Tutorials and Reviews

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NINETY PLUS THIRTY YEARS OF NONLINEAR DYNAMICS: LESS IS MORE AND MORE IS DIFFERENT*

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To David Chillingworth, John Guckenheimer, Jerry Marsden and David Rand

I review the early (1885–1975) and more recent history of dynamical systems theory, identifying key principles and themes, including those of dimension reduction, normal form transformation and unfolding of degenerate cases. I end by briefly noting recent extensions and applications in nonlinear fluid and solid mechanics, with a nod toward mathematical biology. I argue throughout that this essentially mathematical theory was largely motivated by nonlinear scientific problems, and that after a long gestation it is propagating throughout the sciences and technology.

Keywords: Bifurcation; chaos; dimension reduction; homoclinic orbit; hyperbolic set; normal form; unfolding.

1. Introduction

Nonlinear dynamics, more grandly called "nonlinear science" or "chaos theory," is a rapidly-growing but still ill-defined field, and in this article I can only offer my own view of (a small) part of it. With this in mind, I hope that the reader's indulgence will allow me to begin on a personal note.

When I was finishing my doctoral work at the Institute of Sound and Vibration Research (ISVR) of Southampton University in 1973 I chanced to see a poster notifying that a course of lectures on catastrophe theory would be offered by David Chillingworth, then a recently-appointed Lecturer in Mathematics. At that time, at least in the UK, there were near-impenetrable walls between and even within engineering, the sciences, and pure and applied mathematics. It was therefore with some trepidation that I crossed University road, found the Maths Department, and sat down near the back of the classroom, trying to appear as if I knew where I was. I was soon asking my classmate, David Rand, what on earth diffeomorphisms, k-jets, and the implicit function theorem were. This led to our first joint paper [Holmes & Rand, 1976], but more critically the lecture course, and the two Davids, changed my research life. The course would become a book [Chillingworth, 1976] that was particularly useful to those wishing to penetrate the arcana of catastrophes, bifurcations, and chaos, and if the language and methods of dynamical systems have now entered almost every branch of the sciences, it is largely due to such early propagators of these ideas from within mathematics, including Jerry Marsden, John Guckenheimer, Floris Takens and others. (Contrary to some reports, nonlinear dynamics was not solely invented by physicists.) I had the good fortune to meet and begin collaborating with some

*With apologies to Ludwig Mies van der Rohe and Philip Anderson.

of these pioneers in the mid 1970's and I am happy to dedicate this article accordingly.

At about the same time that such (originally pure) mathematicians began exporting dynamical ideas to the sciences, physicists such as Harry Swinney and Jerry Gollub began appealing to bifurcation theory and dynamics in interpreting experimental data [Swinney & Gollub, 1987], and two remarkable multi-disciplinary meetings took place at the New York Academy of Science in 1977 and 1979 [Gurel & Rössler, 1979; Helleman, 1980]. The time was ripe to tear down walls. Some thirty years later we can claim partial success: many of the walls are lower, and we have built new science from their fragments. Certainly the EUROMECH ENOC-05 meeting, to which this issue of IJBC is devoted, is a local proof of the near (or asymptotic) nonexistence of walls between nonlinear mechanics and mathematics in 2005.

In this brief article I sketch key contributions to the early (≈ 1885–1975) history of dynamical systems theory, and outline four central themes that have emerged as it has spread throughout the sciences and engineering since then. I focus on mathematical developments (as noted in the next section, I believe that the field has coherence only as a part of mathematics), but emphasize the importance of applications in motivating and even defining mathematical developments. I provide an idiosyncratic, but perhaps representative, set of references, highlighting papers that were influential in determining central themes (sometimes long after their publication) and that may be unfamiliar. At appropriate points I note some more personal reminiscences by contributors to the field.

This paper is not a tutorial: I shall largely eschew technical detail and precise statements, referring the reader to the many textbooks and monographs. These range in level from introductory [Hirsch *et al.*, 2004] to advanced [Guckenheimer & Holmes, 1983; Wiggins, 2003; Kuznetsov, 2003], and from applied [Strogatz, 1997] to abstract [Arnold, 1983; Katok & Hasselblatt, 1995] in their approach. This partial list has probably already offended the authors of numerous other fine books: I apologise to them, and to the many contributors to the research literature whose work I am unable to note due to lack of space. In the mid 1980's a dynamical systems bibliography containing 4405 items was produced by K. Shiraiwa of Nagoya University. Since then the output must have multiplied at least tenfold; indeed, I doubt that a single individual could google and paste fast enough to keep abreast of the current literature.

2. The Qualitative Theory of Dynamical Systems

Before beginning the story I should declare a prejudice and in doing so explain the title chosen for this central section. I do not believe that "chaos theory" has the status of, say, the quantum or relativity theories, or that "nonlinear science" is a science in the manner of physics, chemistry or biology. Dynamical systems theory neither addresses specific phenomena nor proposes particular models of (parts of) reality; it is, rather, a loosely-related set of methods for analyzing a broad class of differential equations and iterated mappings. These comprise a *mathematical theory* largely built from natural developments in analysis, geometry and topology: areas of mathematics which in turn had their origins in Newtonian mechanics. With the goal of classifying dynamical models, the theory's topological viewpoint provides a unifying structure for a riot of detail, but it does not help one *formulate* models *per se*, although knowing the nature of the beasts in the mathematical forest is certainly useful. Moreover, while it might allow one to prove that models of the solar system, planetary weather, or the world economy are chaotic, any conclusions drawn regarding asteroid impacts, hurricanes or stockmarket crashes will depend on the validity of the models themselves. Thus, I am not persuaded by hyperbolic claims of scientific revolution, although I am delighted by the increasing reach of dynamical systems beyond the mathematical sciences themselves.

2.1. *Some early history: Homoclinic points and global behavior*

The modern theory of dynamical systems derives from the work of Poincaré on the three-body problem of celestial mechanics [Poincaré, 1892, 1893, 1899], and primarily from a single, massive and initially-flawed paper [Poincaré, 1890] which, in 1889, won a prize offered by King Oscar II of Sweden and Norway. In it Poincaré laid the foundations for the local and global analysis of nonlinear differential equations, including the use of first return (Poincaré) maps for the study of periodic motions; he defined stable and unstable manifolds, discussed stability issues at length, proved the (Poincaré) recurrence theorem, and much more.

Finally, in correcting proofs of the paper, and without benefit of computer workstations or color graphics, Poincaré realized that certain differential equations describing "simple" mechanical systems with two or more degrees of freedom were not integrable in the classical sense, that obstructions to integrability were due to the presence of "doubly-asymptotic" points, now called homo- and heteroclinic orbits, and that this had profound implications for the stability of motion. In December 1889 and January 1890 he created the perturbative and geometric theory to detect chaos, and provided the first explicit example of it. See [Barrow-Green, 1997; Holmes, 1990] for historical and (some) technical details, and [Diacu & Holmes, 1996] for a broader picture and sketches of other key contributors to the theory.

Following Poincaré's work, Hadamard considered the dynamics of certain geodesic flows, but the next major thrust was due to G. D. Birkhoff in the US, who had earlier proved Poincaré's "last theorem" [Birkhoff, 1913] regarding fixed points for annulus maps. Among much else, in a notable book on two degree-of-freedom Hamiltonian systems that still bears reading [Birkhoff, 1927], he showed that, close to any homoclinic point of a two-dimensional mapping, there is an infinite sequence of periodic points with periods approaching infinity. Birkhoff would go on to prove that annulus maps having points of two distinct periods also contain complex limit sets separating their domains of attraction [Birkhoff, 1932], thus providing a key clue for Cartwright and Littlewood in their attempts to understand the van der Pol equation. But this is getting ahead in the story of homoclinic points and their implications, which would not be settled for a further 33 years.

Before resuming that story I will move continents again to the Soviet Union, where the "Andronov school" in Gorki (now again named Nizhni-Novgorod) introduced the key idea of structural stability under the name "systèmes grossieres" (coarse systems) [Andronov & Pontryagin, 1937]. This notion, now a central theme of the theory, asks what properties are necessary and sufficient for the qualitative behavior of the flow comprising *all* solutions of a given ODE to survive a small perturbation to the vectorfield defining it. Here "survive" implies that the flows of the original and perturbed system must be topologically equivalent (homeomorphic). While subtle technical questions remain on function space topologies and norms that define "small" and "equivalent," this approach launched a grand project to describe not only structurally stable systems, but also degenerate or *bifurcation* points at which arbitrarily small perturbations can produce qualitatively different behaviors. The classification and *universal unfoldings* of such points provided for the first time lists of behaviors that one might expect when studying families of ODEs or maps depending on one or more (control) parameters. In the special case of gradient systems (vectorfields derived from a potential function), this culminated in René Thom's catastrophe theory [Thom, 1975], which is where our story started. But we have got ahead of ourselves again.

The Andronov work led to detailed studies of specific nonlinear oscillators, mostly with radio and electronic applications, summarized in a classic text [Andronov, Vitt & Khaiken, 1966]. (Vitt's name was missing from the title page of the first (1937) Russian edition, appearing only in the second with the enigmatic prefatory note that it had been omitted "by an unfortunate mistake." In fact the "mistake" was Vitt's death in a prison camp in Kolyma, Siberia in the winter of 1936-7.[1]) This book focussed on planar ODEs, starting with conservative (Hamiltonian) systems, moving on to dissipative systems and discussing bifurcations of fixed points and limit cycles, including global (homoclinic) bifurcations, and, unlike most of the Western mathematical literature, containing practical examples.

A more abstract approach to dynamical systems developed from the mid 1930's onward at Moscow University, gaining attention outside the USSR via the translation of the 1946 book [Nemytskii & Stepanov, 1960] introduced by Lefshetz, who had himself published a key text a few years earlier [Lefschetz, 1957]. It was in [Nemytskii & Stepanov, 1960] that a strange attractor, in the form of the solenoid construction, made perhaps its first textbook appearance. The work of Anosov, Arnold and Sinai grew out of this "Moscow school" in the 1950's and simultaneously Peixoto generalized the Andronov–Pontryagin results to flows on two-dimensional manifolds [Peixoto, 1962]. His theorem states that a flow on a compact

[1] I am indebted to an article by C. Bissell in the Times Higher Educational Supplement of January 28th, 1994 for this information.

two-dimensional manifold is structurally stable if and only if it has a finite number of fixed points and periodic orbits, all of which are hyperbolic, there are no orbits connecting saddle points, and the nonwandering set consists of fixed points and periodic orbits alone.

Stephen Smale brought topological ideas to these problems in the late 1950's and began to generalize to $n > 2$ dimensions, defining gradient-like Morse–Smale systems[2] and seeking structural stability results. Among other things he conjectured that a system is structurally stable if and only if it is Morse–Smale. On hearing of this, Norman Levinson drew Smale's attention to a brief, very dense paper on the periodically-forced van der Pol equation that had emerged from a remarkable collaboration between two Cambridge mathematicians motivated by the British effort to develop radar in the second World War [Cartwright & Littlewood, 1945]. Levinson had himself worked on a simplified version [Levinson, 1949], and he suggested that it might provide a counterexample in the form of a structurally stable ODE with infinitely many periodic orbits. This led to Smale's discovery of the horseshoe in 1960, on the Leme beach of Rio de Janiero, while visiting Peixoto's Institute of Pure and Applied Mathematics. The story has now been told numerous times: in Smale's own words [Smale, 1980], in the biography [Batterson, 2000], and in [Diacu & Holmes, 1996]. The contributions of Cartwright and Littlewood are, however, not as well known: see [Cartwright, 1972; McMurran & Tattersall, 1996]. Mary Cartwright gave her paper at a conference in Southampton during my second year there; had I read the appropriate notices, I would have learned of this in time to hear it. Perhaps we should all take more time to explore our environments,

Smale's work appeared after a considerable delay and in an obscure publication [Smale, 1965], and became widely known only after a survey article was published [Smale, 1967], but that article, now a classic, would launch a thousand PhD theses, including that of at least one of the dedicatees of this paper. Jürgen Moser subsequently gave a beautiful exposition of the horseshoe [Moser, 1973], providing explicitly-testable criteria to prove its presence in two-dimensional maps and explaining clearly how the presence of dense orbits precludes the existence of additional integrals of motion. He thereby showed that the problem for which King Oscar's prize was awarded was in essence insoluble. A nice pictorial account of the horseshoe has recently appeared [Shub, 2005].

It is interesting to point out a footnote in [Cartwright & Littlewood, 1945] which remarks that the authors' "faith in [their] results was at one time sustained only by the experimental evidence that stable subharmonics of two distinct orders did occur," referring to [van der Pol & van der Mark, 1927] and (implicitly) to [Birkhoff, 1932]. Smale was almost certainly ignorant of this work and of [Poincaré, 1890], but in proving that diffeomorphisms containing transverse homoclinic points possess nearby hyperbolic invariant sets on which the dynamics is conjugate to a shift on a finite alphabet of symbols, he completed the story that Poincaré had begun over seventy years earlier, connecting ODEs and deterministic maps with probabilistic Markov processes and showing that, in a deep sense, their orbits are indistinguishable. This is now referred to as the Smale–Birkhoff homoclinic theorem [Guckenheimer & Holmes, 1983]. The full story of the van der Pol equation remains to be told, although significant progress was made on the piecewise-linear Levinson version [Levi, 1981], and the existence of strange attractors and invariant measures has been proved for related problems [Wang & Young, 2002]. Also, new results on the original cubic equation have recently been obtained using multiple time scale analysis and unfolding methods [Guckenheimer *et al.*, 2002; Bold *et al.*, 2003].

A final trip back to the USSR will close this partial early history. Most of the advances in structural stability and bifurcation theory, including those of Peixoto and Smale, were made from a topological perspective. The classical analytical tools of perturbation theory had not been used extensively in dynamical systems, apart from the averaging theory of Krylov and Bogoliubov, as generalized by [Hale, 1969] and others. However, in the early 1960's in Moscow regular pertubation methods were used to prove the existence of transverse homoclinic orbits to periodic motions in periodically-forced oscillators [Melnikov, 1963] and in two- and three degree-of-freedom Hamiltonian systems [Arnold, 1964]. This provided the final link

[2] A Morse–Smale system has a finite set of fixed points and periodic orbits, all of which are hyperbolic and all of whose stable and unstable manifolds intersect transversely, but no other nonwandering or recurrent points.

in a chain of methods and results that allows one to prove the existence of chaotic invariant sets in specific ODEs. Since then, "Melnikov's method" has been significantly extended, to multi- and even infinite dimensional systems (PDEs) [Wiggins, 1988; Holmes & Marsden, 1981], and related ideas have been used to approximate Poincaré return maps near homoclinic orbits to equilibria and find similar chaotic sets [Silnikov, 1965]. In a pleasing return to the origins of Poincaré's work, it has been suggested that heteroclinic connections among unstable n-body orbits might provide routes for low energy space missions [Koon *et al.*, 2000].

Some readers may be wondering how the famous discovery of chaos in a three-dimensional ODE modeling Rayleigh–Bénard convection of [Lorenz, 1963] fits into this story. In a strict sense, it does not, although in presenting his remarkable discovery of sensitive dependence on initial conditions Lorenz appealed to Birkhoff's work and, thanks to a perceptive reviewer, that of [Nemytskii & Stepanov, 1960]. But it was not until 1971, when Lorenz heard Ruelle speak on the proposal of [Ruelle & Takens, 1970] that structurally stable strange attractors and the solenoid in particular might describe turbulence, that connections began to be made. See [Lorenz, 1993] for the full story and a nice introduction to chaos, and, for a view from the mathematical side [Ruelle, 1991].

In the period between Smale's construction of the horseshoe and its publication, chaos was independently discovered by a second scientist. In 1960–64 Yoshisuke Ueda was a graduate student in Electrical Engineering at Kyoto University working on a periodically-forced mixture of the van der Pol and Düffing equations. While using an analog computer to compare perturbative approximations of periodic solutions with exact ones, he discovered motions that were neither periodic nor quasiperiodic. However, his advisor, C. Hayashi, was sceptical and, while these solutions were mentioned as "complicated phenomena," in Ueda's PhD thesis, they did not appear in journal form until considerably later [Hayashi *et al.*, 1970; Ueda *et al.*, 1973]. These papers include the first rather accurate pictures of homoclinic tangles in specific ODEs. I am personally indebted to Professor Ueda for sending me additional computations of stable and unstable manifolds for Düffing's equation in the late 1970's, while I was trying to check estimates for the appearance of homoclinic tangencies derived by Melnikov's method; versions of these figures subsequently found their way into [Guckenheimer & Holmes, 1983]. For more on Ueda's work, with reprints of key papers including those cited above, see [Ueda, 2001].

2.2. *Some more recent history: Unfolding local behavior*

Poincaré's recognition that stable and unstable invariant manifolds could "organize" dynamical behavior had important implications for the study of local behavior, especially near degenerate equilibrium points. The Hartman–Grobman theorem justifies studying local solutions near hyperbolic equilibria and periodic orbits via linearized flows and return maps, but if one or more eigenvalues have zero real part (for an ODE) or unit modulus (for a map), simple examples such as

$$\dot{x}_1 = \alpha x_1^3, \quad \dot{x}_2 = -x_2 + \mathcal{O}(|x_j|^2) \tag{1}$$

show that the behavior of a nonlinear system can differ qualitatively from that of its linearization for arbitrarily small $|x_j|$. The key idea that in studying such equilibria one need only examine higher order terms in the "degenerate" directions by restricting to an invariant *center* manifold tangent to the eigenspaces, was arrived at near-simultaneously [Pliss, 1964; Kelley, 1967], and the whole framework of invariant manifolds was thereafter generalized to include invariant sets other than fixed points [Hirsch *et al.*, 1977]. A related theory was developed independently in [Fenichel, 1971], and this has now been extended and applied to singularly-perturbed (fast-slow) systems [Jones, 1994; Guckenheimer, 2002]. I first learned of the practical power of these ideas from the monograph [Carr, 1981], which showed how to iteratively compute Taylor series approximations to local center manifolds. The finite (albeit often lengthy) computation necessary to achieve this is essentially a coordinate change that straightens the stable, unstable and center manifolds and decouples the hyperbolic (exponentially stable and unstable) modes.

Center manifold theory has also been extended to stochastic ODEs [Boxler, 1991; Arnold, 1998] and inertial manifolds have been shown to exist for certain infinite-dimensional evolution equations [Constantin *et al.*, 1989; Temam, 1997]. These are essentially global center-unstable manifolds with a finite set of weakly stable modes. This extensive body of invariant manifold theory provides a rigorous foundation for such physically-based notions as the "slaving principle," which posits that a (possibly infinite) set of modes in an extended system is driven, or determined, by a finite (and often small) subset of them [Haken, 1983].

In determining the behavior near a degenerate equilibrium, even after reduction to a d-dimensional center manifold, one must ostensibly consider the coefficients of every term in a vector-valued Taylor series up to, say, the third order: a matter of 18 for $d = 2$ and 54 for $d = 3$! However, just as center manifolds reduce the number of state variables that must be considered, normal form theory reduces the number of coefficients. A *normal form* is a nonlinear version of the similarity transformations that put matrices into Jordan or diagonal form: it retains key properties of the object in question (the spectrum of eigenvalues and their algebraic and geometric multiplicities, in the matrix case) while shedding less important ones (the original coordinate representation). Thus, in the $n \times n$ case one drops from n^2 matrix coefficients to n. The normal form theory of dynamical systems provides a systematic framework in which to perform near-identity *nonlinear* coordinate changes that successively remove "nonresonant" terms that do not influence the qualitative behavior at each order. These ideas originally arose in celestial mechanics: a field long-interested in clever coordinate choices.

However, it soon became clear that normal forms offer far more than simplification of the functions defining degenerate vectorfields; they also allow the introduction of "minimal" perturbations that *unfold* the degeneracy, just as one might perturb a matrix to split a real eigenvalue of multiplicity two into a pair of distinct real or complex ones by adding a single new parameter. The notion of the *codimension* of a degenerate singularity — roughly the number of parameters required to reveal *all* topologically-distinct phase portraits that perturbations can elicit — had been used in [Sotomayor, 1973] and the generic codimension-one local bifurcations — the saddle-node and Hopf — were well known. (The n-dimensional generalization of the planar Hopf bifucation had been analyzed years earlier [Hopf, 1942]: see [Marsden & McCracken, 1976], which contains a translation of and commentary on Hopf's paper.) At around the same time Arnold made important contributions to unfolding theory [Arnold, 1972].

Shortly thereafter the first unfoldings of a codimension two singularity, the double-zero eigenvalue, appeared [Takens, 1974; Bogdanov, 1975]. (In fact it was the chance discovery of Takens' duplicated set of lecture notes from Utrecht in the ISVR library that led to my second paper in dynamical systems [Holmes & Rand, 1978].) Actually Takens did not only unfold the degenerate fixed point of the vectorfield and find branches of Hopf, homoclinic and saddle-node bifurcations emanating from it (as Bogdanov did independently at around the same time), he also showed that the time-1 flow maps of these vectorfields approximated the Poincaré maps for periodically-forced nonlinear oscillators having fixed points with eigenvalue one and multiplicity two, and he argued on generic grounds and by reference to [Smale, 1967] that near the homoclinic bifurcation curve one would expect transverse homoclinic points and chaos for the maps. He was probably unaware that Ueda had already described examples of this [Ueda *et al.*, 1973].

In the past 30 years a rather complete analysis of local bifurcations of codimension two has been carried out (see e.g. [Guckenheimer & Holmes, 1983; Kuznetsov, 2003]), and unfoldings in constrained contexts including Hamiltonian vector fields [van der Meer, 1985] and systems equivariant under symmetry groups [Golubitsky & Schaeffer, 1985; Golubitsky *et al.*, 1988] have been studied.

2.3. *Four central themes*

A number of common themes have emerged in the study on nonlinear dynamical systems. Here I note four; readers can probably supply additional ones.

2.3.1. *Dimension reduction*

I start with an observation that explains my choice of subtitle. I have already noted that the reduction to center, center-unstable and inertial manifolds affords a vast conceptual simplification, and, coupled with effective computational methods (symbolic manipulation and computer algebra), expansion of analytical and predictive power. More globally, the topological equivalence between orbits of chaotic hyperbolic sets and subshifts of finite type effectively reduces the study of a continuum of solutions to a combinatorial problem involving sequences drawn from a finite alphabet of symbols. Less is more! However, even for codimension two bifurcations, unfolding theory remains incomplete due partly to the presence of homoclinic tangles and persistent nontransverse intersections or wild hyperbolic sets [Palis & Takens, 1993]. More is different!

2.3.2. *Judicious linearization*

Stability of fixed points and periodic orbits in nonlinear systems is typically proven or disproven by studying a linearized system and appealing to the Hartman–Grobman theorem. But the local theory, in which one uses the fact that the linearized flow or mapping dominates the dynamics near hyperbolic invariant sets, extends to the study of global behavior. Smale's major insight in his construction of the horseshoe (as clearly revealed by [Moser, 1973] was that, *restricted to the invariant set*, the horseshoe mapping is a perturbation of a linear map. This allows the proof of stable and unstable manifolds for the entire set at one fell swoop, not just for fixed points or periodic orbits within it, and also yields structural stability. In a related manner, in studying homoclinic bifurcations, one uses the linearized flow near a hyperbolic saddle to provide an explicit estimate of a local flow map, combined with an affine approximation to the finite-time flow map that returns orbits to the saddle's neighborhood, to estimate the full return map [Silnikov, 1965; Guckenheimer & Holmes, 1983].

2.3.3. *Good coordinates*

The normal form theorem provides a reduction in complexity near a fixed point, but global coordinate changes were introduced earlier, in the form of action-angle variables and Hamilton–Jacobi (HJ) theory in classical mechanics [Goldstein, 1980]. An n degree-of-freedom Hamiltonian system possessing n independent constants of motion is completely integrable and the n configuration and conjugate momentum variables can in general be transformed into action-angle coordinates (I_j, θ_j) so that the actions I_j are conserved and the Hamiltonian takes the form $H(I_1, \dots, I_n)$. Hamilton's equations are then trivially integrable:

$$\begin{aligned} \dot{I}_j = 0 &\Rightarrow I_j(t) \equiv I_j(0). \\ \dot{\theta}_j = \frac{\partial H}{\partial I_j}(I_1, \dots, I_n) &\Rightarrow \theta_j(t) = \theta_j(0) + \Omega(\mathbf{I})t. \end{aligned} \tag{2}$$

These coordinates make the topological structure of phase space clear: it is filled with an n-parameter family of invariant tori, on each of which the flow is quasiperiodic. However, finding the transformation that puts the system in these coordinates is far from simple, even if the required set of integrals can be found, for the HJ equation is, in general, a nonlinear PDE. (The transformation yielding the form (2) may not apply globally — singular submanifolds, the stable and unstable manifolds of saddle type fixed points, periodic orbits and invariant tori — may divide the phase space into open sets within which different transformations are necessary. The simple pendulum, with its distinct classes of libration (oscillation) and rotation solutions divided by a saddle separatrix provides the simplest example.)

Since solutions of the HJ equation are hard to come by, most textbooks restrict themselves to linear systems, but in the case of weakly-nonlinear or otherwise near-integrable systems, perturbation methods based on Hamilton–Jacobi theory, couched in the language of Lie transforms, can be used to generate *approximate* integrals [Lichtenberg & Lieberman, 1983]. These provide formal approximations to solutions, and can usefully reveal global solution structures. It was in essence this method that Poincaré was following and extending in his 1890 memoir. As we have seen, he ended by exhibiting a two degree-of-freedom system that is nonintegrable, and for which series solutions to the HJ equation, or for coordinate transformations to action-angle variables, diverge.

The opposite case, that series solutions *converge* for certain initial conditions, is the subject of the famous Kolmogorov–Arnold–Moser (KAM) theorem, which proves that a metrically-large set of invariant tori survive for sufficiently small perturbations of integrable systems [Arnold, 1978; Gallavotti, 1983]. This was first announced at the International Congress of Mathematicians in 1954 [Kolmogorov, 1957], but the full proof was not published for some time, and improved versions and generalizations are still appearing. The surviving tori occupy "thick" Cantor sets, and are separated by chaotic zones in which (at least generically) transverse heteroclinic cycles to lower-dimensional hyperbolic (saddle type) tori exist. In the three-dimensional constant energy manifolds of two degree-of-freedom systems, the invariant tori separate regions of phase space, preventing solutions from traveling far from their initial action values (it was such a result that Poincaré incorrectly claimed in the first version of [Poincaré, 1890]). However, for $n \geq 3$-degrees-of-freedom, the n-tori do not separate the $2n-1$-dimensional energy manifolds into disjoint sets, and solutions can diffuse throughout phase space, as first realized by

[Arnold, 1964]. Again, see [Diacu & Holmes, 1996] for the history.

2.3.4. *Structural stability and generic properties*

I end this section with a plea for appreciation of the abstract viewpoint adopted by Smale and his students. While applied scientists in general, and applied mathematicians in particular, are faced with specific families of differential equations derived from the details of the phenomena they study, the abstract questions of which classes of systems are structurally stable, and which families exhibit generic bifurcations are nonetheless relevant. Not only do answers to them provide a unifying setting in which to observe and deduce relationships among models derived in different fields, or describing quite different phenomena, but they provide classifications of typical properties and behaviors, and thus are especially useful in model building. Data is inevitably noisy and imprecise, and if a model or family of models is not robust under small perturbations, then its value is surely questionable. Of course, in appealing to these ideas one must first determine the appropriate spaces of systems (conservative, dissipative, symmetric, etc.), and whether the relevant notion of genericity is topological (open dense sets) or metrical (sets of full measure), since these do not coincide.

3. Some More Recent Applications and Extensions of the Theory

I now remark briefly on some extensions of dynamical systems methods, and on ideas from other areas of mathematics and science that have influenced and interacted with dynamical systems theory.

3.1. *Infinite-dimensional evolution equations*

I have already mentioned the extension of invariant manifold methods to PDEs [Constantin *et al.*, 1989; Temam, 1997]. Local bifurcations of other evolution equations, including differential delay equations, have also been studied by center manifold and normal form methods [Faria & Magalhães, 1995]. Delay equations arise in numerous areas of mechanics, including control theory (due to sensor, processing and actuator delays), and in milling and other machining processes in which a tool follows its previous path cyclically. For a recent example involving drilling, see [Campbell & Stone, 2004]. Machining models in which contacts between tool and workpiece are intermittent also give rise to hybrid dynamical systems [Szalai *et al.*, 2004].

3.2. *Completely integrable partial differential equations*

The idea of globally-defined transformations that yield "good" coordinates was significantly extended to nonlinear wave equations by the discovery of completely integrable PDEs, starting in the West with work on solitons in the Korteweg–deVries (KdV) equation [Gardner *et al.*, 1967]. The inverse scattering transform provides a generalization of action-angle variables in the form of the Lax operator, whose spectrum of eigenvalues, invariant under the flow of the PDE, plays the role of the conserved actions in the classical theory. One finds families of invariant tori and a beautiful topological description of the infinite-dimensional phase space ensues. For an introduction, see [Drazin & Johnson, 1989].

Kruskal's work on the KdV equation was motivated by earlier numerical studies of anharmonic oscillator chains [Fermi *et al.*, 1955] using the Los Alamos computer MANIAC, which had indicated close quasiperiodic returns to initial data rather than an ergodic mixing of all the modes. In spite of Truesdell's (justified, albeit harsh) criticism of the authors' confusions regarding continuum limits and their evident ignorance of earlier analytical work and of the importance of shock waves [Truesdell, 1984], this work motivated at least part of the integrable PDE movement. Good mathematics, and good science, can be stimulated by questionable or even flawed work.

3.3. *Low dimensional models of turbulence*

I now turn to an opposite sort of limit from integrable PDEs: highly dissipative systems whose behavior is governed by (relatively) few modes. In turbulence a lot of things happen, but not everything: many fluid flows of practical and technological importance organize themselves into *coherent structures*: large scale concentrations of vorticity or shear of which hurricane cloud patterns in satellite weather maps are a well-known example. In [Lumley, 1967] it was suggested that principal components analysis (PCA) or the proper orthogonal (Karhunen–Loève) decomposition (POD) might be used to provide unbiased descriptions of such coherent structures. Briefly, one derives, from an ensemble of spatial flow observations or CFD simulations, an ordered subset of mutually orthogonal *empirical eigenfunctions* that are optimal in that finite linear combinations of them capture a greater fraction of kinetic energy (L^2 norm) than any other linear representation of the same order. PCA in various incarnations had been used extensively in the compression, analysis and presentation of data, but it was only in the mid 1980's that it was combined with Galerkin projection into subspaces spanned by empirical eigenfunctions, to yield low dimensional (ODE) truncations of PDEs [Sirovich, 1987; Aubry *et al.*, 1988]. For a (relatively) recent introductory survey see [Holmes *et al.*, 1996], and for a deeper approach to fluid mechanics from a dynamical viewpoint, [Gallavotti, 2002].

The work of [Aubry *et al.*, 1988] showed that attracting heteroclinic cycles, rendered structurally stable due to the translation and reflection symmetries of the Navier–Stokes equations over a flat plate (or in a circular pipe), might play an important role in describing the burst-sweep regeneration cycle of boundary layer turbulence. Subsequent studies with John Guckenheimer (e.g. [Guckenheimer & Holmes, 1988; Armbruster *et al.*, 1988]) of such O(2)-equivariant heteroclinic cycles, periodic- and quasi-periodic orbits and their bifurcations emphasized the importance of working in the group-theoretic [Golubitsky & Schaeffer, 1985] framework for both POD reduction [Aubry *et al.*, 1988; Smith *et al.*, 2005b] and analysis of the resulting dynamical systems [Holmes *et al.*, 1996]. However, even when fluid flows are energetically dominated by as few as two or three modes, it has become clear that many more modes must be included to reproduce the correct dynamics and modal energy budgets in general [Gibson, 2002]. Nonetheless, careful modeling of energy transfer to neglected modes, and inclusion of key modal interactions, can yield O(10)-dimensional models of flows in small domains that reproduce key behaviors, and help explain them [Smith *et al.*, 2005a].

3.4. *Stochastic differential equations*

The study of stochastic models originated in probability theory [Gardiner, 1985] and developed independently of dynamical systems theory, which focused on deterministic ODEs and mappings, although, as noted earlier, the ergodic theory of iterated maps has played an important role in the abstract theory [Katok & Hasselblatt, 1995]. Indeed, the Sinai–Ruelle–Bowen (SRB) invariant measure is a central characteristic of a strange attractor [Ruelle, 1989]. But it was only with the work of a group around Ludwig Arnold that the ideas of invariant manifolds, bifurcations, and unfolding theory began to impact the study of stochastic ODEs (e.g. [Boxler, 1991]). The difference in viewpoint is nicely expressed in the two books [Arnold, 1974, 1998]. Thus far, most of the applications have been to low dimensional problems with one or two degrees-of-freedom (see [Sri Namachchivaya & Ramakrishnan, 2003] for an example that involves symmetry groups and a Hamiltonian limit), but it will be important to connect these with mean field and other analyses of many degree-of-freedom systems done from the viewpoint of statistical physics.

3.5. *Numerically-assisted proofs and integration algorithms*

Shortly after he learned of the Lorenz equations in the mid 1970's R. F. (Bob) Williams proposed how a numerically-assisted proof of the attractor might be given. Although this did not come for almost twenty years [Mischaikow & Mrozek, 1995; Hastings & Troy, 1996; Tucker, 1998], the use of computers in rigorous argument, as well as illustration and simulation, is becoming more common. Perhaps the most famous example in dynamical systems is Lanford's proof of the deductions of [Feigenbaum, 1980] on period-doubling sequences [Lanford, 1982], but now, for example, sophisticated algorithms are being developed to implement C. Conley's index theory [Kaczynski *et al.*, 2004].

The original algorithms of Doedel and Keller for following branches of equilibria and detecting local bifurcations have been significantly extended [Doedel *et al.*, 1997] and algorithms for the computation of multidimensional invariant manifolds, including invariant tori, are being developed, e.g. [Krauskopf & Osinga, 1999]. Guckenheimer and his students have extended their DsTool software to hybrid systems [Back *et al.*, 1993], and more recently have applied automatic differentiation and computer algebra to revive classical power series approximations to ODE solutions [Phipps, 2003]. Their algorithms are more economical and more accurate than direct integrations, especially in the case of stiff, singularly-perturbed systems.

3.6. *Nonlinear mechanics of solids and structures*

Much of the ENOC-05 conference concerns solid and structural mechanics, and so I will mention only one of the instances of many applications. Advances in both theory and numerical methods to compute homoclinic and heteroclinic orbits have been used to study buckled states of long, slender rods, building on Kirchhoff's recognition that, in body coordinates, the equilibrium equations for an inextensible rod are identical to the dynamical equations of a rigid body [Love, 1927]. There is a substantial literature on "chaotic buckled states," and a nice recent example applies these ideas to "kinks" in the tendrils that support climbing plants [McMillen & Goriely, 2002].

3.7. *Nonlinear dynamics in biology*

It is becoming a commonplace that, if the 20th was the century of physics, the 21st will be the century of biology, and, more specifically, mathematical biology. In fact it is over 50 years since Hodgkin and Huxley proposed their model for the generation and propagation of an action potential: the neural "spike" [Hodgkin & Huxley, 1952]. This has led to a substantial literature in mathematical neuroscience, mostly focussing on the dynamics and bifurcations of single cells subject to externally-imposed currents and small networks such as those of central pattern generators. All the methods of dynamical systems theory, and singular and regular perturbation theory, have been brought to bear on the nonlinear ionic current models of the type introduced in [Hodgkin & Huxley, 1952], and various simplifications to linear integrate, fire and reset models, and phase models have been proposed [Hoppensteadt & Izhikevich, 1997]. A recent application of phase response curves to the study of the response to stimuli of a brainstem area involved in cortical neurotransmitter release suggests that these methods may also assist in the development of brain science [Brown *et al.*, 2004], but a large gap remains to be filled between the relatively fine scales of single cells and small circuits, and models of brain areas involved in decision-making and motor planning. The recent work of [Cai *et al.*, 2004a; Cai *et al.*, 2004b] on the visual cortex — a rather well-characterized region — shows promise for the development of general kinetic and averaging theories that might preserve sufficient low level detail in reduced higher level models of brain functions such as decision-making.

I will not comment on applications in genomics and molecular biology (areas about which I know very little), but will close by noting that dynamical systems theory, with its natural relations to classical mechanics and control theory, might also play an central role in another project in which one wants to understand how neural spikes give rise to behavior: animal locomotion [Dickinson *et al.*, 2000]. Ending, as I began, on a personal note, I draw the reader's attention to a substantial forthcoming review article [Holmes *et al.*, 2006].

4. Epilogue, and Part of What Is Missing

This article surveys a large field, and I have been able to include only a few topics and examples, mentioning some of them, such as hybrid dynamical systems, only in passing. Much of importance is missing or slighted: nonlinear methods for time series data analysis and phase space reconstruction; experimental techniques informed by nonlinear dynamics; control of chaos and more general relations to control theory; hybrid and nonsmooth dynamical systems; numerical methods for bifurcation and computation of invariant manifolds; asymptotic methods, and many areas of application. Fortunately, the other plenary lectures and the minisymposia at ENOC-05 cover a substantial number of these topics. I apologise to the other speakers and organizers for not noting their work in areas that I know, and thank them in advance for informing me on much more of which I am ignorant.

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