotating waves. Otherwise, $Y(\lambda)$ consists of equilibria.*

Proof. We prove that if $\Sigma_y = \langle S^1, \xi\zeta\rangle$ then $\dim N(\Sigma_y) = 2$ and otherwise $\dim N(\Sigma_y) = 1$. The proposition will then follow from Theorem 4.1.

By compactness of Γ it suffices to show that the normalizer of Σ_y in Γ_0 has the indicated dimension. Recall that Γ_0 consists of the elements of Γ of the form $(1, p')$, $p' \in \mathbb{T}^2$. Suppose that $(\sigma, 0) \in \Sigma_y$, $\sigma \in D_2$. The identity (8.3) implies that

$$(8.5) \qquad \sigma \cdot p' - p' \in S^1.$$

Recall that S^1 is the image of the y-axis under the natural projection $R^2 \to \mathbb{T}^2$. Hence (8.5) implies that $\sigma p - p = (0, q)$, $q \in R$.

Suppose that $p = (p_1, p_2)$. The element $\xi\zeta$ acts on R^2 as reflection through the x-axis; that is, $\xi\zeta(p_1, p_2) = (p_1, -p_2)$. It follows that $\xi\zeta p - p = (0, -2p_2)$. Hence (8.4) holds for all p' in S^1. It follows that if $\Sigma_y = \langle S^1, \xi\zeta\rangle$, then $\dim N(\Sigma_y) = 2$.

The element ξ acts on R^2 as rotation by π; that is, $\xi(p_1, p_2) = (-p_1, -p_2)$. It follows that $\xi p - p = (-2p_1, -2p_2)$. Hence (8.5) holds if $p' \in S^1$. It follows that if $\Sigma_y = \langle S^1, \xi\rangle$, then $\dim N(\Sigma_y) = 1$.

The element ζ acts on R^2 as reflection through the x-axis, that is, $\zeta(p_1, p_2) = (-p_1, p_2)$. It follows that $\zeta p - p = (-2p_1, 0)$. Hence (8.5) holds if $p' \in S^1$. It follows that if $\Sigma_y = \langle S^1, \zeta\rangle$, then $\dim N(\Sigma_y) = 1$.

Case (2). We show that if S^1 acts nontrivially, then generically the flow on the bifurcating relative equilibria is trivial. Namely, we prove the following proposition.

PROPOSITION 8.4. *A generic family F has a unique branch of bifurcating relative equilibria $Y(\lambda)$. The flow on the sets $Y(\lambda)$ is trivial.*

Proof. We first consider bifurcations of the family G. Let $\mathcal{K}(\Sigma_r)$ be the kernel of the action of Σ_r on E. We show that $\mathcal{K}(\Sigma_r)$ must be nontrivial. If ζ acts on E as identity, then $(\zeta, 0) \in \mathcal{K}(\Sigma_r)$, which implies the assertion. Suppose that ζ acts on E as minus identity. Let p_0 be the element of S^1 acting on E as rotation by π. Then $(\zeta, p_0) \in \mathcal{K}(\Sigma_r)$, so $\mathcal{K}(\Sigma_r)$ is nontrivial.

Since the action of S^1 is nontrivial it follows that the action of ξ on E is a reflection. Hence $\mathcal{K}(\Sigma_r)$ is generated by a cyclic subgroup of S^1 and either $(\zeta, 0)$ or (ζ, p_0). It follows that $\Sigma_r/\mathcal{K}(\Sigma_r)$ is isomorphic to $O(2)$. In the analysis of the bifurcations of G we replace the action of Σ_r on E by the action of $\Sigma_r/\mathcal{K}(\Sigma_r)$ on E. By standard results on $O(2)$-equivariant bifurcation G has a unique branch of equilibria $y(\lambda)$ and Σ_y contains a reflection. Since ζ acts as reflection we can assume (by possibly replacing $y(\lambda)$ by a conjugate branch) that $(\zeta, 0) \in \Sigma_y$.

We now prove that the normalizer of Σ_y in Γ_0 is discrete. As we argued earlier, this implies that the normalizer of Σ_y in Γ is discrete. The group Σ_y is generated by $(\zeta, 0)$ and the elements of $\mathcal{K}(\Sigma_r)$. Suppose that $(1, p') \in N(\Sigma_y)$, $p = (p_1, p_2)$. Recall that ξ acts on R^2 as rotation by π; that is, $\xi(p_1, p_2) = (-p_1, -p_2)$. The identity (8.3) implies that $\zeta \cdot p' - p' \in \Sigma_y$. Also $\xi p - p = (-2p_1, -2p_2) = -2p'$. It follows that $2p'$ must be in $\mathcal{K}(\Sigma_r) \cap S^1$, which, by assumption is a discrete group. It follows that $N(\Sigma_y)$ is discrete and by Theorem 4.1 the relative equilibria $Y(\lambda) = \Gamma y(\lambda)$ consist of equilibria of F.

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