# A Mathematical Example Displaying Features of Turbulence

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## *Introduction*

Before entering upon the study of the example in question we want to make some introductory remarks about the actual hydrodynamic problems, in particular, about what is known and what is conjectured concerning the future behavior of the solutions. Consider an incompressible and homogeneous viscous fluid within given material boundaries under given exterior forces. The boundary conditions and the outside forces are assumed to be stationary, i.e. independent of time. For that, it is not necessary that the walls be at rest themselves. Parts of the material walls may move in a stationary movement provided that the geometrical boundary as a whole stays at rest. An instance is a fluid between two concentric cylinders rotating with prescribed constant velocities or a fluid between two parallel planes which are translated within themselves with given constant velocities. As to the stationarity of the exterior forces we may cite the case of a flow through an infinitely long pipe with a pressure drop (regarded as an outside force). In this case the pressure drop is required to be a given constant independent of time.

Each motion of the fluid that is theoretically possible under these conditions satisfies the Navier-Stokes equations ($\rho = 1$)

$$\frac{\partial u_i}{\partial t} = -\sum_\nu u_\nu \frac{\partial u_i}{\partial x_\nu} - \frac{\partial p}{\partial x_i} + \mu \Delta u_i \tag{0.1}$$

together with the incompressibility condition

$$\sum \frac{\partial u_\nu}{\partial x_\nu} = 0 \tag{0.2}$$

and the given stationary boundary conditions. To an arbitrarily prescribed initial velocity field $u(x, 0)$ satisfying (0.2) and the boundary conditions there is expected to belong a unique solution $u(x, t; \mu)$ ($t \geq 0$) of (0.1) and (0.2) that fulfills these boundary conditions. The pressure $p(x, t; \mu)$ may be considered as an auxiliary variable which, at every moment $t$ is (up to an additive constant) perfectly well determined by the instantaneous velocity field $u(x, t)$ (solution of a Neumann problem of potential theory). If $p$ is eliminated in this manner the Navier-Stokes equations appear in the form of an integrodifferential space-time system for the $u_i$ alone where the right hand sides consist of first and second degree terms in the $u_\nu$.

It is convenient to visualize the solutions in the phase space $\Omega$ of the problem. A phase or state of the fluid is a vector field $u(x)$ in the fluid space that satisfies (0.2) and the boundary conditions. The totality $\Omega$ of these phases is therefore a functional space with infinitely many dimensions. A flow of the fluid represents a point motion in $\Omega$ and the totality of these phase motions forms a stationary flow in the phase space $\Omega$, which, of course, is to be distinguished from the fluid flow itself. What is the asymptotic future behavior of the solutions, how does the phase flow behave for $t \to \infty$? And how does this behavior change as $\mu$ decreases more and more? How do the solutions which represent the observed turbulent motions fit into the phase picture? The great mathematical difficulties of these important problems are well known and at present the way to a successful attack on them seems hopelessly barred. There is no doubt, however, that many characteristic features of the hydrodynamical phase flow occur in a much larger class of similar problems governed by nonlinear space-time systems. In order to gain insight into the nature of hydrodynamical phase flows we are, at present, forced to find and to treat simplified examples within that class. The study of such models has been originated by J. M. Burgers in a well known memoir.[1] His principal example is essentially

$$\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} + v - w + \mu\frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial w}{\partial t} = w\frac{\partial v}{\partial x} + v\frac{\partial w}{\partial x} + v + w + \mu\frac{\partial^2 w}{\partial x^2}$$

where $0 \leq x \leq 1$ and where the boundary conditions are $v = w = 0$ at $x = 0$ and $x = 1$. Though simpler in form than the hydrodynamic equations this example presents essentially the same difficulties and the future behavior of the solutions for small values of $\mu$ still is an unsolved problem.

In this paper another nonlinear example is presented and studied that differs from Burgers' model in that the future behavior of its solutions can be completely determined. In this respect our example seems to us to be the first of its kind. The detailed study of this space-time system reveals geometrical features of the phase flow which come close to the qualitative picture we believe to prevail in the hydrodynamic cases. It must, however, be said that, for reasons to appear later in the paper, the analogy does not extend to the quantitative relations found to hold in turbulent fluid flow.

The observational facts about hydrodynamic flow reduced to the case of fixed side conditions and with $\mu$ as the only variable parameter are essentially these: For $\mu$ sufficiently large, $\mu > \mu_0$, the only flow observed in the long run is a stationary one (laminar flow). This flow is stable against arbitrary initial

[1]J. M. Burgers, *Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion*. Akademie van Wetenschappen, Amsterdam, Eerste Sectie, Deel XVII, No. 2, pp. 1–53, 1939.

disturbances. Theoretically, the corresponding exact solution is known to exist for every value of $\mu > 0$ and its stability in the large can be rigorously proved, though only for sufficiently large values of $\mu$. The corresponding phase flow in phase space $\Omega$ thus possesses an extremely simple structure. The laminar solution represents a single point in $\Omega$ invariant under the phase flow. For $\mu > \mu_0$, every phase motion tends, as $t \to \infty$, toward this laminar point. For sufficiently small values of $\mu$, however, the laminar solution is never observed. The turbulent flow observed instead displays a complicated pattern of apparently irregularly moving "eddies" of varying sizes. The view widely held at present is that, for $\mu > 0$ having a fixed value, there is a "smallest size" of eddies present in the fluid depending on $\mu$ and tending to zero as $\mu \to 0$. Thus, macroscopically, the flow has the appearance of an intricate chance movement whereas, if observed with sufficient magnifying power, the regularity of the flow would never be doubted.

The qualitative mathematical picture which the author conjectures to correspond to the known facts about hydrodynamic flow is this: To the flows observed in the long run after the influence of the initial conditions has died down there correspond certain solutions of the Navier-Stokes equations. These solutions constitute a certain manifold $\mathfrak{M} = \mathfrak{M}(\mu)$ in phase space invariant under the phase flow. Presumably owing to viscosity $\mathfrak{M}$ has a finite number $N = N(\mu)$ of dimensions. This effect of viscosity is most evident in the simplest case of $\mu$ sufficiently large. In this case $\mathfrak{M}$ is simply a single point, $N = 0$. Also the complete stability of $\mathfrak{M}$ is in this simplest case obviously due to viscosity. On the other hand, for smaller and smaller values of $\mu$, the increasing chance character of the observed flow suggests that $N(\mu) \to \infty$ monotonically as $\mu \to 0$. This can happen only if at certain "critical" values

$$\mu_0 > \mu_1 > \mu_2 > \cdots \to 0$$

the number $N(\mu)$ jumps. The manifold $\mathfrak{M}(\mu)$ itself presumably changes analytically as long as no critical value is passed. Now we believe that when $\mu$ decreases through such a value $\mu_k$ a continuous branching phenomenon occurs. The manifold $\mathfrak{M}(\mu)$ of motions observed in the long run (more precisely its analytical continuation for $\mu < \mu_k$) loses its stability. The notion of stability here refers to the whole manifold and not to the single motions contained in it. The loss of stability implies that the motions on the analytically continued $\mathfrak{M}$ are no longer observed. What we observe after passing $\mu_k$ is not the analytical continuation of the previous $\mathfrak{M}$ but a new manifold $\mathfrak{M}(\mu)$ continuously branching away from $\mathfrak{M}(\mu_k)$ and slightly swelling in a new dimension. This new $\mathfrak{M}(\mu)$ takes over stability from the old one. Stability here means that the "majority" of phase motions tends for $t \to \infty$ toward $\mathfrak{M}(\mu)$. We must expect that there is a "minority" of exceptional motions that do not converge toward $\mathfrak{M}$ (for instance the motions on the analytical continuation of the old $\mathfrak{M}$ and of all the other manifolds left over from all the previous branchings). The simplest case of such a bifurcation with corresponding change of stability is the branching of a periodic motion from a stationary one. This case is clearly observed in the flow around an obstacle (transition from the laminar flow to a periodic one with periodic discharges of eddies from the boundary). The next simplest case is the branching of a one-parameter family of almost periodic solutions from a periodic one. The new solutions are expressed by functions

$$u(\phi_1, \phi_2; \mu)$$

periodic in each $\phi$ with period $2\pi$ where

$$\phi_i = a_i t + c_i, \qquad a_i = a_i(\mu),$$

and where the $c_i$ are arbitrary constants (we can without loss of generality assume $c_1 = 0$). The functions $f$ with $\phi_i$ arbitrary describe the manifold $\mathfrak{M}(\mu)$ which, in our case, is of the type of a torus. If $\mathfrak{M}$, quite generally, continuously develops out of the laminar point there is a reasonable expectation that $\mathfrak{M}$ is a multidimensional torus-manifold described by functions

$$u(\phi_1, \cdots, \phi_N; \mu)$$

with period $2\pi$ in each of the $\phi$ and that the turbulent solutions are given by linear functions $\phi_i = a_i(\mu)t + c_i$ as before. This is what happens in our example which precisely exhibits this phenomenon of continuous growth of almost periodic solutions out of the laminar one with an infinite succession of branchings of the type described above.

The geometrical picture of the phase flow is, however, not the most important problem of the theory of turbulence. Of greater importance is the determination of the probability distributions associated with the phase flow, particularly of their asymptotic limiting forms for small $\mu$. In the case of our example these distributions have limiting forms (normal distribution). Recent investigations, however, suggest that there are essential deviations from normality in the hydrodynamic case. It seems that the influence of the second degree terms is in this case essentially different and much more complicated than in the case of our over-simplified model.

Another observation on our model case is this: If we proceed to the limit $\mu \to 0$ within the "observed," i.e. the turbulent solutions the turbulent fluctuations are found to disappear and we obtain a special stationary solution in the "ideal case" (equations with $\mu = 0$). This shows, by way of analogy, how important a role viscosity plays in turbulence.

## *Formulation of the Problem*

The space of our model is a one-dimensional circular line and our space variable is an angular variable $x$ mod $2\pi$. All space functions are thus periodic functions of $x$ with period $2\pi$. For two arbitrary space functions $f$, $g$ we denote