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Relativistic Quantum Mechanics

Wave Equations

With a Foreword by D.A. Bromley

With 62 Figures and 89 worked examples and problems

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. A Wave Equation for Spin-$\frac{1}{2}$ Particles — The Dirac Equation

'e follow the historical approach of ***Dirac*** who, in 1928, searched for a relativistic ovariant wave equation of the Schrödinger form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{2.1}$$

ith positive definite probability density. At that time there were doubts concerning the lein-Gordon equation, which did not yield such probability density [see (1.29)]. The harge density interpretation was not known at that time and would have made little hysical sense, because π^+ and π^- mesons as charged spin-0 particles had not yet been scovered.

Since an equation in the form (2.1) is linear in the time derivative, it is natural to try construct a Hamiltonian that is also linear in the spatial derivatives (equality of spatial nd temporal coordinates). Hence, the desired equation (2.1) has to be of the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \left[\frac{\hbar c}{\mathrm{i}}\left(\hat{\alpha}_1\frac{\partial}{\partial x^1} + \hat{\alpha}_2\frac{\partial}{\partial x^2} + \hat{\alpha}_3\frac{\partial}{\partial x^3}\right) + \hat{\beta}m_0c^2\right]\psi \equiv \hat{H}_\mathrm{f}\psi \quad . \tag{2.2}$$

he – yet unknown – coefficients $\hat{\alpha}_i$ cannot be simple numbers, otherwise (2.2) would ot be form invariant with respect to simple spatial rotations. We suspect that the $\hat{\alpha}_i$ are atrices and indicate this by the operator sign $\wedge$. Then ψ cannot be a simple scalar, but as to be a column vector

$$\psi = \begin{pmatrix} \psi_1(\boldsymbol{x}, t) \\ \psi_2(\boldsymbol{x}, t) \\ \vdots \\ \psi_N(\boldsymbol{x}, t) \end{pmatrix} \quad , \tag{2.3}$$

om which a positive definite density of the form

$$\varrho(x) = \psi^\dagger\psi(x) = (\psi_1^*, \psi_2^*, \dots, \psi_N^*)\begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_N \end{pmatrix} = \sum_{i=1}^{N}\psi_i^*\psi_i(x) \tag{2.4}$$

an be constructed immediately. We still have to show that $\varrho(x)$ is the temporal component f a four-vector (current) for which a continuity equation must exist so that the spatial tegral $\int \varrho d^3x$ becomes constant in time. Only then is the probability interpretation of (x) ensured. It is clear that the wave function ψ in (2.3) is a column vector analogous to the spin wave functions of the Pauli equation[1]. Hence, we shall call them spinors, specifying this name later. The dimension N of the spinor is not yet known, but we will be able to decide this soon. The coefficients $\hat{\alpha}_i$ and $\hat{\beta}$ must obviously be quadratic $N \times N$ matrices so that a column vector of dimension N stands on the lhs as well as on the rhs (2.2). Thus the Schrödinger-like equation (2.1) and (2.2) represents a system of N coupled first-order differential equations of the spinor components ψ_i, $i = 1, 2, \ldots, N$. We also indicate this point in the notation and write (2.2) in the form

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial\psi_\sigma}{\partial t} &= \frac{\hbar c}{\mathrm{i}}\sum_{\tau=1}^{N}\left(\hat{\alpha}_1\frac{\partial}{\partial x^1}+\hat{\alpha}_2\frac{\partial}{\partial x^2}+\hat{\alpha}_3\frac{\partial}{\partial x^3}\right)_{\sigma\tau}\psi_\tau + m_0c^2\sum_{\tau=1}^{N}\hat{\beta}_{\sigma\tau}\psi_\tau \\ &\equiv \sum_{\tau=1}^{N}(\hat{H}_\mathrm{f})_{\sigma\tau}\psi_\tau \quad . \end{aligned} \tag{2.5}$$

Equation (2.2) is a short form of (2.5), in which the four $N \times N$ matrices $(\hat{\alpha}_i)_{\sigma\tau}$ $(i = 1, 2, 3)$ and $\hat{\beta}_{\sigma\tau}$ are expressed in the usual abbreviated form for matrices by $\hat{\alpha}_i$ $(i = 1, 2, 3)$ and $\hat{\beta}$ respectively. To continue, we demand the following natural properties:

a) the correct energy – momentum relation for a relativistic free particle

$$E^2 = \boldsymbol{p}^2c^2 + m_0^2c^4 \quad , \tag{2.6}$$

b) the continuity equation for the density (2.4) and

c) the Lorentz covariance (i.e. Lorentz form-invariance) for (2.2) and (2.5), respectively.

To fulfill requirement a), every single component ψ_σ of the spinor ψ has to satisfy the Klein-Gordon equation[2], i.e.

$$-\hbar^2\frac{\partial^2\psi_\sigma}{\partial t^2} = \left(-\hbar^2c^2\nabla^2 + m_0^2c^4\right)\psi_\sigma \quad . \tag{2.7}$$

On the other hand, from (2.2) it follows by iteration that

$$\begin{aligned} -\hbar^2\frac{\partial^2\psi}{\partial t^2} =& -\hbar^2c^2\sum_{i,j=1}^{3}\frac{\hat{\alpha}_i\hat{\alpha}_j+\hat{\alpha}_j\hat{\alpha}_i}{2}\frac{\partial^2\psi}{\partial x^i\partial x^j} \\ &+\frac{\hbar m_0c^3}{\mathrm{i}}\sum_{i=1}^{3}\left(\hat{\alpha}_i\hat{\beta}+\hat{\beta}\hat{\alpha}_i\right)\frac{\partial\psi}{\partial x^i} + \hat{\beta}^2m_0^2c^4\psi \quad . \end{aligned}$$

[1] See Vol. 1 of this series, *Quantum Mechanics – An Introduction* (Springer, Berlin, Heidelberg 1989) Chaps. 12, 13.

[2] Notice that the analogy to classical electrodynamics, where the six electromagnetic fields E_x, E_y, E_z, H_x, $\mathrm{H_y}$, H_z satisfy the first-order differential equations (Maxwell equations)

$$(\nabla\times\boldsymbol{H}) = \frac{\partial\boldsymbol{E}}{c\partial t}, \quad (\nabla\times\boldsymbol{E}) = -\frac{\partial\boldsymbol{H}}{c\partial t}, \quad \nabla\cdot\boldsymbol{E}=0, \quad \nabla\cdot\boldsymbol{B}=0$$

in a vacuum. Each single component E_i and H_i satisfies simultaneously the differential equation of the second order (wave equation)

$$\left(\boldsymbol{\nabla}^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E_i=0 \quad \text{and} \quad \left(\boldsymbol{\nabla}^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)H_i=0 \quad .$$

For further discussion of this analogy we refer to Exercise 2.1.

Comparison with (2.7) shows the following requirements for the matrices $\hat{\alpha}_i$, $\hat{\beta}$:

$$\begin{aligned}\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i &= 2\delta_{ij}\mathbb{1} \quad ,\\ \hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i &= 0 \quad ,\\ \hat{\alpha}_i^2 = \hat{\beta}^2 &= \mathbb{1} \quad .\end{aligned} \tag{2.8}$$

These anticommutation relations define an algebra for the ψ matrices. In order to establish hermiticity of the Hamiltonian $\hat{H}_\mathrm{f}$ in (2.2), the matrices $\hat{\alpha}_i$, $\hat{\beta}$ also have to be Hermitian; thus,

$$\hat{\alpha}_i^\dagger = \hat{\alpha}_i \quad , \quad \hat{\beta}^\dagger = \hat{\beta} \quad . \tag{2.9}$$

Therefore, the eigenvalues of the matrices are real. Since, according to (2.8), one has $\hat{\alpha}_i^2 = 1$ and $\hat{\beta}^2 = 1$, it follows that the eigenvalues can only have the values ± 1. Because the eigenvalues are independent of the special representation[3] this can best be shown in the diagonal representation of the single matrices. For example, $\hat{\alpha}_i$ in its eigenrepresentation has the form

$$\hat{\alpha}_i = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_N \end{pmatrix} \quad ,$$

with the eigenvalues $A_1, \ldots, A_N$, and (2.8) now yields

$$\hat{\alpha}_i^2 = \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ .. & .. & .. & \ddots \end{pmatrix} = \begin{pmatrix} A_1^2 & 0 & \cdots\cdots \\ 0 & A_2^2 & \cdots\cdots \\ \vdots & & \ddots \\ .. & .. & ...A_N^2 \end{pmatrix} \quad ,$$

from which

$$A_k^2 = 1 \quad , \quad \text{i.e.} \quad A_k = \pm 1 \quad . \tag{2.10}$$

Furthermore, from the anticommutation relations (2.8) it follows that the trace (i.e. the sum of the diagonal elements of the matrix) of each $\hat{\alpha}_i$ and of $\hat{\beta}$ has to be zero. Namely, according to (2.8) one has

$$\hat{\alpha}_i = -\hat{\beta}\hat{\alpha}_i\hat{\beta} \quad .$$

Because of the identity

[3] This follows, because $\hat{A}\psi_\alpha = \alpha\psi_\alpha$ implies that

$$\hat{U}\hat{A}\hat{U}^{-1}\hat{U}\psi_\alpha = \alpha\hat{U}\psi_\alpha \quad ,$$

and, therefore,

$$\hat{A}'\left(\hat{U}\psi_\alpha\right) = \alpha\left(\hat{U}\psi_\alpha\right) \quad .$$

The solutions of the rotated matrix $\hat{A}' = \hat{U}\hat{A}\hat{U}^{-1}$ are just the rotated vectors $\psi'_\alpha = \hat{U}\psi_\alpha$ with the same eigenvalues α.

$$\operatorname{tr} \hat{A}\hat{B} = -\operatorname{tr} \hat{B}\hat{A} \quad ,$$

one concludes that

$$\operatorname{tr} \hat{\alpha}_i = \operatorname{tr} \hat{\beta}^2\hat{\alpha}_i = \operatorname{tr} \hat{\beta}\hat{\alpha}_i\hat{\beta} = -\operatorname{tr} \hat{\alpha}_i \Rightarrow \operatorname{tr} \hat{\alpha}_i = 0 \quad . \tag{2.11}$$

The trace of a matrix is always equal to the sum of its eigenvalues, which can be seen if $\hat{U}$ transforms the matrix $\hat{\alpha}_i$ into its diagonal form,

$$\begin{pmatrix} A_1 & 0 & \cdots\cdots \\ 0 & A_2 & \cdots\cdots \\ \vdots & & \vdots \\ .. & .. & ...A_N \end{pmatrix} = \hat{U}\hat{\alpha}_i\hat{U}^{-1} \quad .$$

Then

$$\operatorname{tr} \begin{pmatrix} A_1 & 0 & \cdots\cdots \\ 0 & A_2 & \cdots\cdots \\ \vdots & & \ddots \\ .. & .. & ...A_N \end{pmatrix} = \sum_{k=1}^{N} A_k = \operatorname{tr} \hat{U}\hat{\alpha}_i\hat{U}^{-1} \quad = \operatorname{tr} \hat{\alpha}_i\hat{U}\hat{U}^{-1} \quad = \operatorname{tr} \hat{\alpha}_i \quad ,$$

which proves the above statement. Because the eigenvalues of $\hat{\alpha}_i$ and $\hat{\beta}$ are equal to ± 1, each matrix $\hat{\alpha}_i$ and $\hat{\beta}$ has to possess as many positive as negative eigenvalues, and therefore has to be of even dimension. The smallest even dimension, $N = 2$, can not be right, because only three anticommuting matrices exist, namely the three Pauli[4] matrices $\hat{\sigma}_i$. Therefore, the smallest dimension for which the requirements (2.8) can be fulfilled is $N = 4$. We now study this case in more detail and indicate immediately one possible explicit representation of the Dirac matrices, i.e.

$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \quad , \quad \hat{\beta} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad , \tag{2.12}$$

where $\hat{\sigma}_i$ are Pauli's 2×2 matrices and 1 is the 2×2 unit matrix. With the explicit form of the Pauli matrices of (1.65), we have, in detail,

$$\hat{\alpha}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \hat{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix} \quad ,$$

$$\hat{\alpha}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad , \quad \hat{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad . \tag{2.13}$$

Indeed, we can easily check the validity of the relations (2.8). For example,

[4] See Vol. 1 of this series, *Quantum Mechanics – An Introduction* (Springer, Berlin, Heidelberg 1989), Chaps. 12, 13 and especially Exercise 13.1.

$$\begin{aligned}\hat{\alpha}_i\hat{\alpha}_j+\hat{\alpha}_j\hat{\alpha}_i &= \begin{pmatrix}0 & \hat{\sigma}_i\\ \hat{\sigma}_i & 0\end{pmatrix}\begin{pmatrix}0 & \hat{\sigma}_j\\ \hat{\sigma}_j & 0\end{pmatrix}+\begin{pmatrix}0 & \hat{\sigma}_j\\ \hat{\sigma}_j & 0\end{pmatrix}\begin{pmatrix}0 & \hat{\sigma}_i\\ \hat{\sigma}_i & 0\end{pmatrix}\\ &= \begin{pmatrix}\hat{\sigma}_i\hat{\sigma}_j & 0\\ 0 & \hat{\sigma}_i\hat{\sigma}_j\end{pmatrix}+\begin{pmatrix}\hat{\sigma}_j\hat{\sigma}_i & 0\\ 0 & \hat{\sigma}_j\hat{\sigma}_i\end{pmatrix}\\ &= \begin{pmatrix}\hat{\sigma}_i\hat{\sigma}_j+\hat{\sigma}_j\hat{\sigma}_i & 0\\ 0 & \hat{\sigma}_i\hat{\sigma}_j+\hat{\sigma}_j\hat{\sigma}_i\end{pmatrix}=\begin{pmatrix}2\delta_{ij}\mathbb{1} & 0\\ 0 & 2\delta_{ij}\mathbb{1}\end{pmatrix}\\ &= 2\delta_{ij}\begin{pmatrix}\mathbb{1} & 0\\ 0 & \mathbb{1}\end{pmatrix}\quad ,\end{aligned}$$

holds. Here we have used the relation for the Pauli matrices[5]

$$\hat{\sigma}_i\hat{\sigma}_j+\hat{\sigma}_j\hat{\sigma}_i=2\delta_{ij}\mathbb{1}\quad . \tag{2.14}$$

We also notice that (2.12) describes just one possible choice of the Dirac matrices $\hat{\alpha}_i$, $\hat{\beta}$. Each set $\hat{\alpha}'_i=\hat{U}\hat{\alpha}_i\hat{U}^{-1}$, $\hat{\beta}'=\hat{U}\hat{\beta}\hat{U}^{-1}$, which is obtained from the original $\hat{\alpha}_i$, $\hat{\beta}$ of (2.13) by a unitary transformation $\hat{U}$, can be used equally as well as the one introduced here [see (2.21)]. In Example 3.1 it will be shown that all representations of the Dirac algebra are unitarily equivalent to each other. Therefore, physical results do not depend on the special choice of the Dirac matrices $\hat{\alpha}_i$ and $\hat{\beta}$, but the calculations can become particularly simple in a certain representation.

Next we want to construct the four-current density and the equation of continuity. For that we multiply (2.2) from the left by $\psi^\dagger=(\psi_1^*,\psi_2^*,\psi_3^*,\psi_4^*)$ and obtain

$$\mathrm{i}\hbar\psi^\dagger\frac{\partial}{\partial t}\psi=\frac{\hbar c}{\mathrm{i}}\sum_{k=1}^{3}\psi^\dagger\hat{\alpha}_k\frac{\partial}{\partial x^k}\psi+m_0c^2\psi^\dagger\hat{\beta}\psi\quad . \tag{2.15a}$$

Furthermore, we form the Hermitian conjugate of (2.12), i.e.

$$-\mathrm{i}\hbar\frac{\partial\psi^\dagger}{\partial t}=-\frac{\hbar c}{\mathrm{i}}\sum_{k=1}^{3}\frac{\partial\psi^\dagger}{\partial x^k}\hat{\alpha}_k^\dagger+m_0c^2\psi^\dagger\hat{\beta}^\dagger\quad ,$$

and multiply this equation from the right by ψ, taking into consideration the hermicity of the Dirac matrices ($\hat{\alpha}_i^\dagger=\hat{\alpha}_i$, $\hat{\beta}^\dagger=\hat{\beta}$), to give

$$-\mathrm{i}\hbar\frac{\partial\psi^\dagger}{\partial t}\psi=-\frac{\hbar c}{\mathrm{i}}\sum_{k=1}^{3}\frac{\partial\psi^\dagger}{\partial x^k}\hat{\alpha}_k\psi+m_0c^2\psi^\dagger\hat{\beta}\psi\quad . \tag{2.15b}$$

Then, subtraction of (2.15b) from (2.15a) yields

$$\mathrm{i}\hbar\frac{\partial}{\partial t}(\psi^\dagger\psi)=\frac{\hbar c}{\mathrm{i}}\sum_{k=1}^{3}\frac{\partial}{\partial x^k}(\psi^\dagger\hat{\alpha}_k\psi) \tag{2.16}$$

or

$$\frac{\partial\varrho}{\partial t}+\operatorname{div}\boldsymbol{j}=0\quad , \tag{2.17}$$

[5] This relation is covered in detail in Vol. 1 of this series, *Quantum Mechanics – An Introduction* (Springer, Berlin, Heidelberg 1989).

where

$$\varrho = \psi^\dagger \psi = \sum_{i=1}^{4} \psi_i^* \psi_i \tag{2.18a}$$

is the positive definite density (2.4) and

$$j^k = c\psi^\dagger \hat{\alpha}^k \psi \quad \text{or} \quad \boldsymbol{j} = c\psi^\dagger \hat{\boldsymbol{\alpha}} \psi \tag{2.18b}$$

is the *current density*. Here we have symbolically introduced the three-vector

$$\hat{\alpha} = \{\hat{\alpha}^1, \hat{\alpha}^2, \hat{\alpha}^3\} = \{-\hat{\alpha}_1, -\hat{\alpha}_2, -\hat{\alpha}_3\} \tag{2.19}$$

and introduced the upper and lower indices according to our former convention [see (1.5) and (1.6)]. From (2.17) the conservation law follows immediately in the usual way

$$\frac{\partial}{\partial t} \int_V d^3x \psi^\dagger \psi = -\int_V \operatorname{div} \boldsymbol{j} d^3x = -\int_F \boldsymbol{j} \cdot d\boldsymbol{f} = 0 \tag{2.20}$$

where V denotes a certain volume and F its surface. Since ϱ is positive definite and because of the conservation law (2.17) we can accept the interpretation of ϱ as a probability density [in contrast to the density ϱ obtained for the Klein-Gordon equation, see (1.29) which was not positive definite]. Accordingly, we call $\boldsymbol{j}$ the *probability current density*. Here we have presumed that $\boldsymbol{j}$ is a vector, i.e. that its components (2.18b) transform under spatial rotations as the components of a three-vector. This still has to be shown. Furthermore, $\{c\varrho, \boldsymbol{j}\}$ should form a four-vector. Hence, it should transform from one inertial system into another one by a Lorentz transformation. This point and, in addition, the form invariance of the Dirac equation (2.2) with respect to Lorentz transformations (we also call the form invariance *covariance*) have still to be shown, before we can regard the Dirac equation as an acceptable relativistic wave equation.

We also notice that we have achieved a special representation with (2.12). The choice of the matrices (2.12) is not unequivocal. One recognizes immediately that each unitary transformation $\hat{S}$ yields the matrices

$$\hat{\alpha}_i' = \hat{S}\hat{\alpha}_i\hat{S}^{-1} \quad , \quad \hat{\beta}' = \hat{S}\hat{\beta}\hat{S}^{-1} \tag{2.21}$$

which also satisfy the algebra (2.8). We check this for the first commutator (2.8), as an example:

$$\begin{aligned} &\hat{S}\hat{\alpha}_i\hat{S}^{-1}\hat{S}\hat{\alpha}_j\hat{S}^{-1} + \hat{S}\hat{\alpha}_j\hat{S}^{-1}\hat{S}\hat{\alpha}_i\hat{S}^{-1} = 2\delta_{ij}\hat{S}\mathbb{1}\hat{S}^{-1} \\ &\quad \Rightarrow \hat{\alpha}_i'\hat{\alpha}_j' + \hat{\alpha}_j'\hat{\alpha}_i' = 2\delta_{ij}\mathbb{1} \end{aligned} \qquad \text{(q.e.d.)}$$

Maxwell Equations
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$$H - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j} \quad , \qquad \text{(a)}$$

in the form analogous to the Dirac equation (spinor equation):

$$-\frac{1}{\mathrm{i}}\sum_{j=0}^{3}\hat{\alpha}^j \frac{\partial}{\partial x^j}\psi = -\frac{4\pi}{c}\Phi \quad . \qquad \text{(b)}$$

2.1 Free Motion of a Dirac Particle

We examine the solution of the free Diraç equation (2.2) (that is, the Dirac equation without potentials) and again write it in the form

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = \hat{H}_{\mathrm{f}}\psi = \left(c\hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}} + m_0c^2\hat{\beta}\right)\psi \quad . \tag{2.22}$$

Its stationary states are found with the ansatz

$$\psi(\boldsymbol{x}, t) = \psi(\boldsymbol{x})\exp[-(\mathrm{i}/\hbar)\varepsilon t] \quad , \tag{2.23}$$

which transforms (2.2) into

$$\varepsilon\psi(\boldsymbol{x}) = \hat{H}_{\mathrm{f}}\psi(\boldsymbol{x}) \quad . \tag{2.24}$$

Again the quantity ε describes the time evolution of the stationary state $\psi(\boldsymbol{x})$. For many applications it is useful to split up the four-component spinor into two two-component spinors ϕ and χ, i.e.

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}\varphi\\ \chi\end{pmatrix} \quad \text{with} \tag{2.25a}$$

$$\varphi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \quad \text{and} \quad \chi = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix} \quad . \tag{2.25b}$$

Using the explicit form (2.12) for the $\hat{\alpha}$ and $\hat{\beta}$ matrices (2.24) can be written as

$$\varepsilon\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = c\begin{pmatrix}0 & \hat{\boldsymbol{\sigma}}\\ \hat{\boldsymbol{\sigma}} & 0\end{pmatrix}\cdot\hat{\boldsymbol{p}}\begin{pmatrix}\varphi\\ \chi\end{pmatrix} + m_0c^2\begin{pmatrix}1\!\!1 & 0\\ 0 & -1\!\!1\end{pmatrix}\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$$

or

$$\begin{aligned}\varepsilon\varphi &= c\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}\chi + m_0c^2\varphi \quad ,\\ \varepsilon\chi &= c\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}\varphi - m_0c^2\chi \quad .\end{aligned} \tag{2.26}$$

States with definite momentum $\boldsymbol{p}$ are

$$\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = \begin{pmatrix}\varphi_0\\ \chi_0\end{pmatrix}\exp[(\mathrm{i}/\hbar)\boldsymbol{p}\cdot\boldsymbol{x}] \quad . \tag{2.27}$$

The equations (2.26) are transformed into the same equations for ϕ_0 and χ_0, but replacing the operators $\hat{\boldsymbol{p}}$ by the eigenvalues $\boldsymbol{p}$. Ordering with respect to ϕ_0 and χ_0 results in the system of equations

$$\begin{aligned}&\left(\varepsilon - m_0c^2\right)1\!\!1\varphi_0 - c\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{p}\chi_0 = 0 \quad ,\\ &-c\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{p}\varphi_0 + \left(\varepsilon + m_0c^2\right)1\!\!1\chi_0 = 0 \quad .\end{aligned} \tag{2.28}$$

This linear homogenous system of equations for ϕ_0 and χ_0 has nontrivial solutions only in the case of a vanishing determinant of the coefficients, that is

$$\begin{vmatrix} \left(\varepsilon - m_0c^2\right)\mathbb{1} & -c\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{p} \\ -c\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{p} & \left(\varepsilon + m_0c^2\right)\mathbb{1} \end{vmatrix} = 0 \quad . \tag{2.29}$$

Using the relation[6]

$$(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{A})(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{B}) = \boldsymbol{A}\cdot\boldsymbol{B}\mathbb{1} + \mathrm{i}\hat{\boldsymbol{\sigma}}\cdot(\boldsymbol{A}\times\boldsymbol{B}) \quad , \tag{2.30}$$

equation (2.29) transforms into

$$\begin{aligned} &\left(\varepsilon^2 - m_0^2c^4\right)\mathbb{1} - c^2(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{p})(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{p}) = 0 \quad , \\ &\varepsilon^2 = m_0^2c^4 + c^2\boldsymbol{p}^2 \quad , \end{aligned}$$

from which follows

$$\varepsilon = \pm E_p \quad , \quad E_p = +c\sqrt{\boldsymbol{p}^2 + m_0^2c^2} \quad . \tag{2.31}$$

The two signs of the time-evolution factor ε correspond to two types of solutions of the Dirac equation. We call them *positive* and *negative* solutions, respectively. From (2.28), for fixed ε,

$$\chi_0 = \frac{c(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{p})}{m_0c^2 + \varepsilon}\varphi_0 \quad . \tag{2.32}$$

Let us denote the two-spinor φ_0 in the form

$$\varphi_0 = U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \quad , \tag{2.33}$$

with the normalization

$$U^\dagger U = U_1^*U_1 + U_2^*U_2 = 1 \quad ,$$

where U_1, U_2 are complex. Using (2.27) and (2.23) we obtain the complete set of *positive and negative free solutions of the Dirac equation as*

$$\Psi_{\boldsymbol{p}\lambda}(\boldsymbol{x},t) = N\begin{pmatrix} U \\ \dfrac{c(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{p})}{m_0c^2 + \lambda E_p}U \end{pmatrix} \frac{\exp\left[\mathrm{i}(\boldsymbol{p}\cdot\boldsymbol{x} - \overbrace{\lambda E_p}^{\varepsilon}t)/\hbar\right]}{\sqrt{2\pi\hbar}^3} \quad . \tag{2.34}$$

Here $\lambda = \pm 1$ characterizes the positive and negative solutions with the time evolution factor $\varepsilon = \lambda E_p$. The normalization factor N is determined from the condition

$$\int \Psi_{\boldsymbol{p}\lambda}^\dagger(\boldsymbol{x},t)\Psi_{\boldsymbol{p}'\lambda'}(\boldsymbol{x},t)d^3x = \delta_{\lambda\lambda'}\delta(\boldsymbol{p}-\boldsymbol{p}') \quad . \tag{2.35}$$

Hence,

[6] Encountered previously in Vol. 1 of this series, *Quantum Mechanics – An Introduction* (Springer, Berlin, Heidelberg 1989), Exercise 13.2.