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Reduction of three-dimensional, volume-preserving flows with symmetry

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Abstract. We develop a general, coordinate-free theory for the reduction of volume-preserving flows with a volume-preserving symmetry on three-manifolds. The reduced flow is generated by a one-degree-of-freedom Hamiltonian which is the generalization of the Bernoulli invariant from hydrodynamics. The reduction procedure also provides global coordinates for the study of symmetry-breaking perturbations. Our theory gives a unified geometric treatment of the integrability of three-dimensional, steady Euler flows and two-dimensional, unsteady Euler flows, as well as quasigeostrophic and magnetohydrodynamic flows.

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1. Introduction

In this paper we study three-dimensional flows which admit a continuous symmetry. Motivated by applications to incompressible fluid flows, both the flow and symmetry are assumed to be volume preserving. The three main questions we are interested in answering are the following. First, under what conditions can we construct a first integral, i.e. a quantity that is preserved by the flow? Secondly, is it possible to reduce the dimension of the problem by 1 so that the reduced two-dimensional flow also preserves some volume? Thirdly, can we use this reduction to construct coordinates in which symmetry-breaking perturbations are conveniently studied?

It turns out that if the flow admits a symmetry group that has a volume-preserving infinitesimal generator w, and the flow is not everywhere tangent to the orbits of the symmetry group, then it always admits a nontrivial invariant. This invariant B can be constructed explicitly based on the vector field v that generates the flow, the volume Ω that is preserved by the flow, and the generator w of the symmetry group (see formula (8)). This result generalizes Bernoulli's theorem in hydrodynamics to arbitrary volume-preserving flows on three-manifolds.

With the above invariant at hand, it is tempting to reduce the three-dimensional flow just by simply restricting the vector field v to the level surfaces of B. This procedure, however, has several disadvantages. First, the reduced flow in general does not conserve any two-dimensional volume, i.e. there is no symplectic structure with respect to which it is Hamiltonian (see section 7 for an example). As a result, there is no systematic way to derive the reduced equations. In addition, the reduced equations admit no nontrivial

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invariant besides B, which is now a single constant for the whole reduced flow. Finally, since the construction of B depends on the given volume-preserving field v, the reduction procedure depends heavily on v. This makes it impossible to study the structure of the reduced problem for a class of flows with the same symmetry, as the corresponding vector field v is different for each member of the class.

To remedy all these problems, we develop a reduction procedure which bears some similarities with the symplectic reduction of Hamiltonian systems with symmetry (see, e.g. Abraham and Marsden [1]). The reduced phase space here is not a level surface of the invariant B, but rather the space of orbits of the underlying symmetry group action. More precisely, it is the quotient space M/G, where M is the three-dimensional ambient manifold and G is the Lie group generating the symmetry. The orbit space M/G turns out to inherit a natural symplectic structure from the volume form Ω, and this structure only depends on the symmetry, not on the given field v. This enables one to study whole classes of symmetric flows on the same reduced phase space. The projection of the flow on the reduced phase space is now Hamiltonian, hence the reduced flow is also volume preserving. Remarkably, the underlying Hamiltonian is just the generalized Bernoulli invariant B described above.

In applications, one is often concerned with the effect of perturbations on the original flow. These perturbations may not preserve the volume, nor break the symmetry of the unperturbed vector field. To study the fate of unperturbed structures, one needs an appropriate coordinate representation for the flow which facilitates the application of perturbation methods. Such methods include Melnikov-type methods for the continuation of homoclinic orbits, or KAM-type methods for the continuation of invariant tori. Our reduction procedure renders these coordinates as a side result, as we show in section 5. The original flow is represented in a set of (y, s) coordinates, where y denotes a coordinate on the reduced phase space, and s labels elements of the symmetry group G. The (y, s) coordinates have two main advantages: their evolution depends only on y before symmetry-breaking perturbations, and their construction depends solely on the symmetry group G and the generator w of its action. Finally, these coordinates highlight the relation of our reduction procedure to contact geometry, as we show in section 5.

The volume-preserving reduction described in this paper is purely geometric and avoids the usage of local coordinates on the underlying three-manifold M. It generalizes and extends the local, coordinate-dependent theory in Mezić and Wiggins [22] (see also Sposito [25] for an extension) by rendering the reduced phase space with its symplectic structure, as well as the reduced Hamiltonian B. This general approach enables us to give a unified, geometric treatment of the integrability of several classes of fluid flows. These flows include three-dimensional, steady Euler flows and steady magnetohydrodynamic flows, as well as two-dimensional, unsteady Euler flows and quasigeostrophic flows. In the case of two-dimensional unsteady flows, the role of the manifold M is played by the three-dimensional extended phase space of the variables (x, y, t), and the preserved volume is the 'space-time volume' $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t$.

2. Notation and definitions

In this section we collect the tools from the calculus on manifolds that we shall need later. In order to emphasize the similarities with symplectic reduction, we will use the notation customary in the theory of reduction of Hamiltonian systems with symmetry (see, for example Abraham and Marsden [1] or Marsden and Ratiu [18]).

Let M be a three-dimensional manifold on which closed differential forms are exact. The class of such manifolds includes, for example contractible manifolds, and in particular,

$\mathbb{R}^3$. If v is a smooth vector field on M and γ is a differential k-form on M with $0 \leqslant k \leqslant 3$, then the *inner product* of v and γ is the $k-1$-form $i_v\gamma$, which is defined at a point $x \in M$ as

$$i_v\gamma[x](u_1, \ldots, u_{k-1}) = \gamma[x](v(x), u_1, \ldots, u_{k-1})$$

for all $u_i \in T_xM$. Let N be another manifold and η be a differential k-form on N. Then any smooth map $f: M \to N$ can be used to define the *pull-back* of η to M as the k-form $f^*\eta$ given by

$$f^*\eta[x](u_1, \ldots, u_k) = \eta[f(x)](\mathrm{d}f_x u_1, \ldots, \mathrm{d}f_x u_k)$$

for all $u_i \in T_xM$. Clearly, $f^*\eta$ can only be nondegenerate if f is a submersion, i.e. its derivative $\mathrm{d}f_x$ is surjective at any point $x \in M$. Note that if η is a function (i.e. a zero-form), then we simply have $f^*\eta = \eta \circ f$.

If f is a diffeomorphism between M and N, then the *push-forward* of any k-form γ on M can be defined as a k-form $f_*\gamma$ given by

$$f_*\gamma[y](v_1, \ldots, v_k) = \gamma[f^{-1}(y)](\mathrm{d}f_y^{-1}v_1, \ldots, \mathrm{d}f_y^{-1}v_k)$$

for all $v_i \in T_yN$.

If the vector field v is smooth, it generates a local flow on M which we denote by $F^t: M \to M$. Then the *Lie derivative of a function* $f : M \to N$ with respect to v is defined as

$$L_v f(x) = \frac{\mathrm{d}}{\mathrm{d}t} f(F^t(x))|_{t=0}.$$

The *Lie bracket* $[v, w]$ of two smooth vector fields v and w on M is the unique vector field which satisfies

$$L_{[v,w]} = L_v \circ L_w - L_w \circ L_v.$$

Furthermore,

$$L_v w = [v, w] = -L_w v$$

is called the *Lie derivative of* w *with respect to* v. Finally, the *Lie derivative of a form* γ with respect to v is defined as

$$L_v\gamma = \frac{\mathrm{d}}{\mathrm{d}t}(F^{t*}\gamma)|_{t=0}.$$

A useful formula for this derivative is given by

$$L_v\gamma = \mathrm{d}i_v\gamma + i_v\mathrm{d}\gamma \tag{1}$$

where the operator d refers to the exterior derivative (see, e.g. Abraham and Marsden [1]). Another formula that we will use is

$$i_{[v,w]}\gamma = L_v i_w\gamma - i_w L_v\gamma \tag{2}$$

(see Abraham *et al* [2, p 445]).

3. Volume-preserving flows with symmetry

Let Ω be a volume form on M, i.e. a differential form that restricts to a non-degenerate three-form $\Omega[x](\cdot, \cdot, \cdot): T_xM \times T_xM \times T_xM \to \mathbb{R}$ at any point $x \in M$. Let v be a smooth vector field on M. We say that v is *volume preserving* if

$$L_v \Omega = 0.$$

This definition is equivalent to

$$\mathrm{div}_\Omega\, v = 0$$

where the divergence of v with respect to the volume Ω is defined through the formula

$$L_v \Omega = (\mathrm{div}_\Omega\, v)\Omega. \tag{3}$$

If F^t is the flow generated by v, then we call F^t volume preserving if v is volume preserving. In that case, we have

$$F^{t*}\Omega = \Omega. \tag{4}$$

In this paper we are interested in volume-preserving flows that admit a symmetry. To this end, we consider a one-parameter family of diffeomorphisms on M denoted by $g^s: M \to M$ with $s \in G$. Here G is a one-dimensional Lie group, which is assumed to be $\mathbb{R}$ or S^1 for simplicity. We also assume that the vector field v is equivariant with respect to the action of this group, i.e.

$$v(g^s(x)) = \mathrm{d}g^s v(x) = i_v \mathrm{d}g^s(x) \tag{5}$$

for all $s \in G$. The condition of equivariance can also be written as

$$[v, w] = 0 \tag{6}$$

where the infinitesimal generator of the action of G is given by

$$w(x) = \frac{\mathrm{d}}{\mathrm{d}s} g^s(x)|_{s=0}.$$

We say that w generates a *volume-preserving symmetry* for v if w is a volume-preserving vector field, i.e.

$$L_w \Omega = 0. \tag{7}$$

In the following we will assume that (7) holds. Note that this implies $g^{s*}\Omega = \Omega$ for all $s \in G$.

The following theorem states that all volume-preserving flows with a volume-preserving symmetry admit an integral.

Theorem 3.1 (generalized Bernoulli theorem). *Suppose that w generates a volume-preserving symmetry for the volume preserving vector field v. Then:*

(i) *The flow generated by v admits a first integral $B: M \to \mathbb{R}$ which satisfies*

$$\boxed{\mathrm{d}B = -i_v i_w \Omega.} \tag{8}$$

(ii) $L_w B = 0$, *i.e. B is constant along the orbits of the vector field w.*

Proof. By our basic assumption on the manifold M, to prove the existence of a function B satisfying (8), it suffices to show that $-i_v i_w \Omega$ is closed, i.e. $\mathrm{d}i_v i_w \Omega = 0$. Using (1) we can write

$$\begin{aligned} \mathrm{d}i_v i_w \Omega &= \mathrm{d}i_v(i_w\Omega) = L_v(i_w\Omega) - i_v\, \mathrm{d}i_w\Omega = L_v(i_w\Omega) - i_v L_w \Omega + i_v i_w d\Omega \\ &= L_v(i_w\Omega) = i_{[v,w]}\Omega = 0 \end{aligned} \tag{9}$$

where we also used (2), (4), (7), and the fact that $\mathrm{d}\Omega$ four-form on a 3-manifold, hence it vanishes identically. To show that B is a first integral for both v and w, it is enough to observe that

$$L_v B = i_v \,\mathrm{d}B = -i_v i_v i_w \Omega = 0$$
$$L_w B = i_w \,\mathrm{d}B = -i_w i_v i_w \Omega = 0.$$

This completes the proof of the theorem. □

We note that in the case of $M = \mathbb{R}^3$ and $\Omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, we have $-i_v i_w \Omega(u) = (v \times w) \cdot u$ for any vector u. Hence formula (8) for the gradient of the invariant B takes the form

$$\nabla B = v \times w. \tag{10}$$

Our notation for first integral is motivated by the fact that for steady, incompressible, inviscid fluid flows with velocity v and vorticity $w = \nabla \times v$, the above theorem simplifies to the well-known Bernoulli theorem of fluid mechanics (see section 6). In fact, we can decompose B into the sum of kinetic and potential energy-type terms to make the analogy with the Bernoulli invariant clearer.

Proposition 3.2. *Let a one-form α and a function p on M be defined by*

$$\mathrm{d}\alpha = i_w \Omega \tag{11}$$
$$\mathrm{d}p = \tfrac{1}{2}\mathrm{d}i_v \alpha - L_v \alpha. \tag{12}$$

Then the invariant B of theorem 3.1 can be written as the sum of a kinetic energy-type term and a pressure-type term:

$$\boxed{B = \tfrac{1}{2} i_v \alpha + p.} \tag{13}$$

Proof. First note that both α and p are well defined since the right-hand sides of (11) and (12) are closed and hence exact by our assumption on M. Furthermore, from (11) and (12) we obtain

$$\mathrm{d}(\tfrac{1}{2} i_v \alpha + p) = \mathrm{d}i_v \alpha - L_v \alpha = -i_v \mathrm{d}\alpha = -i_v i_w \Omega$$

hence by (8), B and $\frac{1}{2} i_v \alpha + p$ are equal up to a constant. □

We close this section by noting that there is a degenerate case in which formula (8) of theorem 3.1 may give a trivial invariant, i.e. a constant. This happens when the vector field v is everywhere parallel to the generator w of the symmetry, hence the form $i_v i_w \Omega$ vanishes identically. In such cases trajectories of v are not confined to lower-dimensional level surfaces of B.

4. Reduction of volume-preserving flows

We will now use the presence of the symmetry to reduce our three-dimensional flow on M to a two-dimensional flow on an abstract two-manifold, the reduced phase space. The main result is that the reduced phase space can be endowed with a symplectic structure through the volume form Ω defined on M. The reduced flow is Hamiltonian with respect to this symplectic structure. Furthermore, the corresponding Hamiltonian is precisely the projection of the invariant B onto the reduced phase space.

Consider the Lie group G whose action is generated by the vector field w. The group action g^s is said to be *proper* if for any $s \in G$, the map $(s, x) \mapsto (x, g^s(x))$ is proper, i.e. the pre-image of any compact set in $M \times M$ under this map is compact in $G \times M$. The action g^s is said to be *free* if for all $x \in M$, the map $s \mapsto g^s(x)$ is one-to-one. Finally, the action is *regular* if it is proper and free.

We define the orbit space M/G as a set of equivalence classes of points that lie on the same orbit of the group action g^s. More precisely, we define the equivalence class containing $x \in M$ as

$$[x] = \{y \in M | \exists s \in G \colon g^s(y) = x\}$$

and the orbit space M/G as

$$M/G = \cup_{x \in M}[x].$$

The usual quotient projection $\pi\colon M \to M/G$ is defined as $\pi(x) = [x]$. We now recall a result which is well known in the theory of Lie groups.

Lemma 4.1. *If the group action $g^s\colon M \to M$ is regular, then M/G is a smooth, two-dimensional manifold and π is a submersion, i.e. $\mathrm{d}\pi_x\colon T_xM \to T_{\pi(x)}M/G$ is surjective for all $x \in M$.*

Proof. The lemma follows directly from an identical result for general Lie groups G and general manifolds M (see, e.g. Abraham and Marsden [1] or Olver [24]). □

By construction, the orbits of the group action g^s in M correspond to points in the orbit space M/G. By formula (6), the flow F^t commutes with the group action g^s. This fact will enable us to 'project' the flow F^t onto the orbit space M/G, which will therefore play the role of a reduced phase space. For this reduction to make sense, we have to argue that orbits of the full flow can be uniquely reconstructed from orbits of the reduced flow. The following two lemmas present the main elements for this argument.

First, we show that M/G can be endowed with a symplectic structure through the volume form Ω defined on M.

Lemma 4.2. *The two-form ω defined as*

$$\boxed{\pi^*\omega = -i_w\Omega} \tag{14}$$

is a symplectic form on the orbit space M/G.

Proof. First we argue that ω is well defined. Let $y \in M/G$ and $u_1, u_2 \in T_y(M/G)$. Since π is onto, there exists $x \in M$ such that $y = [x] = \pi(x)$. Furthermore, by lemma 4.1, $\mathrm{d}\pi_x$ is onto, hence there exists $v_1, v_2 \in T_xM$ such that $\mathrm{d}\pi_x\, v_i = u_i$. Then, by definition,

$$\omega[y](u_1, u_2) = -\Omega[x](w(x), v_1, v_2).$$

Now let $\bar{x} \neq x$ be another point such that $y = [\bar{x}] = \pi(\bar{x})$. This means that x and $\bar{x}$ lie on the same orbit of the group action g^s. As a result, there exists $\bar{s} \in G$ such that $g^{\bar{s}}(x) = \bar{x}$. This implies that $\pi(g^{\bar{s}}(x)) = \pi(x)$, from which we obtain the relation $\mathrm{d}\pi_{\bar{x}}\mathrm{d}g_x^{\bar{s}} = \mathrm{d}\pi_x$. This in turn implies

$$\mathrm{d}\pi_{\bar{x}}\mathrm{d}g_x^{\bar{s}} v_i = \mathrm{d}\pi_{\bar{x}}\bar{v}_i \tag{15}$$

for any two vectors $\bar{v}_1, \bar{v}_2 \in T_{\bar{x}}M$ which satisfy $\mathrm{d}\pi_{\bar{x}}\bar{v}_i = \mathrm{d}\pi_x v_i = u_i$. For ω to be well defined, the identity

$$\Omega[x](w(x), v_1, v_2) = \Omega[\bar{x}](w(\bar{x}), \bar{v}_1, \bar{v}_2) \tag{16}$$

must hold. To show this, we first observe that the volume preservation formula $g^{s*}\Omega = \Omega$ implies

$$\Omega[x](w(x), v_1, v_2) = \Omega[\bar{x}](\mathrm{d}g_x^{\bar{s}} w(x), \mathrm{d}g_x^{\bar{s}} v_1, \mathrm{d}g_x^{\bar{s}} v_2) = \Omega[\bar{x}](w(\bar{x}), \mathrm{d}g_x^{\bar{s}} v_1, \mathrm{d}g_x^{\bar{s}} v_2). \tag{17}$$

Now note that the kernel of the operator $\mathrm{d}\pi_{\bar{x}}$ is precisely $\mathrm{span}\{w(\bar{x})\}$. Thus for any two-dimensional subspace S that forms a direct sum $S \oplus \mathrm{span}\{w(\bar{x})\} = T_{\bar{x}}M$, we obtain that $\mathrm{d}\pi_{\bar{x}}|S \colon S \to T_y(M/G)$ is an isomorphism. Therefore, if $p_S \colon T_{\bar{x}}M \to S$ denotes the canonical projection on the first component of the direct sum, (15) shows that

$$p_S(\mathrm{d}g_x^{\bar{s}} v_i) = p_S(\bar{v}_i). \tag{18}$$

Introducing the canonical projection $p_G \colon T_{\bar{x}}M \to \mathrm{span}\{w(\bar{x})\}$ on the second component of the direct sum and using (18), we can write

$$\begin{aligned}
\Omega[\bar{x}](w(\bar{x}),\ \mathrm{d}g_x^{\bar{s}} v_1,\ \mathrm{d}g_x^{\bar{s}} v_2)
&= \Omega[\bar{x}](w(\bar{x}),\ p_S(\mathrm{d}g_x^{\bar{s}} v_1) + p_G(\mathrm{d}g_x^{\bar{s}} v_1),\ p_S(\mathrm{d}g_x^{\bar{s}} v_2) + p_G(\mathrm{d}g_x^{\bar{s}} v_2)) \\
&= \Omega[\bar{x}](w(\bar{x}),\ p_S(\mathrm{d}g_x^{\bar{s}} v_1),\ p_S(\mathrm{d}g_x^{\bar{s}} v_2))) = \Omega[\bar{x}](w(\bar{x}),\ p_S(\bar{v}_1),\ p_S(\bar{v}_2)) \\
&= \Omega[\bar{x}](w(\bar{x}), \bar{v}_1, \bar{v}_2).
\end{aligned}$$

But this last expression and (17) together imply (16), hence ω is well defined.

Next we show that ω is nondegenerate. Suppose that for some $y \in M/G$ and $u_1 \in T_y(M/G)$, $\omega[y](u_1, u_2) = 0$ holds for all $u_2 \in T_y(M/G)$. We have to show that this implies $u_1 = 0$. Again, by lemma 4.1, for any $x \in \pi^{-1}(y)$ we have two vectors $v_1, v_2 \in T_xM$ with $\mathrm{d}\pi_x v_i = u_i$. Then, by the definition of ω,

$$\Omega[x](w(x), v_1, v_2) = 0$$

must hold for all $v_2 \in T_xM$. Since Ω is nondegenerate, this can only hold if

$$v_1 \in \mathrm{span}\{w(\bar{x})\} = \ker(\mathrm{d}\pi_x)$$

which implies $u_1 = \mathrm{d}\pi_x v_1 = 0$ as claimed. Finally, the smoothness of ω follows from its definition by the chain rule. □

Next we show that the invariant B induces a well-defined Hamiltonian on the orbit space M/G.

Lemma 4.3. *The function $H \colon M/G \to \mathbb{R}$ defined through the relationship*

$$\boxed{\pi^* H = B} \tag{19}$$

is well-defined and smooth.

Proof. Consider any point $y \in M/G$. Since π is surjective by definition, there exists a point $x \in M$ such that $y = [x]$. Then $H(y) = H(\pi(x)) = B(x)$. Now suppose there exists $\bar{x} \neq x$ such that $y = [\bar{x}]$ holds. By the definition of the quotient space M/G, this implies that $\bar{x}$ and x lie on the same orbit of the group action g^s. But then statement (ii) of theorem 3.1 shows that $B(x) = B(\bar{x})$, hence H is well defined. The smoothness of H follows from the smoothness of B and from lemma 4.1 by the chain rule. □

We are now in the position to prove our main result about the relation between the flow generated by the Hamiltonian H on $(M/G, \omega)$ and the original volume-preserving flow F^t.

Theorem 4.4 (volume-preserving reduction). *Suppose that w generates a volume-preserving symmetry for the volume-preserving vector field v. Assume that the associated group action $g^s\colon M \to M$ is regular. Let us define the Hamiltonian vector field v_H on $(M/G, \omega)$ as*

$$\boxed{i_{v_H}\omega = \mathrm{d}H} \tag{20}$$

where ω and H are defined in (14) and (19).

Then the following hold:

(i) *The reduced vector field v_H is related to the vector field v through the formula*

$$v_H(\pi(x)) = \mathrm{d}\pi_x v(x). \tag{21}$$

(ii) *The reduced flow F_H^t generated by v_H commutes with the flow F^t through the smooth semiconjugacy π, i.e.*

$$\pi \circ F^t = F_H^t \circ \pi.$$

Proof. Since ω is nondegenerate and H is well defined, the Hamiltonian vector field v_H is uniquely determined by (20). Therefore, to prove (i), it is enough to verify that $\mathrm{d}\pi \cdot v$ satisfies (20). Let us fix a point $x \in M$ and select an arbitrary vector $u_0 \in T_y(M/G)$ with $y = \pi(x)$. If v_0 is a vector such that $\mathrm{d}\pi_x v_0 = u_0$, then

$$i_{\mathrm{d}\pi\cdot v}\omega[y](u_0) = \omega[y](\mathrm{d}\pi_x v(x), u_0) = \pi^*\omega[x](v(x), v_0) = -\Omega[x](w(x), v(x), v_0) \tag{22}$$

by the definition of ω. On the other hand, form the definition of H and (8) we obtain that

$$\pi^*\mathrm{d}H = \mathrm{d}\pi^* H = \mathrm{d}B = -i_v i_w \Omega$$

which shows that

$$\mathrm{d}H_y u_0 = \mathrm{d}H_y \mathrm{d}\pi_x v_0 = \pi^*\mathrm{d}H[x]v_0 = -\Omega[x](w(x), v(x), v_0).$$

But the vector u_0 was arbitrary, hence this last equation together with (22) proves that $v_H = \mathrm{d}\pi \cdot v$ satisfies (20).

To prove statement (ii), we note that for any point $x \in M$ with $\pi(x) = y \in M/G$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(F^t(x)) = \mathrm{d}\pi_{F^t(x)} v(F^t(x)) = v_H(\pi(F^t(x)))$$

hence the projection $\pi(F^t(x))$ of the solution $F^t(x)$ on the quotient space M/G satisfies the Hamiltonian equations generated by the Hamiltonian H. By uniqueness of solutions for the reduced flow, this projected solution must coincide with the solution of the reduced system that starts from the same point, i.e.

$$\pi(F^t(x)) = F_H^t(\pi(x))$$

must hold. But this proves statement (ii) of the lemma since x was arbitrary. $\square$

5. Symmetry-breaking perturbations

In most applications the system under consideration is only approximately symmetric and volume preserving, i.e. the corresponding vector field v_p can be written in the form

$$v_p = v + \epsilon v_1 \qquad 0 \leqslant \epsilon \ll 1$$

with

$$L_w v = 0 \qquad L_v \Omega = 0$$
$$L_w v_1 \not\equiv 0 \qquad \text{and/or} \qquad L_{v_1}\Omega \not\equiv 0.$$

Since ϵ is small, v_p is a small perturbation of the vector field v, and one hopes that some features of the flow generated by v_p can be understood based on the knowledge of the flow of v. In practice, this can be achieved by applying some perturbation method, which typically requires a suitable coordinate representation of v_p. In this section we show how the volume-preserving reduction performed for v yields coordinates which are ideal for perturbation methods.

We begin by recalling that the pre-image of any point $y \in M/G$ is a whole group orbit in the phase space M, hence the quotient projection $\pi\colon M \to M/G$ is clearly not invertible. However, π becomes invertible if we restrict it to a suitable two-dimensional submanifold of M. Suppose that after possibly shrinking the domain M, there exists a two-dimensional submanifold $\mathcal{S} \subset M$ which has a unique, transverse intersection with every group orbit in M. We then define the map

$$\mathcal{P} = \pi|\mathcal{S}$$

and observe that $\mathcal{P}$ is diffeomorphism between $\mathcal{S}$ and the quotient space. Indeed, the map $\mathrm{d}\mathcal{P}_x = \mathrm{d}\pi_x|T_x\mathcal{S}$ is an isomorphism since by the construction of $\mathcal{S}$, we have $\ker \mathrm{d}\pi_x = \mathrm{span}\{w(x)\} \not\subset T_x\mathcal{S}$. Then the inverse function theorem guarantees that $\mathcal{P}$ is a local diffeomorphism. But $\mathcal{P}$ is one-to-one and onto, hence it is also a global diffeomorphism between $\mathcal{S}$ and M/G.

Next we define the 'orbit projection map' $P\colon M \to \mathcal{S}$ through the commutative diagram

$$\begin{tikzcd} & M/G & \\ M \arrow[ur, "\pi"] \arrow[rr, "P"] & & \mathcal{S}. \arrow[ul, "\mathcal{P}"'] \end{tikzcd} \tag{23}$$

Clearly, P takes any point $x \in M$ to the intersection of the orbit of the group action g^s through x with the surface $\mathcal{S}$, as shown in figure 1. Note that P is surjective by definition and smooth by the chain rule. We define the group orbit through a point $x_0 \in M$ as

$$\gamma(x_0) = \cup_{s\in G}\, g^s(x_0).$$

Since the action g^s assumed to be regular and hence free in M, for any point $x \in \gamma(x_0)$ there exists a unique group element $\tau(x) \in G$ such that

$$g^{\tau(x)}(x_0) = x. \tag{24}$$

The map $\tau\colon M \to G$, $x \mapsto \tau(x)$ is smooth by the implicit function theorem (here we used the transverse intersection of group orbits with $\mathcal{S}$). The following observation is fundamental in our construction of coordinates.

Lemma 5.1. *The map $\mathcal{C}\colon M \to M/G \times G$ defined as*

$$\mathcal{C}(x) = (\pi(x), \tau(x))$$

is a diffeomorphism.

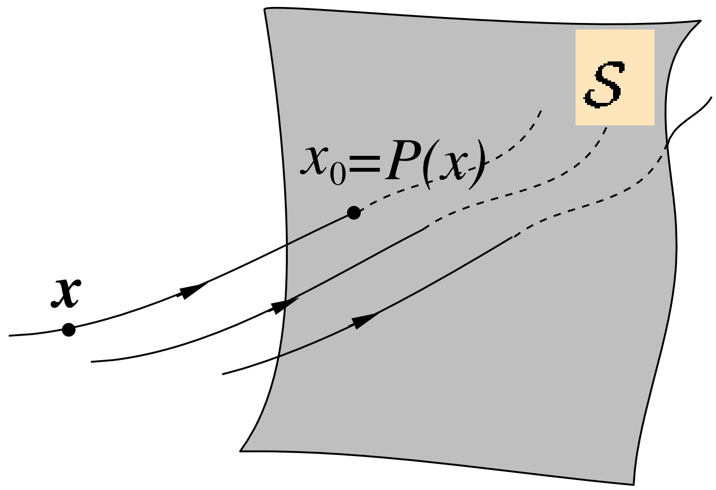

Figure 1. The geometric meaning of the map P.

Proof. The map $\mathcal{C}$ is clearly a smooth bijection, so we only need to show that $\mathcal{C}$ is a local diffeomorphism, which in turn implies the smoothness of $\mathcal{C}^{-1}$ by the implicit function theorem. We have

$$\mathrm{d}\mathcal{C}_x = \begin{pmatrix} \mathrm{d}\pi_x \\ \mathrm{d}\tau_x \end{pmatrix}.$$

We will show that $\mathrm{d}\mathcal{C}_x$ is an isomorphism by establishing that its kernel is trivial. Consider a vector $u \in \ker \mathrm{d}\mathcal{C}_x$. Then we have

$$\mathrm{d}\pi_x u = 0$$
$$\mathrm{d}\tau_x u = 0.$$

The first of these equations implies $u \in \mathrm{span}\{w(x)\}$. The second equation than gives $\mathrm{d}\tau_x w(x) = 0$. But, $\mathrm{d}\tau_x w(x)$ must be nonzero because the vector field w is nonzero and tangent to the orbits of the symmetry group, whose action is assumed to be free—thus the function τ has a nonzero derivative along group orbits. Consequently, we obtain that $u = 0$, hence the kernel of $\mathrm{d}\mathcal{C}_x$ is trivial. This completes the proof of lemma 5.1. □

Lemma 5.1 allows us to think of the map $\mathcal{C}$ as a change of coordinates on M. The new coordinates split into a two-dimensional part which is the coordinate y on the reduced phase space, and a one-dimensional part which is the global coordinate s on the Lie group G. The following theorem shows that in these coordinates, the flow generated by the perturbed vector field v_p takes a particularly simple form which is suitable for the application of perturbation methods.

Theorem 5.2. *Suppose that there exists a two-dimensional manifold $\mathcal{S} \subset M$ which is a global transversal surface to the orbits of the regular group action $g^s : M \to M$, and each group orbit has a unique intersection with $\mathcal{S}$. Then $(y, s) = \mathcal{C}(x)$ defines a smooth change of coordinates, which transforms the flow of v_p to the form*

$$\dot{y} = v_H(y) + \epsilon i_{v_1} \mathrm{d}\pi(y, s) \tag{25}$$
$$\dot{s} = i_v \mathrm{d}\tau(y) + \epsilon i_{v_1} \mathrm{d}\tau(y, s)$$

where the Hamiltonian vector field v_H is defined in (21), and the map τ is defined in (24).

Proof. The first equation of (25) follows immediately by projecting the vector field v_p to the reduced phase space M/G with the map π, and using the results of theorem 4.4. Similarly, the basic structure of the second equation follows from the calculation

$$\dot{s} = \mathrm{d}\tau_x \cdot \dot{x} = i_{v_p}\mathrm{d}\tau = i_v\mathrm{d}\tau + \epsilon i_{v_1}\mathrm{d}\tau. \tag{26}$$

We only have to argue that the function $i_v\mathrm{d}\tau$ does not depend on the variable s explicitly. To show this, we first note that for $\epsilon = 0$, the transformed equations (25) are invariant with respect to the transformed group action

$$\mathcal{G}^s = \mathcal{C} \circ g^s \circ \mathcal{C}^{-1}.$$

This can be seen by introducing the unperturbed transformed vector field $\nu = (v_H, i_v\mathrm{d}\tau)$. For $p \in M/G \times G$, we have

$$\begin{aligned}\nu(\mathcal{G}^s(p)) &= \mathrm{d}\mathcal{C}_{\mathcal{C}^{-1}(\mathcal{G}^s(p))} \cdot v(\mathcal{C}^{-1}(\mathcal{G}^s(p))) = \mathrm{d}\mathcal{C}_{g^s(x)} \cdot v(g^s(x)) = \mathrm{d}\mathcal{C}_{g^s(x)}\mathrm{d}g^s_x \cdot v(x)\\ &= \mathrm{d}\mathcal{C}_{g^s(x)}\mathrm{d}g^s_x\mathrm{d}\mathcal{C}^{-1}_y\mathrm{d}\mathcal{C}_x \cdot v(x)\\ &= \mathrm{d}\mathcal{C}_{g^s(x)}\mathrm{d}g^s_x\mathrm{d}\mathcal{C}^{-1}_p\nu(p) = \mathrm{d}\mathcal{G}^s_p\nu(p)\end{aligned}$$

where we used the equivariance of v under g^s (see (5)). But this equation means that the vector field ν is equivariant with respect to $\mathcal{G}^s$. By construction, for any group element $s_0 \in G$ the representation of $\mathcal{G}^{s_0}$ is very simple:

$$\mathcal{G}^{s_0}\begin{pmatrix} y \\ s \end{pmatrix} = \begin{pmatrix} y \\ s + s_0 \end{pmatrix}.$$

But equivariance with respect to this action means that the right-hand side of equation (26) cannot depend on s explicitly for $\epsilon = 0$. □

The above theorem provides a global coordinate representation for the perturbed flow on the space $M/G \times G$, as long as we have a global coordinate system defined on the reduced phase space M/G. For $\epsilon = 0$, equation (25) is volume preserving, as its flow preserves the volume $\tilde{\Omega} = \mathcal{C}_*\Omega$. In this limit, the y equations decouple and yield the reduced system on M/G. Taking the Cartesian product of reduced orbits with the Lie group G, we obtain diffeomorphic copies nonsingular level surfaces of the invariant B. These surfaces form invariant manifolds for the unperturbed problem, and one is usually interested in their fate under perturbation. If the reduced orbit in question is closed, then the corresponding level surface of B is either a cylinder or a two-torus, depending on the nature of the group G. In both cases, these level surfaces typically occur in families. In the case of two-tori, one can expect the majority of the tori to survive if the unperturbed system satisfies certain nondegeneracy conditions and the perturbed vector field $v + \epsilon v_1$ preserves the volume $\tilde{\Omega}$ even for $\epsilon > 0$. Details of the related KAM-type results can be found in Cheng and Sun [10], Herman [13], and Xia [28]. If the reduced orbit in question is a homoclinic orbit, then the corresponding level set of B is a two-dimensional homoclinic manifold asymptotic to an orbit γ of v which is diffeomorphic to the symmetry group G. The question is then the survival of homoclinic orbits to an orbit γ_ϵ near γ. This problem can be studied using the appropriate version of Melnikov's method, which can be found, for example in Wiggins [27]. The application of these two perturbation methods to three-dimensional vector fields (with one equation decoupling in the unperturbed limit) is surveyed in Mezić and Wiggins [22]. Theorem 5.2 above gives conditions under which such a coordinate representation is globally attainable, and also provides an explicit, geometric construction for the coordinates.

The coordinates developed in this section can also be used to endow the manifold M with a contact structure (see, e.g. Arnold [5] or Abraham and Marsden [1] for definitions).

Proposition 5.3. *Let us define the two-form $\tilde{\omega}$ on the manifold M as $\tilde{\omega} = \pi^*\omega = -i_w\Omega$. Then $(M, \tilde{\omega})$ is a contact manifold. Furthermore, assume that ω is an exact symplectic form, i.e. $\omega = \mathrm{d}\theta$ holds for some one-form θ. Define the one-form $\tilde{\theta}$ as*

$$\boxed{\tilde{\theta} = \mathrm{d}\tau + \pi^*\theta} \tag{27}$$

where τ is the function defined in (24). Then $(M, \tilde{\theta})$ is an exact contact manifold and $\mathrm{d}\tilde{\theta} = \tilde{\omega}$.

Proof. To prove the first statement of the theorem, we recall that $(M, \tilde{\omega})$ is a contact manifold if the linear map $\tilde{\omega}^\flat[x]\colon T_xM \to T_xM^*$, $\tilde{\omega}^\flat[x](a) \cdot b = \tilde{\omega}[x](a, b)$ has maximal rank (i.e. rank 2) for any $x \in M$. But this is clearly true, since the kernel of $\omega^\flat[x]$ is spanned by the infinitesimal generator $w(x)$, hence it is one-dimensional.

By definition, $(M, \tilde{\theta})$ is an exact contact manifold with the one-form $\tilde{\theta}$ if $\tilde{\theta} \wedge \mathrm{d}\tilde{\theta}$ is a volume form on M. As shown for example in Abraham and Marsden [1], a necessary condition for this is that $\mathrm{d}\tilde{\theta}$ is nondegenerate on the subbundle

$$R_{\tilde{\theta}} = \{a \in TM \mid \tilde{\theta}(a) = 0\}.$$

Now $\mathrm{d}\tilde{\theta} = \pi^*\omega = \tilde{\omega} = -i_w\Omega$, hence $\mathrm{d}\tilde{\theta}[x]$ is degenerate only on the subspace $\mathrm{span}(w(x))$. But this subspace is not contained in $R_{\tilde{\theta}}$, because

$$\tilde{\theta}(w) = \mathrm{d}\tau \cdot w + \pi^*\theta(w) = \mathrm{d}\tau \cdot w + \pi^*\theta(\mathrm{d}\pi \cdot w) = \mathrm{d}\tau \cdot w \neq 0. \tag{28}$$

But (28) implies that $\mathrm{span}\{w(x)\} \not\subset R_{\tilde{\theta}}$, hence $\mathrm{d}\tilde{\theta}$ is nondegenerate on $R_{\tilde{\theta}}$. □

6. Applications

In this section we discuss several problems in which volume-preserving reduction can be used. Many of the results listed below are known, but were obtained through different procedures or in *ad hoc* ways. Here we present a unified construction of invariants for all these problems and describe the structure of the reduced equations.

6.1. Three-dimensional, steady Euler flows

The velocity field of a three-dimensional, inviscid fluid satisfies the equation

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v = -\frac{1}{\rho}(\nabla\psi + \nabla p) \tag{29}$$

where ρ is the density, p is the pressure, and ψ is the potential energy. Taking the curl of both sides yields the vorticity equation

$$\frac{\partial w}{\partial t} = (w \cdot \nabla)v - (v \cdot \nabla)w = [v, w] \tag{30}$$

where $w = \nabla \times v$. For steady, incompressible flows, equation (29) can be rewritten as

$$v \times w = \frac{\nabla|v|^2}{2} + \nabla\psi + \frac{1}{\rho}\nabla p. \tag{31}$$

Furthermore, for steady flows (30) implies that

$$L_w v = -[v, w] = 0.$$

Hence, if $g^s: \mathbb{R}^3 \to \mathbb{R}^3$ denotes the flow generated by the vorticity field w, then g^s commutes with the flow F^t generated by the velocity field v. By incompressibility, F^t preserves the standard volume form $\Omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ on $\mathbb{R}^3$. Furthermore, using formula (3), we obtain

$$L_w \Omega = (\operatorname{div} w)\Omega = 0.$$

This shows that the vorticity generates a volume-preserving symmetry for the volume-preserving velocity field v (see Arnold [5] for more detail and references).

By formula (10), the invariant B guaranteed by theorem 3.1 satisfies

$$\nabla B = v \times w$$

which, combined with equation (31), gives that

$$B = \frac{1}{2}|v|^2 + \psi + \frac{p}{\rho}$$

is a first integral for the flow F^t, i.e. it is conserved along streamlines. This is just Bernoulli's theorem from hydrodynamics.

By theorem 4.4, equation (31) can be reduced to a one-degree-of-freedom Hamiltonian system on the quotient space M/G. This space is a manifold if the open region M is selected in a way such that the *vorticity field* defined by the equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \\ \frac{\mathrm{d}z}{\mathrm{d}s} &= \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \end{aligned} \tag{32}$$

generates a regular group action on M. This means that either all orbits in M are nontrivial periodic orbits, or all of them are nonperiodic and nondense. Then M/G can be identified with an open set of a two-dimensional plane $\Lambda \subset \mathbb{R}^3$ that has a unique point of intersection with every orbit of (32) in M. Then the quotient projection $\pi: M \to \Lambda, (x, y, z) \mapsto \eta = (\pi_1(x, y, z), \pi_2(x, y, z))$ is just the map that maps a point $(x, y, z) \in M$ to the unique intersection η of the group orbit through (x, y, z) with the plane Λ. (Here $\eta \in \mathbb{R}^2$ denotes an arbitrary coordinate system on Λ.) The reduced symplectic form ω on M/G then takes the form $\omega = f(\eta)\mathrm{d}\eta_1 \wedge \mathrm{d}\eta_2$. Since ω is nondegenerate, $f(\eta)$ is nonzero on M/G, hence after rescaling time by $t \to f(\eta)t$, the reduced system can be written as a canonical Hamiltonian system. This canonical system is generated through the symplectic form $\mathrm{d}\eta_1 \wedge \mathrm{d}\eta_2$ by the Hamiltonian H defined in (19). Concretely, we obtain the reduced system

$$\dot{\eta} = J\mathrm{D}_\eta \left[\frac{1}{2}|v(\eta)|^2 + \psi(\eta) + \frac{p(\eta)}{\rho(\eta)}\right] \tag{33}$$

where D_η denotes differentiation with respect to η, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that this reduction is only meaningful if the vorticity field is not everywhere parallel to the velocity field. Otherwise, as we discussed at the end of section 3, we obtain $\mathrm{d}B = 0$, hence the reduced flow is just a set of equilibria. In that case the reduction is equivalent to arranging the orbits of the velocity field into an orbit space on which all particle motions appear as relative equilibria. Classic examples of this degenerate case are the ABC-flows first studied by Hénon [14]. These flows violate Arnold's integrability condition (see Arnold [5]) for three-dimensional, steady Euler flows, which requires v and w not to be collinear everywhere. The numerical experiments of Hénon seem to produce trajectories that densely fill up three-dimensional regions in the phase space. As a result, the vorticity flow (32) cannot be regular on invariant open sets, because the orbit space cannot be a two-dimensional manifold. These numerical results suggest that the vorticity flow does not generate a proper group action in ABC flows.

The global coordinates η on the reduced phase space are in fact the Clebsch coordinates of classical fluid dynamics. This is discussed in more detail in [22].

6.2. Two-dimensional, unsteady Euler flows

If the velocity field v appearing in the Euler equation (29) is in fact two-dimensional, then the incompressibility condition $\nabla \cdot v = 0$ implies the existence of stream function $\Psi(x, y, t)$ for the corresponding two-dimensional flow. Then the Lagrangian particle motions satisfy the Hamiltonian equations

$$\begin{aligned} \dot{x} &= v_x = \frac{\partial \Psi(x, y, t)}{\partial y} \\ \dot{y} &= v_y = -\frac{\partial \Psi(x, y, t)}{\partial x}. \end{aligned} \tag{34}$$

Taking the curl of both sides of (29) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\nabla \times v) = \frac{\mathrm{d}}{\mathrm{d}t}(\nabla \times J\nabla\Psi) = 0.$$

This equation reflects the well-known fact that the only nonzero component of the vorticity,

$$\xi = (\nabla \times v)_z = -\Delta\Psi$$

is conserved along particle motions, i.e.

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = \frac{\partial \xi}{\partial t} + v_x \frac{\partial \xi}{\partial x} + v_y \frac{\partial \xi}{\partial y} = 0. \tag{35}$$

Consider now the space $\mathbb{R}^3$ with coordinates (x, y, t). On this space, we will use the velocity vector

$$v = \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}.$$

We observe that the vector field

$$w = \begin{pmatrix} \frac{\partial \xi}{\partial y} \\ -\frac{\partial \xi}{\partial x} \\ 0 \end{pmatrix}$$

preserves the volume $\Omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}t$ as

$$\mathrm{div}_\Omega w = 0. \tag{36}$$

Moreover,

$$[v, w] = \mathrm{D}w \cdot v - \mathrm{D}v \cdot w = \begin{pmatrix} \frac{\partial \xi}{\partial y}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) - \frac{\partial}{\partial y}\frac{\mathrm{d}\xi}{\mathrm{d}t} \\ -\frac{\partial \xi}{\partial x}\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) + \frac{\partial}{\partial x}\frac{\mathrm{d}\xi}{\mathrm{d}t} \\ 0 \end{pmatrix} = 0 \tag{37}$$

where we used (34) and (35). Hence w generates a volume-preserving symmetry for the volume-preserving vector field v on the extended phase space $(x, y, t) \in \mathbb{R}^3$ of the two-dimensional, nonautonomous equation (34). The action of the underlying symmetry group is given by the three-dimensional flow

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= \frac{\partial \xi}{\partial y} \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= -\frac{\partial \xi}{\partial x} \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= 0 \end{aligned} \tag{38}$$

which is generated by w. We pick an open domain $M \subset \mathbb{R}^3$ which is invariant under the flow of this equation and is filled entirely with either nonclosed orbits or nontrivial closed orbits of (38). (The nonperiodic orbits cannot be dense by $\mathrm{d}t/\mathrm{d}s = 0$, hence the properness of the group action cannot fail, only the freeness.) As a result, the flow of (38) defines a regular action of the group $G = \mathbb{R}^1$ or $G = S^1$ on the domain M. By (36), this action is volume preserving, and by (37), it commutes with the flow generated by v in the extended phase space. In that case, theorem 3.1 just repeats the fact that ξ is a first integral, as formula (8) implies

$$\mathrm{d}B = -i_v i_w \Omega = -\frac{\partial \xi}{\partial y}(v_y \mathrm{d}t - \mathrm{d}y) - \frac{\partial \xi}{\partial x}(v_x \mathrm{d}t - \mathrm{d}x) = \frac{\partial \xi}{\partial x}\mathrm{d}x + \frac{\partial \xi}{\partial y}\mathrm{d}y + \frac{\mathrm{d}\xi}{\mathrm{d}t}\mathrm{d}t = \mathrm{d}\xi.$$

Finally, the reduced phase space M/G is a manifold which is diffeomorphic to an open set of $\mathbb{R}^2$. For example, it can be identified with the intersection of a two-dimensional plane of the extended phase space Λ with M given by $y = f(x)$, provided all orbits of (38) in M have a unique intersection point with $M \cap \Lambda$. Then the same argument that led to (33) gives that the reduced system on M/G can be written as a Hamiltonian system of the form

$$\dot{\eta} = J\mathrm{D}_\eta \left[\frac{\partial v_y(\eta)}{\partial x} - \frac{\partial v_x(\eta)}{\partial y} \right].$$

Here the corresponding Hamiltonian is therefore the vorticity ξ, which must be considered as a function of the coordinates η on the plane Λ.

6.3. Flows on a sphere

In the β-plane approximation, the velocity field of an incompressible, inviscid fluid moving on a rotating sphere with small Rossby number is given by

$$\begin{aligned} v_x &= -\frac{\partial \Psi}{\partial y} \\ v_y &= \frac{\partial \Psi}{\partial x} \end{aligned}$$

where $\Psi(x, y, t)$ is the quasigeostrophic stream function. It is well known that the potential vorticity

$$q(x, y, t) = \Delta \Psi(x, y, t) + \beta y$$

with the planetary β-plane parameter $\beta > 0$ satisfies the conservation law

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial q}{\partial t} + v_x \frac{\partial q}{\partial x} + v_y \frac{\partial q}{\partial y} = 0.$$

By direct analogy with the two-dimensional Euler equations discussed above, we obtain the following result: let $M \subset \mathbb{R}^3$ be an open set in the space of (x, y, t) coordinates which contains entire trajectories of the 'potential-vorticity flow' generated by the equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= -\frac{\partial q}{\partial y} \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= \frac{\partial q}{\partial x} \\ \frac{\mathrm{d}t}{\mathrm{d}s} &= 0. \end{aligned} \tag{39}$$

Assume further that either all trajectories in M are nontrivial periodic orbits or none of them are periodic. Then the flow of (39) is the regular action of the group $G = S^1$ or $G = \mathbb{R}$, respectively, on M. The reduced phase space M/G can again be taken as an open subset of a plane $\Lambda \subset \mathbb{R}^3$ which has a unique intersection point with every group orbit in M. The reduced flow on M/G is Hamiltonian and after a rescaling of time, it satisfies the equation

$$\dot{\eta} = J\mathrm{D}_\eta[\Delta\Psi(x(\eta), y(\eta), t(\eta)) + \beta y(\eta)]$$

where η denotes coordinates on Λ.

The above β-plane approximation is a simplification of the equations of motion of fluid in a thin layer on a rotating sphere (see, e.g. Batchelor [6]). The full equations of motion of the potential vorticity which is here defined as $(\omega + f)/H$ where ω is the radial component of vorticity, $f = 2\Omega_r \cos\theta$, with Ω_r the magnitude of rotation, θ the variable on the sphere that is 0 at the north pole and changes along the latitude, and H the depth of the layer of fluid which is assumed constant. The evolution equation for $(\omega + f)/H$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}((\omega + f)/H) = 0$$

and thus $(\omega + f)/H$ is conserved on trajectories of the velocity field. This invariant is easily seen to correspond to a spatial, volume-preserving symmery in the same way as in the cases treated above. The motion of fluid on a sphere is interesting in the context of geophysical applications (see e.g. [7]). We remark that Kirwan [15] has done previous work on the flows on a sphere using concepts of symplectic reduction [19] to study the motion of vortices.

6.4. Three-dimensional, steady, magnetohydrodynamic flows

The equations of an inviscid, incompressible, magnetic fluid are given by the equations

$$\begin{aligned} \frac{\partial v}{\partial t} + (v \cdot \nabla)v &= -\frac{1}{\rho}(\nabla\psi + \nabla p) - \frac{1}{4\pi}\boldsymbol{B} \times (\nabla \times \boldsymbol{B}) \\ \frac{\partial \boldsymbol{B}}{\partial t} + (v \cdot \nabla)\boldsymbol{B} &= (\boldsymbol{B} \cdot \nabla)v \end{aligned} \tag{40}$$

where the vector $\boldsymbol{B}$ denotes the flux, and the other quantities are the same as in the Euler equation (29). By Maxwell's equations, we have

$$\operatorname{div} \boldsymbol{B} = 0. \tag{41}$$

We can rewrite the second equation (40) as

$$\frac{\partial \boldsymbol{B}}{\partial t} = [v, \boldsymbol{B}]. \tag{42}$$

It is customary to introduce the vector potential $\boldsymbol{A}$ through the formula

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

As argued for example in Kuzmin [16], $\boldsymbol{A}$ can be selected in a way so that its evolution satisfies the equation

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{\partial \boldsymbol{A}}{\partial t} + (v \cdot \nabla)\boldsymbol{A} = -(\nabla v)^T \boldsymbol{A}. \tag{43}$$

Now equation (42) shows that for steady flows we must have

$$L_{\boldsymbol{B}} v = -[v, \boldsymbol{B}] = 0.$$

This equation and (41) imply that $\boldsymbol{B}$ generates a volume-preserving symmetry for the velocity field v. To construct a first integral for v, we use formula (10) to write

$$\nabla B = v \times (\nabla \times \boldsymbol{A}). \tag{44}$$

Based on the identity

$$\nabla(v \cdot \boldsymbol{A}) = v \times (\nabla \times \boldsymbol{A}) + (v \cdot \nabla)\boldsymbol{A} + (\nabla v)^T \boldsymbol{A}$$

we obtain from equations (43) and (44) that steady magnetohydrodynamic flows admit a Bernoulli-type invariant of the form

$$B = \boldsymbol{A} \cdot v.$$

As in the case of three-dimensional, steady Euler flows, we obtain the reduced flow

$$\dot{\eta} = J\mathrm{D}_\eta[\boldsymbol{A}(\eta) \cdot v(\eta)] \tag{45}$$

in appropriate coordinates η after a rescaling of time.

7. An example

In this section we show how our results can be applied to Hill's spherical vortex problem amended with a line vortex at the z-axis. This problem was studied in Mezić and Wiggins [22], where the reduction of the flow was accomplished via a local, coordinate-dependent theory. Here we reconsider the same example and give an intrinsic, geometric meaning to the reduction. Furthermore, theorem 3.1 enables us to construct a first integral for the flow before performing the reduction.

Consider the three-dimensional flow generated by the vector field

$$v(x, y, z) = \begin{pmatrix} xz - 2cy/(x^2 + y^2) \\ yz + 2cx/(x^2 + y^2) \\ 1 - 2(x^2 + y^2) - z^2 \end{pmatrix}. \tag{46}$$

Here $p = (x, y, z) \in \mathbb{R}^3$, hence the manifold M is just the usual three-dimensional Euclidean space. Passing to cylindrical coordinates, it is easy to see that the vector field is equivariant under rotations around the z-axis. Hence, the corresponding symmetry group is $G = S^1$ with the group action

$$g^s(p) = \begin{pmatrix} \cos s & \sin s & 0 \\ -\sin s & \cos s & 0 \\ 0 & 0 & 1 \end{pmatrix} p \qquad s \in S^1. \tag{47}$$

This action is proper but not free, because it leaves any point of the z-axis fixed for any $s \in S^1$. As a result, the reduced phase space M/G may not be a manifold. Indeed, M/G can be identified with the closed half plane of $\mathbb{R}^2$, which is a manifold with boundary:

$$M/G = \{(r, \zeta) \in \mathbb{R}^2 \,|\, r \geqslant 0\}.$$

The associated quotient projection π is given by

$$\pi : M \to M/G$$
$$(x, y, z) \mapsto (\sqrt{x^2 + y^2}, z).$$

As is easily seen, this map is not differentiable on the z-axis, a fact which is again related to the degeneracy of the group action. Note, however, that if we exclude the z-axis from M, then M/G becomes a manifold (the open half plane of $\mathbb{R}^2$) and π becomes a smooth map onto M/G. Nonetheless, we choose M to be the whole of $\mathbb{R}^3$, because the reduced flow will turn out to be nonsingular on the boundary of M/G (although the reduced symplectic form does become degenerate on $\partial(M/G)$).

From (47) we obtain that the infinitesimal generator of the group action is given by the vector field

$$w = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}.$$

We now use theorem 3.1 to construct an invariant for the flow. We have

$$\mathrm{d}B = xv_z\mathrm{d}x + yv_z\mathrm{d}y - (yv_y + xv_x)\mathrm{d}z = x(1 - 2(x^2 + y^2) - z^2)\,\mathrm{d}x$$
$$+ y(1 - 2(x^2 + y^2) - z^2)\,\mathrm{d}x - 2xyz\,\mathrm{d}z$$

which yields that

$$B = \tfrac{1}{2}(x^2 + y^2) - \tfrac{1}{2}(x^2 + y^2)^2 - \tfrac{1}{2}(x^2 + y^2)z^2$$

is a first integral for the vector field v. Then from the definition $\pi^* H = B$, we obtain the reduced Hamiltonian

$$H(r, \zeta) = \tfrac{1}{2}(r^2 - r^4 - r^2\zeta^2).$$

Furthermore, for any two vectors $a, b \in T_{(x,y,z)}M$ we have

$$\pi^*\omega[x, y, z](a, b) = -i_w\Omega[x, y, z](a, b) = (y\mathrm{d}y \wedge \mathrm{d}z + x\mathrm{d}x \wedge \mathrm{d}z)(a, b)$$
$$= y(a_y b_z - a_z b_y) + x(a_x b_z - a_z b_x)$$

on the other hand

$$\pi^*\omega[x, y, z](a, b) = \omega[r, \zeta](\mathrm{d}\pi a, \mathrm{d}\pi b) = f(r, \zeta) \cdot \mathrm{d}r \wedge \mathrm{d}\zeta(\mathrm{d}\pi a, \mathrm{d}\pi b)$$
$$= \frac{f(r, \zeta)}{\sqrt{x^2 + y^2}}[y(a_y b_z - a_z b_y) + x(a_x b_z - a_z b_x)].$$

These two equations show that $f(r, \zeta) = \sqrt{x^2 + y^2} = r$, hence the symplectic form on the reduced phase space M/G is given by

$$\omega = r\,\mathrm{d}r \wedge \mathrm{d}\zeta.$$

We remark that the same symplectic form appeared in Broer [8, 9] in the study of local bifurcations of three-dimensional, rotationally symmetric vector fields.

Our calculations imply that the reduced flow on M/G is given by the Hamiltonian equations

$$\dot{r} = \frac{1}{r}\frac{\partial H}{\partial \zeta} = -r\zeta \tag{48}$$
$$\dot{\zeta} = -\frac{1}{r}\frac{\partial H}{\partial r} = 2r^2 + \zeta^2 - 1.$$

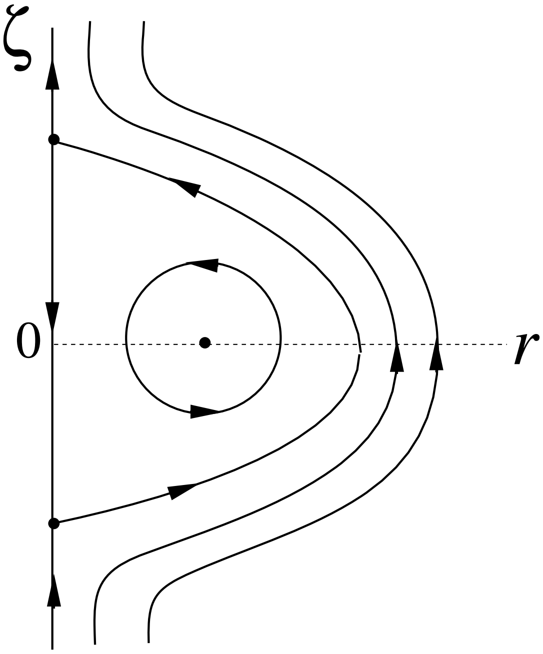

Figure 2. The reduced flow for the example.

As we noted earlier, although the symplectic structure becomes degenerate at the $r = 0$ boundary of the reduced phase space, the reduced flow extends smoothly to $r = 0$. The phase portrait of the reduced system is shown in figure 2. Note that, as we noted in the introduction, the dynamics on the level surfaces of B is does not necessarily preserve any volume. This is quite transparent on the 'bubble' that corresponds to the rotation of the heteroclinic orbit of (48) around the z-axis. Any open set on this surface shrinks asymptotically to the lower fixed point on the z-axis. The closed orbits give rise to invariant two-tori for the full flow. The flow on them is volume preserving but it is not Hamiltonian (if there were a smooth Hamiltonian defined on one of these tori, the flow on the torus would have at least two fixed points).

To obtain a representation of the full, three-dimensional flow in a form suitable for perturbation theory, we can use theorem 5.2. Since any open half plane $\{x = ay, x > 0\}$ is globally transverse to the vector field v, we can pick the transverse surface

$$\mathcal{S} = M/G - \partial M/G$$

in which case the map $\mathcal{P}$ is just the identity map. The map $\tau\colon M \to G$ can be defined locally as

$$\tau(x, y, z) = \tan^{-1} \frac{y}{x}$$

for $x \neq 0$. However, its differential is globally defined for $x^2 + y^2 \neq 0$:

$$\mathrm{d}\tau = -\frac{y}{x^2 + y^2}\mathrm{d}x + \frac{x}{x^2 + y^2}\,\mathrm{d}y$$

which together with (46) gives

$$i_v \mathrm{d}\tau = \frac{2c}{r^2}.$$

Hence, by theorem 5.2, $\mathcal{O}(\epsilon)$ perturbations of the full flow can be written in the form

$$\begin{aligned}\dot{r} &= -r\zeta + \mathcal{O}(\epsilon)\\ \dot{\zeta} &= 2r^2 + \zeta^2 - 1 + \mathcal{O}(\epsilon)\\ \dot{s} &= \frac{2c}{r^2} + \mathcal{O}(\epsilon)\end{aligned}$$

where the $\mathcal{O}(\epsilon)$ terms in this equation are of the form shown in (25).

8. Conclusions

We have constructed a coordinate-free theory for the reduction of volume-preserving flows on three-manifolds by volume-preserving symmetries. The reduced phase space is a symplectic two-manifold, on which the motion derives from a Hamiltonian. The Hamiltonian is a direct generalization of the Bernoulli integral from ideal hydrodynamics, and can be represented as a sum of pressure-like and kinetic-energy-like parts. Aided by our reduction procedure, we found globally defined coordinates in which the three-dimensional vector field takes a particularly simple form. Based on this, we found a contact form which makes the three-manifold in question an exact contact manifold. We have illustrated the utility of all these concepts in three-dimensional steady and two-dimensional unsteady ideal hydrodynamics, geostrophic flows and ideal magnetohydrodynamics.

Our results can be extended to the case when the vector fields v and w are defined on a manifold with boundary. The situation in which v is tangent to the boundary is natural in fluid mechanics, a typical example being the rotationally symmetric flow in a body of revolution. In such a case, we require the boundary of the flow to be invariant under the action of both v and w. As a result, one can take the quotient space of the boundary with respect to the symmetry and perform the reduction separately on the boundary.

The above approach works in the case of two-dimensional boundaries, but naturally fails for singular, one-dimensional boundaries, such as an axis of revolution. Since such boundaries often occur in applications, there is certainly a need for the extension of our theory to cover singularities. We believe that some of the ideas of Arms *et al* [3], Cushman and Sjamaar [12], and Lerman and Sjamaar [17] can be adopted to establish the analogue of singular symplectic and Poisson reductions for the volume-preserving case.

It is known that geometric phases in mechanics can be viewed as coming from the reconstruction process of the vector field from the reduced phase space [18, 20]. The work on geometric phases in two-dimensional (i.e. translation-invariant) flows was done by Newton [23]. As these flows are translationally symmetric and time dependent, the occurrence of geometric phases in them could possibly be linked with our theory. Another possible development would be to link symmetry considerations developed here with the issues of stability of inviscid fluid flows, a topic discussed in Chern and Marsden [11]. The theory presented here may also have interesting consequences for the statistical mechanics of three-dimensional Euler flows.

Finally, let us mention that the theory developed here might fit into a more general framework of multisymplectic geometry [21].

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