Dynamics and Symmetry

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CHAPTER 1

Groups

The topic of this chapter is groups: finite, topological and Lie groups. We start with a quick review of basic definitions from the theory of groups and then describe some of the foundational theory of topological and, more particularly, Lie groups. As far as the theory is concerned, our focus will largely be on basic examples, constructions and definitions. We usually omit proofs. A background in differential manifolds – especially Lie brackets and integration – is required for the latter parts of the chapter. With the possible exception of one or two of the examples (only needed in chapter 4), readers are likely to be familiar with most of what we survey. A good strategy is probably to skim through the chapter noting the notational conventions which are established and used throughout the book. Finally, we remark that there are many good introductory texts on finite groups – one we recommend is Scott [**157**]. A concise graduate level text, which covers much of what we need in this and the following chapter, is Thomas's book [**169**] which is angled towards the representation theory of finite and Lie groups and includes basic material on induced representations and Lie algebras.

1.1. Definition of a group and examples

DEFINITION 1.1.1. A group consists of a set G with an *identity* element, denoted by $e_G = e$, together with operators of composition (or *multiplication*)

$$G \times G \to G;\ (g, h) \mapsto gh,$$

and *inversion*

$$G \to G;\ g \to g^{-1},$$

which satisfy the following properties

(Id) (Identity) $ge = eg = g$, for all $g \in G$.
(In) (Inverse) For all $g \in G$, $gg^{-1} = g^{-1}g = e$.
(As) (Associativity) $(gh)k = g(hk)$, for all $g, h, k \in G$.

The group is *Abelian* or *commutative* if $gh = hg$, for all $g, h \in G$.

REMARK 1.1.2. As simple consequences of the definition we have

(1) (Cancellation law) If $gh = g\bar{h}$ (or $hg = \bar{h}g$) then $h = \bar{h}$.
(2) The identity element of G is unique. (That is, if $e' \in G$ satisfies (Id) then $e = e'$).
(3) Every $g \in G$ has a unique inverse g^{-1}.

(4) $(gh)^{-1} = h^{-1}g^{-1}$, for all $g, h \in G$.

If G is finite, the *order* of G, denoted $|G|$, is the number of elements in G.

DEFINITION 1.1.3. Let G be a group. A non-empty subset H of G is a *subgroup* of G if for all $g, h \in H$, $gh^{-1} \in H$.

REMARK 1.1.4. If H is a subgroup then $e \in H$ and H inherits the structure of a group from G.

Many interesting examples of groups, both finite and infinite, are obtained as *transformation groups*. That is, as sets of transformations of a space, often preserving some preassigned structure. Group multiplication is then composition of transformations and so is automatically associative. We usually denote the identity map of a transformation group of the set X by I or I_X.

EXAMPLES 1.1.5. (1) Let X be a set. If we let $\mathcal{B}(X)$ denote the set of bijections of X, then $\mathcal{B}(X)$ is a group with identity element equal to the identity transformation of X. If X is finite, $\mathcal{B}(X)$ may be identified with the group $\mathrm{Sym}(X)$ of all permutations of X. We refer to $\mathrm{Sym}(X)$ as the *symmetric group* of X. If $X = \mathbf{n} = \{1, \ldots, n\}$, we write $\mathcal{B}(X) = S_n$ – the *symmetric group* on n-symbols – and have $|S_n| = n!$ The symmetric groups are of special importance in finite group theory; in part this is because every finite group G can be represented as a subgroup of $\mathrm{Sym}(G) \cong S_{|G|}$ (Cayley's theorem).
(2) Let V be a vector space (over $\mathbb{R}$ or $\mathbb{C}$). The *general linear group* of V, $\mathrm{GL}(V)$, is the group of invertible linear transformations of V. If $V = \mathbb{R}$ then $\mathrm{GL}(\mathbb{R}) \approx \mathbb{R}^\star$ (the multiplicative group of nonzero real numbers). Similarly, $\mathrm{GL}(\mathbb{C}) \approx \mathbb{C}^\star$ (the multiplicative group of nonzero complex numbers). If V is of dimension d then, after choosing a basis for V, we may identify $\mathrm{GL}(V)$ with an open subset of the space $M(d, d)$ of $d \times d$-matrices. The group operations of composition and inverse are then smooth – in fact, rational – functions in the components of the matrices. We often write $\mathrm{GL}(n, \mathbb{R})$, instead of $\mathrm{GL}(\mathbb{R}^n)$, and $\mathrm{GL}(n, \mathbb{C})$, instead of $\mathrm{GL}(\mathbb{C}^n)$.
(3) Let V be a vector space over $\mathbb{R}$ and let $(\ ,\ )$ be a (positive definite) inner product on V. Let $\mathrm{O}(V)$ denote the subgroup of $\mathrm{GL}(V)$ consisting of linear maps A preserving $(\ ,\ )$: $(Av, Aw) = (v, w)$, all $v, w \in V$. We refer to $\mathrm{O}(V)$ as the *orthogonal* group of $(V, (\ ,\ ))$. If V is finite dimensional, we may choose an orthonormal basis of V and thereby identify $(V, (\ ,\ ))$ with $\mathbb{R}^n$ (standard Euclidian inner product). We always write the orthogonal group of $\mathbb{R}^n$ as $\mathrm{O}(n)$. If instead V is a $\mathbb{C}$-vector space, and $(\ ,\ )$ is an Hermitian inner product on V, we obtain the *unitary* group $\mathrm{U}(V)$ of V. We write $\mathrm{U}(\mathbb{C}^d) = \mathrm{U}(d)$. Both $\mathrm{O}(n)$ and $\mathrm{U}(d)$ are compact subgroups of the corresponding general linear group. The *special orthogonal* group $\mathrm{SO}(n)$ is the subgroup of $\mathrm{O}(n)$ consisting of linear maps of determinant $+1$. We similarly define the special unitary group $\mathrm{SU}(n)$.
(4) Let (X, d) be a metric space. An *isometry* of X is a map $f : X \to X$ preserving distance: $d(f(x), f(y)) = d(x, y)$, all $x, y \in X$. Let $\mathrm{Iso}(X)$ denote the set of isometries of X. The identity map I of X always lies in $\mathrm{Iso}(X)$ and $\mathrm{Iso}(X)$

has the structure of a group where multiplication is given by composition.
(5) The group of isometries of $\mathbb{R}^n$, denoted $\mathbf{E}(n)$, is called the *Euclidean* group. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then we may write $f(x) = Ax + b$, where $A \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$ are uniquely determined by f. In particular, $\mathrm{O}(n)$ (orthogonal rotations about the origin) and $T(n)$ (the group of all translations of $\mathbb{R}^n$) are naturally defined as subgroups of $\mathbf{E}(n)$. The group $\mathbf{E}(n)$ may be represented as $\mathrm{O}(n) \times T(n)$ (that is, all pairs (A, b)), but the group structure on $O(n) \times T(n)$ is not the obvious one defined componentwise. We return to this point shortly. We let $\mathbf{SE}(n)$ (the *special Euclidean group*) be the subgroup of $\mathbf{E}(n)$ consisting of orientation preserving isometries. We may identify $\mathbf{SE}(n)$ with $\mathrm{SO}(n) \times T(n)$.
(6) For $n \geq 3$, let $\mathbf{D}_n$ denote the dihedral group of order $2n$ defined (up to isomorphism) as the subgroup of $\mathbf{O}(2)$ consisting of isometries of a regular n-gon centered at the origin. Denote the subgroup of $\mathbf{D}_n$ consisting of orientation preserving symmetries by $\mathbb{Z}_n$ (or $\mathbb{Z}/n\mathbb{Z}$). Since $\mathbf{D}_n$ permutes the n vertices of a regular n-gon, $\mathbf{D}_n$ naturally embeds as a subgroup of S_n. In case $n = 2$, we define $\mathbf{D}_2$ to be the group of isometries of a (non-square) rectangle. The groups $\mathbb{Z}_n$, $n \geq 2$, and $\mathbf{D}_2$ are Abelian; $\mathbf{D}_n$ is not Abelian, $n \geq 3$.

EXERCISE 1.1.6. Prove that $\mathrm{Iso}(\mathbb{R}^n) = \mathbf{E}(n)$. (Hint: In case $n = 2$, show that an isometry is uniquely determined by its values at three non-collinear points.)

1.2. Homomorphisms, subgroups, cosets and quotient groups

DEFINITION 1.2.1. A *homomorphism* $T : G \to K$ of groups G, K is a mapping satisfying

$$T(gg') = T(g)T(g'),\ (g, g' \in G).$$

An *isomorphism* is a bijective homomorphism.

REMARKS 1.2.2. (1) If $T : G \to K$ is a homomorphism, then $T(e_G) = e_K$.
(2) If $T : G \to K$ is an isomorphism, then $T^{-1} : K \to G$ is a homomorphism.
(3) An isomorphism $T : G \to G$ is usually referred to as an *automorphism* (of G). If there exists $h \in G$ such that $T(g) = hgh^{-1}$, T is an *inner automorphism*. The set of automorphisms $\mathrm{Aut}(G)$ of G is a group under composition which contains the set of inner automorphisms as a subgroup.

DEFINITION 1.2.3. Let H be a subgroup of G.

(1) H is a *normal* subgroup if
$$gHg^{-1} = H,\ (g \in G).$$
If H is a normal subgroup of G, we write $H \lhd G$.
(2) The *normalizer* $N(H)$ of H is the subgroup of G defined by $N(H) = \{g \in G \mid gHg^{-1} = H\}$.
(3) The centralizer of a subgroup H of G is the subgroup of G defined by $C_G(H) = \{g \in G \mid gh = hg \quad \forall h \in H\}$. We call $C_G(G) = Z(G)$ the *centre* of G.

REMARK 1.2.4. Let H be a subgroup of G. Then $H \lhd N(H)$, $C_G(H) \subset N(H)$ and $C_G(H) \cap H = Z(H)$ – the centre of H.

LEMMA 1.2.5. *If $T : G \to K$ is a homomorphism then kernel$(T) \lhd G$ and image$(T) \lhd K$.*

LEMMA 1.2.6. *Let $G/H = \{gH \mid g \in G\}$ denote the space of (left) cosets. If $H \lhd G$, then G/H has the natural structure of a group with respect to which the quotient map $q : G \to G/H$ is a homomorphism.*

EXAMPLES 1.2.7. (1) Give $\mathbb{R}$ the structure of an (additive) group under $+$ (so the identity is 0). For $a \in \mathbb{R}^\star$, the map $M_a : \mathbb{Z} \to \mathbb{R}$ defined by $M_a(n) = an$ is a group monomorphism with image $a\mathbb{Z}$. The Abelian group $\mathbb{R}/a\mathbb{Z}$ is isomorphic to SO(2) by the map

$$\theta \mapsto \begin{pmatrix} \cos(\frac{2\pi\theta}{a}) & -\sin(\frac{2\pi\theta}{a}) \\ \sin(\frac{2\pi\theta}{a}) & \cos(\frac{2\pi\theta}{a}) \end{pmatrix}.$$

We often identify $\mathbb{R}/2\pi\mathbb{Z}$ (or $\mathbb{R}/\mathbb{Z}$) with SO(2). Another representation of SO(2) is as the subgroup $S^1 \subset \mathbb{C}^\star$ consisting of complex numbers of unit modulus. We tend to use the symbol S^1, as opposed to SO(2), when there is a direct connection to scalar multiplication by complex numbers of unit modulus. For $n \geq 1$, the n-torus $\mathbb{T}^n$ is defined to be $\mathbb{R}^n/2\pi\mathbb{Z}^n \approx \mathrm{SO}(2)^n \approx (S^1)^n$.
(2) If V is a finite dimensional vector space over $\mathbb{R}$, then the determinant defines a homomorphism $\det : \mathrm{GL}(V) \to \mathbb{R}^\star$. The kernel of det is the normal subgroup SL(V) consisting of linear maps of determinant 1. In case $V = \mathbb{R}^n$, we set $\mathrm{SL}(V) = \mathrm{SL}(n, \mathbb{R})$ and refer to $\mathrm{SL}(n, \mathbb{R})$ as the *special linear* group (of degree n). $\mathrm{SL}(n, \mathbb{R})$ is the group of orientation and volume preserving linear isomorphisms of $\mathbb{R}^n$. We have $GL(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{R}) \approx \mathbb{R}^\star$. We may similarly define $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{Z})$ (the group of integer $n \times n$-matrices with determinant $+1$).
(3) Let $n \geq 2$. Identify $\mathbb{C}^\star$ with the group of all non-zero multiples of the identity map of $\mathbb{C}^n$. Then $\mathbb{C}^\star \lhd \mathrm{GL}(n, \mathbb{C})$. We define $\mathrm{PGL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C})/\mathbb{C}^\star$ to be the *projective linear group*. In case $n = 2$, $\mathrm{PGL}(n, \mathbb{C})$ is isomorphic to the group of invertible Mobius transformations $z \mapsto \frac{az+b}{cz+d}$ of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$. We may similarly define the real projective linear group $\mathrm{PGL}(n, \mathbb{R})$.
(4) Many important examples of finite groups come from geometries defined over a finite field (see [**98**] for basic material on finite fields). Let p be a prime, $n \geq 1$ and $\mathbb{F} = \mathbb{F}_{p^n}$ denote the finite field of order p^n. Let $\mathrm{GL}(n, \mathbb{F}_{p^n}) = \mathrm{GL}(n, p^n)$ denote the group of invertible $n \times n$ matrices with entries in $\mathbb{F}$. Setting $p^n = q$, $|\mathrm{GL}(n, p^n)| = \Pi_{j=0}^{n-1}(q^n - q^j)$ [**157**, 5.7.20]. We define the projective group $\mathrm{PGL}(n, p^n)$ to be $\mathrm{GL}(n, p^n)/\mathbb{F}^\star$. Let $\mathrm{Aff}_1(\mathbb{F})$ denote the group of affine isomorphisms of $\mathbb{F}$. That is, $\mathrm{Aff}_1(\mathbb{F}) = \{(a, b) \mid a \in \mathbb{F}^\star, b \in \mathbb{F}\}$. An element $(a, b) \in \mathrm{Aff}_1(\mathbb{F})$ acts on $\mathbb{F}$ by $x \mapsto ax + b$ and it is easily shown that $|\mathrm{Aff}_1(\mathbb{F})| = q(q - 1)$.

1.2.1. Generators and relations for finite groups. Let $g_1, \ldots, g_k$ be non-identity elements of the finite group G. Let $\langle g_1, \ldots, g_k\rangle$ denote the subset of G

consisting of all finite products of the g_i. Since G is finite, every $g \in G$ has finite order. In particular, if g has order d then g has inverse g^{d-1}. Consequently, $\langle g_1, \dots, g_k \rangle$ defines a subgroup of G. We say G is *generated* by $g_1, \dots, g_k$ if $G = \langle g_1, \dots, g_k \rangle$. A finite group G may be specified precisely by a (minimal) set $g_1, \dots, g_k$ of generators together with a set of monomial *relations* $R_j(g_1, \dots, g_k) = e$, $j = 1, \dots, \ell$. The set of relations includes the order relations $g_i^{d_i} = e$, $i = 1, \dots, k$, as well as relations between generators. A homomorphism $h : G \to J$ is uniquely determined by the set of values $h(g_1), \dots, h(g_k)$ and is well defined provided that $R_j(h(g_1), \dots, h(g_k)) = e$, $j = 1, \dots, \ell$.

REMARK 1.2.8. If G is infinite, we define $\langle g_1, \dots, g_k \rangle$ to be the subgroup of G generated by all products of g_i and g_i^{-1}. We caution the reader that later, when we come to define topological groups, we use the notation $\langle g_1, \dots, g_k \rangle$ to denote the *closure* (in G) of the subgroup generated by $g_1, \dots, g_k$.

EXAMPLES 1.2.9. (1) The cyclic group $\mathbb{Z}_n \subset \mathrm{SO}(2)$ of order n can be generated by one element (for example, rotation through $2\pi/n$); the dihedral group $\mathbf{D}_n \subset \mathrm{O}(2)$, $n \geq 2$, can be generated by two elements one at least of which must reverse orientation. For example, if α corresponds to reflection in the x-axis and β to rotation through $2\pi/n$, then $\mathbf{D}_n = \langle \alpha, \beta \rangle$ and the defining relations are $\alpha^2 = \beta^n = e$, $(\alpha\beta)^2 = e$ (see also chapter 2).
(2) Let $\mathbb{F} = \mathbb{F}_{p^n}$, where $p \geq 3$ is prime. Then $\mathrm{Aff}_1(\mathbb{F})$ may be represented as a subgroup of S_{p^n} of order $p^n(p^n - 1)$ [**157**, Chapter 10].
(3) The projective group $\mathrm{PGL}(2, p^n)$ may be represented as a subgroup of S_{p^n+1}. This follows by noting that the associated projective space $P^1(\mathbb{F})$ is identified with $\mathbb{F} \cup \{\infty\}$ (for details see [**157**, 10.6.7–8]).

1.3. Constructions

DEFINITION 1.3.1. Let G, K be groups. The *direct product* of groups G and K is the group $G \times K$ with composition defined by

$$(g_1, k_1)(g_2, k_2) = (g_1 g_2, k_1 k_2), \quad (g_1, g_2 \in G, k_1, k_1 \in K).$$

EXAMPLE 1.3.2. Let $\mathbb{T}^n$ denote the n-fold direct product of $\mathrm{SO}(2)$. Then $\mathbb{T}^n$ is an Abelian group – the n-torus (see examples 1.2.7(1)).

DEFINITION 1.3.3. Let H, J be subgroups of G such that $H \lhd G$. The group G is the *semidirect product* of H and J if $G = HJ$ and $H \cap J = \{e\}$. We write $G = H \rtimes J$ (or $J \ltimes H$).

REMARKS 1.3.4. (1) If $G = H \rtimes J$ then every $g \in G$ can be written uniquely as $g = hj$, $h \in H$, $j \in J$.
(2) If G is the semidirect product of H and J, we say G *splits* over H. If $G = H \rtimes J$, then $G/H \cong J$ and the exact sequence

$$e \to H \to G \xrightarrow{q} G/H \to e$$

splits (that is, there is an isomorphism $\sigma : G/H \to J \subset G$ such that $qj = e_{G/H}$.

EXAMPLES 1.3.5. (1) Let $\kappa \in \mathrm{O}(2)$ reverse orientation. Then $\kappa^2 = e$ (κ is an *involution*). Since $\mathrm{SO}(2) \lhd \mathrm{O}(2)$, $\mathrm{O}(2) = \mathrm{SO}(2) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2 = \langle\kappa\rangle$. The product is *not* direct. Similarly $\mathrm{O}(n) = \mathrm{SO}(n) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is a subgroup of $\mathrm{O}(n)$ generated by any orientation reversing involution. If n is odd, $\mathrm{O}(n) \approx \mathrm{SO}(n) \times \mathbb{Z}_2$ (take $\kappa = -I$).
(2) For $n \geq 2$, $\mathbf{D}_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by an orientation reversing element of $\mathbf{D}_n$.
(3) The Euclidean group $\mathbf{E}(n)$ consists of all pairs $(A, b) \in \mathrm{O}(n) \times T(n)$. We define group composition by

$$(A, b)(C, d) = (AC, Ad + b).$$

This is compatible with the transformation group action of $\mathbf{E}(n)$ on $\mathbb{R}^n$ defined by $(A, b)x = Ax + b$. The group $T(n) = \{(I, b) \mid b \in \mathbb{R}^n\}$ is a normal subgroup of $\mathbf{E}(n)$ and consequently $\mathbf{E}(n) = T(n) \rtimes \mathrm{O}(n) \cong \mathbb{R}^n \times \mathrm{O}(n)$. A similar result holds for $\mathbf{SE}(n)$ with $\mathrm{SO}(n)$ replacing $\mathrm{O}(n)$.
(4) Let $q = p^n$, p prime. The group $\mathrm{Aff}_1(\mathbb{F}_q)$ (examples 1.2.9(2)) is the semidirect product $\mathbb{Z}_q \rtimes \mathbb{Z}_{q-1}$, where $\mathbb{Z}_q \lhd \mathrm{Aff}_1(\mathbb{F}_q)$ is the group of translations $x \mapsto x + b$ and $\mathbb{Z}_{q-1} \approx \mathrm{GL}(\mathbb{F}_q) = \mathbb{F}_q^\star$.

The semidirect product may be defined as a product between groups – that is, without assuming the groups are subgroups of a given group. Specifically, suppose that H, J are groups and $\rho : J \to \mathrm{Aut}(H)$ is a homomorphism ($\mathrm{Aut}(H)$ is the group of automorphisms of H). Define a group operation on $H \times J$ by

$$(h, j)(h', j') = (h\rho(j)(h'), jj'), \ \ (h, h' \in H, j, j' \in J).$$

With this group operation, we denote the product by $H \times_\rho J$ and refer to $H \times_\rho J$ as the *semidirect product of H and J with respect to ρ*. We can identify H and J with the subgroups $\{(h, e_J) \mid h \in H\}$ and $\{(e_H, j) \mid j \in J\}$ respectively. With these identifications, $H \times_\rho J = H \rtimes J$.

EXAMPLES 1.3.6. (1) Let $H = \mathbb{R}^n$, $J = \mathrm{O}(n)$ and define $\rho : \mathrm{O}(n) \to \mathrm{Aut}(\mathbb{R}^n)$ by $\rho(A)(b) = Ab$. We have $\mathbb{R}^n \times_\rho \mathrm{O}(n) = \mathbf{E}(n)$. If instead we take $\rho(A) = I_{\mathbb{R}^n}$, we obtain the direct product.
(2) Suppose $H = \mathrm{SO}(2)$ and $J = \mathbb{Z}_2$. The automorphism group of $\mathrm{SO}(2)$ consists of the identity and the involution $r(\theta) = -\theta$, $\theta \in \mathrm{SO}(2)$. If $J = \langle\kappa\rangle$, define $\rho : J \to \mathrm{Aut}(\mathrm{SO}(2))$ by $\rho(\kappa) = r$. We have $\mathrm{SO}(2) \times_\rho \mathbb{Z}_2 \approx \mathrm{O}(2)$.
(3) Suppose $G = H \rtimes J$ and define $\rho : J \to \mathrm{Aut}(H)$ by $\rho(j)(h) = hjh^{-1}$. Then $H \rtimes J \cong H \times_\rho J$. We leave it to the reader to verify that this is consistent with the previous two examples.
(4) Suppose that J is a subgroup of the symmetric group S^n and let H be a group. Define $\rho : J \to \mathrm{Aut}(H^n)$ by $\rho(j)(h_1, \ldots, h_n) = (h_{j^{-1}(1)}, \ldots, h_{j^{-1}(n)})$, $(h_1, \ldots, h_n) \in H^n$. The *wreath product* $H \wr J$ of H and J is the semidirect product $H \times_\rho J$. Let H_n denote the group of $n \times n$ *signed permutation matrices*.

Each element of P_n is an $n \times n$ permutation matrix where we allow the non-zero entries to be ± 1. If we let Δ_n denote the group of all $n \times n$ diagonal matrices, entries ± 1, then $H_n = \Delta_n \rtimes S_n \approx \mathbb{Z}_2 \wr S_n$.

1.4. Topological groups

DEFINITION 1.4.1. A group G has the structure of a *topological group* if G is a Hausdorff topological space and

$$G \times G \to G;\ (g,h) \mapsto gh^{-1},\ (g,h \in G)$$

is continuous.

REMARK 1.4.2. So as to avoid all topological issues, we generally assume that the topology of a topological group has countable base (satifies the second Axiom of countability). In particular, the group will be compact if and only if it is sequentially compact.

EXAMPLES 1.4.3. (1) Every group may be given the structure of a topological group by taking the discrete topology on G. However, this is not a particularly interesting topology and plays no role in what follows. The discrete topology is the only Hausdorff topology on finite groups.
(2) Euclidean space $\mathbb{R}^n$ and Hermitian space $\mathbb{C}^n$ have the structure of Abelian topological groups (usual topology, group composition addition).
(3) Let V be a d-dimensional vector space over either $\mathbb{R}$ or $\mathbb{C}$. Then $\mathrm{GL}(V)$ is a topological group. This is easily seen by choosing a basis for V and regarding $\mathrm{GL}(V)$ as the group of $d \times d$ invertible matrices.
(4) The groups $\mathrm{O}(n)$, $\mathrm{SO}(n)$ are compact subgroups of $\mathrm{GL}(n, \mathbb{R})$ and therefore inherit the structure of compact topological groups. Similar statements hold for the unitary and special unitary groups $\mathrm{U}(n)$ and $\mathrm{SU}(n)$.
(5) The groups $\mathbf{E}(n)$ and $\mathbf{SE}(n)$ have the topology induced from the product topology on $\mathrm{O}(n) \times \mathbb{R}^n$. Both groups have the structure of (non-compact) topological groups.

LEMMA 1.4.4. *Suppose that G, K are topological groups and that $\rho : G \to K$ is a continuous homomorphism. Then kernel(ρ) is a closed normal subgroup of G. If ρ is proper (inverse images of compact sets are compact), then image(ρ) is a closed normal subgroup of K.*

EXERCISE 1.4.5. (1) Show that if the group G is a topological space such that the $(g,h) \mapsto gh^{-1}$ is continuous then G is Hausdorff if and only if $\{e_G\}$ is a closed subset of G.
(2) Show that the connected component G_0 of the identity of a topological group G is an open normal subgroup of G.
(3) Show that if H is a subgroup of a topological group G then the closure $\bar{H}$ of H in G is a closed subgroup of G.
$(4)^\star$ Find an example of a homomorphism $\rho : \mathrm{SO}(2) \to \mathrm{SO}(2)$ which is not

continuous.
(5) Find an example of a continuous homomorphism $\rho : \mathbb{R} \to \mathbb{T}^2$ such that image(ρ) is *not* a closed subgroup of $\mathbb{T}^2$.

Let H be a closed normal subgroup of the topological group G. We give G/H the quotient topology (the largest or *finest* topology on G/H which makes the quotient map $q : G \to G/H$ continuous). It follows from the definition that the group operations on G/H are continuous relative to the quotient topology.

LEMMA 1.4.6. *Let H be a closed normal subgroup of the topological group G. Then G/H has the structure of a topological group. In particular, G/H is Hausdorff.*

PROOF. Use Exercise 1.4.5(1). □

EXAMPLE 1.4.7. We conclude this section with an exotic example of a compact Abelian topological group that cannot be represented as a closed subgroup of $\mathrm{GL}(V)$ for any finite dimensional vector space V. Although the group theoretic aspects of this example will not play a role in the sequel, we show later that the group can be represented as a hyperbolic attractor of a smooth diffeomorphism of $\mathbb{R}^3$. Topologically, the group is locally the product of an open interval with a Cantor set.

We start with a quite general construction. Let G be a compact topological group. Under componentwise multiplication, the (countable) infinite product

$$G_\infty = \Pi^\infty G = G^{\mathbb{N}} = \{(g_n) \mid g_n \in G, n \geq 0\}$$

has the structure of a compact topological group. If G is Abelian, so is G_∞. Suppose that $\rho : G \to G$ is a continuous surjective homomorphism of G. We are interested in the case when ρ is not 1:1 and ρ has nontrivial kernel. Define

$$\Sigma = \{(g_j) \in G_\infty \mid \rho(g_{j+1}) = g_j, j \geq 0\}.$$

Clearly Σ is a closed subgroup of G_∞. We refer to Σ as the *inverse limit* of $\rho : G \to G$. The homomorphism ρ extends to a continuous group automorphism $\hat{\rho} : \Sigma \to \Sigma$ defined by

$$\hat{\rho}(g_0, g_1, \ldots) = (\rho(g_0), g_0, g_1, \ldots).$$

(The inverse of $\hat{\rho}$ is given by $\hat{\rho}^{-1}(g_0, g_1, \ldots) = (g_1, \ldots)$.) If ρ is an automorphism of G, then $\Sigma \approx G$. If not, we can expect Σ to be much 'bigger' than G.

If $G = \mathrm{SO}(2)$, $p \geq 2$ and ρ is the p-fold covering map $\rho(\theta) = p\theta$, then Σ is the *p-adic solenoid*. We show in chapter 9 that Σ is topologically the product of an open interval with a Cantor set.

More examples can be manufactured by taking nonsingular matrices $A \in \mathrm{GL}(n, \mathbb{R})$ with integer entries. These maps induce endomorphisms of the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ which will be surjective but not bijective if $\det(A) > 1$. For example, if $n = 2$, take

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Both eigenvalues of T are real and greater than 1. The map T induces a $5:1$ covering homomorphism of $\mathbb{T}^2$. Taking the inverse limit, we obtain a compact Abelian subgroup Σ of $\Pi^\infty \mathbb{T}^2$ which may be realized as a 2-dimensional expanding attractor of a smooth diffeomorphism of $\mathbb{R}^5$ (see [**180**] and chapter 9).

EXERCISE 1.4.8. Let $g_1, \ldots, g_n$ be non-identity elements of the topological group G. Let $\langle g_1, \ldots, g_n \rangle$ denote the closure in G of the set of all finite words in $g_1, g_1^{-1}, \ldots, g_n, g_n^{-1}$. We say $g_1, \ldots, g_n$ are a set of (topological) generators for G if $\langle g_1, \ldots, g_n \rangle = G$.

(1) Show that $\langle g_1, \ldots, g_n \rangle$ is a topological subgroup of G.
(2) Show that if G is compact, then $\langle g_1, \ldots, g_n \rangle$ is the closure of the set of all finite words in g_i.
(3) Let $\alpha \in [0, 1)$. Show that $\langle \alpha \rangle = \mathrm{SO}(2) = \mathbb{R}/\mathbb{Z}$ if and only if α is irrational.
(4) Show that SO(3) can be generated by two elements but not one element. Noting Euler's theorem and the classification of closed subgroups of SO(3), find conditions on a pair $g, h \in \mathrm{SO}(3)$ that imply $\langle g, h \rangle = \mathrm{SO}(3)$.

1.5. Lie groups

In this section we review some basic facts about Lie groups. Our emphasis will be on compact Lie groups and our approach will be similar to that given in the book by Thomas [**169**, Chapter 7]. Bröcker and Dieck [**30**] and Kobayashi and Nomizu [**103**] (for the Lie bracket theory) are good alternate references. Our emphasis will be on the differential rather than the algebraic theory (Lie algebras). A fair slice of what we discuss will be developed further in subsequent chapters.

DEFINITION 1.5.1. A Lie group is a topological group G such that

(1) G has the structure of a smooth differential manifold.
(2) The composition map $G \times G \to G; (g, h) \mapsto gh^{-1}$ is smooth.

REMARKS 1.5.2. (1) We emphasize that by 'smooth' we always mean C^∞.
(2) It may be shown that every Lie group admits the structure of a real analytic manifold such that the operations of group composition are real analytic maps. This 1952 result follows from theorems of Gleason [**78**] and Montgomery and Zippin [**128**] which imply that every connected locally Euclidean topological group is Lie (Hilbert's fifth problem). The result for G compact was proved by von Neumann [**133**] in 1933 and for Abelian groups by Pontryagin [**144**] in 1939.

EXAMPLES 1.5.3. (1) Since the general linear groups $\mathrm{GL}(m, \mathbb{R})$ and $\mathrm{GL}(m, \mathbb{C})$ may be represented as open subsets of $\mathbb{R}^{m^2}$ and $\mathbb{C}^{m^2}$, it is obvious that both groups have the structure of a Lie group. In this case group operations are rational functions of the matrix entries. The orthogonal groups $\mathrm{O}(m),\ \mathrm{SO}(m) \subset \mathrm{GL}(m,\mathbb{R})$ and the unitary groups $\mathrm{U}(m),\ \mathrm{SU}(m) \subset \mathrm{GL}(m,\mathbb{C})$ also have the structure of compact Lie groups. This may be shown either directly by using Lie algebras to construct charts or by noting that both groups are closed subgroups of a Lie group and therefore Lie by a result we prove later in the chapter (theorem 1.5.24), (2) If G is compact but not connected then the identity component G_0 of G is a normal open subgroup of G and G/G_0 is finite.

Exercise 1.5.4. Show that a non-empty open subgroup H of the identity component G_0 of a Lie group is equal to G_0 (Hint: The cosets of H in G_0 are all open and so H must be closed.)

Remark 1.5.5. Using the Peter-Weyl theorem, it may be shown that every compact Lie group is isomorphic to a subgroup of $\mathrm{O}(m)$, for large enough m (for a proof see [**30**, Chapter III, §§3,4]). A consequence is that every compact group has the (unique) structure of a real algebraic variety. Indeed, a compact Lie group may be represented as the set of real points of a (complex) algebraic group. If G is *finite*, G may be represented as as subgroup of $\mathrm{O}(m)$, where $m = |G|$ – this follows from Cayley's theorem since G is isomorphic to a subgroup of the symmetric group S_m and $S_m \subset \mathrm{O}(m)$ if regard each element of S_m as a permutation of coordinates in $\mathbb{R}^m$. On the other hand there exist examples of non-compact connected Lie groups which cannot be represented as a subgroup of $\mathrm{GL}(m,\mathbb{R})$ for any $m \in \mathbb{N}$. As an example, we sketch why the universal cover $\mathrm{SL}(2,\mathbb{R})^\star$ of $\mathrm{SL}(2,\mathbb{R})$ cannot be represented as a matrix group. First note that the universal cover $G^\star$ of a Lie group G carries the natural structure of a Lie group for which the covering homomorphism $\pi : G^\star \to G$ is smooth. It is straightforward to show that $\pi_1(\mathrm{SL}(2,\mathbb{R})) = \mathbb{Z}$. Using a complexification argument ($\mathrm{SL}(2,\mathbb{C})$ is simply connected and the inclusion $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{SL}(2,\mathbb{C})$ induces the zero map on π_1), it may be shown that every homomorphism $h : \mathrm{SL}(2,\mathbb{R})^\star \to \mathrm{GL}(m,\mathbb{R})$ factors through $\pi : \mathrm{SL}(2,\mathbb{R})^\star \to \mathrm{SL}(2,\mathbb{R})$. Hence $\mathrm{SL}(2,\mathbb{R})^\star$ cannot be represented as a subgroup of $\mathrm{GL}(m,\mathbb{R})$ for any $m \in \mathbb{N}$. ($\mathrm{SL}(2,\mathbb{R})^\star$ is not a *linear* Lie group.) This example, due to Birkoff, appears in [**129**, page 191].

Exercise 1.5.6. Show that the universal cover $G^\star$ of a connected Lie group G carries the natural structure of a Lie group for which the covering homomorphism $\pi : G^\star \to G$ is smooth. Show that the kernel of π is a (discrete) normal subgroup of $Z(G^\star)$.

The group G is *semisimple* if G has finite centre and is *simple* if G contains no proper normal subgroups. The classical compact Lie groups $\mathrm{SO}(n)$, (special orthogonal group, $n \geq 3$), $\mathrm{SU}(n)$, (special unitary group, $n \geq 2$), and $\mathrm{Sp}(n)$ (symplectic group, $n \geq 2$), provide examples of compact connected semisimple groups. The quotient of any one of these groups by its centre defines a simple group (thus $\mathrm{SO}(2n+1)$ is simple, all $n \geq 1$). The only other compact simple groups are the exceptional simple Lie groups G_2, F_4, E_6, E_7, E_8. If G is simple, then the universal cover $G^\star$ is compact semisimple.

If G is any compact connected Lie group, there exist $m \geq 0$, compact simply connected groups $G_1, \ldots, G_p$, with each $G_i/Z(G_i)$ simple, and a finite covering homomorphism

$$\phi : \mathbb{T}^m \times \tilde{G}_1 \times \ldots \times \tilde{G}_p \to G,$$

Up to order, $\mathbb{T}^m$, $G_1, \ldots, G_p$ are uniquely determined by G. (See [**30**, Chapter V, Theorem 8.1].) In case G is Abelian, the proof of this result is elementary and we give it shortly.

It follows from Kronecker's theorem that there is a dense full measure subset of $\mathcal{G}$ of $\mathbb{T}^m$ consisting of topological generators: if $g \in \mathcal{G}$, then $\langle g \rangle = \mathbb{T}^m$. For general compact connected Lie groups, two generators suffice. For compact connected semisimple Lie groups, it can be shown that there is a (Zariski) open and dense subset of G^2 consisting of pairs which topologically generate G (for a proof of this result, which is due to Ulam, see [**63**]).

1.5.1. The Lie bracket of vector fields. Before we discuss the Lie algebra of a Lie group, we review some facts about the Lie bracket of vector fields (for more details, see [**103**, Chapter 1] or [**169**, Chapter 7]).

Let $C^\infty(TM)$ denote the space of smooth vector fields on the manifold M. If $X \in C^\infty(TM)$, let ϕ_t^X denote the flow of X (ϕ_t^X will be defined and smooth on a non-empty open neighbourhood of $M \times \{0\}$ in $M \times \mathbb{R}$).

The space $C^\infty(TM)$ is naturally isomorphic to the space of $\mathbb{R}$-linear derivations[1] of $C^\infty(M)$. Each $X \in C^\infty(TM)$ determines the derivation (or Lie derivative) $L_X : C^\infty(M) \to C^\infty(M)$ by $L_X f = \langle df, X \rangle$. The Lie bracket $[X, Y]$ of $X, Y \in C^\infty(TM)$ is the unique vector field associated to the derivation $L_Y L_X - L_X L_Y$. We recall some basic properties of the Lie bracket which may be proved either directly from the definition or by using local coordinates.

LEMMA 1.5.7. (1) *$(C^\infty(TM), [\ ,\ ])$ has the structure of a real Lie algebra. In particular, we have $[X, Y] = -[Y, X]$ and Jacobi's identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z.[X, Y]] = 0$, $X, Y, Z \in C^\infty(TM)$.*
(2) *If, in local coordinates, $X = (X_1, \ldots, X_m)$, $Y = (Y_1, \ldots, Y_m)$, then $[X, Y]_i = \sum_j \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right)$.*
(3) *If $f : M \to N$ is a diffeomorphism, $X \in C^\infty(TM)$ and we define $f_\star X = TfX \circ f^{-1} \in C^\infty(TN)$, then $f_\star[X, Y] = [f_\star X, f_\star Y]$, all $X, Y \in C^\infty(TM)$.*
(4) *If $X, Y \in C^\infty(TM)$ have respective flows ϕ_t^X, ϕ_t^Y, then*

$$[X, Y] = \frac{d}{dt}(\phi_t^X)_\star Y|_{t=0} = -\frac{d}{dt}(\phi_t^Y)_\star X|_{t=0}$$

[1] A derivation δ of $C^\infty(M)$ satisfies $\delta(fg) = g\delta(f) + f\delta(g)$, for all $f, g \in C^\infty(M)$

1.5.2. The Lie algebra of G. Every $h \in G$ determines a smooth diffeomorphism $L_h : G \to G$ defined by left translation $L_h(g) = hg$. Similarly, we define right translation $R_h : G \to G$ by $R_h(g) = gh$. Left and right translation commute

$$L_h R_k = R_k L_h, \quad (h, k \in G).$$

Let $\mathfrak{g} = T_e G$ denote the tangent space to G at the identity.

Let $C^\infty_G(TG)$ denote the space of (smooth) left invariant vector fields on G. That is, $\bar{X} \in C^\infty_G(TG)$ if $\bar{X}(gh) = TL_g \bar{X}(h)$. There is a natural isomorphism $\mathfrak{g} \approx C^\infty_G(TG)$ defined by mapping $X \in \mathfrak{g}$ to the left invariant vector field $\bar{X}$ on G defined by

$$\bar{X}(g) = TL_g X, \quad g \in G.$$

Conversely, every left invariant vector field on G is equal to $\bar{X}$ for a unique $X \in \mathfrak{g}$.

Example 1.5.8. The evaluation map $G \times \mathfrak{g} \to TG$, $(g, X) \to \bar{X}(g)$, defines a natural trivialization $TG \approx G \times \mathfrak{g}$ and so G is *parallelizable*. This simple observation lies at the heart of many geometric properties of compact Lie groups. As an example, we shall see later that the exponential map of the Lie algebra of a *compact* Lie group G (to be defined shortly) is *equal* to the exponential map of an invariant Riemannian metric on G (see chapter 3). We thereby obtain a relationship between the Lie algebraic and differential geometric properties of G.

If $X, Y \in \mathfrak{g}$, then $[\bar{X}, \bar{Y}]$ is left invariant by lemma 1.5.7(3). Hence there exists a unique $Z \in \mathfrak{g}$ such that $[\bar{X}, \bar{Y}] = \bar{Z} = \overline{[X, Y]}$. In this way we may define a Lie algebra structure on $\mathfrak{g}$ by $[X, Y] = Z$. In future we refer to $\mathfrak{g}$ as the *Lie algebra* of G and always assume that $\mathfrak{g}$ is equipped with the Lie bracket $[\ ,\ ]$.

Exercise 1.5.9. Show that the lie algebra $\mathfrak{gl}(m, \mathbb{R})$ of $\mathrm{GL}(m, \mathbb{R})$ is the space of all $m \times m$ matrices and that the Lie algebra structure on $\mathfrak{gl}(m, \mathbb{R})$ is defined by $[A, B] = AB - BA$.

Lemma 1.5.10. *Let $K : G \to H$ be a smooth homomorphism of Lie groups. Then*

$$K_\star = T_e K : \mathfrak{g} \to \mathfrak{h}$$

is a homomorphism of Lie algebras ($K_\star[X, Y] = [K_\star X, K_\star Y]$, for all $X, Y \in \mathfrak{h}$).

Proof. We have $K \phi^X_t = \phi^{K_\star X}_t$ for all $X \in \mathfrak{g}$. Now use lemma 1.5.7(4). ☐

1.5.3. The exponential map of $\mathfrak{g}$. We start by establishing properties of the flow ϕ^X of $\bar{X}$, $X \in \mathfrak{g}$. Given $g, h \in G$, the left invariance of $\bar{X}$ implies that $g\phi^X_t(h)$ is the integral curve of $\bar{X}$ through gh and so

$$\phi^X_t g = g \phi^X_t, \text{ for all } g \in G. \tag{1.1}$$

(This is a general property of the flow of a 'G-equivariant' vector field – see chapter 2).

Lemma 1.5.11. *Let $X \in \mathfrak{g}$.*

(1) *The domain of* ϕ^X *is (all of)* $G \times \mathbb{R}$.
(2) *The map* $\mathbb{R} \to G$, $t \mapsto \phi_t^X(e)$ *is a (smooth) group homomorphism. (*$\{\phi_t^X \mid t \in \mathbb{R}\}$ *is a 1-parameter group of diffeomorphisms of* G*.)*
(3) *For all* $t, s \in \mathbb{R}$, $X \in \mathfrak{g}$, $\phi_t^{sX} = \phi_{st}^X$.

PROOF. Let $\gamma = \phi_e^X : (a,b) \to G$ denote the maximal integral curve of $\bar{X}$ through e. Since, by left invariance, $g\gamma : (a,b) \to G$ is an integral curve through g, we see easily that $g\gamma : (a,b) \to G$ is the maximal integral curve through g for all $g \in G$. Hence the domain of ϕ^X is $G \times (a,b)$. For all $t, s \in (a,b)$ we have

$$\gamma(s)\gamma(t) = \phi_s^X(e)\phi_t^X(e) = \phi_t^X(\phi_s^X(e)),$$

where the second equality uses (1.1). Since $\phi_t^X(\phi_s^X(e))$ is an integral curve through $\phi_s^X(e) = \gamma(s)$, uniqueness of integral curves implies that $t+s \in (a,b)$ for all $t, s \in (a,b)$. Hence $(a,b) = \mathbb{R}$, proving (1). Since $\gamma(t+s) = \phi_{s+t}^X(e) = \phi_s^X(\phi_t^X(e)) = \gamma(t)\gamma(s)$, $\gamma : \mathbb{R} \to G$ is a group homomorphism, proving (2). For the final statement, rescale time ($\gamma(t)$ is a trajectory of X if and only if $\gamma(st)$ is a trajectory of sX). □

EXERCISE 1.5.12. Show that the set of all smooth group homomorphisms $c : \mathbb{R} \to G$ is naturally isomorphic to $\mathfrak{g}$ (we can replace 'smooth' by 'continuous' in this result).

We define the *exponential map* $\exp_G = \exp : \mathfrak{g} \to G$ by $\exp(X) = \phi_1^X(e)$.

LEMMA 1.5.13. (1) $\exp : \mathfrak{g} \to G$ *is smooth.*
(2) $T_e \exp : \mathfrak{g} \to \mathfrak{g}$ *is the identity map.*
(3) $\exp$ *restricts to a diffeomorphism of an open neighbourhood of the origin in* $\mathfrak{g}$ *onto an open neighbourhood of the identity in* G.
(4) $\exp$ *is* natural *in the sense that if* $K : G \to H$ *is a (smooth) homomorphism of Lie groups then*

$$K \exp_G = \exp_H K_\star,$$

where $K_\star = T_e K : \mathfrak{g} \to \mathfrak{h}$.
(5) *If* $X \in \mathfrak{g}$ *and* ϕ_t^X *denotes the flow of* $\bar{X}$, *then* $\phi_t^X(g) = g\exp(tX)$, *all* $g \in G$.

PROOF. Statement (1) is a consequence of standard results on the smooth dependence of solutions of ordinary differential equations on a parameter. For (2), note that by lemma 1.5.11(3) we have

$$D\exp(0)(X) = \frac{d}{dt}\exp(tX)|_{t=0} = \frac{d}{dt}\phi_t^X(e) = X,$$

and so $D\exp(0) = I_{\mathfrak{g}}$. Applying the implicit function theorem, exp is a local diffeomorphism at $0 \in \mathfrak{g}$. The naturality of exp follows from $K\phi_t^X(e_G) = \phi_t^Y(e_H)$, where $Y = K_\star(X)$. The final statement is a consequence of lemma 1.5.11(3) and (1.1). □

EXAMPLE 1.5.14. If G is a Lie subgroup of $\mathrm{GL}(m,\mathbb{R})$ and $X \in \mathfrak{g}$ then $\exp X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$.

LEMMA 1.5.15. *If G is connected then* $\exp : \mathfrak{g} \to G$ *is a homomorphism with respect to the additive structure on* $\mathfrak{g}$ *if and only if G is Abelian.*

PROOF. If G is Abelian the composition map $c : G \times G \to G$, $c(g,k) = gk$, is a homomorphism and $T_{(e,e)}c(X,Y) = X + Y$. So, by lemma 1.5.13(4), $\exp(X + Y) = \exp(X)\exp(Y)$. We leave the converse to the reader (use lemma 1.5.13(3) together with exercise 1.5.4). □

PROPOSITION 1.5.16. *A compact connected Abelian Lie group is isomorphic to a torus.*

PROOF. Let G be a connected compact Abelian group. By lemma 1.5.15, $\exp : \mathfrak{g} \to G$ is a homomorphism. The homomorphism is surjective since the image of the exponential map generates G (lemma 1.5.13(3) and exercise 1.5.4). Since exp is a local diffeomorphism at $0 \in \mathfrak{g}$, the kernel Γ of exp is a discrete subgroup Γ of $\mathfrak{g} \cong \mathbb{R}^n$, $n = \dim(G)$. One may show (see exercise 1.5.17(3)) that there exist linearly independent vectors $g_1, \dots, g_p \in \mathfrak{g}$ generating Γ. If $p = n$, then $\mathfrak{g}/\Gamma \cong \mathbb{T}^n$ and we are done. Otherwise, complete to a basis $g_1, \dots, g_n$ of $\mathfrak{g}$, and observe that $\mathfrak{g}/\Gamma \cong \mathbb{T}^p \times \mathbb{R}^{n-p}$ which cannot be compact unless $p = n$. □

EXERCISE 1.5.17. (1) Regard $\mathrm{SO}(n)$ as a subgroup of $\mathrm{GL}(n,\mathbb{R})$. Verify that $\mathrm{SO}(n)$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ and that the Lie algebra $\mathfrak{so}(n) \subset \mathfrak{gl}(n,\mathbb{R})$ of $\mathrm{SO}(n)$ may be identified with the space of $n \times n$ skew symmetric matrices. (Hint: Let $A \in \mathfrak{gl}(n,\mathbb{R})$ be skew symmetric: $A + A^t = 0$. Show that $\exp(A) \in \mathrm{SO}(n)$. Now use the fact that $\dim(\mathrm{SO}(n)) = n(n-1)/2$ together with the fact that exp is a local diffeomorphism to construct a chart for $\mathrm{SO}(n)$ at the identity. Translate by group elements to get a differential atlas.)
(2) Let X be a non-zero element of the lie algebra of G. Show that $\Gamma_X = \{\exp(tX) \mid t \in \mathbb{R}\}$ is an Abelian subgroup of G. Find (up to isomorphism) Γ_X in case $G = \mathrm{SO}(3)$. Show that for 'most' $X \in \mathfrak{so}(4)$, the closure of Γ_X is isomorphic to $\mathbb{T}^2$. Find the corresponding results for $\mathrm{SO}(2n)$ and $\mathrm{SO}(2n+1)$. (These examples are special cases of the fundamental theorem that every compact connected Lie group has a *maximal torus* $\mathbb{T}^m$ and that the set of conjugates $g\mathbb{T}^m g^{-1}$ fills out G. There are also results for compact disconnected Lie groups [**30**, Chapter IV, §4] and we return to these questions in chapter 8).
(3) Suppose that Γ is a discrete subgroup of $\mathbb{R}^n$. Complete the proof of proposition 1.5.16 by showing that there exists a linearly independent set $g_1, \dots, g_p \in \Gamma$ which generates Γ. (Hints: Prove by induction on n. Assume true for $n-1$. Choose $g_1 \in \Gamma \setminus \{0\}$ to be of shortest length. Write $\mathbb{R}^n = \mathbb{R}g_1 \oplus V$, where $V = (\mathbb{R}g_1)^\perp$. If $\pi : \mathbb{R}^n \to V$ denotes orthogonal projection, prove $\pi(\Gamma)$ is discrete by showing that every nonzero element of $\pi(\Gamma)$ has length at least $\|g_1\|/2$. Now apply the inductive hypothesis to $\pi(\Gamma)$ and thereby find a linearly independent

subset of Γ which projects by π onto a basis of $\pi(\Gamma)$ and extends by g_1 to a basis of Γ.)
(4) Show that if we define $f : \mathfrak{g}^2 \to G$ by $f(X,Y) = \exp(X+Y)\exp(-X)\exp(-Y)$, then $Tf_{(0,0)} = 0$. In particular, $\exp(X+Y)\exp(-Y)\exp(-X) = e_G + o(\|(X,Y)\|)$, relative to any norm $\| \ \|$ on $\mathfrak{g}^2$.)
(5) Show that every connected Abelian Lie group G is isomorphic to $\mathbb{T}^p \times \mathbb{R}^q$, where $p + q = \dim(G)$.
(6) Show that the Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ of $\mathrm{SL}(n,\mathbb{R})$ is the space of $n \times n$-matrices with trace zero.
(7) Show that even if G is connected, the exponential map $\exp : \mathfrak{g} \to G$ need not be onto. (Hint: Take $G = \mathrm{SL}(2,\mathbb{R})$ and show that the diagonal matrix (λ, λ^{-1}) cannot be represented as e^A, $\mathrm{trace}(A) = 0$, if $\lambda < 0$ and $\lambda \neq -1$.)

1.5.4. Additional properties of brackets and exp. In this subsection we give some details on the relationship between the adjoint representation of $\mathfrak{g}$ and the action of $\mathfrak{g}$ on G via the exponential map. While we make rather limited use of these results in the remainder of the book, we include them as they provide a useful guide to computations that relate the bracket structure to the group operations on G.

Given $X \in \mathfrak{g}$, let $c(X) \in \mathrm{Aut}(G)$ be defined by $c(X)(g) = \exp(X)g\exp(-X)$. Let $c_\star(X) = T_e c(X) : \mathfrak{g} \to \mathfrak{g}$ and note that $c_\star(X)$ is a Lie algebra homomorphism (lemma 1.5.10).

LEMMA 1.5.18. *For all $X, Y \in \mathfrak{g}$,*

$$[X,Y] = \frac{d}{dt} c_\star(tX)Y|_{t=0}.$$

PROOF. By lemma 1.5.7(4), $[\bar{X},\bar{Y}] = \frac{d}{dt}(\phi_t^X)_\star \bar{Y}|_{t=0}$. Since $\phi_t^X(g) = g\exp(tX)$, we have

$$(\phi_t^X)_\star \bar{Y}(e) = T\exp(tX)\bar{Y}(\exp(-tX))(e) = c_\star(tX)\bar{Y}(e).$$

Hence $[X,Y] = [\bar{X},\bar{Y}](e) = \frac{d}{dt} c_\star(tX)Y|_{t=0}$. $\square$

The *adjoint Lie algebra representation* of $\mathfrak{g}$ is the map $\mathrm{ad} : \mathfrak{g} \to L(\mathfrak{g},\mathfrak{g})$ defined by

$$\mathrm{ad}(X)(Y) = [X,Y], \ X, Y \in \mathfrak{g}.$$

EXERCISE 1.5.19. Using the Jacobi identity, show that ad is a Lie algebra homomorphism: $[\mathrm{ad}(A), \mathrm{ad}(B)] = \mathrm{ad}([X,Y])$ (the bracket on $L(\mathfrak{g},\mathfrak{g})$ is the commutator).

Given $A \in L(\mathfrak{g},\mathfrak{g})$, we define $e^A = \sum_{j=0}^\infty A^j/j! \in \mathrm{GL}(\mathfrak{g}) \subset L(\mathfrak{g},\mathfrak{g})$.

LEMMA 1.5.20. *For $X, Y \in \mathfrak{g}$, $c_\star(X)Y = e^{ad(X)}Y$.*

PROOF. Define the smooth curve $X(t)$ in $\mathfrak{g}$ by $X(t) = c_\star(tX)Y$. Then $X(0) = Y$ and $X(s+t) = c_\star(sX)c_\star(tX)Y$, since $c(tX + sX) = c(sX)c(tX)$.

Differentiating with respect to s and setting $s = 0$ it follows from Lemma 1.5.18 that

$$X'(t) = [X, X(t)] = \mathrm{ad}(X)(X(t)).$$

The solution of this ordinary differential equation with $X(0) = Y$ is given by $X(t) = e^{t\mathrm{ad}(X)}Y$. Taking $t = 1$, the result follows. □

LEMMA 1.5.21. *Let $X(t)$ be a smooth curve in $\mathfrak{g}$. Then*

$$T\exp(X(t))\frac{d}{dt}\exp(-X(t)) = -f(ad(X(t))X'(t),$$

where $f(z) = (e^z - 1)/z$.

PROOF. For $s, t \in \mathbb{R}$, define $B(s,t) \in \mathfrak{g}$ by

$$B(s,t) = T\exp(sX(t))\frac{d}{dt}\exp(-sX(t)).$$

Differentiating with respect to s, we find after some work that

$$\frac{\partial B}{\partial s} = [X, B] - X'(t).$$

The solution to this inhomogeneous linear equation in s is given by

$$B(s,t) = e^{s\mathrm{ad}(X)}\left(B(0,t) + \int_0^s e^{-u\mathrm{ad}(X)}X'(t)\,du\right).$$

Setting $s = 1$, and noting that $B(0,t) = 0$, the result follows. □

REMARK 1.5.22. Choose an open neighbourhood U of $0 \in \mathfrak{g}$ so that exp restricts to a diffeomorphism of U onto an open neighbourhood V of $e_G \in G$. Let $\log = (\exp|U)^{-1}$. Lemma 1.5.21 is the key step towards proving the Cambell-Baker-Hausdorff formula which gives an explicit formula for $\log(\exp(X)\exp(Y))$ in terms of the Lie algebra structure on $\mathfrak{g}$:

$$\log(\exp(X)\exp(Y)) = X + \int_0^1 \Psi(e^{\mathrm{ad}(X)}e^{t\mathrm{ad}(Y)})(Y)\,dt,$$

where $\Psi(z) = \log z/(z-1)$ and is analytic near $z = 1$. Using this result one may derive a power series expansion for $\log(\exp(X)\exp(Y))$ in terms of Lie bracket operations:

$$\log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[Y,X]]) + \dots.$$

The Cambell-Baker-Hausdorff formula implies that the group multiplication of G is determined uniquely by the Lie algebra structure on $\mathfrak{g}$. From this it can be shown fairly easily that every linear Lie algebra (that is, Lie subalgebra of the space of $n \times n$-matrices) is the linear algebra of a Lie group (the group will not be unique).

EXERCISE 1.5.23. Suppose the neighbourhoods U, V are chosen as in the previous remark. Let $C(t) = \log e^{tA}e^B$, where $A, B \in U$ are chosen sufficiently small so that $e^{tA}e^B \in V$, $|t| < 1$. Show that $e^{C(t)}\frac{d}{dt}e^{-C(t)} = -A$ and hence $A = f(\mathrm{ad}(C)(t))C'(t)$ (lemma 1.5.21). Now use the fact that $f(\log z)\Psi(z) = 1$ to find a differential equation for C and deduce the Cambell-Baker-Hausdorff formula.

1.5.5. Closed subgroups of a Lie group.

THEOREM 1.5.24. *If H be a closed subgroup of the Lie group G, then H is a Lie subgroup of G (that is, H is a closed submanifold of G).*

PROOF. It suffices to find an open neighbourhood D' of $e \in G$ such that $H \cap D'$ is a submanifold of G since we can translate charts for $H \cap D'$ by elements of H to obtain a differential atlas of H. The proof proceeds by constructing the Lie algebra $\mathfrak{h}$ of H (strictly H_0) and using the exponential map restricted to $\mathfrak{h}$ to construct a chart with domain D'.

Choose an open neighbourhood U of $0 \in \mathfrak{g}$ such that exp restricts to a diffeomorphism of U onto an open neighbourhood D of $e \in G$. Let $\log : D \to U$ denote the inverse map.

Fix a norm $\| \ \|$ on $\mathfrak{g}$. Set $U' = \log(H \cap D)$. If $(v_n) \subset U' \setminus \{0\}$ is a sequence converging to 0, choose a subsequence so that $v_n/\|v_n\|$ converges to $X \in \mathfrak{g}$, $\|X\| = 1$. We claim $\exp(tX) \in H$, all $t \in \mathbb{R}$. To this end, fix $t \neq 0$. Let m_n denote the integer part of $t/\|v_n\|$. Then $\lim_{n\to\infty} m_n\|v_n\| = t$ and so $\lim_{n\to\infty} m_n v_n = tX$. Hence $\lim_{n\to\infty} \exp(m_n v_n) = \exp(tX)$. But $\exp(m_n v_n) = \exp(v_n)^{m_n} \in H$ and so, since H is closed, $\exp(tX) \in H$.

Let $L = \{tX \mid X = \lim (v_n/\|v_n\|),\ (v_n) \subset U',\ t \in \mathbb{R}\}$. We claim L is a linear subspace of $\mathfrak{g}$. Since L is closed under scalar multiplication, we must show L is closed under addition. Suppose $X, Y \in L$. We have $\exp(X/n)\exp(Y/n) \in H$ for all $n \in \mathbb{N}$ and so $h_n = \log(\exp(X/n)\exp(Y/n)) \in U'$, for all sufficiently large n. By exercise 1.5.17(4), $\exp(X/n)\exp(Y/n) = \exp((X+Y)/n + O(1/n))$ and so $h_n = (X+Y)/n + O(1/n)$. Hence $\lim_{n\to\infty} h_n/\|h_n\| = (X+Y)/\|X+Y\|$ and $X + Y \in L$.

Let L' be an open neighbourhood of the origin in L. We claim that $\exp(L')$ is a neighbourhood of the identity in H. Certainly $\exp(L') \subset \exp(L) \subset H$. Let E be a vector space complement to L in $\mathfrak{g}$ and define $f : L \oplus E \to G$ by $f(X, Y) = \exp(X)\exp(Y)$. Since $T_{(0,0)}f(A, B) = A + B$, $(A, B) \in L \oplus E$, f is a local diffeomorphism at the origin of $\mathfrak{g}$. Suppose that every neighbourhood of $e \in G$ contains points of H not in $\exp(L')$. Then we may choose a sequence (X_n, Y_n) of points in $L' \oplus E$ such that $(X_n, Y_n) \to 0$, $Y_n \neq 0$, and $f(X_n, Y_n) \in H$. Choosing a subsequence, we may suppose that $Y_n/\|Y_n\| \to Y$, $\|Y\| = 1$. Since $\exp(X_n) \in H$ and H is a subgroup, we have $\exp(Y_n) \in H$ and so $Y \in L$, a contradiction. Since $\exp|L$ is a local diffeomorphism at the origin, we may choose L' so that exp maps L' diffeomorphically onto an open neighbourhood D' of $e \in H$. $\square$

1.6. Haar measure

We conclude the chapter by recalling the simple proof of the existence of *Haar measure* for a compact Lie group.

THEOREM 1.6.1. *Let G be a compact Lie group. There exists a unique Borel probability measure on G which is invariant under both left and right translations.*

PROOF. We prove existence and leave uniqueness to the reader. Suppose that $\dim(G) = m$. Let $\tau_G^\star : T^\star G \to G$ denote the cotangent bundle of G (bundle of 1-forms). Then $\wedge^m T^\star G \to G$ is a line bundle over G. The fibre of $\wedge^m T^\star G \to G$ over the identity $e \in G$ is precisely $\wedge^m \mathfrak{g}^\star$. Let $\phi \in \wedge^m \mathfrak{g}^\star$ be non-zero. We define the smooth left invariant section ω of $\wedge^m T^\star G$ by

$$\omega(g) = (\wedge^m T L_{g^{-1}}^\star)(\phi), \quad g \in G.$$

Since ω is non-vanishing it defines a volume form on G. Multiplying ϕ by a non-zero constant we may assume that $\int_G \omega = 1$. Since ω is left invariant, the associated Borel probability measure dh on G is left invariant. To prove right invariance of dh, it suffices to show ω is right invariant. Let $k \in G$ and set $\bar{\omega} = R_k^\star \omega$. Since $\wedge^m \mathfrak{g}^\star$ is 1-dimensional, there exists $a \in \mathbb{R}$ such that $\omega(e) = a\bar{\omega}(e)$. Since left and right multiplication commute and ω is left invariant, $\bar{\omega}$ is left invariant and so $\omega(g) = a\bar{\omega}(g)$, for all $g \in G$. Certainly $\int_G \bar{\omega} = a \int_G \omega$. On the other hand, it follows from standard properties of the integral of m-forms that $\int_G \bar{\omega} = \int_G R_k^\star \omega = \int_G \omega$. Hence $a = 1$ and ω is right invariant. □

REMARKS 1.6.2. (1) If G is finite, then $dh = \frac{1}{|G|} \sum_{g \in G} \delta(g)$, where $\delta(g)$ is the Dirac probability measure supported at g.
(2) If G is a compact topological group then G admits a unique left and right invariant probability measure. If we allow G to be non-compact then it can be shown that there exist both left and right invariant Borel measures on G but not necessarily a measure which is left and right invariant. Proofs can be found in any text on topological groups (a proof for compact topological groups may be found in [**169**, Appendix A]).

CHAPTER 2

Group Actions and Representations

2.1. Introduction

The topic of this chapter is group actions and the geometry of linear G-spaces (representations). Our emphasis will be on basic constructions, such as the twisted product, representation theory, and on the stratification of a linear G-space by isotropy type. We also include some preliminary definitions and results on smooth invariant and equivariant maps (a topic we develop much further in subsequent chapters). As the chapter is intended to be introductory, most of the examples we give will involve linear actions by *finite* groups and we defer the more general non-linear theory of smooth actions by compact Lie groups to the next chapter.

2.2. Groups and G-spaces

Let X be a set and $\mathcal{B}(X)$ denote the group of all bijections of X. We have an *action* of $\mathcal{B}(X)$ on X defined by

$$\mathcal{B}(X) \times X \to X;\ (g, x) \mapsto g(x) = gx.$$

More generally, if G is a group then a homomorphism $\rho : G \to \mathcal{B}(X)$ determines an action of G on X by

$$G \times X \to X;\ (g, x) \mapsto \rho(g)(x).$$

Usually, we just set $\rho(g)(x) = gx$. Note that the identity $e \in G$ always acts as the identity transformation of X and that for all $g, h \in G$, $x \in X$ we have

$$(gh)x = g(hx).$$

We call X, together with an action of G on X, a *G-set*.

DEFINITION 2.2.1. Given G-sets X, Y, a map $f : X \to Y$ is *G-equivariant* if $f(gx) = gf(x)$ for all $x \in X$, $g \in G$.

DEFINITION 2.2.2. Let $\rho_1, \rho_2 : G \to \mathcal{B}(X)$ be homomorphisms. The associated G-actions on X are *isomorphic* or *equivalent* if there is a G-equivariant $T \in \mathcal{B}(X)$. That is, a bijection $T : X \to X$ such that

$$T\rho_1(g) = \rho_2(g)T,\ \ (g \in G).$$

We say T *intertwines* ρ_1 and ρ_2. More generally, the G-sets X and Y are equivalent if there is a G-equivariant bijection between X and Y.

EXAMPLES 2.2.3. (1) Let (X,d) be a metric space and $\mathrm{Iso}(X) \subset \mathcal{B}(X)$ denote the group of isometries of X (see chapter 1). Every isometry is a homeomorphism of X. If X is compact, we can define a metric ρ on $\mathrm{Iso}(X)$ by $\rho(f,g) = \sup_{x\in X} d(f(x), g(x))$. The group operations of multiplication and inversion are continuous with respect to ρ and $\mathrm{Iso}(X)$ has the structure of a *topological group*. The associated action $\mathrm{Iso}(X) \times X \to X$ of $\mathrm{Iso}(X)$ on X is continuous.
(2) Let $X = S^1$, the unit circle in $\mathbb{C}$, and $\mathrm{O}(2)$ denote the orthogonal group of transformations of $\mathbb{C} \approx \mathbb{R}^2$. As metric on S^1 we take arc length. Then $\mathrm{Iso}(S^1) \approx \mathrm{O}(2)$. Represent $\mathrm{SO}(2) \subset \mathrm{O}(2)$ as the group of rotations R_θ through angle θ, $\theta \in [0, 2\pi)$. Define actions $\rho_1, \rho_2 : \mathrm{SO}(2) \to \mathrm{Iso}(S^1)$ by $\rho_1(\theta)(z) = R_\theta(z)$, $\rho_2(\theta)(z) = R_{-\theta}(z)$. The actions ρ_1, ρ_2 are isomorphic since $T\rho_1(\theta) = \rho_2(\theta)T$, where $T : S^1 \to S^1$ is complex conjugation. (The map T is an isometry and we usually require that the intertwining map preserves the underlying structure.)
(3) If we take $X = S^n \subset \mathbb{R}^{n+1}$ with the metric induced from the Euclidean inner product on $\mathbb{R}^{n+1}$, then $\mathrm{Iso}(S^n) = \mathrm{O}(n+1)$ – the orthogonal group. (For general results on isometry groups of (Riemannian) manifolds, we refer to Kobayashi & Nomizu [**103**, pages 306–309].)

2.2.1. Continuous actions and G-spaces. Suppose that X is a topological space and G is a topological group (always assumed Hausdorff). We call X, together with a continuous action $G \times X \to X$, a *G-space*. If X is a G-space, then every $g \in G$ defines a homeomorphism $g : X \to X$. For equivalence of G-spaces we require that the intertwining map is a homeomorphism.

EXAMPLES 2.2.4. (1) Let $f : X \to X$ be a homeomorphism. We define a $\mathbb{Z}$-action on X by

$$\mathbb{Z} \times X \to X,\ (n,x) \mapsto f^n(x).$$

With respect to this action, X has the structure of a $\mathbb{Z}$-space. Conversely, every continuous $\mathbb{Z}$-action on X is generated by a homeomorphism of X.
(2) Let V be a smooth vector field on $\mathbb{R}^n$ and suppose that solutions to the differential equation $x' = V(x)$ are defined for all time. Given $x \in \mathbb{R}^n$, let $\phi_x(t)$ denote the solution with initial condition x. Define $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ by $\phi_t(x) = \phi_x(t)$. Then ϕ_t is a smooth diffeomorphism of $\mathbb{R}^n$ and $\phi_t \circ \phi_s = \phi_{t+s}$. It follows that $\{\phi_t\}$ gives $\mathbb{R}^n$ the structure of a (smooth) $\mathbb{R}$-space. We call $\{\phi_t \mid t \in \mathbb{R}\}$ a *flow* on $\mathbb{R}^n$. Conversely, if $\{\phi_t\}$ gives $\mathbb{R}^n$ the structure of a smooth $\mathbb{R}$-space, we have an associated vector field V on $\mathbb{R}^n$ (the infinitesimal generator of the action) defined by $V(x) = \frac{d}{dt}\phi_t(x)|_{t=0}$.

2.3. Orbit spaces and actions

Throughout this section we assume that X is a metric space (the results hold if X is a Hausdorff topological space). Suppose that X is G-space.

If $Y \subset X$, then Y is *G-invariant* if

$$gY = Y,\ \ (g \in G).$$

Given $A \subset X$, the *G-orbit of A* is defined by

$$G(A) = GA = \{ga \mid g \in G, a \in A\}.$$

If $x \in X$, Gx is the G-orbit through x. A G-orbit is a G-invariant subset of X.

LEMMA 2.3.1. *If G is compact, then the action $\rho : X \times G \to X$ is closed (ρ maps closed sets to closed sets). In particular, if $A \subset X$ is closed, then $GA \subset X$ is closed and the G-orbit through $x \in X$ is compact.*

PROOF. Let $Z \subset X \times G$ be closed and suppose $z \in \overline{\rho(Z)}$. There exists a sequence $(x_n, g_n) \subset Z$ such that $\rho(x_n, g_n) = z_n$ converges to z. Taking subsequences we may assume that $g_n \to g$, since G is compact. It follows that $x_n = \rho(g_n(x_n), g_n^{-1})$ converges to $\rho(z, g^{-1}) = g^{-1}z$. Hence $(x_n, g_n) \to (g^{-1}z, g) \in Z$, since Z is closed. Therefore $z = \rho(g^{-1}z, g) \in \rho(Z)$. $\square$

REMARK 2.3.2. If G is not compact, then Gx will generally not be closed. Examples are easily constructed using $\mathbb{R}$- or $\mathbb{Z}$-actions.

DEFINITION 2.3.3. The orbit space for the action of G on X is the quotient topological space X/G.

LEMMA 2.3.4. *Suppose that X is G-space and G is a compact topological group. Let $q : X \to X/G$ denote the orbit map. Then*

(1) *q is an open, closed and proper mapping (inverse images of compact sets are compact).*
(2) *X/G is Hausdorff.*

PROOF. Let $U \subset X$ be open. Then $p(U)$ is open if and only if $p^{-1}(p(U))$ is open (quotient topology). But $p^{-1}(p(U)) = \cup_{g \in G} gU$ is a union of open sets and therefore open. If $A \subset X$ is closed then GA is closed by lemma 2.3.1 and so, since $GA = p^{-1}(p(A))$, it follows by definition of the quotient topology on X/G that $p(A)$ is closed. Let $A \subset X/G$ be compact and $\{U_i \mid i \in I\}$ be an open cover of $p^{-1}(A)$. Since $p^{-1}(y)$ is compact for all $y \in X/G$ (G-orbits are compact), it follows that for each $y \in A$, we may choose a finite subset $I_y \subset I$ such that the corresponding $\{U_i \mid i \in I_y\}$ cover $p^{-1}(y)$ and $y \in p(U_i)$, each $i \in I_y$. Let $V_y = p(\cap_{i \in I_y} U_i)$. Since p is open, V_y is an open neighbourhood of y for all $y \in A$. Hence $\{V_y \mid y \in A\}$ is an open cover of A. Choose a finite subcover, say $\{V_{y_j} \mid j = 1, \dots, k\}$. Since $\{p^{-1}(V_{y_j}) \mid j = 1, \dots, k\}$ covers $p^{-1}(A)$, it follows that $\{U_i \mid i \in I_{y_j}, j = 1, \dots, k\}$ is a finite subcover of $p^{-1}(A)$.

It remains to prove that X/G is Hausdorff. Suppose $x, y \in X$ and $Gx \neq Gy$. Since Gx, Gy are disjoint compact sets we can choose an open neighbourhood U of x such that $\bar{U} \cap Gy = \emptyset$. It follows that $p(U), X/G \setminus p(\bar{U})$ are disjoint open sets separating $p(x)$ and $p(y)$. $\square$

REMARK 2.3.5. If G is not compact, X/G need not be Hausdorff. Highly pathological examples may be constructed using smooth $\mathbb{R}$-actions.

EXAMPLE 2.3.6. Regard the 2-torus $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z}^2$. Given $\alpha \in \mathbb{R}^\star$, define the $\mathbb{R}$-action $\Phi_t : \mathbb{T}^2 \to \mathbb{T}^2$ by $\Phi_t(\theta, \psi) = (\theta + t, \psi + \alpha t)$, mod $\mathbb{Z}^2$. If α is irrational, then every orbit of Φ_t is dense in $\mathbb{T}^2$. In this case the orbit space $\mathbb{T}^2/\mathbb{R}$ has only one non-empty open set: $\mathbb{T}^2/\mathbb{R}$. For this, observe that if $q : \mathbb{T}^2 \to \mathbb{T}^2/\mathbb{R}$ denotes the orbit map then $U \subset \mathbb{T}^2/\mathbb{R}$ is open if and only if $q^{-1}(U)$ is an open and Φ_t-invariant subset of $\mathbb{T}^2$. Since every orbit of Φ_t is dense in $\mathbb{T}^2$, every orbit must meet $q^{-1}(U)$ and therefore, by Φ_t-invariance, be a subset of $q^{-1}(U)$.

EXERCISE 2.3.7. Verify that if X is a G-space and $f : X \to Y$ is a continuous map which is constant on G-orbits ($f(gx) = f(x)$, all $g \in G$, $x \in X$), then f induces a continuous map $f^\star : X/G \to Y$. Interpret in case $Y = \mathbb{R}$ and X is the $\mathbb{R}$-space defined in the previous example.

DEFINITION 2.3.8. Given $x \in X$, the *isotropy* subgroup at x is the subgroup G_x of G defined by

$$G_x = \{g \in G \mid gx = x\}.$$

REMARK 2.3.9. The isotropy group at x measures the 'symmetry' of the point x. The most symmetric points have isotropy G – for example, the origin of a linear G-space.

EXERCISE 2.3.10. (1) Show that G_x is a closed subgroup of G.
(2) Show that if $y = gx$, then $G_y = gG_xg^{-1}$. (Isotropy groups of points on the same G-orbit are *conjugate* subgroups of G.)
(3) Show that if H is a closed subgroup of the compact group G and we give G/H the quotient topology, then G/H has the structure of a compact G-space where the action of G is given by left translation: $g(k[H]) = gk[H]$. What is the isotropy group at $[H] \in G/H$? (The space G/H is called a *homogeneous space*.)
(4) Show that if X is a G-space and $x \in X$ then the natural map $\alpha : G/G_x \to Gx$ defined by $\alpha(gG_x) = gx$ is a homeomorphism. (This result uses the compactness of G – strictly G/G_x.) Show also that α is G-equivariant with respect to the natural left actions of G on G/G_x and Gx.
(5) Suppose we are given an $\mathbb{R}$-action on X (as in examples 2.2.4(2)). What are the possible isotropy groups for the $\mathbb{R}$-action? Dynamical interpretation?

DEFINITION 2.3.11. The action of G on X is

(1) *free* if G_x is trivial (that is, the identity element) for all $x \in G$,
(2) *effective* (or *faithful*) if $\cap_{x \in X} G_x = \{e\}$. That is, if no element of $G \setminus \{e\}$ acts as the identity on X (equivalently, the map $\rho : G \to \mathcal{B}(X)$ is injective).
(3) *transitive* if $Gx = X$ for any (all) $x \in X$.

EXAMPLE 2.3.12. The action of $\mathrm{SO}(2) = \mathbb{R}/2\pi\mathbb{Z}$ on the unit circle $S^1 \subset \mathbb{C}$ defined for $p \in \mathbb{Z}$ by $(\theta, z) \mapsto e^{\imath p\theta} z$ is free and faithful if and only if $p = \pm 1$. If $p \neq 0$, the action is transitive and the isotropy group of the action is constant and equal to $\mathbb{Z}_{|p|}$. The action of SO(2) on $\mathbb{C}$ is never free or transitive but is always faithful provided $p = \pm 1$.

DEFINITION 2.3.13. Points $x, y \in X$ have the same *isotropy type* if G_x, G_y are conjugate subgroups of G. That is, if there exists $g \in G$ such that $G_y = gG_xg^{-1}$.

REMARKS 2.3.14. (1) If G is Abelian, points have the same isotropy type if and only if they have the same isotropy group.
(2) There are only finitely many isotropy types for the action of a finite group.

EXERCISE 2.3.15. For $p \geq 1$, let SO(2) act on S^1 by $z \mapsto e^{p\imath\theta}z$. Let $X = \Pi S^1$ denote the countable infinite product of S^1 and let SO(2) act on X by

$$e^{\imath\theta}(z_1, \ldots, z_p, \ldots) = (e^{\imath\theta}z_1, \ldots, e^{p\imath\theta}z_p, \ldots).$$

Give X the product topology and note that X is compact by Tychonoff's theorem. Show that the action is continuous and that there are infinitely many different isotropy groups (types) for the action.

The next extended set of examples describes how the Hopf fibration arises from a free SO(2) action on S^3.

EXAMPLES 2.3.16. (1) For $p, q \in \mathbb{Z}$, we define an SO(2)-action $\rho_{p,q}$ on the 2-torus $\mathbb{T}^2 = S^1 \times S^1$ by

$$(\theta, (z_1, z_2)) \mapsto (e^{\imath p\theta}z_1, e^{\imath q\theta}z_2).$$

Just as in examples 2.2.3(2), the actions $\rho_{\pm p, \pm q}$ are all equivalent and so it is no loss of generality to assume $p, q \geq 0$. The action $\rho_{p,q}$ is free if either $p = 1$ or $q = 1$. In case $p = q = 1$, we refer to the action as the *diagonal action* on $\mathbb{T}^2$. The actions $\rho_{1,1}$, $\rho_{1,0}$, $\rho_{0,1}$ are all equivalent. For example, if we define the homeomorphism T of $\mathbb{T}^2$ by $T(z_1, z_2) = (z_1, z_2z_1^{-1})$, then

$$T\rho_{1,1} = \rho_{1,0}T.$$

Geometrically speaking, what we are doing here is cutting the torus along $z_1 = 1$, untwisting through an angle of 2π and then rejoining.
(2) Let D^2 denote the unit disk in $\mathbb{C}$. For $p, q \in \mathbb{N}$, we define an SO(2)-action $\rho_{p,q}$ on the solid 2-torus $\mathbf{T}^2 = D^2 \times S^1$ by

$$(\theta, (z_1, z_2)) \mapsto (e^{\imath p\theta}z_1, e^{\imath q\theta}z_2).$$

Just as in the previous example, the actions $\rho_{1,1}$ and $\rho_{0,1}$, are equivalent. It follows that the orbit space of $D^2 \times S^1$ under the action defined by $\rho_{1,1}$ is homeomorphic to the orbit space of the action defined by $\rho_{0,1}$ which is D^2. Note that the action $\rho_{1,1}$ is *not* equivalent to $\rho_{1,0}$: the action $\rho_{1,0}$ is not free.
(3) For $p, q \in \mathbb{Z}$, we define an SO(2)-action $\eta_{p,q}$ on $\mathbb{C}^2$ by

$$(\theta, (z_1, z_2)) \mapsto (e^{\imath p\theta}z_1, e^{\imath q\theta}z_2).$$

The actions $\eta_{\pm p, \pm q}$ are all equivalent so it is no loss of generality to assume $p, q \geq 0$. In this case, $\eta_{1,1}$ is not equivalent to $\eta_{0,1}$. Indeed, $\eta_{1,1}$ acts freely on

$\mathbb{C}^2 \setminus \{0\}$, while $\eta_{0,1}$ fixes all points lying on the z_1-axis. Let S^3 denote the unit sphere $|z_1|^2 + |z_2|^2 = 1$ in $\mathbb{C}^2$. All of the actions $\eta_{p,q}$ restrict to give actions on S^3. Of these actions only $\eta_{1,1}$ is free. If $q \neq 1$, the action $\eta_{1,q}$ has two isotropy groups – the trivial isotropy group and $\mathbb{Z}_q$. If $(p,q) = 1$, the action $\eta_{p,q}$ has three isotropy groups, $\mathbb{Z}_p$, $\mathbb{Z}_q$, and $\mathbb{Z}_{pq}$.

(4) (The Hopf fibration) The free-action $\eta_{1,1}$ on S^3 is of particular interest in topology. We shall shortly show that the orbit space $S^3/\mathrm{SO}(2)$ is diffeomorphic to S^2. The resulting quotient (orbit) map $h : S^3 \to S^2$ is called the *Hopf fibration*.

In order to understand the free-action $\eta_{1,1}$, it helps to start with the observation that S^3 can be viewed as the union of two solid tori $D^2 \times S^1$ identified along their boundaries – $\mathbb{T}^2$. More precisely,

$$S^3 = D^2 \times S^1 \cup_\alpha D^2 \times S^1,$$

where $\alpha : \partial(D^2 \times S^1) \to \partial(D^2 \times S^1)$ identifies boundary points and is defined by $\alpha(z_1, z_2) = (z_2, z_1)$ (note the change in order, without the change in order we would get $S^2 \times S^1$ since the union of two copies of D^2 along their boundary is just S^2). Observe that the SO(2) action $\rho_{1,1}$ extends from $D^2 \times S^1$ to a free SO(2)-action on $D^2 \times S^1 \cup_\alpha D^2 \times S^1$. Rather than give a geometric proof of this result, we give an analytic proof. For this, we start by defining the following parameterization of $S^3 \subset \mathbb{C}^2$:

$$P(\theta, s, \psi) = (\cos(s)e^{\imath(u+v)}, \sin(s)e^{\imath(u-v)}) = e^{\imath u}(\cos(s)e^{\imath v}, \sin(s)e^{-\imath v}),$$

where $\theta \in [0, 2\pi)$, $s \in [0, \pi/2]$, $\psi \in [-\pi, \pi]$. We remark that multiplication $e^{\imath u}$ corresponds to the $\rho_{1,1}$-action of SO(2). If we take $s = 0$ or $\pi/2$, the image of P is a circle (in the z_1-plane and z_2-plane respectively). For $s \neq 0, \pi/2$, the image of P is the torus $\mathbb{T}^2_s \subset S^3$ given by $|z_1|^2 = \cos^2 s$, $|z_2|^2 = \sin^2 s$. When $s = \pi/4$ we obtain the *Clifford torus* in S^3. This is a (flat) torus which divides S^3 into two (isometric) solid tori (the $D^2 \times S^1$ tori we defined above). Specifically, one solid torus is $P([0, 2\pi] \times [0, \pi/4] \times [-\pi, \pi])$, the other is $P([0, 2\pi] \times [\pi/4, \pi/2] \times [-\pi, \pi])$.

The orbit space $S^3/\mathrm{SO}(2)$ is homeomorphic to S^2. This can be seen in several ways. First of all we can consider the map $F : \mathbb{C}^2 \to \mathbb{C}$ defined by $F(z_1, z_2) = z_2/z_1$, $z_1 \neq 0$. This restricts to a map on S^3 which is defined everywhere except $z_1 = 0$. Note that F is *constant* on SO(2)-orbits. It follows that F induces a map $\tilde{F} : S^3/\mathrm{SO}(2) \to \mathbb{C} \cup \{\infty\} = S^2$ which is easily seen to be a homeomorphism. Alternatively, observe that the $\rho_{1,1}$-action on $D^2 \times S^1$ is isomorphic to the $\rho_{0,1}$-action on $D^2 \times S^1$ (example (1) above). Obviously the orbit space of $D^2 \times S^1$ by SO(2) is just D^2. Since $S^3 = D^2 \times S^1 \cup_\alpha D^2 \times S^1$, it follows that the orbit space is two copies of D^2 identified along their boundary. That is, S^2.

The *Hopf fibration* $h : S^3 \to S^2$ can be given quite explicitly by the Hopf formula

$$h(z_1, z_2) = (|z_1|^2 - |z_2|^2, z_1\bar{z}_2 + \bar{z}_1 z_2, -\imath(z_1\bar{z}_2 - \bar{z}_1 z_2)).$$

We leave it as an exercise to check that h is constant on SO(2)-orbits and that h maps onto S^2. Finally we remark that each pair of distinct SO(2)-orbits in S^3 is *linked*. This is rather easy to see from the geometric picture of S^3 as a union of two solid tori.

REMARK 2.3.17. The $\rho_{1,1}$-action is induced from scalar multiplication by complex numbers of unit modulus. Noting our earlier convention on the use of S^1 (see examples 1.2.7(1)), we can consider the Hopf fibration as coming from the free S^1-action on S^3 and then write $S^3/S^1 \cong S^2$.

2.4. Twisted products

For the remainder of the chapter we assume that G is compact (many of the results we prove continue to hold if G is not compact provided that the action is *proper* – see chapter 3).

Suppose that H is a closed subgroup of G and X is an H-space. In this section we describe the twisted product construction that naturally associates to (X, H) a a G-space – the 'twisted product of G and X'. This construction plays an important role in the development of the theory of smooth G-manifolds as G-orbits have a base of neighbourhoods which can be represented as twisted products. The construction is also intimately related to the theory of induced representations. We indicate some of the relevant theory, including Frobenius reciprocity, in the following subsection.

Suppose that H is a (closed) subgroup of the topological group G and that (X, H) is an H-space. We define a free action of H on $G \times X$ by

$$h(g, x) = (gh^{-1}, hx), \quad (h \in H, g \in G, x \in X).$$

This action commutes with the free G-action on $G \times X$ defined by left multiplication on G:

$$g(g', x) = (gg', x), \quad (g, g' \in G, x \in X).$$

The *twisted product* $G \times_H X$ is the orbit space $(G \times X)/H$. Since the G and H actions on $G \times X$ commute, $G \times_H X$ inherits the structure of a G-space from that on $G \times X$. Let $q : G \times X \to G \times_H X$ denote the orbit map. Then $q(g, x) = q(g', x')$ if and only if there exists $h \in H$ such that

$$g' = gh^{-1}, \quad x' = hx.$$

It is convenient to write $q(g, x) = [g, x]$ with the understanding that $[g, x] = [gh^{-1}, hx]$, for all $h \in H$.

LEMMA 2.4.1. *The isotropy subgroup of $[g, x] \in G \times_H X$ is given by*

$$G_{[g,x]} = gH_xg^{-1}.$$

In particular, the G-action on $G \times_H X$ is free if and only if X is a free H-space.

PROOF. We leave the proof as an easy exercise for the reader. $\square$

EXAMPLE 2.4.2. Let $X = [-1, +1]$ and $\mathbb{Z}_2$ act on X as multiplication by ± 1. We identify $\mathbb{Z}_2$ with the subgroup $\{0, \pi\}$ of $\mathrm{SO}(2) = \mathbb{R}/2\pi\mathbb{Z}$. The twisted product $\mathrm{SO}(2) \times_{\mathbb{Z}_2} X$ is homeomorphic to the Mobius band. The induced $\mathrm{SO}(2)$-action on $\mathrm{SO}(2) \times_{\mathbb{Z}_2} X$ has one singular orbit – the centre circle of the Mobius band.

EXERCISE 2.4.3. Construct an $\mathrm{SO}(2)$-action on the Klein bottle which has precisely two singular orbits, both consisting of points with isotropy $\mathbb{Z}_2$.

EXERCISE 2.4.4. Show that the projection map $\pi_2 : X \times G \to G$ induces a continuous G-equivariant projection $p : G \times_H X \to G/H$, where we take the left G-action on G/H. That is, $p(k[g, x]) = kg[H]$, $k, g \in G$, $x \in X$. The fibre $p^{-1}(g[H])$ is homeomorphic to X for all $g[H] \in G/H$.

REMARK 2.4.5. If G, H are Lie groups then $p : G \times_H X \to G/H$ is a locally trivial fibre bundle over G/H, fibre X. The local triviality follows from the existence of local sections of $G \to G/H$ (see also chapter 3).

2.4.1. Induced G-spaces. Let H be a closed subgroup of G and X be a G-space. Noting exercise 2.4.4, let $i_H^G X = C^0(G \times_H X)$ – the space of continuous sections of the map $p : G \times_H X \to G/H$. Give $i_H^G X$ the compact open topology (a base of open neighbourhoods for $i_H^G X$ is given by $\{s \in i_H^G X \mid s(K) \subset U\}$, where K and U range over all compact subsets of G/H and open subsets of $G \times_H X$ respectively). We define a G-action on $i_H^G X$ by

$$gs(k[H]) = gs(g^{-1}k[H]), \quad (k[H] \in G/H, g \in G).$$

The G-action is continuous and so $(i_H^G X, G)$ has the structure of a G-space.

There is a natural identification of $i_H^G X$ with the space $C_H^0(G, X)$ of continuous functions $f : G \to X$ satisfying

$$f(gh) = h^{-1} f(g), \quad (h \in H, g \in G).$$

Indeed, if $f \in C_H^0(G, X)$, define $s_f \in i_H^G X$ by

$$s_f(g[H]) = [g, f(g)].$$

Define a G-action on $C_H^0(G, X)$ by $(gf)(k) = f(g^{-1}k)$, $g, k \in G$. We leave it as an exercise for the reader to check that the correspondence $f \mapsto s_f$ defines a G-equivariant homeomorphism between $C_H^0(G, X)$ and $i_H^G X$.

We define an 'evaluation' map $\mathrm{ev} : i_H^G X \to X$ by $s([H]) = [e_G, \mathrm{ev}(s)]$. Equivalently, if $s = s_f$, $f \in C_H^0(G, X)$, then $\mathrm{ev}(s) = f(e_G)$.

If Z is a G-space, let $\mathrm{res}_H^G Z$ denote the H-space obtained by restricting the action of G to H.

PROPOSITION 2.4.6. *Let H be a closed subgroup of G, X be an H-space and Z be a G-space. There is a canonical homeomorphism*

$$C_G^0(Z, i_H^G X) \approx C_H^0(\mathrm{res}_H^G Z, X).$$

Here $C_G^0(Z, i_H^G X)$ consists of the continuous maps $F : Z \to i_H^G X$ such that $Fg = gF$. Similarly, $C_H^0(\mathrm{res}_H^G Z, X)$ consists of the continuous maps $K : \mathrm{res}_H^G Z \to X$

which commute with H. Both spaces of functions are given the compact open topology.

PROOF. Suppose $\alpha \in C^0_G(Z, i^G_H X)$. We define $\hat{\alpha} \in C^0_H(\mathrm{res}^G_H Z, X)$ by $\hat{\alpha}(z) = \mathrm{ev}(\alpha)$. Conversely, given $\hat{\beta} \in C^0_H(\mathrm{res}^G_H Z, X)$, we define $\beta : Z \to C^0(G, X)$ by $\beta(z)(g) = \hat{\beta}(g^{-1}z)$. We leave it to the reader to check that $\beta \in C^0_G(Z, i^G_H X)$, and that the maps $\alpha \mapsto \hat{\alpha}$, $\hat{\beta} \mapsto \beta$ are continuous and inverses of each other. $\square$

REMARK 2.4.7. Proposition 2.4.6 is a general case of *Frobenius reciprocity*. It is particularly important in the theory of induced representations: we assume X and Z are representations and replace spaces of continuous maps by spaces of continuous linear maps. We give a simple, but useful, application of this result to representations of finite groups later in this chapter.

2.5. Isotropy type and stratification by isotropy type

Given $x \in X$, let (G_x) denote the conjugacy class of G_x in G. We call (G_x) the *isotropy type* or the *orbit type* of x. If G is finite, the number of isotropy types for an action of G on X is finite. As we shall see, finiteness will hold for a large class of actions by compact Lie groups.

We regard the isotropy type of a point as a measure of the 'symmetry' of the point. If $x \in X$, let $\iota(x)$ denote the isotropy type of x (that is, $\iota(x) = (G_x)$). Let $\mathcal{O}(X, G)$ denote the set of isotropy types for the action of G on X. For each $\tau \in \mathcal{O}(X, G)$, define

$$X_\tau = \{x \in X \mid \iota(x) = \tau\}.$$

The collection $\{X_\tau \mid \tau \in \mathcal{O}(X, G)\}$ partitions X into points of the same isotropy type. We refer to X_τ as an *orbit stratum* and the decomposition of X into orbit strata as the *stratification of X by isotropy type* or the *orbit stratification of X*.

We define a partial order on $\mathcal{O}(X, G)$ by requiring that $\tau > \mu$ if there exists $H \in \tau$, $K \in \mu$ such that $H \supsetneq K$. The maximal elements in the partial order correspond to points with the greatest isotropy.

REMARK 2.5.1. The partial order is a relation between subgroups of G. Later, we will be able to interpret the relation in terms of spatial relationships between points in X.

Given a closed subgroup H of G, let

$$X^H = \{x \in X \mid Hx = x\}.$$

We call X^H the *fixed point space* of H. Clearly, X^H is a closed subset of X. In case $H = G$, X^G is the set of all points in X fixed by G.

If $G_x \in \tau$, $X^{G_x}_\tau$ is the set of points in X with isotropy group G_x. We have

$$X^{G_x}_\tau = X_\tau \cap X^{G_x} \subset X^{G_x}.$$

EXAMPLES 2.5.2. (1) We identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way. For $n \geq 2$, $\beta = \exp(\frac{2\pi\imath}{n})$ defines a counter-clockwise rotation of $\mathbb{R}^2$ through $2\pi/n$. Let $\mathbb{Z}_n = \langle\beta\rangle \subset \mathrm{SO}(2)$. We have a linear action of $\mathbb{Z}_n$ on $\mathbb{R}^2 \approx \mathbb{C}$ defined by

$$z \mapsto \beta^j z = \exp(\frac{2\pi\imath j}{n})z, \quad j = 0, \ldots, n-1.$$

The orbit stratification of $\mathbb{R}^2$ is given by $\mathbb{R}^2 = (\mathbb{R}^2 \setminus \{0\}) \cup \{0\}$. Every point in $\mathbb{R}^2 \setminus \{0\}$ has trivial isotropy and the origin has isotropy $\mathbb{Z}_n$.
(2) Let $\mathbf{D}_n$ denote the dihedral group of order $2n$, $n \geq 2$. We define the standard linear action of $\mathbf{D}_n$ on $\mathbb{R}^2 \approx \mathbb{C}$. Let κ denote the reflection in the x-axis ($z \mapsto \bar{z}$) and $\beta = \exp(\frac{2\pi\imath}{n})$ as above. Then $\mathbf{D}_n = \langle\kappa, \beta\rangle$. We claim that as a transformation group, $\mathbf{D}_n$ consists of the rotations in $\mathbb{Z}_n$ together with the reflections in the lines $\theta = \pi j/n$, $j = 0, \ldots, n-1$. Obviously, $\mathbf{D}_n \supset \langle\beta\rangle = \mathbb{Z}_n$. Since $\kappa, \beta \in \mathrm{O}(2)$, $\mathbf{D}_n \subset \mathrm{O}(2)$. If $\zeta \in \mathrm{O}(2)$, then ζ is a rotation if determinant$(\zeta) = +1$; otherwise ζ is a reflection in a line through the origin and determinant$(\zeta) = -1$. From this it follows that if $\zeta \in \langle\kappa, \beta\rangle$ then ζ will be a reflection if κ occurs an odd number of times in the word defining ζ. Suppose $\zeta = \beta^j\kappa$. Then ζ is a reflection and fixes a line in $\mathbb{R}^2$. We claim that $\beta^j\kappa$ fixes the line L_j defined by $\theta = \frac{\pi j}{n}$. This follows since $\kappa(L_j) = L_{n-j}$ and $\beta^j L_{n-j} = L_j$ (lines are invariant under rotations through π – not just 2π). See figure 1 for the case $n = 3$, $j = 1, 2$.

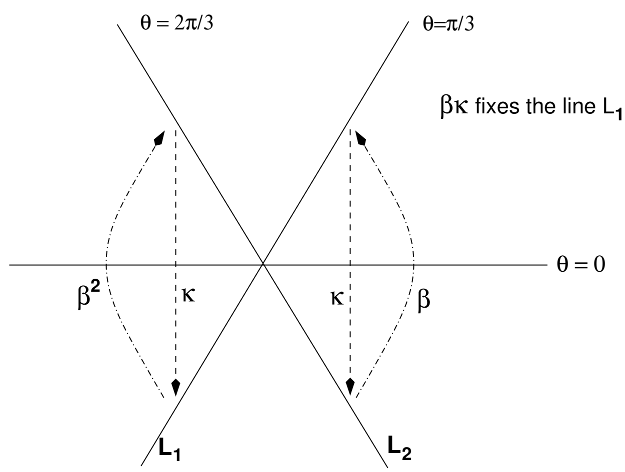


FIGURE 1. Compositions $\beta^j\kappa$ in $\mathbf{D}_3$

Since the square of a reflection is the identity, all that remains to be checked is that composition of reflections in lines L_j, L_k, $j \neq k$, lies in $\langle\beta\rangle$. This follows either directly or from the generating relation $\kappa\beta = \beta^{n-1}\kappa$. We leave details to the reader.

In order to describe the orbit stratification of $\mathbb{R}^2$, we need to distinguish the cases n odd and even.

When n is odd we have the orbit stratification

$$\mathbb{R}^2 = (\mathbb{R}^2 \setminus \cup_{j=0}^{n-1} L_j) \cup (\cup_{j=0}^{n-1} L_j \setminus \{0\}) \cup \{0\}.$$

All points in $\mathbb{R}^2 \setminus \cup_{j=0}^{n-1} L_j$ have trivial isotropy, points in $\cup_{j=0}^{n-1} L_j \setminus \{0\}$ have isotropy conjugate to $\langle\kappa\rangle$ and $\{0\}$ has isotropy $\mathbf{D}_n$. We refer to figure 2 for the case $n = 3$. Since $\mathbf{D}_n$ (in fact $\mathbb{Z}_n$) transitively permutes the axes L_j, the isotropy of points

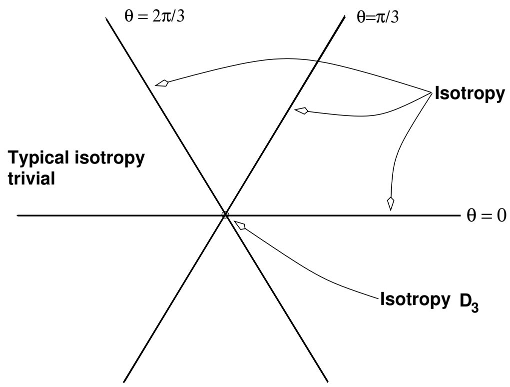


FIGURE 2. Orbit stratification for standard $\mathbf{D}_3$ action

on different axes is conjugate (see exercise 2.3.10(2)).

When n is even we have the orbit stratification

$$\mathbb{R}^2 = (\mathbb{R}^2 \setminus \cup_{j=0}^{n-1} L_j) \cup (\cup_{j=0}^{n/2-1} L_{2j} \setminus \{0\}) \cup (\cup_{j=0}^{n/2-1} L_{2j+1} \setminus \{0\}) \cup \{0\}.$$

The main point to notice here is that the 'odd' and 'even' axes have different isotropy. In order to verify this, observe that $\mathbf{D}_n$ (or $\mathbb{Z}_n$) transitively permutes the set of axes $\{L_0, L_2, \ldots, L_{n-2}\}$. It follows that non-zero points on these axes have isotropy conjugate to $\langle\kappa\rangle$. Similarly, $\mathbf{D}_n$ transitively permutes the set of axes $\{L_1, L_3, \ldots, L_{n-1}\}$ and non-zero points on these axes have isotropy conjugate to $\langle\beta\kappa\rangle$. We refer to Figure 3 for the case $n = 4$.

We claim that $\langle\beta\kappa\rangle$ and $\langle\kappa\rangle$ are *not* conjugate subgroups of $\mathbf{D}_n$. It suffices to show that for all $g \in \mathbf{D}_n$ $\beta\kappa \neq g\kappa g^{-1}$. If g is a rotation, we may assume $g = \beta^j$, where $1 \leq j \leq n-1$. Dividing on the left by β, we show $\kappa \neq \beta^{j-1}\kappa\beta^{n-j}$. Computing we see that $\beta^{j-1}\kappa\beta^{n-j}L_0$ is the line $\theta = \frac{2\pi}{n}(2j-1)$ and, since n is even, this line is always distinct from L_0. Hence $\beta^{j-1}\kappa\beta^{n-j} \neq \kappa$. If g is a reflection, then $g = \beta^j\kappa$, where $1 \leq j \leq n-1$. Substitution and repetition of the previous argument shows that $g\kappa g^{-1} \neq \beta\kappa$. Note that $\beta\kappa$ and κ are conjugate in $\mathrm{O}(2)$ (in fact in $\mathbf{D}_8$).

REMARK 2.5.3. The group $\mathbf{D}_n$ is the symmetry group of the regular n-gon. If we drop the edges from the regular n-gon we see that each $\mathbf{D}_n$ embeds as a subgroup of the symmetric group S_n, (elements of $\mathbf{D}_n$ permute the vertices of the regular n-gon). In case $n = 3$, we have $\mathbf{D}_3 = S_3$. For $n > 3$, $\mathbf{D}_n \neq S_n$.

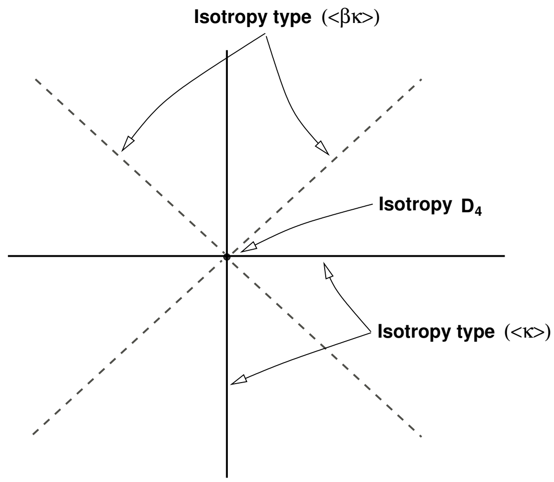


FIGURE 3. Orbit stratification for standard $\mathbf{D}_4$ action

(The equilateral triangle is the only regular n-gon, $n > 2$, such that the distance between all distinct vertices is equal. Hence permutations of vertices induce permutations of edges.)

2.6. Representations

We continue to assume that G is a compact topological group. Let $\mathbb{F}$ denote the field of real or complex numbers and V be a finite dimensional vector space over $\mathbb{F}$. Let $\mathrm{GL}(V)$ denote the group of invertible $\mathbb{F}$-linear transformations of V.

DEFINITION 2.6.1. An $\mathbb{F}$-representation (V, G) of G on V is an $\mathbb{F}$-linear action of G on V. Equivalently, an action of G on V defined by a continuous homomorphism $\rho : G \to \mathrm{GL}(V)$. The *degree* of the representation is the dimension of V (over $\mathbb{F}$). The representation is *faithful* if ρ is injective (a monomorphism).

EXAMPLES 2.6.2. (1) Every group G has at least one representation on a vector space V. We define $\rho : G \to \mathrm{GL}(V)$ by $\rho(g) = I_V \in \mathrm{GL}(V)$, for all $g \in G$. We call this representation the *trivial* representation of G on V.
(2) Suppose G is finite. Let $\mathbb{C}[G]$ denote the complex vector space $\mathbb{C}$-valued functions on G. Obviously, $\mathbb{C}[G] \cong \mathbb{C}^{|G|}$. The *regular representation* of G on $\mathbb{C}[G]$ is defined by

$$gf = f \circ g^{-1},\ f \in \mathbb{C}[G], g \in G.$$

The regular representation of G is faithful (exercise). (If G is a compact group and not finite, it is more fruitful to look at the space $L^2(G, dh)$.)
(3) If G is a subgroup of $\mathrm{GL}(n, \mathbb{R})$, then we have an associated $\mathbb{R}$-representation of G on $\mathbb{R}^n$ with action defined by the restriction of the natural action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ to G. If G is a subgroup of $\mathrm{O}(n)$, we refer to the associated representation of G as an *orthogonal representation* of G. The dihedral groups $\mathbf{D}_n$ discussed in

the previous section were all defined as subgroups of $\mathrm{O}(2)$ and therefore define faithful orthogonal representations on $\mathbb{R}^2$. Note that reflections in $\mathrm{O}(2)$ are not complex linear and so these actions do not define complex representations.
(4) If G is a subgroup of $\mathrm{GL}(n,\mathbb{C})$ we have an associated $\mathbb{C}$-representation of G on $\mathbb{C}^n$. In case $G \subset \mathrm{U}(n)$, we refer to the associated representation of G on $\mathbb{C}^n$ as a *unitary representation* of G. The group $\mathrm{SO}(2)$, acting on $\mathbb{R}^2 \approx \mathbb{C}$ by rotation defines a unitary representation of $\mathrm{SO}(2)$ (sometimes denoted $\mathrm{U}(1)$). The cyclic groups $\mathbb{Z}_n$ discussed in the previous section were all defined as subgroups of $\mathrm{SO}(2)$ and therefore define faithful unitary representations.

DEFINITION 2.6.3. The $\mathbb{F}$-representations (V, G), (W, G) are *isomorphic* or *equivalent* if there exists an $\mathbb{F}$-linear isomorphism $T : V \to W$ such that

$$gT(v) = T(gv), \quad g \in G, v \in V.$$

(Equivalently, in terms of the defining homomorphisms $\rho : G \to \mathrm{GL}(V)$, $\eta : G \to \mathrm{GL}(W)$, we require $\eta(g)T(v) = T(\rho(g))v$, for all $g \in G$, $v \in V$.)

REMARKS 2.6.4. (1) The isomorphism T is called an *intertwining map*.
(2) Suppose that G is a finite group with generators $\alpha_1, \ldots, \alpha_k$ and relations $R_i(\alpha_1, \ldots, \alpha_k) = e$, $1 \le i \le \ell$. In order to construct an orthogonal representation $\rho : G \to \mathrm{O}(n)$ it suffices to define $\rho(\alpha_j) \in \mathrm{O}(n)$, such that $R_i(\rho(\alpha_1), \ldots, \rho(\alpha_k)) = I$ (in $\mathrm{O}(n)$), $1 \le i \le \ell$.

EXAMPLES 2.6.5. (1) Let $\beta = \exp(2\pi\imath/3)$ and set $\mathbb{Z}_3 = \langle\beta\rangle$. In the previous section, we defined the standard complex representation $\rho_3^1 : \mathbb{Z}_3 \to U(1)$ by $\rho(\beta) = \beta \in U(1)$. We may define a different complex representation $\rho_3^2 : \mathbb{Z}_3 \to U(1)$ by $\rho(\beta) = \beta^2$. As complex representations, ρ_3^1 is not equivalent to ρ_1^2. However, if we view the representations as real representations (on $\mathbb{R}^2$) then ρ_3^1 is equivalent to ρ_3^2: $T\rho_3^1 = \rho_3^2 T$, where T is complex conjugation on $\mathbb{C}$.
(2) Let κ denote complex conjugation on $\mathbb{C} \approx \mathbb{R}^2$ and recall that $\mathbf{D}_3 = \langle\kappa, \beta\rangle$. We defined the standard real representation ρ of $\mathbf{D}_3$ by $\rho(\kappa) = \kappa \in \mathrm{O}(2)$, $\rho(\beta) = \beta \in \mathrm{O}(2)$. We may define the real representation η of $\mathbf{D}_3$ on $\mathbb{R}^2$ by $\eta(\kappa) = \kappa$, $\eta(\beta) = \beta^2$. As in the first example, η and ρ are equivalent representations with intertwining operator complex conjugation. We may also define a non-trivial one-dimensional real representation of $\mathbf{D}_3$ by $\zeta(\kappa) = -1$, $\zeta(\beta) = 1$ (note that ζ maps all reflections to -1 and all rotations to $+1$.

EXERCISE 2.6.6. (1) Find eight inequivalent 1-complex dimensional representations of $\mathbb{Z}_8$. Which of these representations are equivalent as real representations?
(2) Find four inequivalent real representations of $\mathbf{D}_4$. (Three of these representations will be 1-dimensional, one will be 2-dimensional), Which, if any, of the representations will be complex?

(3) Show that if we define $\eta : \mathbf{D}_4 \to \mathrm{GL}(\mathbb{C}, 2)$ by

$$\rho(\kappa) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \rho(\beta) = \begin{pmatrix} e^{\frac{\pi}{2}\imath} & 0 \\ 0 & e^{-\frac{\pi}{2}\imath} \end{pmatrix}$$

then η defines a complex representation of $\mathbf{D}_4$. Find four other inequivalent complex representations of $\mathbf{D}_4$, all of which should be of (complex) dimension 1.
(4) Find an infinite set of inequivalent 1-complex dimensional representations of SO(2). Which of these representations become equivalent when viewed as real representations?

2.6.1. Averaging over G. Suppose that (V, G) is a real representation and G is compact Lie group (or finite). Let $(\ ,\ )$ be a (positive definite) inner product on V. Let dh denote Haar measure on G. Define

$$(x, y)^\star = \int_G (gx, gy)\, dh, \ \ (x, y \in V).$$

(If G is finite, $(x, y)^\star = \frac{1}{|G|}\sum_{g\in G}(gx, gy)$.) We leave it as an exercise for the reader to verify that $(\ ,\ )^\star$ is a *G-invariant* (positive definite) inner product on V:

$$(gx, gy)^\star = (x, y)^\star, \ \ (x, y \in V, g \in G). \tag{2.1}$$

It follows that (V, G) is an orthogonal representation with respect to the inner product $(\ ,\ )^\star$ and that $\rho : G \to \mathrm{O}(V)$ – the orthogonal group of V. If we choose an orthonormal basis of V with respect to $(\ ,\ )^\star$, we may identify V with $\mathbb{R}^n$ and regard (V, G) as an orthogonal representation of G on $\mathbb{R}^n$. In future, we usually regard real representations of compact Lie groups as orthogonal representations on some $\mathbb{R}^n$. Similarly, complex representations of a compact Lie group G may be viewed as unitary representations on some $\mathbb{C}^n$. Since $\mathrm{U}(n) \subset \mathrm{O}(2n)$ (under the natural identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$), every unitary representation of G on $\mathbb{C}^n$ defines an orthogonal (real) representation of G on $\mathbb{R}^{2n}$.

EXAMPLES 2.6.7. (1) If $G = \mathrm{SO}(2) = \mathbb{R}/2\pi\mathbb{Z}$, we may define Haar measure on SO(2) to be normalized Lebesgue measure (arc length). If $A \subset [0, 2\pi) \subset \mathrm{SO}(2)$ is a Borel set, then

$$h(A) = \frac{1}{2\pi}\int_A d\lambda,$$

where $d\lambda$ denotes Lebesgue measure on $\mathbb{R}$.
(2) Averaging is a powerful tool in the theory of compact Lie groups. For example, suppose that (V, G) is an $\mathbb{F}$-representation. Define the $\mathbb{F}$-linear map $\Pi : V \to V$ by

$$\Pi(v) = \int_{g\in G} gv\, dh(g) = \left(\int_{g\in G} g\, dh(g)\right)(v).$$

If $k \in G$ then $k\Pi(v) = \int_G kgv\, dh(g) = \int_G gv\, dh(g)$ by left invariance of Haar measure. Hence $k\Pi(v) = \Pi(v)$, for all $k \in G$, and so $\Pi(V) \subset V^G$. Since

$\Pi|V^G = I$, $\Pi(V) = V^G$ and $\Pi^2 = \Pi$. Hence Π is a projection of V onto V^G. Consequently, $\dim(\Pi(V)) = \mathrm{trace}(\Pi)$ and so (by linearity of the integral)

$$\dim(V^G) = \mathrm{trace}(\int_G g\, dh(g)) = \int_G \mathrm{trace}(g)\, dh(g).$$

This provides a useful formula for calculating the dimension of the fixed point space of a compact group action.

DEFINITION 2.6.8. Given a representation (V, G), the *character* of (V, G) is the function

$$\chi_{(V,G)} = \chi_V : G \to \mathbb{C},$$

defined by $\chi_V(g) = \mathrm{trace}(g : V \to V)$.

EXERCISE 2.6.9. Show that (a) χ_V is smooth, (b) if (V, G) and (W, G) are isomorphic representations then $\chi_V = \chi_W$, (c) $\chi_V(ghg^{-1}) = \chi_V(g)$, all $g, h \in G$, and (d) $\chi_V(e_G) = \dim(V)$. (These results hold for real or complex representations. In terms of characters, examples 2.6.7(2) reads $\dim(V^G) = \int_G \chi_V(g)\, dh(g)$.)

REMARK 2.6.10. Although we do not make much use of characters in this book, we want to emphasize that character theory is a fundamental tool for the analysis of group representations, especially over the complex field.

2.7. Irreducible representations and the isotypic decomposition

DEFINITION 2.7.1. The $\mathbb{F}$-representation (V, G) is *irreducible* if the only G-invariant $\mathbb{F}$-linear subspaces of V are $\{0\}$ and V. If there exist non-trivial invariant subspaces, (V, G) is *reducible*.

EXAMPLES 2.7.2. (1) Every one-dimensional representation of a group G is irreducible.
(2) The assumption of a fixed field in definition 2.7.1 is important. For example, if we take the standard complex representation of $\mathbb{Z}_2$ on $\mathbb{C}$ generated by multiplication by -1, then this representation is irreducible as a complex representation. However, if we forget the complex structure and identify $\mathbb{C}$ with $\mathbb{R}^2$, then the resulting real representation of $\mathbb{Z}_2$ on $\mathbb{R}^2$ is not irreducible (as a real representation) since every 1-dimensional linear subspace of $\mathbb{R}^2$ is $\mathbb{Z}_2$-invariant. $\diamondsuit$

EXERCISE 2.7.3. (1) Show that the standard representation of $\mathbf{D}_n$ on $\mathbb{R}^2$ is irreducible if $n > 2$ and reducible if $n = 2$.
(2) Show that the standard $\mathbb{C}$-representation of $\mathbb{Z}_n$ on $\mathbb{C}$ is irreducible, $n \geq 2$. Which of these representations are reducible if viewed as real representations?
(3) Show that $\mathbf{D}_4$ has at least four non-isomorphic real irreducible representations (see examples 2.6.6(2)).

PROPOSITION 2.7.4. *Let V have G-invariant inner product $(\ ,\ )$. Suppose that $\rho : G \to O(V)$ is an orthogonal representation. If $W \subset V$ is a proper G-invariant subspace of V, let $\pi : V \to W$, $\pi^\perp : V \to W^\perp$ denote the orthogonal* *projections of V on W and $W^\perp$ respectively. Then $\pi\rho : G \to O(W)$, $\pi^\perp : G \to O(W^\perp)$ define orthogonal representations of G. The same result holds for unitary representations of G.*

PROOF. If W is a G-invariant subspace of V, then $(\ ,\ )$ defines a G-invariant inner product on W and it follows immediately that $\pi\rho : G \to \mathrm{GL}(W)$ defines an orthogonal representation of G with respect to $(\ ,\ )$. In order to show that $\pi^\perp$ defines an orthogonal representation of G, it suffices to show that $W^\perp$ is G-invariant. By definition, $x \in W^\perp$ if and only if $(w, x) = 0$ for all $w \in W$. The result follows by noting that $(w, gx) = (g^{-1}w, x) = 0$, by G-invariance, and so $(w, gx) = 0$, all $g \in G$, $w \in W$. □

COROLLARY 2.7.5. *Let $\rho : G \to GL(V)$ be an $\mathbb{F}$-representation. If W is a non-trivial G-invariant subspace of V, there exists a G-invariant subspace U of V such that $V = W \oplus U$. In particular, (V, G) may be regarded as the direct sum of the representations (W, G) and (U, G).*

PROOF. Since $\rho(G)$ is a compact subgroup of $\mathrm{GL}(V)$, and therefore Lie, we may take Haar measure on $\rho(G)$ and thereby define a G-invariant inner product on V. The result follows from proposition 2.7.4. □

An important consequence of corollary 2.7.5 is that every representation $\rho : G \to \mathrm{GL}(V)$ can be written as a finite direct sum of irreducible representations. This decomposition will be unique up to order. This result is quite easy to prove if V is a $\mathbb{C}$-representation but takes a little more care if V is an $\mathbb{R}$-representation (this is somewhat analogous to the difference between complex and real Jordan form).

We start by introducing some new notation. Let $\rho : G \to \mathrm{GL}(V)$, $\eta : G \to \mathrm{GL}(W)$ be $\mathbb{F}$-representations of G. Let $L_G^{\mathbb{F}}(V, W)$ denote the space of $\mathbb{F}$-linear maps $A : V \to W$ which commute with G:

$$A(gv) = gA(v), \quad (g \in G, v \in V).$$

If $\mathbb{F} = \mathbb{R}$, we often drop the superscript and just write $L_G(V, W)$.

LEMMA 2.7.6. *Let (V, G) be an irreducible $\mathbb{F}$-representation. Then*

(1) *If $\mathbb{F} = \mathbb{C}$, then $L_G^{\mathbb{C}}(V, V) = \mathbb{C} = \{\lambda I_V \mid \lambda \in \mathbb{C}\}$.*
(2) *If $\mathbb{F} = \mathbb{R}$, then $L_G^{\mathbb{R}}(V, V) \supset \mathbb{R} = \{\lambda I_V \mid \lambda \in \mathbb{R}\}$.*

PROOF. Suppose $A \in L_G^{\mathbb{C}}(V, V)$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Then kernel$(A - \lambda I)$ is a non-trivial G-invariant subspace of V and hence, by irreducibility of V, kernel$(A - \lambda I) = V$. That is, $A = \lambda I_V$. Since $\mathbb{C} I_V \subset L_G^{\mathbb{C}}(V, V)$, (1) follows. For (2), observe that $\mathbb{R} I_V \subset L_G^{\mathbb{R}}(V, V)$. □

LEMMA 2.7.7 (Schur's lemma). *Let V, W be irreducible $\mathbb{F}$-representations. Then*

(1) *If V is not isomorphic to W, $L_G^{\mathbb{F}}(V, W) = \{0\}$.*
(2) *If $\mathbb{F} = \mathbb{C}$ and V is isomorphic to W, then $L_G^{\mathbb{C}}(V, W) \cong \mathbb{C}$.*

(3) *If $\mathbb{F} = \mathbb{R}$ and V is isomorphic to W, then $\dim_{\mathbb{R}}(L_G^{\mathbb{R}}(V,W)) \geq 1$.*

PROOF. Suppose V is not isomorphic to W and let $A \in L_G^{\mathbb{F}}(V,W)$. Since V, W are irreducible, the G-invariant subspace kernel$(A) \subset V$ must either equal V or be the zero subspace. If kernel$(A) = \{0\}$ then, by the same reasoning applied to W, image(A) must equal W. But this says that A is an intertwining map for the representations V and W contradicting the assumption that V and W are not isomorphic. Hence $L_G^{\mathbb{F}}(V,W) = \{0\}$. Lemma 2.7.6 implies (2,3) (replace the representation $\rho : G \to \mathrm{GL}(W)$ by $T^{-1}\rho : G \to \mathrm{GL}(V)$, where $T : V \to W$ is an intertwining operator). $\square$

Matters are particularly simple when G is Abelian.

LEMMA 2.7.8. *If G is Abelian and (V,G) is an irreducible $\mathbb{C}$-representation, then $\dim_{\mathbb{C}}(V) = 1$. In particular, $L_G^{\mathbb{C}}(V,V) \approx \mathbb{C}$.*

PROOF. Let $g \in G$. Then $g : V \to V$ is $\mathbb{C}$-linear. Since G is Abelian, we have $gh = hg$, all $h \in G$. Therefore $g \in L_G^{\mathbb{C}}(V,V)$. By lemma 2.7.6, $g = \lambda I_V$, for some $\lambda \in \mathbb{C}$. This is true for all $g \in G$ and so every 1-dimensional complex subspace of V is G-invariant. Since V is irreducible, $\dim_{\mathbb{C}}(V) = 1$. $\square$

Given an integer $p \geq 1$ and a representation (V,G), let (V^p, G) denote the G-representation obtained by taking the direct sum of p-copies of V. For $p, q \geq 1$, let $M(p,q;\mathbb{F})$ denote the space of $p \times q$ matrices over $\mathbb{F}$.

LEMMA 2.7.9. *Let $p, q \geq 1$ and V, W be irreducible $\mathbb{C}$-representations.*

(1) *If V, W are not isomorphic, $L_G^{\mathbb{C}}(V^q, W^p) = \{0\}$.*
(2) *If $V \cong W$, then $L_G^{\mathbb{C}}(V^q, V^p) \approx M(p,q;\mathbb{C})$.*

In case V, W are irreducible $\mathbb{R}$-representations, (1) remains true and we modify (2) by requiring that $L_G^{\mathbb{R}}(V^q, V^p) \supset M(p,q;\mathbb{R})$ (naturally).

THEOREM 2.7.10 (Isotypic decomposition). *Let (V,G) be an $\mathbb{F}$-representation. There exist unique (up to isomorphism) inequivalent irreducible $\mathbb{F}$-representations $(V_1, G), \ldots, (V_k, G)$ and unique strictly positive integers $p_1, \ldots, p_k$ such that*

$$V \cong \bigoplus_{i=1}^{k} V_i^{p_i}.$$

This isotypic decomposition of (V,G) as a sum of irreducible $\mathbb{F}$-representations is unique up to order.

PROOF. It follows from corollary 2.7.5 by an easy induction that we can find irreducible $\mathbb{F}$-representations $(V_1, G), \ldots, (V_k, G)$ and unique strictly positive integers $p_1, \ldots, p_k$ such that $V \cong \bigoplus_{i=1}^{k} V_i^{p_i}$. Suppose that we can also write $V \cong \bigoplus_{j=1}^{\ell} W_j^{q_j}$ where the (W_j, G) are inequivalent irreducible $\mathbb{F}$-representations and $q_j \geq 1$, $1 \leq j \leq \ell$. Applying lemma 2.7.9 we have

$$\begin{aligned} L_G^{\mathbb{F}}(V,V) &\cong \bigoplus_{i,j} L_G^{\mathbb{F}}(V_i^{p_i}, W_j^{q_j}) \\ &\cong \bigoplus_{i,j \,\mid\, V_i \cong W_j} L_G^{\mathbb{F}}(V_i^{p_i}, W_j^{q_j}). \end{aligned}$$

Since $\bigoplus_{i=1}^k V_i^{p_i}$ is isomorphic to $\bigoplus_{j=1}^\ell W_j^{q_j}$i, we have by lemma 2.7.9(1) that for every $i \in \{1,\dots,k\}$, there exists a unique $j(i)$ such that (V_i, G) is isomorphic to $(W_{j(i)}, G)$. Similarly, to each $j \in \{1,\dots,\ell\}$, there exists a unique $i = i(j)$ such that (W_j, G) is isomorphic to $(V_{i(j)}, G)$. Consequently, $(W_1, G) \dots, (W_\ell, G)$ is (up to isomorphism) just a permutation of $(V_1, G), \dots, (V_k, G)$. Relabelling, we may assume $\ell = k$ and $W_i = V_i$, $1 \leq i \leq k$. It remains to show that $p_i = q_i$, $1 \leq i \leq k$. If $A : \bigoplus_{i=1}^k V_i^{p_i} \to \bigoplus_{i=1}^k V_i^{q_i}$ is an intertwining operator, we may write $A = \oplus_{i=1}^k A_I$ where $A_i : V_i^{p_i} \to V_i^{q_i}$, $1 \leq i \leq k$. Necessarily, each A_i is a linear isomorphism and so $p_i = q_i$, $1 \leq i \leq k$. □

2.7.1. $\mathbb{C}$-representations. If $\oplus_{i=1}^k V_i^{p_i}$ is the isotypic decomposition of (V,G), then $L_G^{\mathbb{F}}(V,V) \cong \bigoplus_{i=1}^k L_G^{\mathbb{F}}(V_i^{p_i}, V_i^{p_i})$. If V is a $\mathbb{C}$-representation, we have

$$L_G^{\mathbb{C}}(V,V) \cong \bigoplus_{i=1}^k M(p_i, p_i; \mathbb{C}).$$

We recall some useful facts about $\mathbb{C}$-representations of finite groups. A finite group has, up to isomorphism, only finitely many inequivalent irreducible $\mathbb{C}$-representations. If we label the irreducible $\mathbb{C}$-representations of G by (V_i, G), $\text{degree}(V_i, G) = d_i$, $1 \leq i \leq k$, then

$$\Sigma_{i=1}^k d_i^2 = |G| \text{ (order of } G). \tag{2.2}$$

$$d_i \mid |G|, \quad 1 \leq i \leq k. \tag{2.3}$$

(Proofs of both statements are in Scott [**157**, 12.2.24–27]. We indicate proofs of the more elementary (2.2) in the exercises below.)

Examples 2.7.11. (1) For $n \geq 1$, we have previously shown that there are n inequivalent $\mathbb{C}$-representations of $\mathbb{Z}_n$, all 1-dimensional. Since $n1^2 = n = |\mathbb{Z}_n|$, (2.2) implies there are no other $\mathbb{C}$-representations of $\mathbb{Z}_n$.
(2) There are five distinct irreducible $\mathbb{C}$-representations of $\mathbf{D}_4$. In exercise 2.6.6(3), we gave a 2-dimensional irreducible $\mathbb{C}$-representation of $\mathbf{D}_4$. Since the trivial representation is 1-dimensional and $2^2 + 1^2 + 1^2 + 1^2 + 1^2 = 8$, the order of $\mathbf{D}_4$, (2.2) implies there are exactly five irreducible $\mathbb{C}$-representations of $\mathbf{D}_4$.

Exercise 2.7.12. Verify that the proof of proposition 2.4.6 applies to show that if X is an H-representation and Z a G-representation then $L_G(Z, i_H^G X) \approx L_H(\text{res}_H^G Z, X)$ (Frobenius reciprocity). Using Frobenius reciprocity, show that

$\mathbb{C}[G] \cong \oplus V_i^{d_i}$ where the sum is over all irreducible representations of G and $d_i = \text{degree}(V_i)$. (Hint: Apply Frobenius reciprocity in case $H = \{e_G\}$, X is the trivial 1-dimensional H-representation and Z is an irreducible G-representation. In particular, deduce that the multiplicity of V in $\mathbb{C}[G]$ is $\dim(V^H) = \dim(V)$.)

EXERCISE 2.7.13. Suppose that (V, G) is a complex representation. Let $(V^\star, G)$ and $(\bar{V}, G)$ denote the *dual* and *conjugate* representations of (V, G). Thus, $V^\star = L^{\mathbb{C}}(V, \mathbb{C})$ has G-action defined by $g\phi = \phi \circ g^{-1}$, $\phi \in L^{\mathbb{C}}(V, \mathbb{C})$, $g \in G$, and $\bar{V}$ is regarded as equal to V but with complex multiplication of $v \in V$ by $c \in \mathbb{C}$ defined as $\bar{c}v$. Show

(a) $\chi_{V^\star}(g) = \chi_V(g^{-1})$.
(b) $\chi_{\bar{V}}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$ ($(V^\star, G)$ and $(\bar{V}, G)$ are isomorphic – an isomorphism is given by a G-invariant hermitian inner product on V).

Show also that if (W, G) is a complex representation then $\chi_{V \otimes W} = \chi_V \chi_W$. Deduce, using $L(V, W) \approx V^\star \otimes W$, (a,b), and examples 2.6.7(2) that

(1) $\langle \chi_W, \chi_V \rangle = \int_G \overline{\chi_V(g)} \chi_W(g)\, dh(g) = \dim(L_G^{\mathbb{C}}(V, W))$.
(2) If (V, G), (W, G) are irreducible then $\langle \chi_W, \chi_V \rangle = 1$ if (V, G), (W, G) are isomorphic, otherwise $\langle \chi_W, \chi_V \rangle = 0$.

Finally, if G is finite, use these results applied to $\mathbb{C}[G]$ to deduce (2.2).

2.7.2. Absolutely irreducible representations.

THEOREM 2.7.14 (Frobenius). *Suppose that (V, G) is an irreducible representation over $\mathbb{R}$. Then $L_G(V, V)$ is isomorphic to one of*

(A) *The real numbers $\mathbb{R}$ ((V, G) is an* absolutely irreducible *representation.)*
(C) *The complex numbers $\mathbb{C}$ ((V, G) is irreducible of* complex type*.)*
(Q) *The quaternions $\mathbb{H}$ ((V, G) is irreducible of* quaternionic type*.)*

PROOF. Since $I \in L_G(V, V)$, $\lambda I \subset L_G(V, V)$, for all $\lambda \in \mathbb{R}$, Suppose that $A \in L_G(V, V)$. Then kernel(A) is a G-invariant subspace of V. Since (V, G) is irreducible, either kernel$(A) = V$, and A is the zero map, or kernel$(A) = \{0\}$ and A is a linear isomorphism. Hence $L_G(V, V)$ is closed under $+$, composition, scalar multiplication and every non-zero element has an inverse. Since $L_G(V, V)$ is associative, $L_G(V, V)$ is a finite dimensional *real associative division algebra*. It was shown by Frobenius in 1878 such an algebra is isomorphic to one of $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ (a proof may be found in [**98**, 7.7, p 430]). □

EXAMPLES 2.7.15. (1) The standard representations $(\mathbb{R}^2, \mathbb{Z}_n)$, $n > 2$, are all irreducible of complex type.
(2) The standard representations $(\mathbb{R}^2, \mathbf{D}_n)$, $n > 2$, are all absolutely irreducible.
(3) The standard representations $(\mathbb{R}^2, \text{SO}(2))$ and $(\mathbb{R}^2, \text{O}(2))$ are irreducible of complex type and absolutely irreducible respectively.
(4) The standard representations of $\text{SO}(n)$ on $\mathbb{R}^n$ are absolutely irreducible for all $n > 2$.

(5) The special unitary group $\mathrm{SU}(2) \subset \mathrm{U}(2)$ may be identified with the group of complex 2×2 matrices of the form

$$A(a,b) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$$

where $a, b \in \mathbb{C}$ satisfy $|a|^2 + |b|^2 = 1$. (Hence $\mathrm{SU}(2)$ is identified with the unit 3-sphere in $\mathbb{C}^2$.) The reader may verify that $(\mathbb{C}^2, \mathrm{SU}(2))$ is irreducible as an $\mathbb{R}$-representation and that $(\mathbb{C}^2, \mathrm{SU}(2))$ is of quaternionic type.
(5) Continuing with the notation of (4), let Q be the subgroup of $\mathrm{SU}(2)$ generated by $\imath = \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix}$, $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $k = \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}$. We have $|Q| = 8$ and the $\mathbb{R}$-representation $(\mathbb{C}^2, Q)$ is irreducible of quaternionic type. (The elements of the group Q are the unit quaternions ± 1, $\pm i$, $\pm j$, $\pm k$.)

PROPOSITION 2.7.16. *Suppose that (V, G) is an irreducible $\mathbb{R}$-representation.*

(1) *If (V, G) is of complex type then V may be given the structure of a complex vector space in such a way that (V, G) is irreducible as a $\mathbb{C}$-representation.*
(2) *If (V, G) is absolutely irreducible, then the complexification $(V \otimes \mathbb{C}, G)$ is irreducible as a $\mathbb{C}$-representation.*
(3) *If (V, G), (W, G) are absolutely irreducible, then $(V \otimes \mathbb{C}, G)$ is isomorphic to $(W \otimes \mathbb{C}, G)$ as a complex representation if and only if (V, G) is isomorphic to (W, G) as a real representation.*

PROOF. If (V, G) is of complex type, then $L_G^{\mathbb{R}}(V, G) \cong \mathbb{C}$. Therefore there exists $J \in L_G^{\mathbb{R}}(V, V)$ such that $J^2 = -I_V$. Define a complex structure on V by $(a + \imath b)v = av + bJv$. With respect to this complex structure, $L_G^{\mathbb{C}}(V, V) \approx \mathbb{C}$. We leave (2,3) to the reader. $\square$

REMARKS 2.7.17. (1) Proposition 2.7.16 describes a small part of the relationships between real, complex and quaternionic representations. For example, the complexification of an $\mathbb{R}$-representation which is irreducible of complex type is the sum of two non-isomorphic complex representations. We refer the reader to [**2, 30**] for more details and proofs.
(2) Suppose (V, G) is an irreducible $\mathbb{R}$-representation, G compact. There is a useful criterion to determine whether or not (V, G) is absolutely irreducible, of complex type or of quaternionic type. Specifically, the type of the representation is determined by the sign of $\int_G \mathrm{trace}(g^2)\, dg$. We refer to [**2**, Theorem 3.6], [**30**] for details and note this criterion implies that a finite group G admits a non-trivial absolutely irreducible representation only if $|G|$ is even.

PROPOSITION 2.7.18. *Suppose that (V, G) is an irreducible $\mathbb{R}$-representation and $p \geq 1$.*

(1) *Suppose (V, G) is irreducible of complex type. Then $L_G^{\mathbb{R}}(V^p, V^p) \approx M(p, p, \mathbb{C})$.*
(2) *Suppose (V, G) is absolutely irreducible. Then $L_G(V^p, V^p) \cong M(p, p, \mathbb{R})$.*

(3) *Suppose (V, G) is quaternionic irreducible. Then $L_G(V^p, V^p) \cong M(p, p, \mathbb{H})$.*

PROOF. The result follows from proposition 2.7.16. □

We conclude this section with an example illustrating how we can apply the theory to obtain all the irreducible $\mathbb{C}$-representations of a subgroup of S_5 (we refer to [**61**, Appendix] and section 5.4 for more about this example). We start by reviewing the standard representation of S_{n+1} on $\mathbb{R}^n$.

The symmetric group S_{n+1} has a natural orthogonal representation on $\mathbb{R}^{n+1}$ defined by permutation of coordinates. The hyperplane $H \cong \mathbb{R}^n$ defined by $x_1 + \ldots + x_{n+1} = 0$ is S_{n+1}-invariant and it is straightforward to verify that $(\mathbb{R}^n, S_{n+1})$ is absolutely irreducible (the case $n = 2$ gives the standard representation of $\mathbf{D}_3$). Since $H^\perp = \mathbb{R}(1, 1, \ldots, 1)$, S_{n+1} acts trivially on $H^\perp$.

Taking $n = 4$, we have the absolutely irreducible orthogonal $(\mathbb{R}^4, S_5)$. Let $s = (12345), t = (2453) \in S_5$. Then $\langle s \rangle = \mathbb{Z}_5$, $\langle t \rangle = \mathbb{Z}_4$ and $st = ts^2$. Using these relations one shows that $G = \langle s, t \rangle$ has order 20. Let $\omega = e^{\frac{2\pi\imath}{5}}$. We define an action of G on $\mathbb{C}^2 \approx \mathbb{R}^4$ by

$$\begin{aligned} s(z_1, z_2) &= (\omega z_1, \omega^2 z_2), \\ t(z_1, z_2) &= (\bar{z}_2, z_1). \end{aligned}$$

(The definition is consistent with the relation $st = ts^2$). The action is orthogonal and absolutely irreducible. The complexification of the action gives a irreducible $\mathbb{C}$-action of degree 4. Since the trivial representation of G is of degree one, it follows from (2.2) that G has three other irreducible $\mathbb{C}$-representations, all of degree 1. Noting the generating relation $st = ts^2$, the 1-dimensional complex irreducible representations of $\mathbb{Z}_4 = \langle t \rangle$ defined by $\rho_p(t) = e^{\frac{\pi\imath p}{2}}$, $p = 1, 2, 3$, extend to a complex irreducible representations of G if we define $\rho_p(s) = 1$. The representation ρ_2 is the complexification of the absolutely irreducible representation of G on $\mathbb{R}$ defined by $\eta(s) = 1$, $\eta(t) = -1$. Viewed as real representations, ρ_1 and ρ_3 are isomorphic representations of complex type. Summing up, the group G has four inequivalent $\mathbb{R}$-representations, one of degree 4, three of degree 1.

EXERCISE 2.7.19. (1) A subgroup T of S_n is *transitive* if given $i, j \in \{1, \ldots, n\}$, there exists $\sigma \in T$ such that $Ti = j$. The subgroup is *doubly transitive* if given (ordered) pairs (i, j), (ℓ, m), $i \neq j$, $\ell \neq m$, there exists $\sigma \in T$ such that $\sigma i = \ell$, $\sigma j = m$. Show that if a T is a doubly transitive subgroup of S_{n+1} then the action of T on $\mathbb{R}^n$, induced from the standard action of S_{n+1} on $\mathbb{R}^n$, is absolutely irreducible. Show that the action of the group $G \subset S_5$ considered above is doubly transitive.
(2) Show that the group G considered above is isomorphic to $\mathrm{Aff}_1(\mathbb{F}_5)$.

2.8. Orbit structure for representations

LEMMA 2.8.1. *Suppose that (V, G) is an $\mathbb{F}$-representation. If H is a subgroup of G, then V^H is an $\mathbb{F}$-linear subspace of V.*

PROOF. Trivial. □

REMARK 2.8.2. If $\dim(V^H) = 1$, we often refer to V^H as an *axis of symmetryaxis of symmetry* for the G-action.

For representations (V, G) of compact Lie groups, the orbit stratification of V is very well behaved. Thus, the set of isotropy types is *finite* and there is a close relation between the partial order on the set of isotropy types and the relative position of the strata in V. In this chapter we describe the simplest case when G is finite.

THEOREM 2.8.3. *Let G be finite.*

(a) *If J is a subgroup of G, there exist unique $\tau \in \mathcal{O}(V, G)$ and $H \in \tau$ such that $V^J = V^H$.*
(b) *If $H \in \tau \in \mathcal{O}(V, G)$, then V_τ^H is an open and dense subset of V^H.*
(c) *For all $\tau \in \mathcal{O}(V, G)$, $\overline{V_\tau} = \cup_{H\in\tau} V^H = GV^K$, any $K \in \tau$.*
(d) *If $\tau, \mu \in \mathcal{O}(V, G)$ then $\tau > \mu$ if and only if $\partial V_\mu \supset V_\tau$.*

PROOF. Let X be the union of all fixed point spaces of V not containing V^J. If $L \in X$, then $L \cap V^J$ is a proper subspace of V^J. Since there are only finitely many subgroups of G, the set X must be finite and

$$V_0^J = V^J \setminus \cup_{L\in X}(V^J \cap L) = \cap_{L\in X}(V^J \setminus L)$$

is an open and dense subset of V^J. We claim that if $x \in V_0^J$ has isotropy group H, then $V^J = V^H$. Let $g \in H$. If there exists $y \in V^J$ such that $g \notin G_y$, then $V^{\langle g\rangle}$ intersects V^J in a proper subspace and hence $x \notin V_0^J$. Consequently, $g|V^J = I_V$ and so $V^H \supset V^J$. Obviously, $J \subset H$ and so $V^H \subset V^J$. Hence $V^J = V^H$, proving (a). Suppose $H \in \tau$. If $y \in V_0^H$, then our previous argument implies $H = G_y$, proving (b). The first equality in (c) follows from (b) using the finiteness of G. The second equality follows by noting that $gV^H = V^{gHg^{-1}}$. It remains only to prove (d). Let $\tau, \mu \in \mathcal{O}(V, G)$ and suppose $\tau > \mu$. By definition of the partial order relation on $\mathcal{O}(V, G)$, we can find $H \in \tau$, $J \in \mu$ such that J is a proper subgroup of H. We have

$$V^J \setminus V_\mu^J \supset V^H \supset V_\tau^H. \tag{2.4}$$

Since $\partial V_\mu \cap V^J = V^J \setminus V_\mu^J$, we deduce $\partial V_\mu \supset V_\tau$ from (2.4) by taking unions first over $H \supset J$ and then over all $J \in \mu$. It remains to prove the converse. Suppose $x \in V_\tau \cap \partial V_\mu$. Since Gx is finite, we may choose an open neighbourhood U of x such that $gU \cap U = \emptyset$, $g \in G \setminus G_x$. If $y \in U$, then $G_y \subset G_x$. Since $x \in \partial V_\mu$, there exists $y \in V_\mu \cap U$ and so $G_y \subset G_x$. □

2.9. Slices

We describe one of the main technical tools used in the theory of G-manifolds and maps: the differentiable slice theorem. We start with the definition of a slice (no differentiability needed) and then show how to construct slices for orthogonal actions by a finite group. We defer the construction of slices for general compact Lie group actions to the next chapter.

Definition 2.9.1. Let (X, G) be a G-space. Given $x \in X$, a *slice* for the action of G at x is a G_x-invariant subset S_x of x such that

(1) $S_x \cap Gx = \{x\}$.
(2) $gS_x \cap S_x \neq \emptyset$ if and only if $g \in G_x$.
(3) $GS_x = \cup gS_x$ is an open neighbourhood of Gx.

Remarks 2.9.2. (1) If (X, G) is a smooth G-manifold, we usually require that S_x is an embedded $(\dim(M) - \dim(Gx))$-disk which is transverse to Gx (see also the section in the next chapter on slices for general smooth actions).
(2) The neighbourhood GS_x of Gx given by the definition of slice is G-equivariantly homeomorphic to the twisted product $G \times_{G_x} S_x$.

We have the following important consequence of the definition of slice.

Lemma 2.9.3. *Let (X, G) be a G-space and let $x \in X$. Suppose that S_x is a slice for the action of G at x. Then $G_y \subset G_x$ for all $y \in S_x$. In particular, if $(G_x) = \tau$, then $X_\tau \cap GS_x = GS_x^{G_x} = G \times_{G_x} S_x$.*

2.9.1. Slices for linear finite group actions. Let G be a finite group. Suppose that V is a finite dimensional inner product space and $G \subset \mathrm{O}(V)$.

Lemma 2.9.4. *For every $x \in V$, we may choose $r = r(x) > 0$ such that the open r-disk $S_x = S_x(r) = \{y \in V \mid \|x - y\| < r\}$ is a slice for the action of G at x. In particular, we may assume that the action of G_x on S_x is linear.*

Proof. For $r > 0$, let $S_x(r)$ denote the open r-disk, centre x radius r. Since $G_x \subset G \subset \mathrm{O}(V)$, we have $gS_x(r) = S_{gx}(r)$, all $g \in G$, In particular, $gS_x(r) = S_x(r)$, for all $g \in G_x$. Since Gx is finite, we may choose $r > 0$ so that $S_x(r) \cap S_{gx}(r) = \emptyset$, for all $g \in G \setminus G_x$. Setting $S_x = S_x(r)$, S_x is a slice for the action of G at x. Moreover, since $S_x \subset V$ and $G_x S_x = S_x$, the action of G_x on x is the restriction of the linear action of G_x on V to S_x. $\square$

Definition 2.9.5. Let V be a finite dimensional inner product space. An involution $r \in \mathrm{O}(V)$ is a *reflection* if $V^{\langle r \rangle} = \{v \mid rv = v\}$ is a codimension 1 linear subspace of V.

Lemma 2.9.6. *Let $r \in O(V)$. Then $V^{\langle r \rangle}$ is a codimension 1 linear subspace of V if and only if r is a reflection.*

Proof. Set $W = V^{\langle r \rangle}$. Since r fixes W, $W^\perp$ must be r-invariant. If $\dim(W^\perp) = 1$, there are just two possible actions of r on $W^\perp$: multiplication by $+1$ or -1. If r acts by multiplication by $+1$, then r fixes $W^\perp$ contradicting the definition of W. It follows that r must act as multiplication by -1 and from this it follows immediately that r is a reflection. $\square$

The next proposition, part of which repeats the last statement of theorem 2.8.3, is a characteristic application of slices.

PROPOSITION 2.9.7. *Let $G \subset O(V)$, G finite.*

(1) *There exists a unique minimal isotropy type Π for the action of G on V. The corresponding orbit stratum V_Π is open and dense in V and connected if G contains no reflections.*
(2) *If $\tau, \mu \in \mathcal{O}(V, G)$, then $\tau < \mu$ if and only if $\partial V_\tau \supset V_\mu$.*

PROOF. Let $x \in V$ be a point of minimal isotropy. If S_x is a slice at x, then by lemma 2.9.3 $G_y = G_x$ for all $y \in S_x$ (else, G_x would not be minimal). Since S_x is an open neighbourhood of x in V, G_x acts trivially on V. By theorem 2.8.3, $V_{(G_x)} = V_{(G_x)}^{G_x}$ is an open dense subset of V. Hence the minimal isotropy type Π is unique (any two open dense subsets of V have non-trivial intersection) and the orbit stratum $V_\Pi = V_{(G_x)}^{G_x}$ is open and dense in V. If G contains a reflection r, then $V^{\langle r \rangle}$ is of codimension 1 and so $V_\Pi \subset V \setminus V^{\langle r \rangle}$ is not connected. If G has no reflections, then $V^{\langle g \rangle}$ is of codimension at least 2 for all $g \in G \setminus G_x$ and so $V_\Pi = V \setminus \cup_{g \notin G_x} V^{\langle g \rangle}$ is connected.

Suppose that $\tau, \mu \in \mathcal{O}(V, G)$ and $\partial V_\tau \supset V_\mu$. Let $x \in V_\mu$ and choose a slice S_x at x. Then $S_x \cap V_\tau \neq \emptyset$. Hence we can choose $y \in S_x \cap V_\tau$ and then $G_y \subset G_x$. By definition of the order on $\mathcal{O}(V, G)$, $\tau < \mu$. Conversely, suppose $\tau < \mu$. Choose $x, y \in V$ such that $G_x \in \mu$, $G_y \in \tau$ and $G_x \supset G_y$. It follows by linearity of the action that every point $z \in [x, y] = \{tx + (1-t)y \mid t \in [0, 1]\}$ has $G_z \supset G_y$. The slice theorem implies that if $z \in [x, y]$ is sufficiently close to y then $G_z = G_y$. Again by the slice theorem, if $z \in [x, y]$ is a limit of points in $[x, y]$ with isotropy G_y, then all points in $[x, y]$ sufficiently close to z, not equal to z, will have isotropy G_y. Hence there are only finitely many points in $[x, y]$ with isotropy different from G_y. It follows that every point of V_μ lies in ∂V_τ and $\partial V_\tau \supset V_\mu$. □

REMARK 2.9.8. The unique minimal isotropy type given by proposition 2.9.7 is called the *principal* isotropy type. The corresponding isotropy group is constant on V_Π (the principal isotropy group is a normal subgroup of G).

2.10. Invariant and equivariant maps

We conclude with definitions, examples and results about symmetric maps.

DEFINITION 2.10.1. Let (X, G) and (Y, G) be G-spaces. A map $f : X \to Y$ is *G-equivariant* or *equivariant* if for all $x \in X$ and $g \in G$ we have

$$f(gx) = gf(x).$$

The map f is *G-invariant* or *invariant* if for all $x \in X$ and $g \in G$ we have

$$f(gx) = f(x),$$

EXAMPLES 2.10.2. (1) Let $\mathbb{Z}_2$ act on $\mathbb{R}$ as multiplication by ± 1. A map $f : \mathbb{R} \to \mathbb{R}$ is equivariant if $f(-x) = -f(x)$, $x \in \mathbb{R}$. It is invariant if $f(x) = f(-x)$. We note that $\mathbb{Z}_2$-equivariance corresponds to f being odd and $\mathbb{Z}_2$-invariance to f being even.

(2) Take the standard irreducible representation of SO(2) on $\mathbb{C} \approx \mathbb{R}^2$. A map $f : \mathbb{C} \to \mathbb{C}$ is SO(2) equivariant if

$$f(e^{\imath\theta} z) = e^{\imath\theta} f(z), \quad (z \in \mathbb{C}, \theta \in \mathbb{R}).$$

A map $g : \mathbb{C} \to \mathbb{R}$ will be SO(2)-invariant if $f(e^{\imath\theta} z) = f(z)$, all $z \in \mathbb{C}$, $\theta \in \mathbb{R}$. If f is a polynomial (in z, $\bar{z}$), f is SO(2)-equivariant if and only if we can write

$$f(z) = h(|z|^2)z,$$

where $h : \mathbb{R} \to \mathbb{C}$ is a polynomial. A polynomial g is SO(2)-invariant if and only if we can write $g(z) = h(|z|^2)$, where $h : \mathbb{R} \to \mathbb{R}$ is a polynomial.

One way of constructing equivariant maps is to use slices. The following lemma is basic.

LEMMA 2.10.3. *Let (X, G) and (Y, G) be G spaces and S_x be a slice for the action of G at $x \in X$. Suppose that $f : S_x \to Y$ is a G_x-equivariant map. Then f extends uniquely to a G-equivariant map $F : GS_x \to Y$.*

PROOF. Let $g \in G, y \in S_x$. Define $F(gy) = gf(y)$. We verify that F is well-defined and G-equivariant. Suppose that $gy = g'y'$, where $g, g' \in G$, $y, y' \in S_x$. Since S_x is a slice, $g^{-1}g' \in G_x$. Using the G_x-equivariance of f,

$$F(gy) = gf(y) = gf(g^{-1}g'y') = gg^{-1}g'f(y') = g'f(y') = F(g'y').$$

Hence F is well-defined. Suppose that $z = gy$, $g \in G$, $y \in S_x$. For $k \in G$ we have

$$F(kz) = F(kgy) = kgf(y) = k(gf(y)) = kF(z).$$

Hence F is G-equivariant. □

PROPOSITION 2.10.4. *Let $f : X \to Y$ be an equivariant map of G-spaces.*

1. *If $x \in X$, then $G_{f(x)} \supset G_x$, $x \in X$.*
2. *If H is a subgroup of G, then $f(X^H) \subset Y^H$.*
3. *If f is injective, then $G_{f(x)} = G_x$, for all $x \in X$, and $f(X_\tau) \subset Y_\tau$, for all $\tau \in \mathcal{O}(X, G)$.*

PROOF. Let $g \in G_x$. Then

$$\begin{aligned} gf(x) &= f(gx), \text{ (by equivariance)} \\ &= f(x), \text{ (since } g \in G_x), \end{aligned}$$

proving (1). A similar argument proves (2). For (3), suppose that $gf(x) = f(x)$. Then, by equivariance, $f(gx) = f(x)$. Since f is injective, $gx = x$ and so $g \in G_x$. The second part of (3) is immediate. □

EXAMPLE 2.10.5. Consider an SO(2)-equivariant map $f : \mathbb{C} \to \mathbb{C}$, where SO(2) acts on the domain as multiplication by $e^{2\imath\theta}$ and on the range as multiplication by $e^{\imath\theta}$. Since f is equivariant, $f(e^{2\imath\theta}z) = e^{\imath\theta}f(z)$, for all $z \in \mathbb{C}$, $\theta \in \mathbb{R}$. Taking $\theta = \pi$, we see that $f(z) = f(e^{2\imath\pi}) = e^{\imath\pi}f(z) = -f(z)$ and so $f(z) = 0$. Hence every SO(2)-equivariant map $f : \mathbb{C} \to \mathbb{C}$ is identically zero. On the other hand, if we switch the SO(2)-actions on domain and target, there are plenty of non-constant SO(2)-equivariant maps. For example, any polynomial in z^2.

2.10.1. Smooth invariant and equivariant maps on representations. Let G be a compact Lie group and (V,G), (W,G) be orthogonal G-representations. Let $C^\infty(V)^G$ denote the vector space of smooth G-invariant $\mathbb{R}$-valued functions on V. Let $C^\infty_G(V,W)$ denote the space of smooth G-equivariant maps from V to W. If $f \in C^\infty(V)^G$, $F \in C^\infty_G(V,W)$, then $fF \in C^\infty_G(V,W)$ and so $C^\infty_G(V,W)$ has the structure of a $C^\infty(V)^G$-module. Let $P(V)^G \subset C^\infty(V)^G$ denote the vector space of G-invariant polynomial maps on V and $P_G(V,W) \subset C^\infty_G(V,W)$ be the $P(V)^G$-module of G-equivariant polynomial maps from V to W.

Example 2.10.6. If (V,G) is orthogonal then any polynomial in $\|x\|^2$ defines a G-invariant polynomial on V. Since $I_V \in P_G(V,V)$, if $p:\mathbb{R}\to\mathbb{R}$ is a polynomial then $P(x) = p(\|x\|^2)x$ is a G-equivariant polynomial endomorphism of V.

C^∞ *topology.* Let D_r denote the closed r-disk in V, centre the origin. For $n \geq 0$, we define the seminorm $\| \, \|_n$ on $C^\infty(V)^G$ (or $C^\infty(V)$) by

$$\|f\|_n = \sup_{x\in D_n} \sum_{j=0}^{n} \|D^j f(x)\|.$$

(Instead of $\|D^j f(x)\|$ we can sum over all the absolute values of all partial derivatives of f at x of order j.) We define a complete metric on $C^\infty(V)^G$ by

$$d(f,g) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|f-g\|_n}{1+\|f-g\|_n}, \; f,g \in C^\infty(V)^G.$$

The completeness of d follows easily from standard results on uniform convergence of sequences of differentiable functions. We similarly define a complete metric on $C^\infty_G(V,W)$. We refer to the associated topology on $C^\infty(V)^G$ or $C^\infty_G(V,W)$ as the C^∞-topology. Note that the C^∞ topology gives poor control over the behaviour of functions at infinity. Specifically, if f and g are ϵ d-close, their values may differ by $2^n\epsilon$ outside D_n. Later we introduce the Whitney C^∞ topology which gives better control over functions at infinity.

Lemma 2.10.7. *Averaging defines continuous linear projection maps*

(1) $\mathrm{Av} : C^\infty(V) \to C^\infty(V)^G$, $\mathrm{Av}(f)(x) = \int_G f(gx)\,dh$.
(2) $\mathrm{Av} : C^\infty(V,W) \to C^\infty_G(V,W)$, $\mathrm{Av}(f)(x) = \int_G gf(g^{-1}x)\,dh$.

Proof. We prove the first statement; the second is similar. Let $f \in C^\infty(V)$. Using the left invariance of Haar measure and standard properties of the integral (chapter 1, section 1.6) we see that $\mathrm{Av}(f)$ is a smooth G-invariant function on V. It follows from the definition of $\| \, \|_n$ and linearity of $\int_G$ that $\|\mathrm{Av}(f-g)\|_n \leq \|f-g\|_n$, for all $n \in \mathbb{N}$. Hence $d(\mathrm{Av}(f),\mathrm{Av}(g)) \leq d(f,g)$, all $f,g \in C^\infty(V)$. Noting that Av restricts to the identity on $C^\infty(V)^G$, we have shown that $\mathrm{Av} : C^\infty(V) \to C^\infty(V)^G$ is a continuous linear projection. $\square$

THEOREM 2.10.8 (Equivariant Stone-Weierstrass theorem).

(1) *$P(V)^G$ is a dense subset of $C^\infty(V)^G$ (C^∞-topology).*
(2) *$P_G(V,W)$ is a dense subset of $C^\infty_G(V,W)$ (C^∞-topology).*

PROOF. Given $\epsilon > 0$ and $f \in C^\infty(V)^G$ we must find $P \in P(V)^G$ such that $d(f,P) < \epsilon$. By the classical Stone-Weierstrass theorem, $P(V)$ is a dense subset of $C^\infty(V)$ and so there exists $Q \in P(V)$ such that $d(f,Q) < \epsilon$. Let $P = \mathrm{Av}(Q)$. We have

$$d(f,P) = d(\mathrm{Av}(f), \mathrm{Av}(Q)) \leq d(f,Q) < \epsilon.$$

The proof of the second part is similar and omitted. $\square$

EXERCISE 2.10.9. Show that if U is a G-invariant non-empty open subset of V, then $P(V)^G$ is a dense subset of $C^\infty(U)^G$ (C^∞-topology) and $P_G(V,W)$ is a dense subset of $C^\infty_G(U,W)$ (C^∞-topology). (Choose an increasing family of G-invariant compact subsets (K_n) of U such that $K_n \subset \mathrm{interior}(K_{n+1})$, $n \geq 1$, and $\cup_{n\geq 1} K_n = U$. For $n \geq 1$, choose $\phi_n \in C^\infty(V)^G$ such that $\phi_n | K_n \equiv 1$, $\phi_n | V \setminus U \equiv 0$. Now for $n \geq 0$, C^n-approximate $\phi_n f$ on K_n by $F_n \in P_G(V,W)$.)

LEMMA 2.10.10. *Let (V,G) be a finite dimensional orthogonal representation of G. Given $R > r > 0$, there exists a smooth G-invariant function $\Psi_{R,r} : V \to \mathbb{R}$ such that*

(1) $\Psi_{R,r}(x) = 1,\ \|x\| \leq r$.
(2) $\Psi_{R,r}(x) \in (0,1),\ r < \|x\| < R$.
(3) $\Psi_{R,r}(x) = 0,\ \|x\| \geq R$.

PROOF. Let $\rho : \mathbb{R} \to \mathbb{R}$ be the C^∞ map defined by $\rho(t) = 0,\ t \leq 0$, and $\rho(t) = \exp(-1/t),\ t > 0$. Define

$$\Psi_{R,r}(x) = \frac{\rho(R^2 - \|x\|^2)}{\rho(R^2 - \|x\|^2) + \rho(\|x\|^2 - r^2)},\ \ (x \in V).$$

Since the denominator is non-vanishing, $\Psi_{R,r}$ is smooth and satisfies (1,2,3). Since (V,G) is orthogonal, $\|\ \|$ is G-invariant and so $\Psi_{R,r} \in C^\infty(V)^G$. $\square$

LEMMA 2.10.11. *Let (V,G), (W,G) be a finite dimensional orthogonal representations of the finite group G. Let $x \in V$ and suppose that the r-disk neighbourhood $D_r(x)$ of x in V is a slice. Let $f : D_r(x) \to W$ be a smooth G_x equivariant map. Given $0 < s < r$, there exists a smooth G-equivariant extension of $f|\overline{D}_s(x)$ to $F : V \to W$.*

PROOF. Applying lemma 2.10.10, we choose a smooth G_x-invariant map $\phi : D_r(x) \to \mathbb{R}$ such that $\phi|D_s(x) = 1$ and $\phi|D_r(x) \setminus D_{(r+s)/2}(x) \equiv 0$. Set $f_1 = \phi f : D_r(x) \to W$. By lemma 2.10.3, f_1 extends uniquely to a smooth G-equivariant map $f_2 : GD_r(x) \to W$ defined by $f_2(gy) = gf_1(y),\ y \in D_r(x), g \in G$. The closed support of f_2 is a compact G-invariant subset of the open G-invariant set $GD_r(x)$. Hence if we define $F : V \to W$ by $F|GD_r(x) = f_2$, $F|V \setminus GD_r(x) = 0$, F will be a smooth G-invariant map. $\square$

Remark 2.10.12. Lemma 2.10.11 holds for compact Lie groups but we defer the proof to the next chapter – the issue is the smoothness of the extension.

Examples 2.10.13. (1) Let (V, G), (W, G) be finite dimensional representations of the finite group G. Given $p \in V$ and $q \in W$ suppose that $G_p \subset G_q$. Let $D_r(p)$ be a slice at p. By lemma 2.10.11, there exists a smooth G-equivariant map $F : V \to W$ such that $F(p) = q$ and $F|V \setminus GD_r(p) \equiv 0$ (take f to be constant, equal q on $D_r(p)$).
(2) Let (V, G) be a finite dimensional representation of the finite group G. Suppose that $p_1, p_2 \in V$ lie on different G-orbits. There exists $P \in P(V)^G$ such that $P(p) \neq P(q)$. For this we start by constructing $P \in C^\infty(V)^G$ such that $P(p_1) = 1$, $P(p_2) = 0$. (use (1) with (W, G) be the trivial representation $(\mathbb{R}, G)$ and $p_2 \in V \setminus GD_r(p_1)$). Apply theorem 2.10.8.

2.10.2. Equivariant vector fields and flows. Let X be a C^r vector field on $\mathbb{R}^n$, $r \geq 1$. For each $x \in \mathbb{R}^n$, there exists a unique *maximal* integral curve $\phi_x : (a_x, b_x) \to \mathbb{R}^n$ for the vector field through x satisfying

$$\begin{aligned} \phi_x'(t) &= X(\phi_x(t)), \quad (t \in (a_x, b_x)), \\ \phi_x(0) &= x, \end{aligned}$$

where (a_x, b_x) is an open interval containing $0 \in \mathbb{R}$. If we define $\mathcal{D}_X = \cup_{z \in \mathbb{R}^n} \{x\} \times (a_x, b_x)$, then $\mathcal{D}_X$ is an open subset of $\mathbb{R}^n \times \mathbb{R}$ containing $\mathbb{R}^n \times \{0\}$. Define $\phi : \mathcal{D}_X \to \mathbb{R}^n$ by $\phi(x, t) = \phi_x(t)$. It is a standard fact from the theory of ordinary differential equations that ϕ is C^r. If $\mathcal{D}_X = \mathbb{R}^n \times \mathbb{R}$, ϕ defines a C^r-flow on $\mathbb{R}^n$. For each $t \in \mathbb{R}$, the map $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ will be a C^r diffeomorphism of $\mathbb{R}^n$.

Lemma 2.10.14. *Suppose that $\rho : G \to GL(V)$ is a representation of G on V (real or complex), Let $X : V \to V$ be a C^r vector field on V, $r \geq 1$ and let $\phi : \mathcal{D}_X \to \mathbb{R}^n$ denote the corresponding solution map.*

(1) *$\mathcal{D}_X$ is a G-invariant open subset of $\mathbb{R}^n \times \mathbb{R}$.*
(2) *ϕ is G-equivariant.*

In case $\mathcal{D}_X = \mathbb{R}^n$, ϕ defines a G-equivariant flow on $\mathbb{R}^n$ and each ϕ_t is a G-equivariant diffeomorphism of $\mathbb{R}^n$.

Proof. Let $\phi_x : (a_x, b_x) \to \mathbb{R}^n$ be the maximal integral curve through x. For $g \in G$, let $\psi = g\phi_x : (a_x, b_x) \to \mathbb{R}^n$. Then $\psi(0) = gx$ and

$$\psi'(t) = g\phi_x'(t) = gX(\phi_x(t)) = X(g\phi_x(t)) = X(\psi(t)), \quad (t \in (a_x, b_x)).$$

Hence ψ is an integral curve through gx. By maximality, we must have $(a_{gx}, b_{gx}) \supset (a_x, b_x)$. It follows from uniqueness of solutions that on (a_x, b_x), $\phi_{gx} = g\phi_x$. Repeating the argument with x replaced by gx and ψ by $g^{-1}\phi_{gx}$, we deduce that $(a_{gx}, b_{gx}) \subset (a_x, b_x)$ and so $(a_{gx}, b_{gx}) = (a_x, b_x)$ and $\phi_{gx} = g\phi_x$ for all $g \in G$, $x \in \mathbb{R}^n$. Our arguments also prove that ϕ is G-equivariant and $\mathcal{D}_X$ is G-invariant. $\square$

CHAPTER 3

Smooth G-manifolds

The topic of this chapter is smooth (always C^∞) actions of a Lie group G on a differential manifold M. We always assume M is a Hausdorff, paracompact and second countable m-dimensional manifold. In particular, M is metrizable and locally compact. With few exceptions, M will be connected. Lie groups will usually be compact but we allow for proper actions by non-compact Lie groups.

We review those parts of the theory of (Riemannian) G-manifolds that we need when we come to investigate equivariant transversality and equivariant dynamical systems on G-manifolds. Important results proved in this chapter include the differentiable slice theorem and the Whitney regularity of the stratification of a G-manifold by isotropy type. We also prove a number of foundational results; for example, that G-orbits are submanifolds for compact or proper G-actions.

3.1. Proper G-manifolds

Definition 3.1.1. Let G be a Lie group. A (smooth) G-manifold consists of a smooth manifold M together with a smooth action $G \times M \to M$, $(g, m) \mapsto gm$, of G on M.

Remark 3.1.2. The conditions of the definition may be weakened since it follows from a theorem of Montgomery (see [**129**, Chapter V, §5.1]) that if G is a Lie group, M is a manifold which is a G-space and $g : M \to M$ is smooth for all $g \in G$, then M is automatically a smooth G-manifold. See also remarks 3.1.10.

Lemma 3.1.3. *Let M be a G-manifold. The tangent bundle TM and cotangent bundle $T^\star M$ have the natural structure of G-manifolds. Moreover, for each $x \in M$, (T_xM, G_x) and $(T_x^\star M, G_x)$ have the structure of (dual) G_x-representations.*

Proof. For each $g \in G$, let $g : TM \to TM$ be defined as the tangent map Tg of g. Since $T(gh) = TgTh$, and $Te_G = I_{TM}$, TM has the natural structure of a G-manifold. Since Tg is a diffeomorphism of TM, $T_xg : T_xM \to T_{gx}M$ is a linear isomorphism, for all $g \in G$. Hence (T_xM, G_x) has the structure of a G_x-representation. We define the action of G on $T^\star M$ by requiring that $\langle v, g\phi\rangle = \langle g^{-1}v, \phi\rangle$, where $x \in M$, $v \in T_xM$, $\phi \in T_x^\star M$. With this definition, $(T_x^\star M, G_x)$ is the dual of the representation (T_xM, G_x). $\square$

Remark 3.1.4. An easy consequence of lemma 3.1.3 is that all of the tensor and exterior power bundles of TM, $T^\star M$ admit the natural structure of

G-manifolds (in fact G-vector bundles – we give a formal definition later).

Just as we did for G-spaces and representations, we may partition a G-manifold into subsets of the same isotropy type. However, without further conditions on the action, this partition may be quite pathological.

EXAMPLE 3.1.5. Let $A : \mathbb{T}^2 \to \mathbb{T}^2$ be the diffeomorphism of the 2-torus defined by $A(x, y) = (2x + y, x + y)$, $(x, y) \in \mathbb{R}^2 \bmod \mathbb{Z}^2$. Let M be the compact 3-manifold defined by identifying the ends of the cylinder $\mathbb{T}^2 \times [0, 1]$ according to $(t, 1) \sim (At, 0)$. The unit vector field $X(x, t) = (0, 1)$ on $\mathbb{T}^2 \times [0, 1]$ induces a smooth vector field on M. Since M is compact, we can integrate X to obtain a smooth $\mathbb{R}$-action $M \times \mathbb{R} \to M$ (the resulting flow Φ_t^A is called the *suspension* of A – the construction is quite general, see section 9.4). Observe that every periodic point of A lifts to a periodic orbit of Φ_t^A. The periodic orbits of Φ_t^A are the *compact*-orbits of the $\mathbb{R}$-action. It is easily shown that the set of periodic points of A is the set of rational points $(\frac{p}{q}, \frac{r}{s})$ of $\mathbb{T}^2$. Hence the set of compact orbits of the $\mathbb{R}$-action is dense in M. However, the set of non-compact $\mathbb{R}$-orbits is also dense. Therefore, not only is the partition of M by isotropy type highly irregular but (most) $\mathbb{R}$-orbits are not (embedded) submanifolds of M. We remark that Φ_t^f is an example of a transitive Anosov flow – in this case the suspension of the Anosov diffeomorphism (or 'cat map') A of the 2-torus.

The pathology described in the previous example does not occur when the group G is compact. However, rather than make the assumption of compactness at the outset, we prefer to start by making an additional assumption on the action that is satisfied in a number of important cases – including the standard action of the Euclidean group $\mathbf{E}(n)$ on $\mathbb{R}^n$.

We recall that a continuous map $f : X \to Y$ between locally compact metric spaces is *proper* if $f^{-1}(K)$ is a compact subset of X whenever K is a compact subset of Y. The map f is *closed* if f maps closed sets to closed sets.

LEMMA 3.1.6. *Let $f : X \to Y$ be continuous. Then f is proper if and only if f is closed and $f^{-1}(y)$ is compact for all $y \in Y$. If f is injective then f is proper if and only if f is closed and then f is a homeomorphism onto $f(X)$ (induced topology).*

PROOF. We prove that if f is closed and $f^{-1}(y)$ is compact for all $y \in Y$ then f is proper and leave the remaining statements to the reader. Suppose then that K is a compact subset of Y and let $\{U_i \mid i \in I\}$ be an open cover of $f^{-1}(K)$. Since $f^{-1}(y)$ is compact, we may for each $y \in K$ pick a finite subset $I_y \subset I$ such that $\{U_i \mid i \in I_y\}$ covers $f^{-1}(y)$. Since f is assumed closed, it follows that $V_y = Y \setminus f(X \setminus \cup_{i \in I_y} U_i)$ is an open neighbourhood of $y \in Y$. Now use the compactness of K to choose $y_1, \ldots, y_N \in K$ such that $\{V_{y_i} \mid i = 1, \ldots, N\}$ covers K. Then $\{U_i \mid i \in I_{y_j}, j = 1, \ldots, N\}$ is a finite open cover of $f^{-1}(K)$. $\square$

DEFINITION 3.1.7. A smooth action $\rho : M \times G \to M$ of G on M is a *proper action* if the associated map

$$\tilde{\rho} : M \times G \to M \times M,\ (x, g) \mapsto (x, \rho(x, g)) = (x, gx),$$

is proper. We refer to M as a *proper G-manifold*.

PROPOSITION 3.1.8. *Suppose M is a proper G-manifold. Then*

(1) *For all $x \in M$, the map $\rho_x : G \to M$, $\rho_x(g) = gx$, is proper.*
(2) *G_x is a compact subgroup of G, Gx is a closed subset of M and the natural map $\rho'_x : G/G_x \to Gx$, $g[G_x] \to gx$ is a G-equivariant homeomorphism.*
(3) *M/G is Hausdorff.*

PROOF. Since ρ_x is the restriction of the proper map $\tilde{\rho}$ to the closed subspace $\{x\} \times G$, ρ_x is obviously proper. Hence $G_x = \rho_x^{-1}(x)$ is compact and $Gx = \rho_x(G)$ is closed, lemma 3.1.6. Since the natural map $\rho'_x : G/G_x \to Gx$ is proper and injective it follows from the second part of lemma 3.1.6 that ρ'_x is a homeomorphism. The proof of the last statement is similar to that of lemma 2.3.4 using the fact that G-orbits are closed. ☐

EXAMPLES 3.1.9. (1) The natural action of $\mathbf{E}(n)$ on $\mathbb{R}^n$ is proper since since the action of $\mathbb{R}^n$ on $\mathbb{R}^n$ by translation is obviously proper and $\mathbf{E}(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$.
(2) Let H be a closed subgroup of G. Then the left and right H-actions on G are proper: $G \times H \to G$, $(g, h) \mapsto hg, gh$. Note that if H is not a closed subgroup, then $\tilde{\rho} : G \times H \to G \times G$ is not closed – $\tilde{\rho}^{-1}(\{e\} \times G) = \{e\} \times H$ is not closed.

REMARK 3.1.10. If M is a proper G-manifold then Illman [**96**] proved that it is possible to give M, G real analytic structures so that the action of G on M is real analytic. In the case when G is compact, this result was proved earlier by Matumoto and Shiota [**121**] and the real analytic structure on M is unique (the real analytic structure on G is always unique).

3.1.1. Proper free actions.

LEMMA 3.1.11. *If $\rho : M \times G \to M$ is a proper, free, smooth action on M, then for all $x \in M$, Gx is a closed* submanifold *of M.*

PROOF. The mapping $\rho_x : G \to M$, $g \mapsto gx$, is a G-equivariant smooth homeomorphism onto Gx, proposition 3.1.8. In order to prove that ρ_x is an embedding, it is enough to show that ρ_x is an immersion. It follows by G-equivariance that the rank of $T_g\rho_x : T_gG \to T_{gx}M$ is constant on G. Applying the rank theorem [**41**, 10.3], we see that if $T_g\rho_x$ is not of maximal rank then ρ_x is not injective, contradiction. Hence $T_g\rho_x$ is of maximal rank and is therefore injective. Therefore ρ_x is an immersion. Alternatively, it suffices to prove directly that $T_e\rho_x : \mathfrak{g} \to T_xM$ is injective – this is easily done using the exponential map and we introduce the general method shortly. ☐

DEFINITION 3.1.12. Let $\rho : M \times G \to M$ be a proper, free, smooth action on M with orbit map $p : M \to M/G$. A local section of $p : M \to M/G$ over the open subset U of M/G is a continuous map $\xi : U \to M$ such that $p\xi = Id_U$.

We note the following elementary properties of a local section $\xi : U \to M$ of $p : M \to M/G$.

(1) $\xi(U)$ meets each G-orbit $p^{-1}(\bar{x})$, $\bar{x} \in U$, in a unique point.
(2) $g\xi : U \to M$ is a local section for all $g \in G$ and $g\xi(U) \cap h\xi(U) = \emptyset$ unless $g = h$..
(3) $\cup_{g\in G} g\xi(U)$ is an open neighbourhood of $p^{-1}(\bar{x})$, $\bar{x} \in U$.
(4) ξ maps U homeomorphically onto $\xi(U)$ (induced topology).
(5) The map $U \times G \to p^{-1}(U)$ defined by $(u, g) \to g\xi(u)$ is a homeomorphism.
(6) If $\xi_i : U_i \to M$ are local sections, $i = 1, 2$, then $\xi_{12} = \xi_1 \circ p : \xi_2(U_1 \cap U_2) \to \xi_1(U_1 \cap U_2)$ is a homeomorphism.

REMARKS 3.1.13. (1) Without the assumption that the action is free it may not be possible to construct local sections at every point of the orbit space. For example, the standard action of $\mathbb{Z}_2$ on $\mathbb{R}$ does not admit a local section defined on a neighbourhood of $\bar{0} \in \mathbb{R}/\mathbb{Z}_2 \approx [0, \infty)$.
(2) Conditions (1–4) amount to $\xi(U)$ being a slice for all G-orbits through $\xi(U)$.

THEOREM 3.1.14. *Let $\rho : M \times G \to M$ be a proper, free, smooth action on M. Then M/G has a (unique) differentiable structure such that $p : M \to M/G$ is a smooth submersion (and admits local smooth sections).*

PROOF. Let $\dim(G) = g$, $\dim(M) = m$. Given $x \in M$, set $p(x) = \bar{x}$. By lemma 3.1.11, Gx is a g-dimensional closed submanifold of M. We prove the result by showing that there exist local sections of $p : M \to M/G$ which patch together to define the required differential structure on M/G.

Let $\eta : D_r(0) \to M$ be a smooth embedding of the open r-disk, centre the origin, in $\mathbb{R}^{m-n}$, such that $\eta(0) = x$, $\eta(D_r(0)) \cap Gx = \{x\}$ and η is transverse to Gx ($T_x\eta(D_r(0)) + T_xGx = T_xM$). Choosing r smaller if necessary, we may assume $\overline{\eta(D_r(0))}$ is compact. We claim that we can choose $0 < s \leq r$ such that for all $y \in \eta(D_s(0))$, $\eta|D_s(0)$ is transverse to Gy, and $Gy \cap \eta(D_s(0)) = \{y\}$. Suppose the contrary. Then there exist sequences $(x_n) \subset \eta(D_r(0))$, $(g_i) \subset G\backslash\bar{V}$, where V is an open neighbourhood of the identity in G, such that $g_ix_n \in \eta(D_r(0))$ and $x_i \to x$. Let $A = \overline{\{(x_i, g_ix_i) \mid i \geq 1\}}$. Since $A \subset \overline{\eta(D_r(0))} \times \overline{\eta(D_r(0))}$, A is a compact subset of $M \times M$ and so, since the action is proper, $\overline{\{(x_i, g_i) \mid i \geq 1\}}$ is a compact subset of $M \times G$. Choosing subsequences, we may assume that $g_i \to g \in G \setminus V$. Since $g_ix_i, x_i \to x$, we have $gx = x$, contradicting the assumption that the action of G is free.

Set $D_x = \eta(D_s(0))$ and $U_{\bar{x}} = p(D_x)$. It follows from our construction that $U_{\bar{x}}$ is an open neighbourhood of $\bar{x}$. Since $p|D_x$ defines a homeomorphism of D_x onto

$U_{\bar{x}}$, we may define $\xi_x = (p|D_x)^{-1} : U_{\bar{x}} \to D_x$. Clearly, $\xi_x : U_{\bar{x}} \to M$ is a local section of M.

Carrying out this construction for each $\bar{x} \in M/G$, we obtain a set $\mathcal{U} = \{(U_{\bar{x}}, \xi_x) \mid x \in M\}$ of local sections of M with $\xi_{gx} = g\xi_x$, $g \in G$. Using the transversality of the sections ξ_x to G-orbits, we deduce that the transition functions $\xi_{xz} : \xi_z(U_{\bar{x}} \cap U_{\bar{y}}) \to \xi_x(U_{\bar{x}} \cap U_{\bar{y}})$ are smooth. Hence $\mathcal{U}$ defines a differential structure on M/G. With respect to this structure, $p|D_x = \xi_x(U_{\bar{x}})$ is a diffeomorphism onto $U_{\bar{x}}$ and so $p|p^{-1}(U_{\bar{x}})$ is a smooth submersion for all $x \in M$. Hence $p : M \to M/G$ is a smooth submersion. Using the local sections, it is trivial to verify that the differential structure we have constructed is the unique differential structure on M/G for which $p : M \to M/G$ is a smooth submersion. □

REMARK 3.1.15. The previous result also holds if the action of G on M is proper and *monotypic*: there is only one isotropy type.

COROLLARY 3.1.16. *Suppose that H is a closed subgroup of the Lie group G. Then the homogeneous space G/H has a unique differential structure with respect to which the quotient map $p : G \to G/H$ is a submersion. The natural action of G on G/H is smooth. The action is proper if H is compact.*

PROOF. The group H acts properly and freely on G by $g \mapsto gh^{-1}$. It follows from theorem 3.1.14 that G/H has a unique differential structure such that the orbit map $p : G \to G/H$ is a submersion. Using local sections of G/H, we verify that the natural action of G on G/H by left translation is smooth. Finally, if H is compact, we use lemma 3.1.6 to show that the action of G on G/H is proper. □

COROLLARY 3.1.17. *Let $\rho : M \times G \to M$ be a proper smooth action on M. For all $x \in M$, the map $\rho_x : G/G_x \to M$, $\rho_x(g[G_x]) = gx$, is a smooth G-equivariant embedding. In particular, Gx is a closed submanifold G-equivariantly diffeomorphic to G/G_x.*

PROOF. We know from proposition 3.1.8 that ρ_x is a G-equivariant homeomorphism of G/G_x onto Gx – in particular, $\rho_x : G/G_x \to M$ is a topological embedding of G/G_x. It suffices to prove that ρ_x is an immersion. Just as in the proof of lemma 3.1.11, the G-equivariance of ρ_x implies that $T_\alpha \rho_x$ is of constant rank, $\alpha \in G/G_x$. Since ρ_x is injective it follows from the rank theorem that $T_\alpha \rho_x$ must be injective. Hence ρ_x is an injective immersion. Alternatively, we may proceed by noting that the kernel of the tangent map of $g \to gx$ at $g = e$ is equal to the Lie algebra $\mathfrak{g}_x$. □

REMARK 3.1.18. Theorem 3.1.14 and corollary 3.1.16 can be formulated as results about smooth principal G-bundles. We recall that a smooth map $p : E \to B$ is a smooth G-principal bundle if: (a) G acts smoothly and freely on the right on E, (2) $p(xg) = p(x)$, for all $x \in E$, $g \in G$, (3) we can find an open neighbourhood U of every $b \in B$ and smooth G-equivariant trivialization $\phi_U : U \times G \to p^{-1}(U) \subset E$ such that ϕ_U covers the identity map on U. (We assume the right G-action on $U \times G$, $(u, k) \to (u, kg)$.) If, in theorem 3.1.14, we define the right action of G on M by $mg = \rho(g)(m)$, then $p : M \to M/G$ is a principal G-bundle over M/G. Corollary 3.1.16 may be restated as saying that if H is a closed subgroup of G then $p : G \to G/H$ is a principal H-bundle (note that the requirement that H acts on the right is quite natural in this case).

Specializing to the case G is compact, the results of this section show that if M is a G-manifold then

(1) G-orbits are compact G-invariant submanifolds of M.
(2) If $x \in M$, Gx is equivariantly diffeomorphic to G/G_x.
(3) If G acts freely on M, then M/G has a unique smooth structure with respect to which the orbit map $p : M \to M/G$ is a principal G-bundle.

EXAMPLE 3.1.19. Let H be a compact subgroup of the Lie group G and (X, H) be a smooth H-manifold. Then the orbit map $G \times X \to G \times_H X$ has the structure of an principal H-bundle over $G \times_H X$. If we take the $G \times H$-action on $G \times X$ defined by $(g, h)(\gamma, x) = (g\gamma h^{-1}, hx)$, then every $G \times H$-equivariant smooth map $f : G \times X \to G \times X$ induces a smooth G-equivariant map $\tilde{f} : G \times_H X \to G \times_H X$.

3.2. G-vector bundles

In this section we give the definition of a G-vector bundle together with one important construction. We start with a rapid revision of the definition of a smooth vector bundle over M (for fuller details, see any of [**1, 18, 92, 103**]).

Let E be smooth manifold and $p : E \to M$ be a smooth surjective map. Given $x \in M$, we call $p^{-1}(x) = E_x$ the *fibre* of p over x. We are interested in maps $p : E \to M$ such that (a) the fibre E_x has the structure of a vector space for all $x \in M$, and (b) the map p is locally a projection map in a way that preserves the vector space structure on fibres.

DEFINITION 3.2.1. (Notation as above.) An $\mathbb{E}$-*trivialization* (U, ϕ) of $p : E \to M$ over the open subset U of M consists of a smooth diffeomorphism $\phi : E|U = p^{-1}(U) \to U \times \mathbb{E}$ such that $\pi_U \circ \phi = p$, where $\pi_U : U \times \mathbb{E} \to U$ is the projection onto U.

Let (U, ϕ) be an $\mathbb{E}$-trivialization of $p : E \to M$. Given $x \in U$, define the smooth diffeomorphism $\phi(x) : E_x \to \mathbb{E}$ by

$$\phi(x) = \pi_{\mathbb{E}} \circ (\phi|E_x),$$

where $\pi_{\mathbb{E}} : U \times \mathbb{E} \to \mathbb{E}$ is the projection on $\mathbb{E}$.

Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover of M and $\mathcal{T} = \{(U_i, \phi_i) \mid i \in I\}$ be a set of $\mathbb{E}$-trivializations $p : E \to M$. Given $i, j \in I$, suppose that $U_{ij} = U_i \cap U_j \neq \emptyset$. For $x \in U_{ij}$, define $\phi_{ij}(x) : \mathbb{E} \to \mathbb{E}$ by

$$\phi_{ij}(x)(e) = \phi_i(x)\phi_j(x)^{-1}.$$

Since $\phi_{ij}(x)$ is a composite of diffeomorphisms, $\phi_{ij}(x)$ is a diffeomorphism of $\mathbb{E}$.

DEFINITION 3.2.2. (Notation as above.) The set $\mathcal{T} = \{(U_i, \phi_i) \mid i \in I\}$ of $\mathbb{E}$-trivializations gives $p : E \to M$ the structure of a *smooth vector bundle* over M with fibre $\mathbb{E}$ if $\phi_{ij} : U_{ij} \to \mathrm{GL}(\mathbb{E})$, for all $i, j \in I$ and $x \in U_{ij}$. If $p : E \to M$ has the structure of a smooth vector bundle, we typically say that $p : E \to M$, or just E, is a smooth vector bundle.

If $\mathcal{T}$ gives $p : E \to M$ the structure of a vector bundle, there exists a unique linear structure on each fibre E_x such that $\phi_i(x) : E_x \to \mathbb{E}$ is a linear isomorphism for all $i \in I$ such that $x \in U_i$.

DEFINITION 3.2.3. Suppose that M is a proper G-manifold, E is a G-manifold and $p : E \to M$ is G-equivariant. Assume that $p : E \to M$ is a smooth vector bundle. Then $p : E \to M$ is a *smooth G-vector bundle* if for all $g \in G$, $g : E_x \to E_{gx}$ is a linear isomorphism.

REMARKS 3.2.4. (1) If $p : E \to M$ is a smooth G-vector bundle then E is a proper G-manifold.
(2) For all $x \in M$, (E_x, G_x) is a G_x-representation. A good way of thinking of a (proper) G-vector bundle is as a smooth family of compact group representations.

EXAMPLE 3.2.5. Let M be a proper G-manifold. The tangent bundle and cotangent bundles of M are smooth G-vector bundles over M.

PROPOSITION 3.2.6. *Let H be a compact subgroup of the Lie group G and suppose that (V, H) is an H-representation. The twisted product $G \times_H V$ has the natural structure of a smooth G-vector bundle over the proper G-space G/H.*

PROOF. We know from section 2.4 (exercise 2.4.4), that the fibre of the orbit map $p : G \times_H V \to G/H$, $p([g, x]) = g[H]$, is naturally identified with V. In order to construct V-trivializations, it suffices to construct a trivialization over a neighbourhood of $[H] \in G/H$. Indeed, we can use the G-action on G/H to transport the trivialization to a neighbourhood of any coset in G/H. Let $\xi : U \subset G/H \to G$ be a smooth local section of G over some open neighbourhood U of $[H] \in G/H$. Define $\phi : G \times_H V|U \to U \times V$ by $\phi([\xi(u), v]) = (u, v)$, $(u, v) \in U \times V$. Since $p([\xi(u), v]) = u$ and ξ is a local section, ϕ is a trivialization. Finally, the action of G on G/H is proper by corollary 3.1.16. $\square$

3.3. Infinitesimal theory

In this section we look at the relationship between the tangent spaces to group orbits and the Lie algebras of G and the isotropy groups of the action.

Let M be a smooth G-manifold and let $C^\infty(TM)$ denote the space of smooth vector fields on M. We have a natural linear map

$$\mathfrak{g} \to C^\infty(TM), \; X \mapsto \bar{X},$$

defined by

$$\bar{X}(x) = \frac{d}{dt}\exp(tX)(x)|_{t=0}.$$

LEMMA 3.3.1. *For $x \in M$, $\mathfrak{g}_x = \{X \in \mathfrak{g} \mid \bar{X}(x) = 0\}$.*

PROOF. If $X \in \mathfrak{g}_x$, then $\exp(tX) \in G_x$, for all $t \in \mathbb{R}$, and so $\exp(tX)(x) = x$, for all $t \in \mathbb{R}$. Obviously $\bar{X}(x) = 0$. Conversely, suppose that $\frac{d}{dt}\exp(tX)(x)|_{t=0} = 0$. Then for all $s \in \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt}\exp((t+s)X)(x)|_{t=0} &= \exp(sX)\frac{d}{dt}\exp(tX)(x)|_{t=0}, \\ &= 0 \end{aligned}$$

Hence $\exp(tX)(x) = x$, for all $t \in \mathbb{R}$, and so $X \in \mathfrak{g}_x$, □

LEMMA 3.3.2. *Let H be a closed subgroup of G then $T_{[H]}G/H \approx \mathfrak{g}/\mathfrak{h}$.*

PROOF. Let $p : G \to G/H$ be the quotient map. We have $T_e p(X) = \frac{d}{dt}p(\exp(tX))|_{t=0}$. If $X \in \mathfrak{h}$, then $p(\exp(tX)) = [H]$ (constant) and so $T_e p(\mathfrak{h}) = \{0\}$. On the other hand, p is a submersion and $\dim(G/H) = \dim(\mathfrak{g}) - \dim(\mathfrak{h})$. It follows that $T_e p$ maps $\mathfrak{g}$ onto $T_{[H]}G/H$. Hence $T_{[H]}G/H \approx \mathfrak{g}/\mathfrak{h}$. □

LEMMA 3.3.3. *Let $M \times G \to M$ be a smooth proper action. Then*

$$T_x Gx = \{\bar{X}(x) \mid X \in \mathfrak{g}\}.$$

PROOF. The map $G/G_x \to Gx$, $gG_x \mapsto gx$, is an embedding. The result follows from lemma 3.3.1. □

EXERCISE 3.3.4. Show that the map $\mathfrak{g} \to C^\infty(TM)$, $X \mapsto \bar{X}$ is an anti-Lie homomorphism: $\overline{[X,Y]} = -[\bar{X},\bar{Y}]$, for all $X, Y \in \mathfrak{g}$.

3.4. Riemannian manifolds

In order to investigate the stratification of M by isotropy type, we introduce some tools from Riemannian geometry. We start with a quick review of basic definitions and facts about Riemannian manifolds. For more details, the reader may consult one of the many texts on Riemannian geometry (for example [**18, 103**]).

A smooth section ξ of the bundle $T^\star M \otimes T^\star M$ is a *Riemannian metric* on M if (a) ξ is symmetric ($\xi(x)(v,w) = \xi(x)(w,v)$, all $v, w \in T_x M$, all $x \in M$), and (b) positive definite ($\xi(x)(v,v) > 0$, all non-zero $v \in T_x M$). We say that M is a *Riemannian manifold* if M comes equipped with a smooth Riemannian metric ξ. We denote the corresponding metric on M by d. The Riemannian manifold M is *complete* if (M,d) is a complete metric space. It can be shown (see for example, [**103, 18**]) that if M is complete then every pair of points $x, y \in M$ can be joined by a minimizing geodesic (that is, there exists a geodesic $\gamma : [0,1] \to M$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $d(x,y) = \int_0^1 \|\gamma'(t)\|\, dt$). It follows from the Hopf-Rinow theorem [**103**, Theorem 4.1] that the following conditions are equivalent

(1) M is complete.
(2) All metrically bounded subsets of M are relatively compact
(3) Every geodesic on M is defined for arbitrarily large values of its canonical parameter.

All compact Riemannian manifolds are complete.

3.4.1. The exponential map of a complete Riemannian manifold. Suppose that (M, ξ) is a complete Riemannian manifold.

Let $X \in T_xM$. Then there exists a unique (maximal) geodesic $\gamma_X : \mathbb{R} \to M$ such that (a) $\gamma_X(0) = x$, and (b) $\gamma_X'(0) = X$. The *exponential map* of the Riemannian metric is the map $\exp : TM \to M$ defined by

$$\exp(X) = \gamma_X(1),\ X \in TM.$$

It is well-known that geodesics can be represented as the solutions of smooth second order differential equation on M (the geodesic 'spray'). It therefore follows from standard results on differential equations that $\exp : TM \to M$ is a smooth map. Let $TM_0 = \{0_x \mid x \in M\} \subset TM$ denote the zero-section of TM. If we identify TM_0 with M, it follows from the definition of exp that $\exp|TM_0$ can be identified with the identity map of M. On the other hand if we set $\exp|T_xM = \exp_x$, $x \in M$, then it follows from the definition of $\exp(X)$ in terms of γ_X that $T_{0_x}\exp_x : T_xM \to T_xM$ is the identity map. Combining these observations on the restriction of exp to the zero section and tangent space, it follows that $T_{0_x}\exp(v, w) = v + w$, $v, w \in T_xM$.

Let $D_x(r)$ denote the open disk centre 0, radius $r > 0$ in T_xM. It follows from the inverse function theorem that we can choose $r > 0$ so that $D_x(r)$ is mapped diffeomorphically by $\exp_x$ onto the open neighbourhood $\exp_x(D_x(r))$ of x.

3.4.2. The tubular neighbourhood theorem. Let N be a closed submanifold of M. Let T_NM denote the restriction of TM to N. Then TN is a vector sub-bundle of T_NM. Restricting the Riemannian metric ξ to N, let Q denote the orthogonal complement of TN in T_NM. That is, for each $x \in N$, Q_x will be the orthogonal complement of the vector subspace T_xN of T_xM. If we let $q : Q \to N$ denote the projection induced from $\tau_M|T_NM$, then $q : Q \to N$ has the structure of a smooth vector sub-bundle of T_NM and $q : Q \to N$ is isomorphic, as a vector bundle, to the normal bundle T_NM/TN of N.

Given $r > 0$, let $Q(r)$ (respectively, $\overline{Q}(r)$) denote the open r-disk bundle (respectively, closed r-disk bundle) of Q.

PROPOSITION 3.4.1. *Let N be a compact submanifold of M. We may choose $r > 0$ such that* exp *restricts to a smooth embedding of $\overline{Q}(r)$ onto the closed neighbourhood* $\overline{\exp(Q(r))} = \overline{\mathbf{T}}$ *of N in M. Furthermore, $\overline{\mathbf{T}}$ has smooth boundary* $\partial\mathbf{T} = \exp(\partial\overline{Q}(r))$.

PROOF. It follows from our earlier description of $T_{0_x}\exp$ that $T_{0_x}\exp : T_{0_x}Q \to T_xM$ is a linear isomorphism for all $x \in N$. Hence, for each $x \in N$, we may choose $r_x > 0$ and an open neighbourhood U_x of $x \in N$ such that exp maps $\overline{Q}(r_x)|U_x$ diffeomorphically onto a neighbourhood of x in M. Since N is compact, we can cover N by a finite set $U_{x_1}, \dots, U_{x_k}$ and make r independent of x by setting $r = \min\{r_{x_1}, \dots, r_{x_k}\}$. In this way we define a local embedding $\exp : \overline{Q}(r) \to M$. It remains to show that we can choose $r > 0$ sufficiently small so that $\exp|\overline{Q}(r)$ is 1:1. Suppose the contrary. Then there exist a pair of sequences $(y_n), (z_n) \subset M$ such that for $n \geq [1/r]$ (a) $\exp(y_n) = \exp(z_n)$, (b) $y_n = \exp(Y_n)$, $z_n = \exp(Z_n)$, where $Y_n, Z_n \in \overline{Q}(\frac{1}{n})$, and (c) $y_n \neq z_n$. Choosing subsequences, we may assume that $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z$, where $y, z \in N$. Suppose $y = z$. Since $\exp|\overline{Q}(r)$ is a local diffeomorphism it follows by (a) that $y_n = z_n$ for all sufficiently large n, contradicting (c). If $y \neq z$ then it follows by continuity of exp that $\exp(y_n) \neq \exp(z_n)$ for all sufficiently large n, contradicting (a). Hence there exists $n \geq [1/r]$ such that $\exp|\overline{Q}(\frac{1}{n})$ is 1:1. □

REMARKS 3.4.2. (1) The neighbourhood $\mathbf{T} = \exp(Q(r))$ (or its closure $\overline{\mathbf{T}} = \exp(\overline{Q}(r))$ given by proposition 3.4.1 is called a *tubular neighbourhood* of N. Referring to the figure, suppose that $b \in \partial\mathbf{T}$. Then there exists a unique $n \in N$

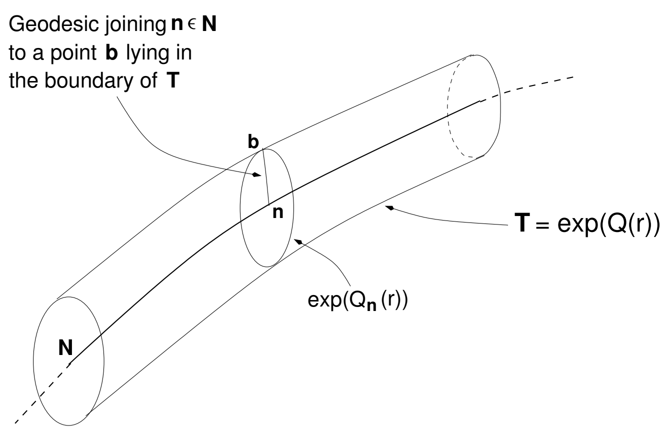


FIGURE 1. Tubular neighbourhood of N

such that $b \in \exp(\overline{Q}_n(r))$. It follows from the definition of exp that there exists a unique $X \in \overline{Q}_n(r)$ such that the curve $g(t) = \exp(tX)$, $t \in [0, 1]$, is a geodesic joining n to b. It is clear from the construction that $g(t)$ minimizes distance between b and N. Since the distance between b and N is independent of $b \in \partial T$, it follows that $g(t)$ minimizes distance between $\partial\mathbf{T}$ and N. Necessarily $g'(1) \perp T_b\partial\mathbf{T}$ (else there would exist a shorter geodesic from N to $\partial\mathbf{T}$). Summarizing, every geodesic starting at a point of N and perpendicular to N is perpendicular to $\partial\mathbf{T}$ at the first point of intersection.
(2) Proposition 3.4.1 holds if N is a closed submanifold of M with the proviso that r will now be a smooth strictly positive function on N.

COROLLARY 3.4.3. *There exists a smooth diffeomorphism* $\rho : Q \to \mathbf{T} = \exp(Q(r))$ *such that*

(1) ρ *covers the identity map* I_N *of* N *(*ρ *is fibre preserving).*
(2) $\rho | Q(r/2)$ *is the identity map (that is,* $q \circ \rho | Q(r/2) = \exp$*.)*

PROOF. Choose a smooth diffeomorphism $\Psi : [0, \infty) \to [0, r)$ such that $\Psi(t) = t$, $0 \le t \le r/2$. For example, we may take Ψ to be of the form

$$\Psi(t) = \frac{t}{(1 + \sigma(t)t^2)^{1/2}},$$

where $\sigma \in C^\infty(\mathbb{R})$ is chosen so that $\sigma(t) = 0$, $t \le r/2$, $\lim_{t\to\infty} \sigma(t) = r^{-2}$ and $|\sigma'(t)|$ is sufficiently small so that $\Psi'(t) > 0$, for all $t \ge 0$. Define $\rho(X) = \Psi(\|X\|)\frac{X}{\|X\|}$. We leave it to the reader to verify that ρ satisfies all the required conditions. $\square$

3.4.3. Riemannian G-manifolds. For the rest of the chapter we assume that G is compact (some of our results hold for proper actions, see [**138, 46**]).

DEFINITION 3.4.4. A *Riemannian G-manifold* consists of a smooth G-manifold M together with a G-invariant Riemannian metric on M.

LEMMA 3.4.5. *Every G-manifold M may be given the structure of a (complete) Riemannian G-manifold.*

PROOF. If ξ is a Riemannian metric on M, then ξ is a smooth section of the bundle $T^\star M \otimes T^\star M$. If we let $\xi_\star$ denote the average over G of ξ, then $\xi_\star$ defines a G-invariant Riemannian metric on M. Alternatively, let $\| \, \| : TM \to \mathbb{R}$ denote the norm on TM determined by ξ. Define

$$\|v\|_\star^2 = \int_G \|gv\|^2 \, dg, \quad (v \in TM).$$

Then $\| \, \|_\star$ defines a G-invariant norm on TM. Clearly $\| \, \|_\star^2 : TM \to \mathbb{R}$ is smooth. Since $\| \, \|$ satisfies the parallelogram law so also does $\| \, \|_\star$ and hence $\| \, \|_\star$ is determined by an inner product on TM. Consequently, $\| \, \|_\star$ defines a unique G-invariant Riemannian metric $\xi_\star$ on M.

If M is compact, $(M, \xi_\star)$ is necessarily complete. Suppose that M is not compact. A result of Nomizu and Ozeki [**135**] shows that $(M, \xi_\star)$ is conformally equivalent to a complete Riemannian manifold. That is, there exists a smooth function $f : M \to \mathbb{R}$ such that the Riemannian metric $f^2\xi_\star$ is complete. It follows easily from the elementary construction used in [**135**] that we may choose f to be smooth and G-invariant so that the resulting metric $f^2\xi_\star$ is G-invariant and complete. We refer the reader to [**135**] for details of the construction of f. $\square$

REMARK 3.4.6. If M admits a G-equivariant (proper) embedding into a finite dimensional representation (V, G) then we may pull back the Euclidean metric on V to give a complete G-invariant metric on M. It follows from the Mostow-Palais theorem [**130, 137**] that there will be such an embedding if (and only if) there are finitely many isotropy types for the action of G on M. In general, a finite dimensional noncompact G manifold may have infinitely many isotropy types.

EXERCISE 3.4.7. Assuming the Mostow-Palais theorem, show that if H is a closed subgroup of the compact group G, then there exists a finite-dimensional G-representation which has (H) as an isotropy type. (Hint: G/H)

Let M be a complete Riemannian G-manifold. The exponential map $\exp : TM \to M$ is G-equivariant. If N is a closed G-invariant submanifold of M then the orthogonal complement of TN in $T_N M$ is a G-invariant subbundle $q : Q \to N$ of $T_N M$ which is isomorphic, as a G-vector bundle, to the normal bundle $T_N M/TN$ of N. All the constructions used in the proof of the tubular neighbourhood theorem may be done equivariantly so as to obtain an equivariant version of the tubular neighbourhood theorem (for more details see [**26**, Chapter VI]).

PROPOSITION 3.4.8 (Equivariant tubular neighbourhood theorem). *Let N be a compact G-invariant submanifold of M. There exists $r > 0$ such that* $\exp : Q \to M$ *restricts to a smooth G-equivariant embedding of $\overline{Q}(r)$ onto a closed G-invariant neighbourhood $\mathbf{T}$ of N in M. The neighbourhood $\mathbf{T}$ has smooth G-invariant boundary $\partial \mathbf{T} = \exp(\partial \overline{Q}(r))$.*

Furthermore, we may choose a G-equivariant diffeomorphism $\rho : Q \to Q(r)$ covering I_N such that $\rho|Q(r/2)$ is the identity map.

As a corollary of proposition 3.4.8 we have Bochner's linearization theorem.

THEOREM 3.4.9. *Let M be a G-manifold and $p \in M$ be a fixed point for the action of G (that is $G_p = G$). Then we may choose local coordinates at p with respect to which the action of G is linear.*

PROOF. Give M the structure of a complete Riemannian G-manifold. Apply proposition 3.4.8 with $N = \{p\}$. Then $f = q \circ \rho : T_p M \to M$ will be a G-equivariant diffeomorphism onto a G-invariant open neighbourhood S of p. That is, $gf = fg$ and so the G-action on S is conjugate via f to the linear G-action on $T_p M$. $\square$

COROLLARY 3.4.10. *If M is a smooth G-manifold then the fixed point set M^G of the action is a closed G-invariant submanifold of M.*

PROOF. An immediate consequence of Bochner's linearization theorem. $\square$

EXERCISE 3.4.11. Let M be a Riemannian G-manifold. Suppose that $\phi : (a, b) \to M$ is a geodesic such that $\phi(0) \in Gx$ and $\phi'(0) \perp T_x Gx$. Show that $\phi'(t) \perp T_{\phi(t)} G\phi(t)$ for all $t \in (a, b)$. What can be said about the variation of isotropy type along a geodesic?

3.5. The differentiable slice theorem

In this section we apply the equivariant version of the tubular neighbourhood theorem to prove the existence of slices for smooth G-manifolds, G-compact. We indicate in the exercises the extension of the slice theorem to proper G-actions.

If H is a closed subgroup of the compact Lie group G and (V, H) is a finite-dimensional H-representation, then the twisted product $G \times_H V$ has the natural structure of a G-vector bundle over G/H (proposition 3.2.6).

LEMMA 3.5.1. *Let $f : X \to G/H$ be a G-equivariant diffeomorphism and suppose that $\bar{q} : E \to X$ is a G-vector bundle over X. Choose $x \in X$ with $G_x = H$ and suppose that (V, H) is an H-representation. If $(\bar{q}^{-1}(x), H)$ is isomorphic as an H-representation to (V, H), then $\bar{q} : E \to X$ is isomorphic as a G-vector bundle to $q : V \times_H G \to G/H$.*

PROOF. Without loss of generality, we may assume that $X = G/H$ (replace $\bar{q} : E \to X$ by the push-forward bundle $f_\star \bar{q} : f_\star E \to G/H$). Since we assume (V, H) is isomorphic to $(\bar{q}^{-1}(x), H)$, there exists an H-equivariant linear isomorphism $A : \bar{q}^{-1}(x) \to V$. Set $W = \bar{q}^{-1}(x)$. Extend A G-equivariantly to $\bar{A} : E \to V \times_H G$ by $\bar{A}(gw) = gA(w)$, $g \in G$, $w \in W$. The map $\bar{A}$ is well-defined since if $gw = g'w'$, where $g, g' \in G$, $w, w' \in G$, then $g^{-1}g' \in H$ and so $gA(w) = g'A(w')$ by H-equivariance of A. Since $\bar{A}$ is G-equivariant and restricts to a linear isomorphism on fibres, it remains only to show that $\bar{A}$ is smooth. Let $\xi : U \subset G/H \to G$ be a smooth local section over an open neighbourhood U of x. The section ξ determines the trivialization $\phi_\xi : U \times W \to \bar{q}^{-1}(U)$, by $\phi_\xi(u, w) = \xi(u)w$. Since the local representative $\bar{A}_\xi = \bar{A}\phi_\xi$ is given explicitly by $\bar{A}_\xi(u, w) = \xi(u)A(w)$, $\bar{A}$ is smooth. $\square$

Let M be a smooth Riemannian G-manifold. Given $x \in M$, set $Gx = \alpha$, $G_x = H$ and $T_xM/T_x\alpha = V$ and note that (V, H) is an H-representation. Let Q denote the orthogonal complement of $T\alpha$ in $T_\alpha M$. It follows from proposition 3.4.8 that there is a smooth G-equivariant embedding $\chi : Q \to M$ of Q onto a G-invariant open neighbourhood of α in M such that $\chi = \exp$ on a disk neighbourhood of the zero section of Q. It follows from lemma 3.5.1 that Q is isomorphic as a G-vector bundle to $q : V \times_H G \to G/H \approx \alpha$. In future, we represent tubular neighbourhoods of G-orbits $\alpha = G/H$ as embeddings $\chi : V \times_H G \to M$ where we assume that χ identifies the zero section of $V \times_H G$ with $\alpha \approx G/H$.

THEOREM 3.5.2 (The differentiable slice theorem). *Let M be a smooth G-manifold. For every $x \in M$, we can choose a smooth family of slices $\mathcal{S} = \{S_y \mid y \in Gx\}$ satisfying the following properties*

(1) *Each S_y is a G_y-invariant smooth submanifold of M.*
(2) *$gS_x \cap S_x \neq \emptyset$ if and only if $g \in G_x$.*
(3) *S_y is G_y-equivariantly diffeomorphic to the representation $(T_yGx^\perp, G_x)$. In particular, G_y acts linearly on S_y if S_y is given the linear structure induced from $T_yGx^\perp$.*

(4) *$GS_x = \cup_{y \in Gx} S_y$ is an open G-invariant neighbourhood of Gx which is G-equivariantly diffeomorphic to $G \times_{G_x} S_x$.*
(5) *If N is a G-manifold and $f : S_x \to N$ is a smooth G_x-equivariant map, then f extends uniquely to a smooth G-equivariant map $\tilde{f} : GS_x \to N$.*

Proof. Set $G_x = H$, $T_x Gx^{\perp} = V$. Let $\chi : V \times_H G \to M$ be a G-equivariant tubular neighbourhood of Gx and define $S_y = \chi(\chi^{-1}(y)$, for all $y \in Gy$. Properties $(1 - 4)$ of the family $\mathcal{S}$ are immediate. It remains to prove (5). By lemma 2.10.3, f has a unique G-equivariant extension $\tilde{f}$ to GS_x. For the smoothness of $\tilde{f}$ we use a local section of $G \to G/H$. We omit the argument which is exactly that of the proof of the final part of lemma 3.5.1. □

Corollary 3.5.3. *Let M be a smooth G-manifold, $x \in M$ and S_x be a slice for the action of G at x. Set $G_x = H$ and suppose that $H \lhd G$. Then for all $y \in S_x$, $N(G_y)y \pitchfork S_x$.*

Proof. Apply Bochner's linearization theorem to the G_y-manifold $N(H)S_x$ at x and use the triviality of the G_y-action on $N(H)x$ and $S_x^{G_y}$. □

Remark 3.5.4. Let M be a smooth G-manifold, $x \in M$ and S_x be a slice for the action of G at x. Set $G_x = H$. Corollary 3.5.3 implies that for all $y \in S_y$, $(N(G_y) \cap N(H))y = N_{N(H)}(G_y)y \pitchfork S_x$ within $N(H)S_x$.

Exercise 3.5.5. Extend the differentiable slice theorem to proper G-actions. (Let $x \in M$. Since G_x is compact, we can choose a G_x-invariant inner product on $T_x M$ and write $T_x M = T_x Gx \oplus W$, where $W = T_x Gx^{\perp}$ is isomorphic to the G_x-representation on the normal space $T_x M / T_x Gx$. Since x is a fixed point for the action of the *compact* group G_x, we can find a G_x-equivariant diffeomorphism L mapping $T_x M$ onto an open neighbourhood of $x \in M$. Setting $f = L|W$, f extends G-equivariantly to a smooth map $\tilde{f}$ of $G \times_{G_x} L$ onto a neighbourhood of Gx. Finally, using the properness of the action, show that one can choose a disk subbundle $D \subset L \times_{G_x} G$ so that $\tilde{f}|D$ is a G-equivariant diffeomorphism onto an open neighbourhood of Gx).

3.6. Equivariant isotopy extension theorem

In chapter 9, we need a version of the equivariant isotopy extension theorem.

Theorem 3.6.1 (cf [**26**, chapter VI]). *Let N be a compact G-invariant m-dimensional submanifold with smooth boundary of the m-dimensional G-manifold M. of Let $h_t : N \to M$, $t \in [0,1]$ be a smooth family of equivariant embeddings such that h_0 is the inclusion $i_N : N \to M$ of N in M. Then h_1 extends to a G-equivariant diffeomorphism H_1 of M. We may require that $H_1 = I_M$ outside of a preassigned open neighbourhood of $\cup_{t \in [0,1]} h_t(N)$.*

Proof. Reparameterizing the isotopy h_t, we may assume that h_t is constant, equal to i_N, for t near 0, and constant equal to h_1, for t near 1. Extend the isotopy

G-equivariantly and smoothly to $N \times \mathbb{R}$ by setting $h_t = h_0$, $t < 0$, and $h_t = h_1$, $t > 1$. We define a smooth G-equivariant vector field Z on $M \times \mathbb{R}$ of the form $Z(x,t) = (Y(x,t),1)$. If $t \notin [0,1]$, we take $Y(x,0) = 0$. For $(x,t) \in h_t(N) \times [0,1]$, we define $Y(h_t(x),t) = \frac{\partial h}{\partial t}(x,t)$. Since h_t is independent of t for t near 0 or 1, Y is well defined, smooth and G-equivariant on $M \times \{0\} \cup \cup_{t\in\mathbb{R}} h_t(N) \times \{t\}$. Since $\bar{N} = \cup_{t\in[0,1]} h_t(N) \times \{t\}$ is a smooth submanifold of $M \times \mathbb{R}$ with smooth boundary, we may extend Y smoothly to $M \times \mathbb{R}$ and then average over G so that Y is smooth and G-equivariant. Since $\bar{N}$ is G-invariant, we do not change the values of Y on $\bar{N}$. We may further require that $Y \equiv 0$ outside of any preassigned compact G-invariant neighbourhood of $\bar{N} \cap (M \times [0,1])$. If we let Φ_t denote the flow of Z, then $\Phi_t(M \times \{s\}) = M \times \{s+t\}$, $t \in \mathbb{R}$. In particular, if we restrict to $M = M \times \{0\}$, we obtain a family $H_t : M \to M$ of G-equivariant diffeomorphisms of M such that $H_0 = I_M$ and $H_1|N = h_1$. $\square$

3.7. Orbit structure for G-manifolds

Let M be a G-manifold, G a compact Lie group. As usual, let $\mathcal{O}(M,G)$ denote the set of isotropy types for the action of G on M. Given $x \in M$, let $\iota(x) = (G_x) \in \mathcal{O}(M,G)$ denote the isotropy type of x. Let $M_\tau \subset M$ denote the set of points of isotropy type τ. If $H \in \tau$, let M^H denote the fixed point set of the action of H on M and $M^H_\tau = M_\tau \cap M^H$.

LEMMA 3.7.1. *Let $x \in M$ and S_x be a differentiable slice for the action of G at x. Then*

(a) *For all $z \in S_x$, $G_y \subset G_x$ with equality if and only if $y \in S_x^{G_x}$.*
(b) *If $y \in GS_x$ then G_y is conjugate to G_z for some $z \in S_x$. In particular, $|\mathcal{O}(GS_x,G)|$ is finite and equal to the number of isotropy types for the (linearized) G_x-action on S_x.*

PROOF. Both statements follow from the differentiable slice theorem. $\square$

PROPOSITION 3.7.2. *Let $\tau \in \mathcal{O}(M,G)$. Then M_τ is a G-invariant submanifold of M. If $H \in \tau$, then*

(1) *M^H_τ is a closed $N(H)$-invariant submanifold of M_τ,*
(2) *M^H is a closed $N(H)$-invariant submanifold of M, and*
(3) *M^H_τ is an open $N(H)$-invariant submanifold of M^H. The induced free action of $N(H)/H$ on M^H_τ gives M^H_τ the structure of an $N(H)/H$-principal bundle over $M^H_\tau/(N(H)/H) \approx M_\tau/G$.*

PROOF. Let S_x be a differentiable slice for the action of G at $x \in M_\tau$. By lemma 3.7.1(a), $M_\tau \cap S_x = S_x^{G_x}$. Since $S_x^{G_x}$ is the fixed point set of a linear action, $S_x^{G_x}$ is a G_x-invariant submanifold of S_x. Hence, by statement (5) of the differentiable slice theorem (or directly using a smooth local cross section of G/G_x), $M_\tau \cap GS_x = GS_x^{G_x}$ is a G-invariant submanifold of M. This proves that M_τ is a G-invariant submanifold of M. Statement (1) is immediate from corollary 3.4.10. If $y \in M^H$, then by lemma 3.7.1 $G_y \supset H$ and so $nG_yn^{-1} \supset H$, for all $n \in N(H)$. Hence $N(H)$ acts on M^H and (2) follows from corollary 3.4.10. The action of $N(H)/H$ on $M_/\tau^H$ is free and, by theorem 3.1.14 and remark 3.1.18, M_τ^H is an $N(H)/H$-principal bundle over $M_\tau^H/(N(H)/H) \approx M_\tau/G$. $\square$

EXAMPLE 3.7.3. Let $\mathbb{Z}_2$ act on $S^2 \subset \mathbb{R}^3$ by restriction of the linear $\mathbb{Z}_2$-action generated by $\kappa(x, y, z) = (-x, -y, z)$. The action of $\mathbb{Z}_2$ has fixed point set $\{(0, 0, 1), (0, 0, -1)\}$. Let $P^2(\mathbb{R})$ be the real projective plane defined as $S^2/\sim$ where $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{x} = \pm\mathbf{y}$. The $\mathbb{Z}_2$-action on S^2 drops to a $\mathbb{Z}_2$-action on $P^2(\mathbb{R})$ since $\mathbf{x} \sim \mathbf{y}$ if and only if $\kappa\mathbf{x} \sim \kappa\mathbf{y}$. The fixed point set of the $\mathbb{Z}_2$-action on $P^2(\mathbb{R})$ consists of an isolated point S together with a circle C of fixed points corresponding to the equator of S^2. In this case $(T_SP^2(\mathbb{R}), \mathbb{Z}_2)$ is *not* isomorphic to $(T_cP^2(\mathbb{R}), \mathbb{Z}_2)$, all $c \in C$. This is in sharp contrast to what happens for representations (V, G) where if $(G_x) = (G_y)$, then we always have $(V, G_x) \cong (V, G_y)$ and the connected components of V_τ have the same dimension. In the next section, we will refine the stratification of (M, G) by isotropy type to take account of possible variation in the isomorphism type of the isotropy representations (T_xM, G_x).

PROPOSITION 3.7.4. *If M is a compact G-manifold or a G-representation, then $\mathcal{O}(M, G)$ is finite.*

PROOF. We prove by induction on $m = \dim(M)$. The result is obvious if $m = 0$. Suppose the result has been proved for all compact G-manifolds and representations of dimension less than m. If M is an m-dimensional orthogonal representation then the number of isotropy types for the action of G on M is one more than the number of isotropy types for the induced action of G on the unit sphere $S(M)$ of M. Since $\dim(S(M)) = m - 1$, it follows by induction that the number of isotropy types for m-dimensional representations is finite. Now suppose M is compact. Given $x \in M$, we can choose coordinates on a differentiable slice S_x at x such that S_x has the structure of a G_x-representation. Since $\dim(S_x) \leq m$, it follows that the number of G_x-isotropy types for S_x is finite and so, by lemma 3.7.1(b), the number of G-isotropy types for the action of G on GS_x is finite. Since M is compact, M can be covered by a finite number of tubular neighbourhoods of this type and so $\mathcal{O}(M, G)$ is finite. $\square$

We define a relation $\prec$ on $\mathcal{O}(M, G)$ by

$$\tau \prec \mu, \text{ if } \bar{M}_\tau \cap M_\mu \neq \emptyset.$$

LEMMA 3.7.5. *Let $\tau, \mu \in \mathcal{O}(M, G)$ and suppose $\tau \prec \mu$. Then there exist $H \in \tau$, $J \in \mu$ such that $H \subsetneq J$. With the notation of section 2.5*

$$\tau \prec \mu \Longrightarrow \tau < \mu.$$

If (V, G) is a G-representation then $\prec$ coincides with the partial order $<$.

PROOF. For the first statement, take a slice at a point $x \in \bar{M}_\tau \cap M_\mu$. It remains to show that if $H \in \tau$, $J \in \mu$ and $H \subsetneq J$ then $\bar{V}_\tau \cap V_\mu \neq \emptyset$. Choose $x \in V_\tau^H$, $y \in V_\mu^H$ and apply the argument of the proof of proposition 2.9.7(2). $\square$

In general, $\prec$ does not define a partial order on $\mathcal{O}(M, G)$ for G-manifolds. However, by lemma 3.7.5, $\prec$ always extends to the partial order $<$ on $\mathcal{O}(M, G)$.

DEFINITION 3.7.6. An isotropy type τ is *maximal* if $\bar{M}_\tau \cap M_\mu = \emptyset$ for all isotropy types $\mu \neq \tau$.

REMARK 3.7.7. (1) If τ is maximal then M_τ is a *closed* G-invariant submanifold of M. Every maximal isotropy subgroup (order subgroups of G by inclusion) defines a maximal isotropy type. If τ is maximal with respect to $\prec$ then τ may not be $<$-maximal.
(2) If (V, G) is a G-representation, then the maximal isotropy type is (G). For representations, we usually define the maximal isotropy type to be the maximal isotropy type for the induced action on the unit sphere $S(V)$. A maximal isotropy type will then be a maximal proper isotropy subgroup of G.

The situation concerning minimal isotropy types is more satisfactory.

THEOREM 3.7.8. *Let M be a connected G-manifold (not necessarily compact). There exists a unique* minimal *isotropy type $\mathbf{\Pi}$. We have*

(1) *$M_{\mathbf{\Pi}}$ is open and dense in M.*
(2) *$\tau > \mathbf{\Pi}$ for all $\tau \in \mathcal{O}(M, G)$, $\tau \neq \mathbf{\Pi}$.*
(3) *If $H \in \mathbf{\Pi}$, then $G(M^H) = M$, that is M^H intersects every G-orbit.*
(4) *If $\dim(M_\tau) < \dim(M) - 1$, all $\tau \neq \mathbf{\Pi}$, then $M_{\mathbf{\Pi}}$ is connected.*

We call $\mathbf{\Pi}$ the principal *isotropy type and $M_{\mathbf{\Pi}}$ the principal orbit stratum.*

PROOF. We start by proving (1). Our proof goes by induction on $\dim(M) = m$. The result is trivial in dimension zero. In the zero-dimensional case the result is also true if G acts either trivially or transitively on M.

Suppose the result proved in dimensions less than or equal to $m-1$. Consider first the case of an m-dimensional orthogonal G-representation (V, G). Restricting the action of G to the unit sphere $S(V)$ and applying the inductive hypothesis, we see that $(S(V), G)$ has a unique minimal isotropy type $\mathbf{\Pi}$ and that $S(V)_{\mathbf{\Pi}}$ is open and dense in $S(V)$ (note the special argument needed in case $\dim(V) = 1$). Hence, by linearity, $V_{\mathbf{\Pi}} = \mathbb{R}^\star S(V)_{\mathbf{\Pi}}$ is open and dense in V and so the result is true for m-dimensional representations. Suppose M is a connected m-dimensional G-manifold. Using the differentiable slice theorem, we may cover M by open sets U_i diffeomorphic to $G \times_{H_i} L_i G$, where H_i is an isotropy group and (L_i, H_i) an H_i-representation on a normal space. By induction (applied to (L_i, H_i)) and lemma 3.7.1, we see that each $G \times_{H_i} L_i$ has a unique minimal isotropy type $\mathbf{\Pi}_i$ and that $(L_i \times_{H_i} G)_{\mathbf{\Pi}_i}$ is open and dense in $L_i \times_{H_i} G$. If $U_i \cap U_j \neq \emptyset$, then $\mathbf{\Pi}_i = \mathbf{\Pi}_j$. Hence, since M is connected, there exists a unique minimal isotropy type $\mathbf{\Pi}$ for the action of G on M and $M_{\mathbf{\Pi}}$ is open and dense in M.

Since $M_{\mathbf{\Pi}}$ is dense in M, (2) is immediate by lemma 3.7.5. Let $H \in \mathbf{\Pi}$. Then $M_{\mathbf{\Pi}} \cap M^H = M_{\mathbf{\Pi}}^H$ is open and dense in M^H. Since G is compact and M^H closed, GM^H is a closed G-invariant subset of M and is therefore equal to M since $G(M_{\mathbf{\Pi}}^H) = M_{\mathbf{\Pi}}$ is open and dense in M. Hence M^H intersects every G-orbit.

Finally, if $\dim(M_\tau) < \dim(M) - 1$, all $\tau \neq \mathbf{\Pi}$, then none of the submanifolds M_τ disconnect M (locally) and so $M \setminus \cup_{\tau \neq \mathbf{\Pi}} M_\tau = M_{\mathbf{\Pi}}$ is connected. □

REMARKS 3.7.9. (1) We could have assumed M/G connected in theorem 3.7.8 and dropped the statement on connectedness of $M_{\mathbf{\Pi}}$.
(2) If $H \in M_{\mathbf{\Pi}}$, then $N(H)$-acts on M^H. Since H-acts trivially on M^H, it follows that we have an induced smooth action of $N(H)/H$ on M^H . The action will be free on $M_{\mathbf{\Pi}}^H$. (See also proposition 3.7.2(4).)

EXERCISE 3.7.10. (1) Let G be finite and M be a connected G-manifold. Show that if $H \in \mathbf{\Pi}$, then $H \lhd G$ and H acts trivially on M. Deduce that it is no loss of generality to replace G by G/H and regard the group as acting freely on $M_{\mathbf{\Pi}}$. Does this argument work if G is not finite?
(2) Let G be finite and M be a connected G-manifold. Assume the principal isotropy is trivial (see (1)) and that $G \neq \{e\}$. Prove that $\dim(M^G) \leq \dim(M) - 1$ with equality only if $G = \mathbb{Z}_2$.
(3) Let G be connected and M be a connected G-manifold. Let $\dim(Gx) = r$, $x \in M_{\mathbf{\Pi}}$. Let $M_r \supset M_{\mathbf{\Pi}}$ denote the union of all orbits of dimension r. Show that $\dim(M \setminus M_r) \leq \dim(M) - 2$ (that is, show that if $\tau \in \mathcal{O}(M, G)$ is the isotropy group of a G-orbit of dimension less than r, then M_τ is a locally finite union of connected manifolds of dimension at most $\dim(M) - 2$). Deduce that if all orbits of top dimension are principal orbits, then $M_{\mathbf{\Pi}}$ is connected. (In fact the connectedness of $M_{\mathbf{\Pi}}$ holds whenever G is connected without any restriction on orbits of top dimension. However, $\dim(M \setminus M_{\mathbf{\Pi}})$ may then equal $\dim(M) - 1$.)

3.7.1. Closed filtration of M by isotropy type.

LEMMA 3.7.11. *Let M be a G-manifold and suppose $\mathcal{O}(M, G)$ is finite. There exists at least one maximal isotropy type.*

PROOF. By the finiteness of $\mathcal{O}(M, G)$ every ascending chain $\mu_1 \prec \mu_2 \prec \dots$ must terminate. □

Let $\mathcal{O}_1$ be the set of maximal isotropy types. Let $\mathcal{O}_2 \subset \mathcal{O}(M, G) \setminus \mathcal{O}_1$ be the set of isotropy types μ such that if $\tau \succ \mu$ then τ is maximal. Observe that, by definition of $\prec$, it follows that if $\mu \in \mathcal{O}_2$, then ∂M_μ consists of points of maximal isotropy type. Continuing inductively, we define $\mathcal{O}_j \subset \mathcal{O}(M, G)$, $j = 1, \dots, N$, so that

(1) If $\mu \in \mathcal{O}_j$, then $\partial M_\mu \subset \cup_{i<j} \cup_{\tau \in \mathcal{O}_i} M_\tau$.
(2) $\mathcal{O}_N = \{\mathbf{\Pi}\}$.

LEMMA 3.7.12. *For $1 \leq j \leq N$, define $M^j = \cup_{i \leq j} \cup_{\tau \in \mathcal{O}_i} M_\tau$. We have*

(1) *M^j is a closed G-invariant subset of M.*

(2) *If $j < N$, then for all $\tau \in \mathcal{O}_{j+1}$, $M^j \cup M_\tau$ is a closed G-invariant subset of M.*
(3) *If $\tau, \mu \in \mathcal{O}_{j+1}$, then $\bar{M}_\tau \cap \bar{M}_\mu \subset M^j$.*

EXERCISE 3.7.13. Extend, as far as possible, the results of section 3.7 to proper G-actions. (Perhaps the main problem is finding conditions on a proper G-manifold M that guarantee finiteness of $\mathcal{O}(M, G)$. Of course, this issue is circumvented in our statement of lemma 3.7.11.)

3.8. The stratification of M by normal isotropy type

Let M be a smooth G-manifold. Given $x \in M$, let $\mathcal{N}(x) = (T_x N/Gx, G_x)$ denote the normal representation at x.

DEFINITION 3.8.1. Points $x, y \in M$ have the same *normal isotropy type* if

(1) x, y have the same isotropy type.
(2) There exists $z \in Gy$ such $G_z = G_x$ and $\mathcal{N}(z)$ is isomorphic as a G_x-representation to $\mathcal{N}(x)$.

REMARKS 3.8.2. (1) Without additional assumptions, for example if G is Abelian, it is not necessarily the case that if $z \in Gx$ and $G_z = G_x$, then $\mathcal{N}(x)$ is isomorphic as a G_x-representation to $\mathcal{N}(z)$.
(2) All points in $M_{\mathbf{\Pi}}$ have the same normal isotropy type.
(3) Although we shall not use the definition here, it is common to require that x, y have the same normal isotropy type if there exists $z \in Gy$ such that $\mathcal{N}(z)$ and $\mathcal{N}(x)$ are isomorphic up to a trivial factor (see [**155**]).

The next lemma is an easy consequence of the differentiable slice theorem.

LEMMA 3.8.3. *Let $\tau \in \mathcal{O}(M, G)$.*

(1) *If A is a connected component of M_τ then normal isotropy type is constant on $G(A)$.*
(2) *If A, B are connected components of M_τ with the same normal isotropy type, then $dim(A) = dim(B)$.*

Let $\mathcal{O}^\star(M, G)$ denote the set of normal isotropy types for the G-manifold M. If $\tau \in \mathcal{O}^\star(M, G)$, we let $\iota(\tau) \in \mathcal{O}(M, G)$ denote the associated isotropy type. Let $M_\tau \subset M$ denote the set of points in M with normal isotropy type τ. Obviously $M_\tau \subset M_{\iota(\tau)}$. We let $d(\tau)$ denote the dimension of M_τ – this is well-defined by lemma 3.8.3

LEMMA 3.8.4. *If $\mathcal{O}(M, G)$ is finite so is $\mathcal{O}^\star(M, G)$.*

PROOF. For all $d \geq 1$, a compact Lie group H has only finitely many isomorphism classes of representations of degree less than or equal to d. Hence, since M is finite dimensional, it follows that for each $\tau \in \mathcal{O}(M, G)$, there are only finitely many associated normal isotropy types. □

PROPOSITION 3.8.5. *Suppose that $\mathcal{O}(M,G)$ is finite. Then normal isotropy type partitions M into finitely many G-invariant manifolds.*

(1) *For each $\tau \in \mathcal{O}^\star(M,G)$, the dimension $d(\tau)$ of the connected components of M_τ is constant.*
(2) *If $\tau \neq \eta \in \mathcal{O}^\star(M,G)$ and $\iota(\tau) = \iota(\eta)$, then $\bar{M}_\tau \cap \bar{M}_\eta = \emptyset$.*
(3) *If $\tau \neq \eta \in \mathcal{O}^\star(M,G)$ and $\tau \succ \eta$ (that is, $\bar{M}_\eta \cap M_\tau \neq \emptyset$), then $d(\tau) > d(\eta)$.*
(4) *If $d(\tau) = dim(M)$, then $\iota(\tau) = \mathbf{\Pi}$ and $M_\tau = M_{\mathbf{\Pi}}$.*

PROOF. Statement (3) follows from the differentiable slice theorem; the remaining statements follow from the preceding discussion and lemmas. □

We refer to $\mathcal{S} = \{M_\tau \mid \tau \in \mathcal{O}^\star(M,G)\}$ as the *stratification of M by normal isotropy type*. Before giving our main result about $\mathcal{S}$, we need to review the theory of stratified sets.

3.9. Stratified sets

Let X be a subset of the differential manifold M. A *stratification* $\mathcal{S} = \{X_\alpha \mid \alpha \in I\}$ of X consists of a locally finite partition of X into connected submanifolds X_α. That is,

$$X = \cup_{\alpha \in I} X_\alpha,$$

where each X_α is a connected submanifold of M and we can find a neighbourhood U of every point $x \in M$ which meets only finitely many X_α. We refer to the sets X_α as *strata*. In what follows, we will sometimes weaken the requirement that the strata are connected but we always insist that X_α consists of submanifolds all of the same dimension. We set $\dim(X_\alpha) = d_\alpha$.

DEFINITION 3.9.1. The stratification $\mathcal{S}$ of $X \subset M$ satisfies the *frontier condition* if

$$\partial X_\alpha \subset \cup_{d_\beta < d_\alpha} X_\beta.$$

EXAMPLE 3.9.2. The stratification of a G-manifold by normal isotropy type satisfies the frontier condition (proposition 3.8.5(3)).

The frontier condition already imposes rather strong conditions on a stratification. It follows from the frontier condition that the higher dimensional strata are in some sense attached to the lower dimensional strata. If we are given a smooth map $f : N \to M$ such that f is transverse to a stratum X_α, then it is natural to require conditions on $\mathcal{S}$ that imply that (A) if $\partial X_\beta \supset X_\alpha$ then f is transverse to X_β near X_α, and (B) that transversality and the local intersection $f^{-1}(X)$ near X_α are preserved under sufficiently smooth perturbations of f. Whitney [**177**] formulated two conditions (a) and (b) on a stratification that imply properties (A) and (B). These conditions are now known as the *Whitney regularity conditions (a) and (b)*. We give a brief description of these conditions here. For a more detailed introduction to stratification theory, we refer the reader either to Mather's original article [**119**] or to the lecture notes by Gibson et al. [**77**]. We discuss in greater detail some of the issues raised here in chapter 6.

DEFINITION 3.9.3 (Whitney's condition (a)). Suppose that $X_\alpha, X_\beta \in \mathcal{S}$ are disjoint strata and that $\bar{X}_\beta \cap X_\alpha \neq \emptyset$. The pair (X_α, X_β) satisfies Whitney's condition (a) if given $x \in \bar{X}_\beta \cap X_\alpha$ and a sequence $(y_i) \subset X_\beta$ such that the sequence of tangent spaces $(T_{y_i}X_\beta)$ converges to a linear subspace $E \subset T_xM$, we have $L \supset T_xX_\alpha$.

REMARK 3.9.4. Condition (a) implies that T_yX_β is close to T_xX_α when y is close to x. It is not surprising therefore that if $f : N \to M$ is transverse to X_α at x then f will be transverse to X_β near x.

DEFINITION 3.9.5 (Whitney's condition (b)). Suppose that $X_\alpha, X_\beta \in \mathcal{S}$ are disjoint strata and that $\bar{X}_\beta \cap X_\alpha \neq \emptyset$. The pair (X_α, X_β) satisfies Whitney's condition (b) if given $x \in \bar{X}_\beta \cap X_\alpha$, a local coordinate system at x, and sequences $(y_i) \subset X_\beta$, $y_i \to x$, $(x_i) \subset X_\alpha$, $x_i \to x$ such that the line joining y_i to x_i converges to a line $\ell \subset T_xM$ and the sequence of tangent spaces $(T_{y_i}X_\beta)$ converges to a linear subspace $E \subset T_xM$, then $E \supset \ell$.

REMARKS 3.9.6. (1) A stratification is *Whitney regular* if it satisfies conditions (a) and (b).
(2) Condition (b) is much stronger than (a). In figure 2 we show a stratification of a surface S that satisfies (a) but not (b). The surface S is defined by the equation $y^2 - t^2x^2 - x^3 = 0$. As one-dimensional stratum S_1 we have taken the t-axis. The two dimensional strata are then the four connected components $S_2^1, \dots, S_2^4$ of $S \setminus S_1$. It is easy to see that (a)-regularity holds for (S_1, S_2^i), $i = 1, \dots, 4$, and that (b)-regularity holds for (S_1, S_2^i), $i = 3.4$. However (b)-regularity fails for (S_1, S_2^i), $i = 1, 2$ at the origin. Although (b)-regularity fails for this example, the topological type of the intersection of X with transversals to X through the origin is constant. This is a characteristic property of points satisfying (a)-regularity (see [**109, 172**]).

EXERCISE 3.9.7. (1) Show that Whitney's condition (b) implies condition (a).
(2) Find a stratification of the surface of figure 2 that satisfies condition (b).
(3) Show that $(\{(0,0)\}, \{e^{-r}(\cos r, \sin r) \mid r > 0\})$ does not satisfy (b)-regularity. Does it satisfy (a)-regularity?

We use the following lemma in the next section. The proof follows easily from the definition of Whitney regularity.

LEMMA 3.9.8. *Let $\mathcal{S} = \{S_\alpha \mid \alpha \in \Lambda\}$ be a Whitney regular stratification of the subset X of the differential manifold M. Then for all $k \geq 1$, $\mathcal{S} = \{S_\alpha \times \mathbb{R}^k \mid \alpha \in \Lambda\}$ is a Whitney regular stratification of $X \times \mathbb{R}^k \subset M \times \mathbb{R}^k$.*

3.9.1. Transversality to a Whitney stratification.

DEFINITION 3.9.9. Let M, N be smooth manifolds. Let X be a closed subset of N and suppose that X has Whitney regular stratification $\mathcal{S}$. A smooth map $f : M \to N$ is transverse*transversality to stratification* to $\mathcal{S}$ at $x \in M$ if either

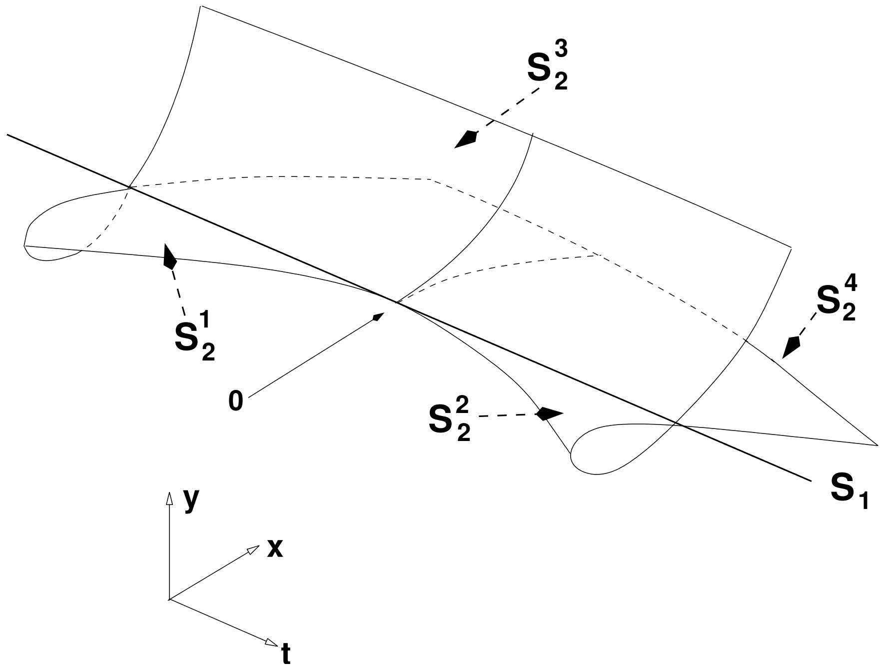


FIGURE 2. A stratification satisfying condition (a) but not (b)

$f(x) \notin X$ or $f(x) \in S_\alpha \in \mathcal{S}$ and f is transverse to the submanifold S_α of N at x (that is, $T_{f(x)}S_\alpha + T_x f T_x M = T_{f(x)}N$.) If f is transverse to $\mathcal{S}$ at all points $x \in M$, we say f is transverse to $\mathcal{S}$. We write this $f \pitchfork_{\mathcal{S}} X$ (or just $f \pitchfork X$ when the stratification is implicit from the context).

The next lemma follows (easily) from Whitney (a)-regularity.

LEMMA 3.9.10. *Let M, N be smooth manifolds and X be a Whitney stratified closed subset of N. If $f : M \to N$ is transverse to $\mathcal{S}$ at $x \in M$, then we can choose an open neighbourhood U of x in M such that $f : M \to N$ is transverse to $\mathcal{S}$ for all $y \in U$.*

PROPOSITION 3.9.11 (Mather [**119**, Corollary 8.8]). *Let M, N be smooth manifolds and let $\mathcal{S}$ be a Whitney stratification of the closed subset $X \subset N$. Suppose that $f \pitchfork X$. Then $f^\star(\mathcal{S}) = \{f^{-1}(S_\alpha) \mid S_\alpha \in \mathcal{S}\}$ is a Whitney stratification of $f^{-1}(X)$.*

THEOREM 3.9.12 (Thom-Mather transversality theorems for stratified sets). *Let M, N be smooth manifolds and let $\mathcal{S}$ be a Whitney stratification of the closed subset $X \subset N$.*

(1) *(Density) The set of smooth maps $f : M \to N$ satisfying $f \pitchfork X$ is dense in $C^\infty(M, N)$ (in the C^∞ or Whitney C^∞-topology).*
(2) *(Openness) If M is compact, then the set of smooth maps $f : M \to N$ satisfying $f \pitchfork X$ is open in $C^\infty(M, N)$ (in the C^2-topology).*

(3) *(Isotopy theorem) If M is compact and $f : M \times [0,1] \to N$, is a smooth family of maps such that $f_t \pitchfork X$, $t \in [0,1]$, then there exists a (continuous) isotopy $F : M \times [0,1] \to M$ of homeomorphisms of M such that*

$$F_t(f_t^{-1}(X)) = f_0^{-1}(X),\ t \in [0,1].$$

Proof. The proof may be found in [**119, 77**]. We remark that density and openness follow easily using standard results from transversality theory. The isotopy theorem is harder and, in particular, we cannot require that the isotopy is C^1 let alone smooth. □

3.9.2. Regularity of the stratification by normal isotropy type.

Proposition 3.9.13. *Let M be a smooth G-manifold. Then the stratification of M by normal isotropy type is Whitney regular.*

Proof. Let $x \in M$. Suppose $\dim(Gx) = k$ and S_x be a smooth slice at x. Using a local section of G over G/G_x, we see that the induced orbit stratification on GU is locally diffeomorphic to the product of $\mathbb{R}^k$ with the stratification of S_x by G_x-isotropy type. Hence, by lemma 3.9.8, it is enough to verify that if (W, H) is an H-representation, H compact, then (W^H, W_τ) satisfies the Whitney regularity conditions for $\tau \in \mathcal{O}(W, H)$, $\tau \neq (H)$. Another application of lemma 3.9.8 allows us to restrict to representations (W, H) for which $W^H = \{0\}$. Let $(u_i) \subset W_\tau$ be a sequence such that $u_i \to 0$ and $T_{u_i}W_\tau \to L$. Set $v_i = u_i/\|u_i\|$ and let $S(W)$ denote the unit sphere of W. Noting that $\lambda W_\tau = W_\tau$, for all $\lambda \in \mathbb{R}^\star$, we see that

$$\begin{aligned} v_i &\to v \in S(W), \\ T_{u_i}W_\tau &= T_{v_i}W_\tau = T_{v_i}(S(W) \cap W_\tau) \oplus \mathbb{R}v_i, \\ &\to L \cap T_vS(W) \oplus \mathbb{R}v. \end{aligned}$$

Hence $L \supset \mathbb{R}v$ and Whitney's condition (b) is satisfied. □

Exercise 3.9.14 (Stratumwise transversality). Let M, N be G-manifolds and P be a G-invariant submanifold of N. Show that for every $H \in \tau \in \mathcal{O}(M, G)$,

$$\mathcal{T}(H) = \{f \in C^\infty_G(M, N) \mid f|M^H \pitchfork P^H \text{ (within } N^H)\}$$

is a residual subset of $C^\infty_G(M, N)$. Deduce that the set $\mathcal{T} = \cap_H \mathcal{T}(H)$ of maps which are stratumwise transverse to P form a residual subset of $C^\infty_G(M, N)$. Even if M and P are compact, $\mathcal{T}$ will not usually be open, see chapter 6.

Exercise 3.9.15. Generalize our results on stratifications of G-manifolds to proper Lie group actions. What problems arise when we consider stratifications and transversality (in particular the isotopy theorem)?

3.10. Invariant Riemannian metrics on a compact Lie group

We conclude this chapter with some more results about compact Lie groups and homogeneous spaces. This work will be useful when we study relative equilibria for equivariant vector fields and maps.

Let G be a compact Lie group. We define composition on $G \times G$ by

$$(a, b)(c, d) = (ac, bada^{-1}), \quad (a, b, c, d \in G).$$

If we set $G_c = G \times \{e_G\}$ and $G_r = \{e_G\} \times G$, then $G_r \lhd G \times G$ and $G \times G$ is the semidirect product $G_c \ltimes G_r$. We define a smooth action of $G \times G$ on G by

$$(g, k)\gamma = g\gamma g^{-1}k^{-1}, \quad ((g, k) \in G \times G, \gamma \in G).$$

If we restrict this action to the subgroups $G_c = G \times \{e_G\}$ and $G_r = \{e_G\} \times G$ we recover the actions by conjugation and right multiplication translation. If we restrict to the subgroup $G_l = \{(g, g^{-1}) \mid g \in G\} \cong G$, we obtain the action of G on G by left translation.

3.10.1. The adjoint representations. Since $G_c e = e$, the isotropy group $(G \times G)_e = G_c$. The associated isotropy representation of $G = G_c$ on $T_eG = \mathfrak{g}$ is the *adjoint representation* of G and is denoted by $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. If we define the homomorphism $c : G \to \mathrm{Aut}(G)$ by $c(g)(h) = ghg^{-1}$, then the adjoint representation is given by

$$\mathrm{Ad}(g)(X) = T_e c(g)(X), \quad (g \in G, X \in \mathfrak{g}).$$

The adjoint representation is trivial if G is Abelian.

REMARKS 3.10.1. (1) Since the Lie algebra of $\mathrm{GL}(\mathfrak{g})$ is $L(\mathfrak{g}, \mathfrak{g})$, we recover the adjoint Lie algebra representation $\mathrm{ad} : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{g})$ defined in chapter 1 by taking $\mathrm{ad}(X) = T_e\mathrm{Ad}(X)$, $X \in \mathfrak{g}$.
(2) Since $\exp : \mathfrak{g} \to G$ is natural – lemma 1.5.13(4), $\exp : \mathfrak{g} \to G$ is G_c-equivariant. That is, if $g \in G$, $\eta \in \mathfrak{g}$, then

$$g \exp(\eta) g^{-1} = \exp(\mathrm{Ad}(g)(\eta)).$$

In the next section we reprove this result by viewing exp as the exponential map of a G_c-invariant Riemannian metric.

3.10.2. The exponential map. Let ξ be a $G \times G$-invariant Riemannian metric on G. In order to construct ξ it suffices to choose an inner product on $\mathfrak{g}$ which is invariant with respect to the adjoint representation of G (average over G) and then extend to all tangent spaces of G by left translation. Alternatively, any Riemannian metric for G can be averaged over $G \times G$. Since G is compact, ξ is a complete Riemannian metric and so geodesics are infinitely extendable and connect all points of G.

It follows by G_l-invariance that if $\gamma : \mathbb{R} \to G$ is a geodesic then so is $g\gamma$, all $g \in G$. Let d denote the associated metric on G. The $G \times G$-invariance of ξ

implies that for all $g, g', h, k \in G$ we have

$$d(ghg', gkg') = d(h, k).$$

It follows that we can choose a base $\mathcal{U}$ of (disk) neighbourhoods for $e \in G$ such that $gUg^{-1} = U$, all $g \in G$, $U \in \mathcal{U}$.

For each $X \in \mathfrak{g}$, there exists a unique geodesic $\gamma_X : \mathbb{R} \to G$ such that $\gamma_X'(0) = X$. (If $X = 0$, then $\gamma_X \equiv e$.) It follows from the parallelizability of G (example 1.5.8 and [**18**, Chapter 6]) that $\gamma_X'(t) = \gamma_X(t)X \in T_{\gamma_X(t)}X$, for all $t \in \mathbb{R}$. That is, geodesics are given by the integral curves of the left-invariant vector fields on G (section 1.5.2). Hence the exponential map $\exp : \mathfrak{g} \to G$ defined by

$$\exp(X) = \gamma_X(1), \quad X \in \mathfrak{g},$$

coincides with the exponential map of G defined in chapter 1. (This is a general fact about the exponential map of a Lie group [**18**, 6.3.2].)

Applying our results for the exponential map of a Riemannian G-manifold, we see that $\exp : \mathfrak{g} \to G$ is G-equivariant with respect to the adjoint action of G on $\mathfrak{g}$ and the action by conjugation on G (cf remarks 3.10.1(2)). Further, we may choose $r > 0$ so that exp maps the open r-disk neighbourhood $D(r)$ of $0 \in \mathfrak{g}$ diffeomorphically onto an open neighbourhood U of the identity $e \in G$ where $gUg^{-1} = U$, for all $g \in G$.

3.10.3. Closed subgroups of a Lie group. Let H be a closed subgroup of G. Since H is an $N(H) \times H$-invariant submanifold of G, TH and $T_H G \to H$ are $N(H) \times H$-vector bundles over H. In particular, the normal bundle $T_H G/TH \to H$ has the structure of an $N(H) \times H$-vector bundle over H.

LEMMA 3.10.2. *Let G be a compact Lie group with $G \times G$-invariant Riemannian metric. Suppose that H is a closed subgroup of G and let $q : Q \to \mathbf{T} \subset G$ be an $N(H) \times H$-invariant tubular neighbourhood of H in G. Set $S_e = q(Q_e)$. If we define $\chi : \mathbf{T} \to S_e$ by $\chi(g) = S_e \cap gH$, then χ is a well-defined smooth submersion satisfying*

1. *For all $g \in \mathbf{T}$, $gH = \chi(g)H$.*
2. *$\chi | S_e$ is the identity map.*
3. *χ is $N(H)$-equivariant: $\chi(ngn^{-1}) = n\chi(g)n^{-1}$, $n \in N(H)$, $g \in \mathbf{T}$.*
4. *If K is a closed subgroup of H, then $\chi(g) \in C_G(K)$ for all $g \in N(K) \cap \mathbf{T}$. In particular, if $n \in \mathbf{T} \cap N(H)$, then $\chi(n) \in C_G(H)$.*

PROOF. The $G \times G$-invariant Riemannian metric on G restricts to an $N(H) \times H$-invariant metric on $Q = T_H G/TH$. Let $q : Q \to \mathbf{T} \subset G$ be an $N(H) \times H$-invariant tubular neighbourhood of H in G. Since $q : Q \to \mathbf{T} \subset G$ is an H-equivariant embedding, where H acts by left translations, $q(Q_e) \cap q(Q_h) = \emptyset$, all $h \in H$, $h \neq e$. Hence for all $V \in Q$, the H-orbit of $g = q(V)$ meets $S_e = q(Q_e)$ in exactly one point, $\chi(g)$. By construction, $gH = \chi(g)H$. Since q is an embedding, each coset gH meets S_e transversally. Hence, $\chi : \mathbf{T} \to S_e$ is a smooth submersion. Statement (2) is trivial. Suppose that $n \in N(H)$, $g \in \mathbf{T}$. Then

$$\chi(ngn^{-1})H = ngn^{-1}H = ngHn^{-1} = n\chi(g)Hn^{-1} = n\chi(g)n^{-1}H.$$

It follows from the $N(H)$-invariance of S_e that $n\chi(g)n^{-1} \in S_e$ and so $\chi(ngn^{-1}) = n\chi(g)n^{-1}$, proving (3). Finally suppose $g \in N(K) \cap \mathbf{T}$, $k \in K$. Since $K \subset H \subset N(H)$, it follows from (3) that $\chi(kgk^{-1}) = k\chi(g)k^{-1}$. But $kgk^{-1} = gk'$ for some $k' \in K$ since $g \in N(K)$. Now $\chi(gk') = \chi(g)$ and so $\chi(g) = \chi(kgk^{-1}) = k\chi(g)k^{-1}$, all $k \in K$. Hence $\chi(g) \in C_G(K)$. □

COROLLARY 3.10.3. *Let H be a closed subgroup of G. Then*

(1) $N(H)_0 = H_0 \cdot C_G(H)_0$.
(2) $(N(H)/H)_0 = N(H)_0/H_0 \approx C_G(H)_0/Z(H)_0$.

PROOF. Since the identity component of a Lie group is generated by any open neighbourhood of the identity, lemma 3.10.2(4) implies that every element of $N(H)_0$ may be written hc, where $h \in H_0$, $c \in C_G(H)_0$, proving (1). Statement (2) follows since $N(H)_0/H_0$ embeds as a connected open subgroup of $N(H)/H$. □

REMARK 3.10.4. In terms of Lie algebras, we have $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h} + \mathfrak{c}(\mathfrak{h})$ and $C_G(H)$ is transverse to H at $e \in G$ within $N(H)$. Note also that $\mathfrak{c}(\mathfrak{h})$, $\mathfrak{n}(\mathfrak{h})$ are invariant under the adjoint action of H on $\mathfrak{g}$. Another way of viewing lemma 3.10.2 is to note that if $g \in \mathbf{T}$ then $d(\chi(g), e) = d(gH, H) = d(gH, e)$. If follows by $H \times H$-invariance that $d(h\chi(g)h^{-1}, e) = d(hgH, H)$, $h \in H$. Now $d(hgH, H) = d(\chi(hg), e)$ and so, by uniqueness of the point in $hgH \cap S_e$ closest to H, we must have $\chi(hg) = h\chi(g)h^{-1}$. If particular, if $g \in N(H)$, so that $\chi(hg) = \chi(g)$, we have $\chi(g) \in C_G(H)$.

EXERCISE 3.10.5. Prove a version of lemma 3.10.2, in particular statement (4), that applies when we only assume H is compact. (For statement (3), replace $N(H)$ by H.)

As a straightforward corollary of lemma 3.10.2 we have the following improved result on the existence of local sections.

LEMMA 3.10.6. *Let H be a closed subgroup of the compact Lie group G. Let $N(H)$ act smoothly on G/H by $g[H] \to ng[H]n^{-1} = ngn^{-1}[H]$. Given an open neighbourhood V of $[H] \in G/H$, we may choose an $N(H)$-invariant open neighbourhood $U \subset V$ of $[H]$ and smooth $N(H)$-equivariant local section $\sigma : U \subset G/H \to G$ of the quotient map $p : G \to G/H$.*

We call a local $N(H)$-equivariant section of $G \to G/H$ an admissible section.

EXERCISE 3.10.7. Let $\sigma : U \subset G/H \to G$ be an admissible local section. Let K be a closed subgroup of H and suppose $g \in N(K)$, $g[H] \in U$. Show that $\sigma(g[H]) \in C_G(K)$.

LEMMA 3.10.8. *Let M be a G-manifold and $\alpha \subset M$ be a G-orbit. Suppose that S_x is a differentiable slice at $x \in \alpha$, set $G_x = H$, and let $\sigma : U \subset G/H \to G$ be an admissible local section. Define $\rho^\sigma : S_x \times U \to \sigma(U)(S_x)$ by $\rho^\sigma(y, u) = \sigma(u)y$.*

(1) *ρ^σ is an H-equivariant diffeomorphism of $S_x \times U$ onto the open H-invariant subset $\sigma(U)(S_x)$ of M.*
(2) *For all $y \in S_x$, $T_y\sigma(U)y$ is a G_y-invariant subspace of T_yM.*
(3) *If $u = g[H] \in U$, $g \in N(G_y)$, then $\rho^\sigma(y, u) \in C_G(G_y)y$.*

PROOF. Since S_x is a slice and σ is a section, ρ^σ is a diffeomorphism of $S_x \times U$ onto the open set $\sigma(U)(S_x)$. Since σ is admissible, we have $\sigma(hu) = h\sigma(u)h^{-1}$ for all $h \in H$ and this gives the H-invariance statements of (1). For (2) we use the H-equivariance of ρ^σ together with $G_y \subset H$ for all $y \in S_x$. Finally (3) is a consequence of exercise 3.10.7. $\square$

REMARK 3.10.9. If $y \in S_x$, then $C(G_x) \subset C(G_y)$ and so the corollary implies that $C_G(G_y)_0 y \pitchfork S_x$ within $N(H)S_x$. See also corollary 3.5.3 and remark 3.5.4.

CHAPTER 4

Equivariant Bifurcation Theory: Steady State Bifurcation

4.1. Introduction and preliminaries

In this chapter we start our investigation of bifurcation of the trivial solution of one-parameter families of G-equivariant vector fields defined on a non-trivial irreducible representation (V, G). We assume throughout that G is finite and (V, G) is an absolutely irreducible orthogonal representation. In particular, V is equipped with a G-invariant positive definite inner product $(\, , \,)$ with associated norm $\| \, \|$. We denote the unit sphere of V by $S(V)$.

If U is an open G-invariant subset of V, let $C^\infty_G(U, V)$ denote the space of smooth G-equivariant vector fields on U. Give $C^\infty_G(U, V)$ the C^∞-topology – uniform convergence of a function and all its derivatives on compact subsets of U (see section 2.10.1). A base of open subsets for $C^\infty_G(U, V)$ consists of all subsets $\mathcal{N}(X, K, r, \varepsilon) = \{Y \in C^\infty_G(U, V) \mid \|X - Y\|^K_r < \varepsilon\}$, where $X \in C^\infty_G(U, V)$, $r \in \mathbb{N}$, $\varepsilon > 0$, K is a compact subset of U, and $\|X - Y\|^K_r$ is the C^r supremum (uniform) norm of $X - Y$ on K. The space $C^\infty_G(U, V)$ is a Fréchet space if we define for $r \geq 0$ the seminorm q_r on $C^\infty_G(U, V)$ by $q_r(f) = \|f\|^{K(r)}_r$, where $(K(r))$ is an increasing sequence of compact subsets of U with $\cup_r K(r) = U$.

We recall some general facts about zeros of smooth vector fields. Let z be a zero of the smooth vector field X.

(1) z is a *simple* zero if $DX(z)$ is non-singular.
(2) z is a *hyperbolic* zero if $DX(z)$ has no eigenvalues with real part zero,

If z is a hyperbolic zero of the vector field X, we define $\mathrm{ind}(X, z) \in \mathbb{N}$ to be the number of eigenvalues of $DX(z)$ with strictly negative real part (counting multiplicities).

The next result is a standard application of the implicit function theorem.

LEMMA 4.1.1. *Let U be a G-invariant open subset of V. Suppose that $X \in C^\infty_G(U, V)$ has a simple zero $z \in U$. Let K be a compact neighbourhood of z in U. Then we may choose an open neighbourhood $U(z) \subset U$ of z and $\varepsilon > 0$ such that if $Y \in \mathcal{N}(X, K, 1, \varepsilon)$, then*

(1) *Y has a unique simple zero $\eta(Y) \in U(z)$.*
(2) *$\eta : \mathcal{N}(X, K, 1, \varepsilon) \to U(z)$ is continuous with respect to the seminorm $\|X - Y\|^K_1$.*

If z is a hyperbolic zero of X, then we may choose $U(z)$, ε so that $\eta(Y)$ is hyperbolic and $\mathrm{ind}(Y, \eta(Y)) = \mathrm{ind}(X, z)$, for all $Y \in \mathcal{N}(X, K, 1, \varepsilon)$.

4.1.1. Normalized families. Let G act on $V \times \mathbb{R}$ by $g(v, \lambda) = (gv, \lambda)$. Let $\mathcal{V} = \mathcal{V}(V, G) = C^\infty_G(V \times \mathbb{R}, V)$ denote the vector space of smooth G-equivariant maps from $V \times \mathbb{R}$ to V. If $X \in \mathcal{V}$, we define the 1-parameter family $\{X_\lambda\}_{\lambda \in \mathbb{R}}$ of smooth G-equivariant vector fields on V by $X_\lambda(v) = X(v, \lambda)$, $\lambda \in \mathbb{R}$. We give $\mathcal{V}$ the C^∞-topology (uniform convergence of a function and all its derivatives on compact subsets of $V \times \mathbb{R}$). Subsets of $\mathcal{V}$ are given the induced topology.

Let $X \in \mathcal{V}$. Since (V, G) is a non-trivial irreducible representation, $V^G = \{0\}$ and so (proposition 2.10.4(2)) $X_\lambda(0) = X(0, \lambda) = 0$, for all $\lambda \in \mathbb{R}$. We refer to this set of zeros of X as the *trivial branch* of zeros of X. We study zeros of X that bifurcate off the trivial branch of zeros as we vary λ. Since $DX_\lambda(0) \in L_G(V, V)$ and (V, G) is absolutely irreducible, $DX_\lambda(0) = \sigma_X(\lambda) I_V$, where $\sigma_X(\lambda) = \det(DX_\lambda(0)) \in \mathbb{R}$ and $\sigma_X \in C^\infty(\mathbb{R})$. If $\sigma_X(\lambda) \neq 0$, we can choose an open neighbourhood U of $(0, \lambda)$ in $V \times \mathbb{R}$ so that the only zeros of $X|U$ are trivial zeros (lemma 4.1.1). Consequently, we only obtain non-trivial zeros near $(0, \lambda)$ if $\sigma_X(\lambda) = 0$ (that is, if $DX_\lambda(0)$ is singular). We call points $(0, \lambda)$ (or just λ) where $\sigma_X(\lambda) = 0$ *bifurcation points*.

We shall assume that X satisfies the generic condition $\sigma'_X(\lambda_0) \neq 0$ at a bifurcation point $(0, \lambda_0)$. (This condition is equivalent to requiring that σ_X is transverse to $0 \in \mathbb{R}$ – an open and dense condition on maps in $C^\infty(\mathbb{R})$.) Without loss of generality, we may take $\lambda_0 = 0$. Using the inverse function theorem, we may reparameterize the λ-variable and assume that $\sigma_X(\lambda) = \lambda$ for λ near 0 (note the orientation reversal if $\sigma'_X(\lambda_0) < 0$). Since we shall only be interested in solutions of $X = 0$ near $(0, 0)$, it is no loss of generality to assume that $\sigma_X(\lambda) = \lambda$, for all $\lambda \in \mathbb{R}$. For the remainder of the chapter we shall restrict attention to the closed affine linear subspace $\mathcal{V}_0 \subset \mathcal{V}$ consisting of *normalized families* defined by

$$\mathcal{V}_0 = \mathcal{V}_0(V, G) = \{X \in \mathcal{V} \mid \sigma_X(\lambda) = \lambda,\ \lambda \in \mathbb{R}\}.$$

If $X \in \mathcal{V}_0$, then we may write

$$X_\lambda(v) = \sigma_X(\lambda) v + Q_\lambda(v), \tag{4.1}$$

where $Q_\lambda(v) = O(\|v\|^2)$, uniformly for λ in compact subsets of $\mathbb{R}$. The origin $(0, 0)$ is a bifurcation point of X and is the only bifurcation point on the λ-axis.

Our aim in this chapter is to study the germ of $X^{-1}(0)$ at $(0, 0)$ for generic $X \in \mathcal{V}_0$ (we give a precise definition of 'generic' later). We establish results that give detailed information on the local zero sets for many representations (V, G). As part of this investigation we need to spend time developing careful and precise definitions of solution branches and what we call the 'branching pattern'.

4.2. Solution branches and the branching pattern

DEFINITION 4.2.1. Let $X \in \mathcal{V}_0$. A *solution branch* of X (more precisely, a solution branch of $X = 0$ at $(0, 0)$) is a C^1-embedding $\gamma = (\mathbf{x}, \lambda) : [0, \delta] \to V \times \mathbb{R}$ such that $\gamma(0) = (0, 0)$ and $X(\gamma(s)) = 0$, $s \in [0, \delta]$. A *solution curve* of X is a C^1-embedding $\gamma : [-\delta, \delta] \to V \times \mathbb{R}$ such that $\gamma|[0, \pm\delta]$ are solution branches.

We define $c^{\pm} : [0, \infty) \to V \times \mathbb{R}$ by $c^{\pm}(s) = (0, \pm s)$. Then $c^{\pm}$ parameterize the trivial solution branches. We say a solution branch $\gamma = (\mathbf{x}, \lambda)$ of X is *non-trivial* if $\mathbf{x}(s) \neq 0$, for all $s > 0$.

LEMMA 4.2.2. *Let $\gamma = (\mathbf{x}, \lambda) : [0, \delta] \to V \times \mathbb{R}$ be a non-trivial solution branch for $X \in \mathcal{V}_0$. Then the* direction of branching $\mathbf{d}(\gamma) = \mathbf{x}'(0)/\|\mathbf{x}'(0)\| \in V$ *is well defined and independent of the parameterization.*

PROOF. Since $X \in \mathcal{V}_0$ we may write

$$X(\mathbf{x}, \lambda) = \lambda \mathbf{x} + F(\mathbf{x}, \lambda), \tag{4.2}$$

where $\|F(\mathbf{x}, \lambda)\| = O(\|\mathbf{x}\|^2)$. Since γ is an embedding, either $\lambda'(0) \neq 0$ or $\mathbf{x}'(0) \neq 0$. It suffices to prove that $\mathbf{x}'(0) \neq 0$. Substituting $\mathbf{x} = \mathbf{x}(s)$, $\lambda = \lambda(s)$ in (4.2) and dividing by $\|\mathbf{x}(s)\|$, we find that

$$|\lambda(s)| = \|F(\mathbf{x}(s), \lambda(s))\|/\|\mathbf{x}(s)\| = O(\|\mathbf{x}(s)\|), \quad s \neq 0.$$

If $\mathbf{x}'(0) = 0$ then $\|\mathbf{x}(s)\| = o(s)$ and so $|\lambda(s)| = o(s)$ and $\lambda'(0) = 0$, contradicting the fact that γ is a C^1- embedding. Therefore $\mathbf{x}'(0) \neq 0$. □

REMARKS 4.2.3. (1) The isotropy type of $\mathbf{d}(\gamma)$ is greater than or equal to that of the isotropy type of the branch.
(2) Suppose $\gamma = (\mathbf{x}, \lambda) : [0, \delta] \to V \times \mathbb{R}$ is a non-trivial solution branch. It follows from lemma 4.2.2 that we may write $\mathbf{x}(s) = r(s)\mathbf{u}(s)$, where $r : [0, \delta] \to \mathbb{R}^+$ and $\mathbf{u} : [0, \delta] \to S(V) \subset V$ are continuous and r is strictly monotone increasing on some interval $[0, \delta'] \subset [0, \delta]$.

Two solution branches are *equivalent* if locally they only differ by a reparameterization. If γ is a solution branch, let $[\gamma]$ denote the equivalence class of γ. Let $\Sigma(X)$ be the set of all equivalence classes of *non-trivial* solution branches of X. We call $\Sigma(X)$ the *branching pattern* of X. Since X is equivariant, γ is a solution branch if and only if $g\gamma$ is a solution branch for all $g \in G$. Hence $\Sigma(X)$ has the structure of a G-set. We emphasize that $[c^{\pm}] \notin \Sigma(X)$.

The solution branch $\gamma = (\mathbf{x}, \lambda)$ is a *forward* or *supercritical* branch if the $\mathbb{R}$-component λ of γ is strictly positive on some interval $(0, \delta')$. We similarly define *backward* or *subcritical* branches. Both forward and backward depend only on the equivalence class of the solution branch. If γ (or $[\gamma]$) is a forward (respectively, backward) solution branch, we set $\text{sgn}([\gamma]) = +1$ (respectively, $\text{sgn}([\gamma]) = -1$).

A solution branch $\gamma : [0, \delta] \to V \times \mathbb{R}$ is a branch of simple zeros if $\mathbf{x}(s)$ is a simple zero of $X_{\lambda(s)}$ for all $s \in (0, \delta]$.

EXERCISE 4.2.4. (1) Show that if $\lambda \equiv 0$, then γ is not a branch of simple zeros.
(2) Show that if γ is a branch of simple zeros, then γ is either a forward or backward branch.

For a branch of simple zeros, isotropy is constant along the branch. This important result is a consequence of the following general lemma.

LEMMA 4.2.5. *Let S be locally compact, connected and Hausdorff space. If $F : S \to C^\infty_G(V, V)$ and $\gamma : S \to V$ are continuous maps such that for every $s \in S$, $\gamma(s)$ is a simple zero of $F(s)$, then $G_{\gamma(s)}$ is constant on S.*

PROOF. Given $s \in S$, set $z = \gamma(s)$ and $X = F(s)$. Since S is connected, it suffices to find an open neighbourhood N of s such that if $t \in N$, then $G_{\gamma(t)} = G_z$. Let C be a compact neighbourhood of s in S and K be a compact neighbourhood of $\gamma(C)$ in V. By lemma 4.1.1, we can choose an open G_z-invariant disk neighbourhood $U(z)$ of z in V and $\varepsilon > 0$ such that for all $Y \in \mathcal{N}(X, K, 1, \varepsilon)$, Y has a unique simple zero $\eta(Y) \in U(z)$. Since F is continuous we can choose a neighbourhood $N \subset C$ of s such that $F(t) \in \mathcal{N}(X, K, 1, \varepsilon)$, $\gamma(t) \in U(z)$ for all $t \in N$. Taking the radius of $U(z)$ smaller if necessary, we may assume that $U(z)$ is a slice for the action of G at z. Suppose that $t \in N$. Since $U(z)$ is a slice, $G_{\gamma(t)} \subset G_z$. If $g \in G_z$, then $\gamma(t)$ and $g\gamma(t)$ are zeros of the equivariant vector field $F(t)$ lying in $U(z) = gU(z)$. But since $F(t)$ has a unique zero in $U(z)$, $g\gamma(t) = \gamma(t)$. Hence $G_z = G_{\gamma(t)}$. $\square$

REMARK 4.2.6. Let γ be a non-trivial branch of simple zeros and set $G_{\gamma(t)} = H$, $t \neq 0$. Since γ is non-trivial, γ has a well-defined direction of branching $\mathbf{d}(\gamma) \in S(V)$. Clearly, $H \supset G_{\mathbf{d}(\gamma)}$. Although equality holds for most of the examples we consider later in this chapter, equality fails in general.

A solution branch γ is a *branch of hyperbolic zeros* if we can find $\delta' > 0$ such that $\mathbf{x}(s)$ is a hyperbolic zero of $X_{\lambda(s)}$ for all $s \in (0, \delta')$. If γ (or $[\gamma]$) is a branch of hyperbolic zeros, we let $\mathrm{ind}([\gamma])$ denote the dimension of the stable manifold of a hyperbolic zero on γ – the *index* of γ.

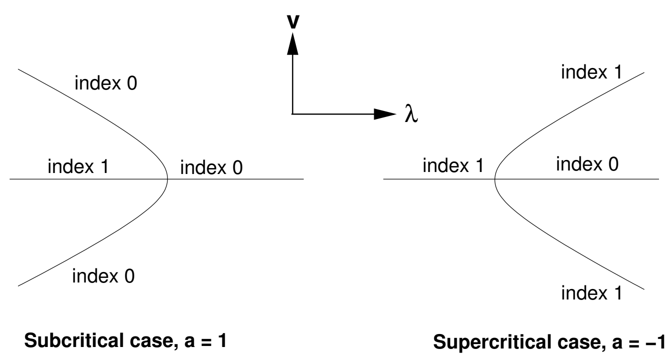


FIGURE 1. Pitchfork bifurcation: $\mathbb{Z}_2$-symmetry

EXAMPLE 4.2.7. Let $G = \mathbb{Z}_2$ act on $V = \mathbb{R}$ as multiplication by ± 1. Define $X_\lambda \in C^\infty_{\mathbb{Z}_2}(V, V)$ by $X_\lambda(v) = \lambda v + av^3$, where $a \in \mathbb{R}^\star$. Non-trivial solution branches of hyperbolic zeros are given by $\gamma_\pm(s) = (-as^2, s)$, $s \in [0, \infty)$. If $a = +1$, we obtain a pair of subcritical branches of index 0; if $a = -1$, we obtain a pair of supercritical branches of index 1. See figure 1.

We consider the following *branching conditions* on a family $X \in \mathcal{V}_0$.

Condition B1

There exists a finite set $\gamma_1, \dots \gamma_{r+2}$ of solution branches, with images $C_1, \dots C_{r+2}$ ($C_j = \text{image}(\gamma_j)$), such that the following conditions hold:

(1) $\Sigma(X) = \{[\gamma_1], ..., [\gamma_r]\}$ and $[\gamma_{r+1}] = [c^+]$, $[\gamma_{r+2}] = [c^-]$.
(2) If $i \neq j$, then $C_i \cap C_j = \{(0,0)\}$.
(3) There is a neighbourhood N of $(0,0)$ in $V \times \mathbb{R}$ such that

$$X^{-1}(0) \cap N = \cup_{j=1}^{r+2} C_j.$$

Condition B2

Every $[\gamma] \in \Sigma(X)$ is a branch of hyperbolic zeros.

Condition B3

Every $[\gamma] \in \Sigma(X)$ is either a forward or backward solution.

Condition B4

Every $[\gamma] \in \Sigma(X)$ is is a branch of simple zeros.

REMARKS 4.2.8. (1) If X satisfies B1, the branches $\gamma_1, \dots, \gamma_r$ are automatically non-trivial (since $C_i \cap C_{r+1}, C_{r+2} = \emptyset$, $i \leq r$).
(2) If X satisfies B1,2,3, the sign function, $\text{sgn} : \Sigma(X) \to \{\pm 1\}$, and the index function, $\text{ind} : \Sigma(X) \to \mathbb{N}$, are G-invariant integer valued functions on the finite G-set $\Sigma(X)$. In this case we say that $\Sigma^\star(X) = (\Sigma(X), \text{sgn}, \text{ind})$ is the *signed indexed branching pattern* of X. We have an obvious notion of isomorphism for signed indexed branching patterns.
(3) Exercise 4.2.4(2) implies that if $X \in \mathcal{V}_0$ satisfies B1 and B4, then the sign function, $\text{sgn} : \Sigma(X) \to \{\pm 1\}$ is well-defined. Under these assumptions on X, we call $(\Sigma(X), \text{sgn})$ the *signed branching pattern* of X.

4.2.1. Stability of branching patterns.

DEFINITION 4.2.9. Let $X \in \mathcal{V}_0$.

(W) X is *weakly stable* and X has a *stable branching pattern* if
  (a) X satisfies the branching condition B1.
  (b) There exists an open neighbourhood $\mathcal{U}$ of X in $\mathcal{V}_0$ (C^∞-topology), such that all $Y \in \mathcal{U}$ satisfy the branching condition B1 and the isomorphism class of $\Sigma(Y)$ is constant on $\mathcal{U}$.

(S) X is *stable* and X has a *stable indexed branching pattern* if
  (a) X satisfies the branching conditions B1-B3.

(b) There exists an open neighbourhood $\mathcal{U}$ of X in $\mathcal{V}_0$ (C^∞-topology), such that all $Y \in \mathcal{U}$ satisfy the branching conditions B1-B3 and the isomorphism class of $\Sigma^\star(Y)$ is constant on $\mathcal{U}$.

We let $\mathcal{S} = \mathcal{S}(V,G)$ (respectively, $\mathcal{S}_w = \mathcal{S}_w(V,G)$) denote the open subsets of $\mathcal{V}_0$ consisting of stable (respectively, weakly stable) families. We prove in chapter 7 that $\mathcal{S}, \mathcal{S}_w$ are dense subsets of $\mathcal{V}_0$ (C^∞-topology). We also show that the branching pattern $\Sigma(X)$, regarded as the germ of a set in $V \times \mathbb{R}$, depends continuously on $X \in \mathcal{S}_w$. The proofs of these genericity and stability theorems are quite technical and depend on ideas from equivariant transversality. However, for the examples discussed in this chapter, the proof of genericity of stable branching patterns is elementary.

Exercise 4.2.10 (Pitchfork bifurcation, see example 4.2.7). Let $\mathbb{Z}_2$ act on $V = \mathbb{R}$ as multiplication by ± 1. Show that $\mathcal{S}(V, \mathbb{Z}_2)$ is the set of $X \in \mathcal{V}_0(V, \mathbb{Z}_2)$ such that $\frac{\partial^3 X}{\partial v^3}(0,0) \neq 0$. (Note we automatically have $\frac{\partial^2 X}{\partial v \partial \lambda}(0,0) = 1 \neq 0$.)

If $X \in \mathcal{S}$, then the signed indexed branching pattern $\Sigma^\star(X)$ contains all of the essential qualitative information on the germ of $X^{-1}(0)$ at $(0,0)$. Roughly speaking, the main aim of steady state equivariant bifurcation theory is to provide a classification and description of all possible (isomorphism classes of) signed indexed branching patterns for elements $X \in \mathcal{S}$. We carry out this program for a number of representations (V, G) including the standard representations of the symmetric, alternating and hyperoctahedral groups.

4.3. Symmetry breaking – the MISC

The branching pattern has the structure of a G-set. The next lemma makes precise the connection between the isotropy of elements in the branching pattern and isotropy of points on the associated solution branches.

Lemma 4.3.1. *Let $X \in \mathcal{V}_0$ and suppose that $\gamma = (\mathbf{x}, \lambda) : [0, \delta] \to V \times \mathbb{R}$ is a branch of simple zeros for X. Then*

(1) *$G_{\mathbf{x}(s)}$ is constant on $(0, \delta]$.*
(2) *$G_{\mathbf{x}(s)} = G_{[\gamma]}$, for all $s \in (0, \delta]$.*

Proof. Statement (1) follows from lemma 4.2.5. If H denotes the common value of $G_{\gamma(s)}$, then $H \subset G_{[\gamma]}$. Write $\mathbf{x}(s) = r(s)\mathbf{u}(s)$, $s \in [0, \delta']$, where $u(s) \in S(V)$, r is strictly monotone increasing on $[0, \delta']$ and $\delta' \in (0, \delta]$ (remark 4.2.3). If $g \in G_{[\gamma]}$, then $g\mathbf{x}(s) = \mathbf{x}(s)$, since otherwise the curve $\mathbf{x}$ would meet the sphere of radius $r(s)$ in more than one point, contradicting the monotonicity of r. Hence $g \in H$ and so $H \supset G_{[\gamma]}$. □

4.3.1. Symmetry breaking isotropy types.

Definition 4.3.2. An isotropy type $\tau \in \mathcal{O}(V, G)$ is *proper* if $\tau \neq (G)$. Let $\mathcal{O}^\star(V, G)$ denote the set of proper isotropy types.

DEFINITION 4.3.3. Let $X \in \mathcal{V}_0$. An isotropy type τ is *symmetry breaking for* X if τ is proper and for every open neighbourhood U of $(0,0)$ in $V \times \mathbb{R}$, there exists (x,λ) in $U \cap X^{-1}(0)$ such that $(G_x) = \tau$.

LEMMA 4.3.4. *Suppose $X \in \mathcal{V}_0$ satisfies the branching conditions B1 and B4. A proper isotropy type τ is symmetry breaking for X if and only if there exists $\alpha \in \Sigma(f)$ such that $(G_\alpha) = \tau$.*

The lemma follows immediately from B1 and lemma 4.3.1.

DEFINITION 4.3.5. An isotropy type τ is *symmetry breaking* (respectively, *generically symmetry breaking*) if τ is proper and there exists a non-empty open subset U of $\mathcal{V}_0$ (respectively, a dense open subset U of $\mathcal{V}_0$) such that τ is symmetry breaking for all $X \in U$.

DEFINITION 4.3.6. Let $\mathcal{F}$ be a nonempty subset of $\mathcal{O}(V,G)$. Then $\mathcal{F}$ is an *admissible family of symmetry breaking isotropy types* if there exists $X \in \mathcal{S}(V,G)$ such that $\mathcal{F} = \{\, (G_\alpha) \mid \alpha \in \Sigma(f) \,\}$.

REMARKS 4.3.7. (1) If $\mathcal{F}$ is an admissible family of symmetry breaking isotropy types, then there exists a non-empty open subset U of $\mathcal{S}(V,G)$ such that if $X \in U$, then $\mathcal{F} = \{\, (G_\alpha) \mid \alpha \in \Sigma(X) \,\}$.
(2) An isotropy type τ is generically symmetry breaking if and only if τ belongs to every admissible family $\mathcal{F}$.

4.3.2. Maximal isotropy subgroup conjecture.

DEFINITION 4.3.8. An isotropy type $\tau \in \mathcal{O}(V,G)$ is *maximal* if τ is a maximal element of the set of proper isotropy types.

EXAMPLE 4.3.9. If (V,G) is an absolutely irreducible G-representation and H is an isotropy group such that V^H is 1-dimensional, then (H) is a maximal isotropy type. Indeed, suppose $\mu > (H)$. By proposition 2.9.7, $\partial V_{(H)} \supset V_\mu$. Since $\partial V_{(H)} = G\partial(V^H \setminus \{0\}) = G0 = \{0\}$, $V_\mu = \{0\}$ and so $\mu = (G)$.

LEMMA 4.3.10 (The equivariant branching lemma [**173, 34**]). *Let (V,G) be an absolutely irreducible G-representation. Let $(H) \in \mathcal{O}(V,G)$ be such that V^H is one dimensional. Then (H) is generically symmetry breaking.*

PROOF. Let $X \in \mathcal{V}_0$. Since X_λ is tangent to V^H for all $\lambda \in \mathbb{R}$, X induces a family X^H_λ of smooth vector fields on V^H. For $(x,\lambda) \in V^H \times \mathbb{R}$, we have

$$X^H(x,\lambda) = \lambda x + g(x,\lambda),$$

where $g(x,\lambda) = O(x^2)$. Since $\frac{\partial g}{\partial x}(0,0) = g(0,0) = 0$, we may write $g(x,\lambda) = x^2 h(x,\lambda)$, where h is smooth. Hence the solutions to $X^H(x,\lambda) = 0$ are given by $\lambda x + x^2 h(x,\lambda) = 0$. That is, either $x = 0$ (the trivial solution) or $F(x,\lambda) = \lambda + xh(x,\lambda) = 0$. Now $F(0,0) = 0$, $\frac{\partial F}{\partial \lambda}(0,0) = 1$ and so, by the implicit function theorem, we have the smooth local solution curve $\lambda = \psi(x)$ to $F = 0$ defined on some neighbourhood of $0 \in V^H$. Note that $\lambda = \psi(x)$ defines two *branches* of solutions according to the definition of solution branch given earlier. The map $s \mapsto (s, \psi(s))$ is a smooth, certainly C^1, embedding. $\square$

Suppose that G is a (maximal) isotropy type. For the examples which were studied in the early stages of equivariant bifurcation theory, it was noticed that the symmetry breaking isotropy types were precisely the maximal isotropy types. It was suggested (see for example [**79**]) that this phenomenon might hold generally. In terms of symmetry breaking isotropy types, the resulting Maximal Isotropy Subgroup Conjecture can be formulated as follows:

Maximal Isotropy Subgroup Conjecture (MISC)

(1) *Every symmetry breaking isotropy type is maximal.*
(2) *Every maximal isotropy type is generically symmetry breaking.*

Eventually we shall show that both parts of the conjecture are false. Somewhat paradoxically, our verification that part (1) fails will start by proving that the conjecture holds for two infinite families of representations given by finite reflection groups. The analysis of these families, as well as related representations, will occupy most of the remainder of the chapter. However, before we start work on these families, we need a few more preliminaries on *determinacy* and polynomial invariants and equivariants.

4.4. Determinacy

We continue to assume that (V, G) is a finite dimensional absolutely irreducible orthogonal representation of the finite group G. To avoid the discussion of trivial special cases, we assume that $\dim(V) \geq 2$.

4.4.1. Polynomial maps. Let $P(V)$ denote the $\mathbb{R}$-algebra of real-valued polynomial functions on V and let $P(V, V)$ be the $P(V)$-module of all polynomial maps of V into V. We have

$$P(V) = \oplus_{k \geq 0} P^k(V) \ \text{ and } P(V, V) = \oplus_{k \geq 0} P^k(V, V),$$

where $P^k(V)$ (respectively, $P^k(V, V)$) is the space of homogeneous polynomials (respectively, homogeneous polynomial maps) of degree k. For $m \in \mathbb{N}$, set

$$P^{(m)}(V) = \oplus_{0 \leq k \leq m} P^k(V) \ \text{ and } P^{(m)}(V, V) = \oplus_{0 \leq k \leq m} P^k(V, V).$$

We let $P(V)^G$ be the $\mathbb{R}$-subalgebra of $P(V)$ consisting of all G-invariant polynomials and $P_G(V, V)$ be the $P(V)^G$-module of G-equivariant polynomial maps. We have

$$P(V)^G = \oplus_{k \geq 0} P^k(V)^G \ \text{ and } P_G(V, V) = \oplus_{k \geq 0} P_G^k(V, V).$$

We set $P(V, V)_0 = \oplus_{k \geq 2} P^k(V, V)$ and, for $m \geq 2$, define $P^{(m)}(V, V)_0$, $P_G(V, V)_0$ and $P_G^{(m)}(V, V)_0$.

A polynomial map $R : V \to V$ is *radial* if $R = hI_V$ for some $h \in P(V)$. We let $R(V,V) = \oplus_{k\geq 0} R^k(V,V)$ be the graded linear subspace of $P(V,V)$ consisting of radial polynomial maps. Define $R_G(V,V)$, $R^k_G(V,V)$, $R(V,V)_0$ similarly.

Since G is finite and $\dim(S(V) \geq 1$, G cannot act transitively on $S(V)$. Hence $P_G(V,V) \neq R_G(V,V)$. Let $d(V,G)$ denote the smallest integer d such that $P^d_G(V,V)$ is not contained in $R^d_G(V,V)$. Since (V,G) is an absolutely irreducible representation, $d(V,G) \geq 2$. We call $d(V,G)$ the *critical degree* of (V,G).

EXAMPLES 4.4.1. (1) Any polynomial in $\|x\|^2$ is G-invariant. Thus $\|x\|^{2n}$ is an invariant, $n \in \mathbb{N}$. If $p : \mathbb{R} \to \mathbb{R}$ is a polynomial, then $R(v) = p(\|v\|^2)v$ is a radial G-equivariant polynomial map.
(2) Take the standard action of $\mathbf{D}_n$ on $\mathbb{R}^2 \approx \mathbb{C}$, $n \geq 3$. The $\mathbb{R}$-algebra $P(\mathbb{C})^{\mathbf{D}_n}$ is generated by $|z|^2$ and $\mathrm{Re}(z^n)$. The $P(\mathbb{C})^{\mathbf{D}_n}$-module $P_{\mathbf{D}_n}(\mathbb{C},\mathbb{C})$ has generators $z \mapsto z$ and $z \mapsto \bar{z}^{n-1}$. In particular, given $P \in P_{\mathbf{D}_n}(\mathbb{C},\mathbb{C})$, there exist *unique* real polynomials $R(x,y)$, $S(x,y)$ such that

$$P(z) = R(|z|^2, \mathrm{Re}(z^n))z + S(|z|^2, \mathrm{Re}(z^n))\bar{z}^{n-1}.$$

The proof of this result is an elementary computation (details may be found in [**84**, Chapter XII, §4]). Both the uniqueness and the fact that the number of generators is equal to the dimension of the underlying space are quite atypical. Generally, one can expect many generators for the algebra of invariants and that these generators are algebraically dependent (see the following exercise).

EXERCISE 4.4.2. (1) Find the minimal number of generators for the $\mathbb{R}$-algebra $P(\mathbb{R}^n)^{\mathbb{Z}_2}$, $n \geq 1$. Find relations between the generators.
(2) Show that if $f \in P(V)^G$, then $\mathrm{grad}(f) \in P_G(V,V)$. (The gradient vector field of f is characterized by $(\mathrm{grad}(f)(v), X) = DF(v)(X)$, $v, X \in V$.)
(3) Find generators for $P(\mathbb{C})^{\mathbb{Z}}_n$, $P_{\mathbb{Z}_n}(\mathbb{C},\mathbb{C})$, standard action of $\mathbb{Z}_n$, $n > 2$.
(4) Show that $R^2_G(V,V) = \{0\}$ if $V^G = \{0\}$.

4.4.2. Finite determinacy. Suppose $X \in \mathcal{V}_0$ and $d \in \mathbb{N}$. We let $J^d(X) = j^d_1 X(0,0) = j^d X_0(0) \in P^{(d)}_G(V,V)_0$ denote the d-jet of X_0 at $0 \in V$. (The linear term of $J^d(X)$ vanishes as $X \in \mathcal{V}_0$. For background on jets see section 6.2.1 and note that we identify $j^d_1 X(0,0)$ with the Taylor polynomial of X_0 of order d at the origin.)

DEFINITION 4.4.3. Let $d \geq 2$. We say that (V,G) is *d-determined* (or, more precisely, that G-equivariant bifurcation problems on V are d-determined) if there exists a dense open subset $\mathcal{R}(d)$ of $P^{(d)}_G(V,V)_0$ such that if $X \in \mathcal{V}_0$ and $J^d(X) \in \mathcal{R}(d)$, then X is stable (that is, $X \in \mathcal{S}(V,G)$, if $J^d(X) \in \mathcal{R}(d)$).

REMARK 4.4.4. Using equivariant transversality, we show later that there exists $d \geq d(V,G)$ such that (V,G) is d-determined. In this chapter, we prove this result for certain families of representations using elementary techniques.

We may also give a definition of weak determinacy which takes account of the branching pattern but ignores stabilities and the direction of branching.

DEFINITION 4.4.5. Let $d \geq 2$. We say that (V, G) is *weakly d-determined* if there exists a dense open subset $\mathcal{R}_w(d)$ of $P_G^{(d)}(V, V)_0$ such that $X \in \mathcal{V}_0$ is weakly stable whenever $J^d(X) \in \mathcal{R}_w(d)$ (that is, $X \in \mathcal{S}_w(V, G)$ if $J^d(X) \in \mathcal{R}_w(d)$).

Suppose that (V, G) is d-determined and let $\mathcal{R}(d)$ be as in Definition 4.4.3. Given $S \in P_G^{(d)}(V, V)_0$, let $J^S \in \mathcal{V}_0$ be defined by

$$J^S(x, \lambda) = \lambda x + S(x), \quad ((x, \lambda) \in V \times \mathbb{R}). \tag{4.3}$$

LEMMA 4.4.6. *Let $X \in \mathcal{V}_0$.*

1. *If $J^d(X) \in \mathcal{R}(d)$, then X is stable and $\Sigma^\star(X)$ is isomorphic to $\Sigma^\star(J^S)$, where $S = J^3(X)$.*
2. *If X is stable, then the signed indexed branching pattern $\Sigma^\star(X)$ is isomorphic to $\Sigma^\star(J^S)$ for some $S \in \mathcal{R}(d)$.*

PROOF. Statement (1) is just the definition of $\mathcal{R}(d)$. Suppose that $X \in \mathcal{S}(V, G)$. Either $J^d(X) \in \mathcal{R}(d)$ and we may take $S = J^d(X)$ or $J^d(X) \notin \mathcal{R}(d)$. In the latter case, since X is stable, there exists an open neighbourhood W of $J^d(X)$ in $P_G^{(d)}(V, V)_0$ such that for all $S \in W \cap \mathcal{R}(d)$, X and J^S have isomorphic signed indexed branching patterns. This proves (2). $\square$

4.5. The hyperoctahedral family

In this section we analyse equivariant bifurcation problems for the hyperoctahedral groups H_k (also known as the *Weyl groups of type B_k*). For these groups we only need to deal with cubic equivariants and the computations are very easy.

4.5.1. The representations $(\mathbb{R}^k, H_k)$. For $k \geq 2$, we let H_k be the subgroup of $\mathrm{O}(k)$ consisting of all maps A of the form

$$A(x_1, \ldots, x_k) = (\pm x_{\sigma(1)}, \ldots, \pm x_{\sigma(k)}), \quad ((x_1, \ldots, x_k) \in \mathbb{R}^k),$$

where σ belongs to the symmetric group S_k. Thus H_k is the group of all signed permutation matrices and $|H_k| = 2^k k!$. We may represent H_k as the semi-direct product $\Delta_k \rtimes S_k$, where the symmetric group S_k is identified with the group of $k \times k$ permutation matrices, and the normal subgroup $\Delta = \Delta_k$ is the group of all diagonal $k \times k$ matrices with diagonal entries ± 1. Let $\pi_k : H_k \to S_k$ denote the surjective homomorphism corresponding to the semi-direct product decomposition $H_k = \Delta_k \rtimes S_k$. The group H_k is a *finite reflection group* – it is generated by reflections.

EXAMPLE 4.5.1. The group H_3 is the full symmetry group of the 3-dimensional cube (or octahedron). It is often denoted by O_h. The action of H_3 on $\mathbb{R}^3$ has three maximal isotropy types $(\mathbf{D}_4)$, $(\mathbf{D}_2)$, and $(\mathbf{D}_3)$ corresponding respectively

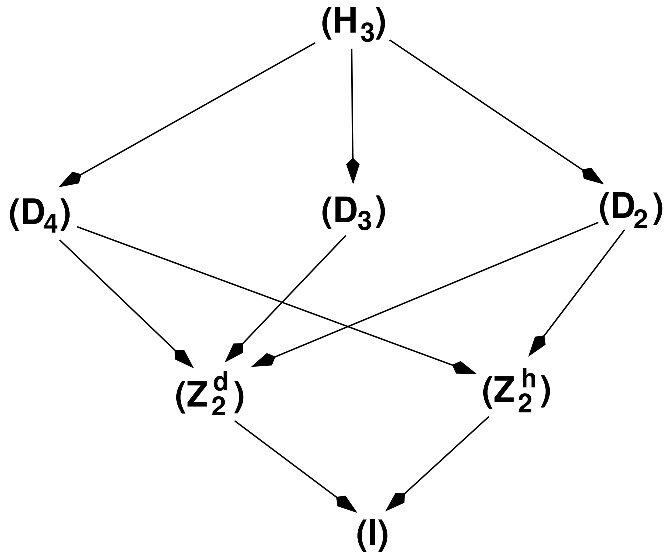


FIGURE 2. Isotropy types for $(H_3, \mathbb{R}^3)$

to axes joining mid-points of opposite faces, mid-points of opposite edges, and opposite vertices.

There are also two types of reflection plane: coordinate planes defined by $x_i = 0$, $i = 1, 2, 3$, and 'diagonal' planes defined by $x_i = x_j$, $1 \leq i < j \leq 3$. Coordinate planes contains axes with isotropy type $(\mathbf{D}_4)$ and $(\mathbf{D}_2)$ but not $(\mathbf{D}_3)$. Diagonal planes contain representatives of each type of axis. For example the plane $P_{112} = \{(s, s, t) \mid s, t \in \mathbb{R}\}$ contains the axes $\mathbb{R}(1, 1, 0)$, $\mathbb{R}(0, 0, 1)$ and $\mathbb{R}(1, 1, 1)$. Although generic points on both types of plane have isotropy groups isomorphic to $\mathbb{Z}_2$, the isotropy groups are *not* conjugate subgroups of H_3 and they define different isotropy types. We denote the isotropy type of generic points on coordinate planes $x_i = 0$ by $(\mathbb{Z}_2^h)$ and on diagonal planes by $(\mathbb{Z}_2^d)$. In figure 2, we show the order relations between the different isotropy types. We write $\tau \to \mu$ if $\tau > \mu$ and there does not exist η such that $\tau > \eta > \mu$.

We need a description of all the axes of symmetry of $(\mathbb{R}^k, H_k)$, $k \geq 2$. To this end, let $\mathcal{E}$ denote the set of all non-zero vectors $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k) \in \mathbb{R}^k$ such that each $\varepsilon_i \in \{0, +1, -1\}$. Given $\boldsymbol{\varepsilon} \in \mathcal{E}$, define the strictly positive integer $|\boldsymbol{\varepsilon}|$ by $|\boldsymbol{\varepsilon}| = \sum_{i=1}^k |\varepsilon_i|$. We call $|\boldsymbol{\varepsilon}|$ the *type* of $\boldsymbol{\varepsilon}$. If $\text{type}(\boldsymbol{\varepsilon}) = s$, we may write $\boldsymbol{\varepsilon} = \sum_{i=1}^s \delta_i \mathbf{e}_{j_i}$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ is the standard basis of $\mathbb{R}^k$, $\delta_i \in \{-1, +1\}$ and $1 \leq j_1 < \ldots < j_s \leq k$. For future reference, we define

$$\boldsymbol{\varepsilon}_j = \sum_{i=1}^j \mathbf{e}_i, \ \ 1 \leq j \leq k.$$

The axes of symmetry for H_k are the lines $L_\varepsilon = \mathbb{R}\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon} \in \mathcal{E}$. Write $\mathcal{E} = \dot{\bigcup}_{j=1}^{k} \mathcal{E}_j$, where $\mathcal{E}_j = \{\boldsymbol{\varepsilon} \mid |\boldsymbol{\varepsilon}| = j\}$. The set $\mathcal{E}$ has the structure of an H_k-set and H_k acts transitively on each $\mathcal{E}_j$. Non-zero points on L_ε have isotropy group isomorphic to $S_j \times H_{k-j}$, where $j = |\boldsymbol{\varepsilon}|$. It follows that $(\mathbb{R}^k, H_k)$ has exactly k maximal isotropy types: $(H_{k-1}), (S_2 \times H_{k-2}), \dots, (S_k)$.

4.5.2. Invariants and equivariants for H_k. Define the elementary symmetric functions $\{\sigma_i\}$ by $\sigma_i = \sum_{1 \le \ell_1 < \dots < \ell_i \le k} x_{\ell_1} \dots x_{\ell_i}$, $1 \le i \le k$. It is a basic result of classical invariant theory that every $p \in P(\mathbb{R}^k)^{S_k}$ can be written uniquely as a polynomial in the elementary symmetric functions. That is, the elementary symmetric functions form a *basis* for the $\mathbb{R}$-algebra $P(\mathbb{R}^k)^{S_k}$. If we define $\sigma_i^\star = \sum_{j=1}^k x_j^i$, $1 \le i \le k$, then it is straightforward to verify that each σ_i can be written (uniquely) as a polynomial in $\sigma_1^\star, \dots, \sigma_i^\star$, $1 \le i \le k$. Hence $\sigma_1^\star, \dots, \sigma_k^\star$ also define a basis for $P(\mathbb{R}^k)^{S_k}$.

Let $\psi_i = \frac{1}{i}\mathrm{grad}(\sigma_i^\star)$, $1 \le i \le k$. It follows from classical invariant theory that $\psi_1, \dots, \psi_k$ define a basis for the $P(\mathbb{R}^k)^{S_k}$-module of S_k-equivariant maps $P_{S_k}(\mathbb{R}^k, \mathbb{R}^k)$. That is, for all $Q \in P_{S_k}(\mathbb{R}^k, \mathbb{R}^k)$, there exists unique $q_i \in P(\mathbb{R}^k)^{S_k}$ such that

$$Q(x) = \sum_{i=1}^k q_i(x)\psi_i(x), \quad (x \in \mathbb{R}^k).$$

For $1 \le i \le k$, define $\eta_i \in P(\mathbb{R}^k)$, $\phi_i \in P(\mathbb{R}^k, \mathbb{R}^k)$ by

$$\begin{aligned}\eta_i(x_1, \dots, x_k) &= \sum_{j=1}^k x_j^{2i},\\ \phi_i(x_1, \dots, x_k) &= (x_1^{2i-1}, \dots, x_k^{2i-1}) = \frac{1}{2i}\mathrm{grad}(\eta_i)(x_1, \dots, x_k).\end{aligned}$$

Clearly, $\eta_i \in P(\mathbb{R}^k)^{H_k}$ and $\phi_i \in P_{H_k}(\mathbb{R}^k, \mathbb{R}^k)$.

LEMMA 4.5.2. (1) *Every $p \in P(\mathbb{R}^k)^{H_k}$ may be written uniquely as a polynomial in $\eta_1, \dots, \eta_k$. ($\eta_1, \dots, \eta_k$ is a* basis *for the $\mathbb{R}$-algebra $P(\mathbb{R}^k)^{H_k}$.)*

(2) *Every $Q \in P_{H_k}(\mathbb{R}^k, \mathbb{R}^k)$ may be written uniquely in the form*

$$Q = \sum_{j=1}^k q_j \phi_j,$$

where $q_j \in P(\mathbb{R}^k)^{H_k}$, $1 \le j \le k$.

PROOF. Let $p \in P(\mathbb{R}^k)^{H_k}$. Since $\Delta_k \subset H_k$, there exists a unique $P \in P(\mathbb{R}^k)$ such that $p(x_1, \dots, x_k) = P(x_1^2, \dots, x_k^2)$. Since $S_k \subset H_k$, p must be a symmetric function of $(x_1, \dots, x_k)$ and so P is symmetric in $x_1^2, \dots, x_k^2$. Hence we may write P uniquely as $Q(\eta_1, \dots, \eta_k)$, where $Q \in P(\mathbb{R}^k)$, proving (1). The proof of (2) is similar, using the corresponding result for S_k-equivariant polynomial maps. $\square$

REMARK 4.5.3. Lemma 4.5.2 implies that $P^{2d}_{H_k}(\mathbb{R}^k, \mathbb{R}^k) = \{0\}$, $d \geq 1$.

4.5.3. Cubic equivariants for H_k. We eventually show that H_k-equivariant bifurcation problems are 3-determined. We assume this for the present and consider cubic H_k-equivariant normalized families on $\mathbb{R}^k$. By lemma 4.5.2, the vector space $P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k)$ is two dimensional with basis R, C where

$$\begin{aligned} R(x) &= ||x||^2 x, \\ C(x) &= (x_1^3, \dots, x_k^3), \quad (x = (x_1, \dots, x_k) \in \mathbb{R}^k). \end{aligned}$$

Since R is radial and C is not radial $d(\mathbb{R}^k, H_k) = 3$. We call C is the *basic cubic equivariant* for $(\mathbb{R}^k, H_k)$.

If $X \in \mathcal{V}_0$, then there exist unique $a, b \in \mathbb{R}$ such that

$$J^3(X) = aR + bC \in P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k).$$

4.5.4. Bifurcation for cubic families. We consider families in $\mathcal{V}_0$ of the form

$$X_{a,b}(x, \lambda) = \lambda x + aR + bC,$$

where $a, b \in \mathbb{R}$. Note that $X_{a,b} = \mathrm{grad}(Q)$, where

$$Q(x) = \lambda \frac{1}{2}\|x\|^2 + \frac{a}{4}\|x\|^4 + \frac{b}{4}(\sum_{i=1}^{4} x_i^4).$$

PROPOSITION 4.5.4. *Let $j \in \{1, \dots, k\}$. If $b, ja + b \neq 0$, then associated to every $\boldsymbol{\varepsilon} \in \mathcal{E}_j$, there exists a curve $\gamma_{\boldsymbol{\varepsilon}}$ of hyperbolic solutions for $X_{a,b} = 0$ along $L_{\boldsymbol{\varepsilon}}$.*

1. *If $b > 0$, and $ja + b > 0$, then $\mathrm{ind}(\gamma_{\boldsymbol{\varepsilon}}) = j - 1$ and $sgn(\gamma_{\boldsymbol{\varepsilon}}) = -1$.*
2. *If $b > 0$, and $ja + b < 0$, then $\mathrm{ind}(\gamma_{\boldsymbol{\varepsilon}}) = j$ and $sgn(\gamma_{\boldsymbol{\varepsilon}}) = 1$.*
3. *If $b < 0$, and $ja + b > 0$, then $\mathrm{ind}(\gamma_{\boldsymbol{\varepsilon}}) = k - j$ and $sgn(\gamma_{\boldsymbol{\varepsilon}}) = -1$.*
4. *If $b < 0$, and $ja + b < 0$, then $\mathrm{ind}(\gamma_{\boldsymbol{\varepsilon}}) = k - j + 1$ and $sgn(\gamma_{\boldsymbol{\varepsilon}}) = 1$.*

In particular, if $ja + b, b \neq 0$, $0 \leq j \leq k$, there are a total of $3^n - 1$ solution branches of $X_{a,b} = 0$. Each branch will have maximal isotropy type, be hyperbolic and either supercritical or subcritical. Aside from the trivial solution curve, there will be no other solutions to $X_{a,b} = 0$.

PROOF. Statements (1–4) follow from a straightforward, if lengthy, computation which we leave to the reader (see [**57**] for details). The assertions about the number and type of solutions are easy computations. ☐

COROLLARY 4.5.5. *Suppose that $ja + b, b \neq 0$, $0 \leq j \leq k$, and let $\boldsymbol{\varepsilon} \in \mathcal{E}$. Then $\gamma_{\boldsymbol{\varepsilon}}$ is is a branch of hyperbolic attractors or repellors only if $\boldsymbol{\varepsilon} \in \mathcal{E}_1 \cup \mathcal{E}_k$. In particular, if $\boldsymbol{\varepsilon} \in \mathcal{E}_j$, $j \neq 1, k$, then $\gamma_{\boldsymbol{\varepsilon}}$ is always a branch of hyperbolic saddles.*

REMARK 4.5.6. Since $X_{a,b}$ is a gradient vector field, $X_{a,b}$ has no recurrent trajectories such as limit cycles. Proposition 4.5.4 and corollary 4.5.5 give a complete qualitative picture of the (global) dynamics before and after the bifurcation.

4.5.5. Subgroups of H_k. We consider subgroups G of H_k satisfying the conditions

(IR) $(\mathbb{R}^k, G)$ is absolutely irreducible.
(C) $P^2_G(\mathbb{R}^k, \mathbb{R}^k) = \{0\}$.

Remark 4.5.7. A sufficient condition for (C) to hold is that $-I \in G$. This condition is definitely not necessary.

Lemma 4.5.8. *Let G be a subgroup of H_k and set $\pi_k(G) = \Gamma \subset S_k$. Then $(\mathbb{R}^k, G)$ is absolutely irreducible if and only if the following two conditions hold.*

1. *Γ is a transitive subgroup of S_k,*
2. *For every j with $1 < j \leq k$, there exists $g \in G$ and $\delta = \pm 1$ such that $g\mathbf{e}_1 = \delta\mathbf{e}_1$ and $g\mathbf{e}_j = -\delta\mathbf{e}_j$.*

Proof. It is clear that if $\mathbb{R}^k$ is absolutely irreducible, then Γ is a transitive subgroup of S_k. Assume that Γ is a transitive subgroup of S_k. Let $H = \{g \in G \mid g\mathbf{e}_1 = \pm\mathbf{e}_1\}$. Since Γ is transitive on $\{1, \ldots, k\}$, $\mathbb{R}^k$ is the induced G-representation $\mathrm{Ind}^G_H(\mathbb{R}\mathbf{e}_1)$ ($\cong \oplus_{\sigma \in G/H} \sigma\mathbb{R}\mathbf{e}_1$). The result now follows from the Mackey criterion for irreducibility [**158**], [**169**, Exercise 6, page 44]. $\square$

Remark 4.5.9. For the examples we consider it will usually be easy to verify conditions (IR,C) directly.

Examples 4.5.10. (1) If G is a subgroup of H_k, let $G' = \{g \in G \mid \det(g) = 1\}$. If $k \geq 3$, $(\mathbb{R}^k, H'_k)$ and $(\mathbb{R}^k, \Delta'_k \rtimes S_k)$ are absolutely irreducible. Neither representation admits any non-trivial quadratic equivariants. Both statements are easy to verify directly without recourse to lemma 4.5.8.
(2) Let G be any transitive subgroup of S_k. Then $(\mathbb{R}^k, \Delta_k \rtimes G)$ satisfies conditions (IR) and (C) for $k \geq 3$. If $k \geq 4$, $(\mathbb{R}^k, \Delta'_k \rtimes G)$ satisfies conditions (IR) and (C). These statements are easy to verify by direct computation. Note that $(\mathbb{R}^3, \Delta'_3 \rtimes \mathbb{Z}_3)$ does not satisfy condition (C) (it does satisfy (IR)).

Theorem 4.5.11. *Suppose that G is a subgroup of H_k and that $(\mathbb{R}^k, G)$ satisfies conditions (IR), (C).*

1. *$(\mathbb{R}^k, G)$ is 3-determined.*
2. *For every $\varepsilon \in \mathcal{E}$, (G_ε) is a symmetry breaking isotropy type.*
3. *If H is a subgroup of G and $(\mathbb{R}^k, H)$ satisfies conditions (IR), (C) then $\mathcal{S}(\mathbb{R}^k, G) \subset \mathcal{S}(\mathbb{R}^k, H)$.*

We give the proof of this result in section 4.8. For the remainder of this section, we show how theorem 4.5.11 yields many counterexamples to the Maximal Isotropy Subgroup Conjecture.

Example 4.5.12. Let $G = \Delta'_k \rtimes S_k$. If $\boldsymbol{\varepsilon} \in \mathcal{E}_{k-1}$, then $(G_{\boldsymbol{\varepsilon}})$ is a *submaximal* symmetry breaking isotropy type for $(\mathbb{R}^k, G)$, $k \geq 4$. To see this, observe that if $k \geq 4$, G acts transitively on $\mathcal{E}_{k-1}$. Hence, by theorem 4.5.11(2), it suffices to check that the isotropy group of $\boldsymbol{\varepsilon} = (1, 1, \ldots, 1, 0)$ is submaximal. Let $P = \{(x, x, \ldots, x, y) \mid x, y \in \mathbb{R}\}$. The plane P contains the three lines $\mathbb{R}(1, \ldots, 1, \pm 1)$ and $\mathbb{R}\mathbf{e}_k$ which are axes of symmetry for G. Non-zero points on these lines have maximal isotropy. On the other hand, $\boldsymbol{\varepsilon}$ has the same isotropy group S_{k-1} as any point $(x, x, \ldots, x, y) \in P$ not lying on these lines. Hence, by proposition 2.9.7, $G_{\boldsymbol{\varepsilon}}$ is submaximal. Alternatively, it is easy to check directly that $G_{\boldsymbol{\varepsilon}}$ is a subgroup of $G_{\mathbf{e}_k}$. We may also show that $(G_{\boldsymbol{\varepsilon}})$ is generically symmetry breaking (see also [**70**]).

For $k \geq 4$, $G = \Delta_k' \rtimes S_k$ is a finite reflection group (the *Weyl group of type* D_k).

EXERCISE 4.5.13. Let $G = \Delta_k' \rtimes S_k$. Show that if $k \geq 4$ and $\boldsymbol{\varepsilon} \in \mathcal{E}_j$, $j \neq k-1$, then $G_{\boldsymbol{\varepsilon}}$ is a maximal isotropy subgroup.

EXAMPLE 4.5.14 (A counterexample to the MISC in dimension 3). Suppose that $G = \Delta_3 \rtimes \mathbb{Z}_3 \subset H_3$. Clearly $(\mathbb{R}^3, G)$ satisfies conditions (C,IR) so theorem 4.5.11 applies. Let $\boldsymbol{\varepsilon}_2 = (1, 1, 0)$ and $\boldsymbol{\varepsilon}_1 = (1, 0, 0)$. One checks easily that $G_{\boldsymbol{\varepsilon}_2} = \{\mathrm{diag}(1, 1, \pm 1)\} \subset \Delta_3$ and $G_{\boldsymbol{\varepsilon}_1} = \{\mathrm{diag}(1, \pm 1, \pm 1)\}$, so that $G_{\boldsymbol{\varepsilon}_2}$ is properly contained in $G_{\boldsymbol{\varepsilon}_1}$. Hence $(G_{\boldsymbol{\varepsilon}_2})$ is a submaximal symmetry breaking isotropy type. This is the only example in $\mathbb{R}^3$ for which the MISC fails.

EXAMPLES 4.5.15 (Examples for which the trivial isotropy type is symmetry breaking). (1) Let $G = \mathbb{Z}_k \rtimes \Delta_k' \subset H_k$, $k \geq 4$. Then G satisfies conditions (IR,C). The isotropy type $(G_{\boldsymbol{\varepsilon}_{k-1}})$ is symmetry breaking. We find that $G_{\boldsymbol{\varepsilon}_{k-1}} = \{1\}$ and so the trivial isotropy type $(\{1\})$ is symmetry breaking for $(\mathbb{R}^k, G)$. Now let $k = 4$. Then the isotropy subgroup $G_{\boldsymbol{\varepsilon}_2}$ is of order two. It is clear that $G_{\boldsymbol{\varepsilon}_2}$ is properly contained in $G_{\boldsymbol{\varepsilon}_1}$. Thus $(G_{\boldsymbol{\varepsilon}_2})$ is another submaximal symmetry breaking isotropy type. Let $\boldsymbol{\varepsilon} = (1, 0, 1, 0)$. Then $\boldsymbol{\varepsilon}$ is of type 2. The isotropy subgroup $G_{\boldsymbol{\varepsilon}}$ is of order four and one can check that it is a maximal isotropy subgroup. Thus $\boldsymbol{\varepsilon}_2$ and $\boldsymbol{\varepsilon}$ are of the same type, but in one case the isotropy subgroup is submaximal and in the other case it is maximal.
(2) Let $K < S_4$ be the Abelian 2-group: $K = \{1, (12)(34), (13)(24), (14)(23)\}$. Set $G = \Delta_4' \rtimes K \subset H_4$. Then G satisfies conditions (IR,C) so that $(G_{\boldsymbol{\varepsilon}})$ is a symmetry breaking isotropy type for every $\boldsymbol{\varepsilon} \in \mathcal{E}$. One checks that $G_{\boldsymbol{\varepsilon}_3} = \{1\}$, so again the trivial isotropy type is symmetry breaking.

EXAMPLE 4.5.16 (A branch of sinks with submaximal isotropy). We consider a slightly more complicated version of examples 4.5.15(1). Let $k = 4$, and set $G = \{g \in \Delta_4 \rtimes \mathbb{Z}_4 \mid \det(g) = 1\}$. Then $(\mathbb{R}^4, G)$ satisfies conditions (IR,C). Let $\boldsymbol{\kappa} = (1, 0, 1, 0)$. Then $\{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_4, \boldsymbol{\kappa}\}$ is a set of representatives for the orbits of G on $\mathcal{E}$. Let $G_j = G_{\boldsymbol{\varepsilon}_j}$, $j \in \mathbf{4}$, and $G_5 = G_{\boldsymbol{\kappa}}$. Then $\{(G_1), \ldots, (G_5)\}$ is an admissible family of symmetry breaking isotropy types for $(\mathbb{R}^4, G)$. Some easy calculations show that : (a) $G_1 \cong \mathbb{Z}_2^2$; (b) $G_2 \subset G_1$ and $G_2 \cong \mathbb{Z}_2$; (c) $G_3 = \{1\}$; (d) $G_4 \cong \mathbb{Z}_2$; and (e) $G_4 \subset G_5$ and $G_5 \cong \mathbb{Z}_2^2$. Thus (G_2), (G_3), and (G_4) are submaximal symmetry breaking isotropy types. It is not difficult to show that (G_1) and (G_5) are maximal isotropy types. Since $\boldsymbol{\varepsilon}_4$ is of type 4, it follows from proposition 4.5.4

and theorem 4.5.11(3) that there exists $X \in \mathcal{S}(\mathbb{R}^4, G)$ and $\beta \in \Sigma(X)$ such that β is of maximal index 4 and $G_\beta = G_4$. Thus we obtain a stable solution branch of sinks with submaximal isotropy type.

4.5.6. Some subgroups of the symmetric group. We recall that a subgroup Γ of the symmetric group S_k is *doubly transitive* if for all ordered pairs (i,j), (p,q) of elements of $\mathbf{k} = \{1,\ldots,k\}$ such that $i \neq j$ and $p \neq q$, there exists $\sigma \in \Gamma$ such that $\sigma(i) = p$ and $\sigma(j) = q$. One can similarly define r-transitive subgroups of S_k for every $r \leq k$. The symmetric group S_k is r-transitive for every $r \leq k$. The alternating group A_k is r-transitive for $r \leq k-2$.

Let $\mathbb{F} = \mathbb{F}_q$ be a finite field with q elements. Recall that $\mathrm{Aff}_1(\mathbb{F})$ is the group of automorphisms of the affine line $\mathbb{F}$ and consists of all bijections σ of $\mathbb{F}$ which are of the form $\sigma(x) = ax + b$, $a, b \in \mathbb{F}$, $a \neq 0$. If we identify $\mathrm{Aff}_1(\mathbb{F})$ with a subgroup of the symmetric group S_q, then $\mathrm{Aff}_1(\mathbb{F})$ is a doubly transitive subgroup of S_q. More precisely, $\mathrm{Aff}_1(\mathbb{F})$ acts simply transitively on the set of all ordered pairs of distinct elements of $\mathbb{F}$. Let $\mathrm{PGL}_2(\mathbb{F})$ denote the group of all automorphisms of the projective line $\mathbb{P}^1(\mathbb{F})$ $(= \mathbb{F} \cup \{\infty\})$. Then $\mathrm{PGL}_2(\mathbb{F})$ can be identified with a subgroup of the symmetric group S_{q+1}. It is known from elementary projective geometry that $\mathrm{PGL}_2(\mathbb{F})$ acts simply transitively on the set of all ordered triples of distinct elements of $\mathbb{P}^1(\mathbb{F})$ [**157**, 10.6.8].

4.5.7. A big family of counterexamples to the MISC. Let $k \geq 4$, Γ be a transitive subgroup of S_k and let $G = \Delta_k \rtimes \Gamma \subset H_k$. Then G satisfies conditions (C,IR). Thus $\{(G_\varepsilon) \mid \boldsymbol{\varepsilon} \in \mathcal{E}\}$ is an admissible family of symmetry breaking isotropy types. If J is a subset of $\mathbf{k}$, we set $\Gamma_J = \{\sigma \in \Gamma \mid \sigma(J) = J\}$.

LEMMA 4.5.17. *Let $G = \Delta_k \rtimes \Gamma$ be as above. Then the following two conditions are equivalent:*

(1) *For every $\boldsymbol{\varepsilon} \in \mathcal{E}$, $G_{\boldsymbol{\varepsilon}}$ is a maximal isotropy subgroup for $(\mathbb{R}^k, G)$.*
(2) *For every subset J of $\mathbf{k}$, the group Γ_J acts transitively on $\mathbf{k} \setminus J$.*

PROOF. (1) $\Rightarrow$ (2) Assume (2) false and let J be a subset of $\mathbf{k}$ such that Γ_J has at least two orbits on $\mathbf{k} \setminus J$. We may write $\mathbf{k} \setminus J$ as the disjoint union of two non-empty subsets, L and K, each of which is stable under the action of Γ_J. Define elements $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k)$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ of $\mathcal{E}$ by: $\varepsilon_i = 0$ if $i \in J$ and $\varepsilon_i = 1$ if $i \notin J$; and $\delta_i = 0$ if $i \notin L$ and $\delta_i = 1$ if $i \in L$. It is easy to check that $G_{\boldsymbol{\varepsilon}}$ is a proper subgroup of G_δ.

(2) $\Rightarrow$ (1) Assume that (2) holds. It will suffice to show that for every $\boldsymbol{\varepsilon} \in \mathcal{E}$, the line $\mathbb{R}\boldsymbol{\varepsilon}$ is an axis of symmetry for G. Let $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k) \in \mathcal{E}$ and let $J = \{i \mid \varepsilon_i \neq 0\}$. Define $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ by: $\delta_i = 0$ for $i \notin J$ and $\delta_i = 1$ for $i \in J$. It is clear that there exists $d \in \Delta_k$ such that $d\boldsymbol{\varepsilon} = \boldsymbol{\delta}$, and thus it suffices to show that $\mathbb{R}\boldsymbol{\delta}$ is an axis of symmetry for G. But this follows easily from (2). $\square$

THEOREM 4.5.18. *Let $k \geq 4$, Γ be a transitive subgroup of S_k and $G = \Delta_k \rtimes \Gamma$. Assume that Γ is not equal to the symmetric group S_k or the alternating group A_k*

and, if $k = 6$, that Γ is not conjugate to $PGL_2(\mathbb{F}_5)$. Then there exists $\boldsymbol{\varepsilon} \in \mathcal{E}$ such that the isotropy subgroup $G_{\boldsymbol{\varepsilon}}$ is not a maximal isotropy subgroup. Thus $(G_{\boldsymbol{\varepsilon}})$ is a submaximal symmetry breaking isotropy type and the MISC fails for $(\mathbb{R}^k, G)$.

Proof. Let Γ be a transitive subgroup of S_k which satisfies the following condition:

(T) *For every subset J of $\mathbf{k}$, the stabilizer Γ_J acts transitively on $\mathbf{k} \setminus J$.*

In order to prove the theorem, we must show that Γ is either S_k, A_k or, for the case $k = 6$, a conjugate to $\mathrm{PGL}_2(\mathbb{F}_5)$. A subgroup Γ of S_k is *p-homogeneous* (where $1 \leq p \leq k$) if Γ acts transitively on the set of p-element subsets of $\mathbf{k}$. An easy argument by induction on p shows that condition (T) implies that Γ is p-homogeneous for every $p \in \mathbf{k}$. By a theorem of Beaumont and Peterson [**9**], this implies that Γ must be one of the following groups: (a) S_k; (b) A_k; (c) $k = 5$ and Γ is conjugate to $\mathrm{Aff}_1(\mathbb{F}_5)$; (d) $k = 6$ and Γ is conjugate to $\mathrm{PGL}_2(\mathbb{F}_5)$; (e) $k = 9$ and Γ is conjugate to $\mathrm{PSL}_2(\mathbb{F}_8)$; (f) $k = 9$ and Γ is conjugate to $\mathrm{PGL}_2(\mathbb{F}_8)$. But it follows easily from condition (T) that p divides $|\Gamma|$ for every $p \in \mathbf{k}$. This rules out A_3 and the groups of (c), (e) and (f). On the other hand, the groups A_k $(k \geq 4)$, S_k and $\mathrm{PGL}_2(\mathbb{F}_5)$ satisfy condition (T). □

Example 4.5.19 (An example where all isotropy types are symmetry breaking). Let $k = 5$ and let $G = \Delta_5' \rtimes \mathbb{Z}_5$. We define vectors $\boldsymbol{\kappa}_i \in \mathcal{E}$, $i \in \mathbf{3}$, by $\boldsymbol{\kappa}_1 = (1, 0, 1, 0, 0)$, $\boldsymbol{\kappa}_2 = (1, 0, 1, 0, 1)$, $\boldsymbol{\kappa}_3 = (1, 1, 1, 1, -1)$. Then $\{\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_5, \boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2, \boldsymbol{\kappa}_3\}$ is a set of representatives for the orbits of G on $\mathcal{E}$. Let $G_j = G_{\boldsymbol{\varepsilon}_j}$ for $j \in \mathbf{5}$ and let $G_{5+j} = G_{\boldsymbol{\kappa}_j}$ for $j \in \mathbf{3}$. A direct computation shows that $(G_5) = (G_8)$. By theorem 4.5.11, the family $\{(G_1), \ldots, (G_7)\}$ is an admissible family of symmetry breaking isotropy types. One can check by direct calculation that, for every $x \in \mathbb{R}^5$, the isotropy type (G_x) is equal to (G_j) for some $j \in \mathbf{7}$. Thus every isotropy type for $(\mathbb{R}^5, G)$ occurs in the admissible family $\{(G_j)\}$. In particular, every isotropy type for $(\mathbb{R}^5, G)$ is symmetry breaking. By straightforward calculations, we obtain: (a) $G_1 \cong \mathbb{Z}_2^3$; (b) $G_2 \cong \mathbb{Z}_2^2$; (c) $G_3 \cong \mathbb{Z}_2$; (d) $G_4 = \{1\}$; (e) $G_5 \cong \mathbb{Z}_5$; (f) $G_6 \cong \mathbb{Z}_2^2$; and (g) $G_7 \cong \mathbb{Z}_2$. The isotropy type (G_4) is the trivial isotropy type. We have the following relations between the other isotropy types: (1) $(G_2) < (G_1)$ and $(G_6) < (G_1)$; (2) $(G_3) < (G_2)$ and $(G_3) < (G_6)$; (3) $(G_7) < (G_2)$ and $(G_7) < (G_6)$. The isotropy types (G_1) and (G_5) are maximal isotropy types and the other isotropy types are submaximal.

4.5.8. Examples where $P^3_G(\mathbb{R}^k, \mathbb{R}^k) = P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k)$.

Proposition 4.5.20. *Let $G \subset H_k$ satisfy conditions (C,IR) and assume that $P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k) = P^3_G(\mathbb{R}^k, \mathbb{R}^k)$. Let $X \in \mathcal{V}_0(\mathbb{R}^k, G)$. Define $X^\times \in \mathcal{V}_0(\mathbb{R}^k, H_k)$ by $X^\times(x, \lambda) = \lambda x + J^3(X)(x)$. Then $X \in \mathcal{S}(\mathbb{R}^k, G)$ if and only if $X^\times \in \mathcal{S}(\mathbb{R}^k, H_k)$. If $X \in \mathcal{S}(\mathbb{R}^k, G)$, then*

1. *The signed indexed branching pattern $\Sigma^\star(X)$ is isomorphic (as a G-set) to $\Sigma^\star(X^\times)$.*

(2) *The family* $\{(G_\varepsilon) \mid \boldsymbol{\varepsilon} \in \mathcal{E}\}$ *is the unique admissible family of symmetry breaking isotropy types for* $(\mathbb{R}^k, G)$ *and each isotropy type* (G_ε) $(\boldsymbol{\varepsilon} \in \mathcal{E})$ *is generically symmetry breaking.*

PROOF. The result follows from theorem 4.5.11. □

The following lemma gives sufficient conditions for the hypotheses of proposition 4.5.20 to be satisfied.

LEMMA 4.5.21. *Let* Γ *be a doubly transitive subgroup of* S_k. *Assume that either: (i)* $k \geq 4$ *and* $G = \Delta_k \rtimes \Gamma$*; or (ii)* $k \geq 5$ *and* $G = \Delta'_k \rtimes \Gamma$. *Then* G *satisfies conditions (C,IR) and* $P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k) = P^3_G(\mathbb{R}^k, \mathbb{R}^k)$.

PROOF. The proof follows by an easy direct computation of cubic equivariants. We leave the details to the reader. □

EXAMPLES 4.5.22. (1) Let $G = \Delta_k \rtimes A_k$, $k \geq 4$. It follows from lemma 4.5.21 that the hypotheses of proposition 4.5.20 hold. In this case, one can verify that each isotropy subgroup G_ε $(\boldsymbol{\varepsilon} \in \mathcal{E})$ is maximal and so $(\mathbb{R}^k, G)$ satisfies the MISC.
(2) If $G = \Delta'_k \rtimes A_k$, $k \geq 5$, then the conclusions of proposition 4.5.20 hold and all of the isotropy subgroups G_ε $(\boldsymbol{\varepsilon} \in \mathcal{E})$ are generically symmetry breaking. However, if $\boldsymbol{\varepsilon} \in \mathcal{E}_{k-1}$ then (G_ε) is submaximal and G does not satisfy the MISC.
(3) Let Γ denote the subgroup $\mathrm{Aff}_1(\mathbb{F}_5)$ of S_5. Let $G = \Delta'_5 \rtimes \Gamma \subset H_5$. Then $(\mathbb{R}^5, G)$ satisfies the hypotheses of proposition 4.5.20. As representatives for the G-orbits on $\mathcal{E}$, we may choose $\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_5$ and $-\boldsymbol{\varepsilon}_5$. Set $G_j = G_{\boldsymbol{\varepsilon}_j}$, $j \in \mathbf{5}$. Clearly $G_5 = G_{-\boldsymbol{\varepsilon}_5}$. Thus $\{(G_1), \dots (G_5)\}$ is the unique admissible family of symmetry breaking isotropy types and each of these isotropy types is generically symmetry breaking. We have: (a) $G_1 \cong \mathbb{Z}_2^3 \rtimes \mathbb{Z}_4$; (b) $G_2 \cong \mathbb{Z}_2^3$; (c) $G_3 \cong \mathbb{Z}_2^2$; (d) $G_4 \cong \mathbb{Z}_4$; and (e) $G_5 \cong \mathbb{Z}_5$. The isotropy types (G_2), (G_3) and (G_4) are submaximal isotropy types. The isotropy types (G_1) and (G_5) are maximal isotropy types.

One can vary this example by letting Γ be as above and taking $H = \Delta_5 \rtimes \Gamma$. Then $(\mathbb{R}^5, H)$ satisfies the hypotheses of proposition 4.5.20. In this case $\boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\varepsilon}_5$ is a set of representatives for the orbits of H on $\mathcal{E}$. Let $H_i = H_{\varepsilon_i}$, $i \in \mathbf{5}$. Then $\{(H_1), \dots, (H_5)\}$ is the unique admissible family of symmetry breaking isotropy types. The isotropy subgroup H_3 is isomorphic to $\mathbb{Z}_2^3$ and (H_3) is a submaximal isotropy type. The other isotropy types (H_i), $i \in \mathbf{4}$, are maximal isotropy types.

One has similar constructions for every finite field $\mathbb{F}_q$. Let $\Gamma = \mathrm{Aff}_1(\mathbb{F}_q)$ and $G = \Delta_q \rtimes \Gamma$. Then Γ is a doubly transitive subgroup of S_q. If $q \geq 4$, $(\mathbb{R}^q, G)$ satisfies the hypotheses of proposition 4.5.20. Thus $\{(G_\varepsilon) \mid \boldsymbol{\varepsilon} \in \mathcal{E}\}$ is the unique admissible family of symmetry breaking isotropy types and each isotropy type (G_ε), $\boldsymbol{\varepsilon} \in \mathcal{E}$ is generically symmetry breaking. Applying theorem 4.5.18, there exists $\boldsymbol{\varepsilon} \in \mathcal{E}$ such that the isotropy type (G_ε) is a submaximal isotropy type (which is generically symmetry breaking).

One can also make these constructions using the finite projective groups. Let q be a prime power and let $\Gamma = \mathrm{PGL}_2(\mathbb{F}_q) \subset S_{q+1}$. Let $G = \Delta_{q+1} \rtimes \Gamma \subset H_{q+1}$. Then $(\mathbb{R}^{q+1}, G)$ satisfies the hypotheses of proposition 4.5.20. It follows from theorem 4.5.18 that if $q > 5$ there exists $\boldsymbol{\varepsilon} \in \mathcal{E}$ such that $(G_{\boldsymbol{\varepsilon}})$ is a submaximal isotropy type which is generically symmetry breaking, so that $(\mathbb{R}^{q+1}, G)$ does not satisfy the MISC. This gives another example (in addition to $\Delta' \rtimes H_k$) of a large subgroup of H_k which does not satisfy the MISC.

4.5.9. Stable solution branches of maximal index and trivial isotropy. We give an example of a group $G \subset H_8$ for which there exists $X \in \mathcal{S}(\mathbb{R}^8, G)$ with a solution branch of maximal index and trivial isotropy. The construction goes as follows. Let K be as in examples 4.5.15(2). Let G_1 denote the subgroup $\Delta_4' \rtimes K$ of H_4 and $G_2 = \mathbf{D}_4$. Let ρ_1 (respectively, ρ_2) denote the standard representation of G_1 (respectively, G_2) on $\mathbb{R}^4$ (respectively, $\mathbb{R}^2$) and $\rho : G_1 \times G_2 \to \mathrm{GL}(\mathbb{R}^4 \otimes \mathbb{R}^2) = \mathrm{GL}(8, \mathbb{R})$ be defined by $\rho(g_1, g_2) = \rho_1(g_1) \otimes \rho_2(g_2)$. Since ρ_1 and ρ_2 are absolutely irreducible representations, the representation ρ is absolutely irreducible [**43**, §2]. An easy argument shows that $G = \rho(G_1 \times G_2)$ is contained in H_8. The kernel of ρ is $\{(1, 1), (-1, -1)\}$ so that G is of order 128. Let $\boldsymbol{\varepsilon} = (1, 1, 1, 1, 1, 1, 1, -1)$. One can show by a straightforward, but lengthy, calculation that $G_{\boldsymbol{\varepsilon}} = \{1\}$. Note that $\boldsymbol{\varepsilon} \in \mathcal{E}_8$. It follows from theorem 4.5.11 and proposition 4.5.4 that there exists $X \in \mathcal{S}(\mathbb{R}^8, G)$ and a solution branch γ of X of maximal index such that the isotropy subgroup $G_{[\gamma]}$ is trivial.

4.5.10. An example with applications to phase transitions. Thus far, all our examples have been based on subgroups of H_k which can be represented as a semidirect product of subgroups of Δ_k and S_k. We conclude this section with an example of a subgroup G of H_4 which cannot be so represented. This example was considered by Jarić [**99**] in the context of continuous phase transitions.

Let a and b denote the linear maps of $\mathbb{R}^4$ defined by

$$\begin{aligned} a(x_1, x_2, x_3, x_4) &= (x_1, x_4, x_2, x_3), \\ b(x_1, x_2, x_3, x_4) &= (x_2, x_1, -x_3, x_4). \end{aligned}$$

We identify a, b with their associated signed permutation matrices. Obviously, $a^3 = b^2 = I$. If we define $G = \langle a, b \rangle \subset H_4$, then it may be shown that $|G| = 48$ and $(\mathbb{R}^4, G)$ is absolutely irreducible. Moreover, G is a *non-split* extension of S_4 by $\mathbb{Z}_2$ and is isomorphic to $\mathrm{SL}(2, 3)$. In particular, G is not a semidirect product of subgroups of Δ_4 and S_4. Set $G_1 = G_{\boldsymbol{\varepsilon}_1}$, $G_2 = G_{\boldsymbol{\varepsilon}_3}$, $G_3 = G_{\boldsymbol{\varepsilon}_4}$ and $G_4 = G_{\boldsymbol{\varepsilon}_2}$. It is easy to verify that $G_1, G_2 \cong S_3$, $G_2 \cong \mathbb{Z}_3$ and $G_3 \cong \mathbb{Z}_2$. A straightforward computation shows that (G_1) and (G_2) are the maximal isotropy types (note that G_1 and G_2 are *not* conjugate subgroups of G). The non-trivial submaximal isotropy types are (G_3) and (G_4). Since $\boldsymbol{\varepsilon}_4 \in \mathcal{E}_4$, it follows in the usual way that there exist $X \in \mathcal{S}(\mathbb{R}^4, G)$ and $\beta \in \Sigma(X)$ such that β is of maximal index 4 and $G_\beta = G_3$. Indeed, we may assume X is a gradient vector field on $\mathbb{R}^4$ of degree 3.

4.6. Phase vector field and maps of hyperbolic type

In the next two sections, we develop the tools needed for the proof of theorem 4.5.11. The proof involves several steps. In this section, we prove results about quadratic and cubic vector fields by relating these vector fields to vector fields defined on the unit sphere of the representation (the *phase vector field*). We introduce the important concept of maps of *hyperbolic type* which we use to show that for an open and dense set of quadratic or cubic vector fields, all zeros of the phase vector field are hyperbolic. In the next section, we take account of higher order terms in families and prove a stability result that allows us to infer the branching pattern of a family by looking at the cubic truncation.

4.6.1. Cubic polynomial maps. We follow the notational conventions of 4.4.1. Throughout this section our focus will be on polynomial mappings of degree two or three with a particular focus on cubic polynomial maps. However, everything we say can be generalized without difficulty to polynomials of arbitrary degree and we refer to [**72**, section 4] for details.

Let V be an $(n+1)$-dimensional inner product space with associated norm $\| \|$. We denote the unit sphere $S(V)$ of V by S^n.

LEMMA 4.6.1. *Let $h \in P(V)$ and suppose that* $\mathrm{grad}(h) \in R(V,V)$. *Then h may be written in the form*

$$h(x) = \sum_{i=0}^{\ell} a_i \|x\|^{2i},$$

where $a_0, \dots, a_\ell \in \mathbb{R}$.

PROOF. The result is classical and is proved by noting that if h is homogeneous then $0 \in \mathbb{R}$ is the only critical value of h (Euler's theorem) and the level sets $h = c > 0$ are spheres. $\square$

Assume that (V,G) is an absolutely irreducible orthogonal representation of the finite group G. Suppose the critical degree $d(V,G) = 2$ or 3 and set $d(V,G) = d$ for the remainder of this section. Note that if $d = 3$, then $P^2_G(V,V) = \{0\}$ (and so condition (C) is satisfied). Lemma 4.6.1 implies that $R^3_G(V,V) = \{a\|x\|^2 x \mid a \in \mathbb{R}\}$. If $Q \in P^{(d)}_G(V,V)$, we may write Q uniquely in the form

$$Q = \nu I + S,$$

where $\nu \in \mathbb{R}$ and $S \in P^d_G(V,V)$.

4.6.2. Phase vector field. Let $Q \in P^d(V,V)$. We associate to Q a vector field $\mathcal{P}_Q$ on $S^n \subset V$ defined by

$$\mathcal{P}_Q(u) = Q(u) - (Q(u),u)u, \quad (u \in S^n).$$

We say that $\mathcal{P}_Q$ is the *phase vector field* of Q. The phase vector field $\mathcal{P}_Q$ is the tangential component of the restriction of Q to S^n.

REMARKS 4.6.2. (1) $\mathcal{P}_Q \equiv 0$ if and only if Q is radial and so if $d = 2$ and $Q \not\equiv 0$, then $\mathcal{P}_Q \not\equiv 0$.
(2) If $Q \in P^d_G(V,V)$, then $\mathcal{P}_Q$ is a G-equivariant vector field on S^n.

Let $\mathbf{Z}(\mathcal{P}_Q)$ denote the set of zeros of $\mathcal{P}_Q$. The homogeneity of Q implies that $z \in \mathbf{Z}(\mathcal{P}_Q)$ if and only if $-z \in \mathbf{Z}(\mathcal{P}_Q)$. By definition of $\mathcal{P}_Q$, $z \in \mathbf{Z}(\mathcal{P}_Q)$ if and only if $Q(z) = \alpha z$ for some $\alpha \in \mathbb{R}$. It follows from Euler's theorem that $DQ(z)(z) = d\alpha z$. In particular, $DQ(z)$ induces an endomorphism $A = A_z : V/\mathbb{R}z \to V/\mathbb{R}z$. We identify the tangent space $E = T_z S^n$ with $V/\mathbb{R}z$ in the obvious manner.

Lemma 4.6.3. *Let $Q \in P^d(V,V)$. Suppose that $z \in S^n$ satisfies $Q(z) = \alpha z$ for some $\alpha \in \mathbb{R}$ (so that $z \in \mathbf{Z}(\mathcal{P}_Q)$). Then the linearization $T_z\mathcal{P}_Q \in L(E,E)$ of $\mathcal{P}_Q$ at z is given by*

$$T_z\mathcal{P}_Q = A - \alpha I_E.$$

In particular, the following conditions are equivalent:

(1) *z is a hyperbolic zero of $\mathcal{P}_Q$.*
(2) *No eigenvalue of A has real part α.*

Proof. Choose an orthonormal basis $\{e_0, \ldots, e_n\}$ of V with $e_0 = z$ and let $\{x_0, \ldots, x_n\}$ be the corresponding coordinate system. The proof is now an exercise in advanced calculus. The matrix of the derivative $DQ(e_0)$ is of the form

$$DQ(e_0) = \begin{pmatrix} 3\alpha & * \\ 0 & A \end{pmatrix},$$

where $A \in M_n(\mathbb{R})$ can be identified with the endomorphism A_z of $V/\mathbb{R}z$. One can calculate $T_z\mathcal{P}_Q$ directly in terms of an appropriate coordinate chart for S^n at e_0. An easy computation shows that $T_z\mathcal{P}_Q$ can be identified with the matrix $A - \alpha I_n$, where I_n is the $n \times n$ identity matrix. $\square$

Remarks 4.6.4. (1) Let the notation be as in lemma 4.6.3 and assume that $\alpha \neq 0$. As a consequence of the proof, (2) is equivalent to: (3) No eigenvalue of $DQ(z)$ has real part α.
(2) Assume that $Q = \mathrm{grad}(h)$ for some $h \in P^{d+1}(V)$. Then $DQ(z) : V \to V$ is self-adjoint linear operator, so that all eigenvalues of A are real. Hence if z is a simple zero of $\mathcal{P}_Q$, it is a hyperbolic zero.

4.6.3. Normalized families. Given $Q \in P^d(V,V)$, define $J^Q : V \times \mathbb{R} \to V$ by $J^Q(x,\lambda) = \lambda x + Q(x)$.

Lemma 4.6.5. *Let $Q \in P^d(V,V)$ and suppose that $z \in \mathbf{Z}(\mathcal{P}_Q)$, Set $\alpha_z = (Q(z), z)$ and define $\beta_z : \mathbb{R} \to \mathbb{R}$ by $\beta_z(s) = -\alpha_z s^2$. Then $J^Q(sz, \beta_z(s)) = 0$, $s \in \mathbb{R}$. Conversely, if (x,λ) is a non-trivial zero of J^Q, then $z = x/\|x\| \in \mathbf{Z}(\mathcal{P}_Q)$ and (x,λ) lies on the solution curve $s \mapsto (sz, \beta_z(s))$.*

Proof. Computing, we see that for all $s \in \mathbb{R}$

$$\begin{aligned} J^Q(sz, \beta_z(s)) &= -\alpha_z s^d z + Q(sz), \\ &= -\alpha_z s^d z + s^d Q(z), \\ &= -\alpha_z s^d z + s^d \alpha_z z, \text{ since } Q(z) = \alpha_z z, \\ &= 0. \end{aligned}$$

The converse is equally simple. ☐

For each $z \in \mathbf{Z}(\mathcal{P}_Q)$, let $\gamma^{(z)} : [-1,1] \to V \times \mathbb{R}$ be the smooth solution curve to $J^Q = 0$ defined by $\gamma^{(z)}(s) = (sz, \beta_z(s))$, $s \in [-1,1]$. Let $C_z = \gamma^{(z)}([0,1])$. Then $C_z \cap C_{z'} = \{(0,0)\}$ if $z \neq z'$. Furthermore, we may choose a neighbourhood A of $(0,0)$ in $V \times \mathbb{R}$ such that the set of non-trivial zeros of J^Q in A equals $\cup_{z \in \mathbf{Z}(\mathcal{P}_Q)} C_z$.

We want conditions on Q, z that imply $\gamma^{(z)}$ is a curve of hyperbolic zeros. Let $\lambda = \beta_z(s) = -\alpha_z s^{d-1}$. An easy calculation gives

$$DJ^Q_\lambda(sz) = s^{d-1}(DQ(z) - \alpha_z \mathrm{I}_V). \tag{4.4}$$

LEMMA 4.6.6. *The following conditions are equivalent:*

(1) $\gamma^{(z)}$ *is a curve of hyperbolic zeros.*
(2) *No eigenvalue of* $DQ(z)$ *has real part* α_z
(3) $\alpha_z \neq 0$ *and* z *is a hyperbolic zero of* $\mathcal{P}_Q$.

PROOF. The equivalence of (1) and (2) follows from (4.4). The equivalence of (2) and (3) follows from (4.4) and lemma 4.6.3. ☐

DEFINITION 4.6.7. Let $Q \in P^d(V,V)$ and $u \in S^n$. We define $\alpha(Q,u) \in \mathbb{R}$ by $\alpha(Q,u) = (Q(u), u)$.

The following proposition is an easy consequence of lemma 4.6.6.

PROPOSITION 4.6.8. *Let* $z \in \mathbf{Z}(\mathcal{P}_Q)$ *and assume that* $\alpha(Q,z) \neq 0$. *Let the solution curve* $\gamma^{(z)}$ *of* J^Q *be defined as above. Let* $\gamma_z = \gamma^{(z)}|[0,1]$.

(1) *If* $\alpha(Q,z) < 0$ *(respectively,* $\alpha(Q,z) > 0$*), then* γ_z *is a supercritical (respectively, subcritical) solution branch.*
(2) *If* z *is a hyperbolic zero of* $\mathcal{P}_Q$, *then* $\gamma^{(z)}$ *is a curve of hyperbolic zeros of* J^Q *and we have*
    (a) $\mathrm{ind}(\mathcal{P}_Q, z) = \mathrm{ind}(A_z - (Q(z),z)I)$, *where* A_z *is as in lemma 4.6.3 and* I *is the identity map of* $V/\mathbb{R}z$.
    (b) $\mathrm{ind}([\gamma_z]) = \mathrm{ind}(\mathcal{P}_Q, z) + 1$, *if* $\alpha(Q,z) < 0$, *and* $\mathrm{ind}([\gamma_z]) = \mathrm{ind}(\mathcal{P}_Q, z)$, *if* $\alpha(Q,z) > 0$.

Let $z \in \mathbf{Z}(\mathcal{P}_Q)$ and suppose $\alpha(Q,z) \neq 0$. We define $\mathrm{sgn}_Q(z) = \pm 1$ by

$$\mathrm{sgn}_Q(z) = -\mathrm{sgn}(\alpha(Q,z)) \quad (= -\alpha(S,z))/|\alpha(Q,z)|).$$

It follows from proposition 4.6.8 that $\mathrm{sgn}_Q(z) = \mathrm{sgn}([\gamma_z])$. If z is a hyperbolic zero of $\mathcal{P}_Q$, then we set $\mathrm{ind}_Q(z) = \mathrm{ind}([\gamma_z])$. Note that $\mathrm{ind}_Q(z)$ can be computed directly from $DQ(z)$ and $\alpha(Q,z)$.

4.6.4. Maps of hyperbolic type. Let $P^d(V,V)_h \subset P^d(V,V)$ denote the set of polynomials Q such that all zeros of $\mathcal{P}_Q$ are hyperbolic and $P^d(V,V)_s \subset P^d(V,V)$ be the set of polynomials such that all zeros of $\mathcal{P}_Q$ are simple. Define

$$P^d(V,V)_H = \{Q \in P^d(V,V)_h \mid \alpha(Q,z) \neq 0, \text{ all } z \in \mathbf{Z}(\mathcal{P}_Q)\}.$$

We let $P^d_G(V,V)_h$, $P^d_G(V,V)_s$ and $P^d_G(V,V)_H$ denote the corresponding spaces of G-equivariant polynomial mappings. (Note that for the definition of $P^d(V,V)_H$, we assume $d = 2, 3$. We use a different definition if $d > 3$, see [**72**, §4].)

EXAMPLE 4.6.9. Let $G = H_k$, $V = \mathbb{R}^k$. A vector space basis for $P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k)$ is $\{R, C\}$, section 4.5.3. We have

$$P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k)_h = P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k)_s = \{aR + bC \mid a, b \in \mathbb{R},\ b \neq 0\}.$$

The spaces $P^d(V,V)_h$ and $P^d_G(V,V)_H$ play an important role in the proof of theorem 4.5.11. However, there are technical difficulties in dealing with these spaces on account of the hyperbolicity condition which is not algebraic. In this subsection, we introduce certain open subspaces of these spaces, which are defined by algebraic conditions and are easier to handle. (Most of the results in this subsection hold without the restriction $d \leq 3$.)

Let $\mathbf{V} = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V. Let $P(\mathbf{V}, \mathbf{V}) = \oplus_{k \geq 0} P^k(\mathbf{V}, \mathbf{V})$ be the complex vector space of all complex polynomial endomorphisms of $\mathbf{V}$. We have $P(\mathbf{V}, \mathbf{V}) = P(V,V) \otimes_{\mathbb{R}} \mathbb{C}$. In particular, we may identify $P(V,V)$ with the $\mathbb{R}$-subspace $P(V,V) \otimes_{\mathbb{R}} 1$ of $P(\mathbf{V}, \mathbf{V})$.

Let $\Sigma^n \subset \mathbf{V}$ denote the complexification of the unit sphere S^n of V. If we choose an orthonormal coordinate system on V so that S^n is defined by the equation $\Sigma_{k=1}^{n+1} x_k^2 = 1$, then Σ^n is defined by the equation $\Sigma_{k=1}^{n+1} z_k^2 = 1$. In particular, Σ^n is a complex algebraic subvariety of $\mathbf{V}$.

For $Q \in P^d(\mathbf{V}, \mathbf{V})$, we define

$$\mathbf{Y}(Q) = \{(u, \alpha) \in \Sigma^n \times \mathbb{C} \mid Q(u) = \alpha u\}.$$

If $Q \in P^d(V,V)$, let $\mathbf{Y}(Q)_{\mathbb{R}}$ denote the set of "real points" of $\mathbf{Y}(Q)$, that is the intersection of $\mathbf{Y}(Q)$ with $S^n \times \mathbb{R}$.

Let $Q \in P^d(\mathbf{V}, \mathbf{V})$ and $(u, \alpha) \in \mathbf{Y}(Q)$. The $\mathbb{C}$-linear map $DQ(u) - \alpha I_{\mathbf{V}}$ has radial eigenvector u with corresponding eigenvalue $(d-1)\alpha$. Let $\mathbf{A}(Q, u, \alpha)$ denote the linear operator on the quotient space $\mathbf{V}/\mathbb{C}u$ induced by $DQ(u) - \alpha I_{\mathbf{V}}$. If $Q \in P^d(V,V)$ and $(u, \alpha) \in \mathbf{Y}(Q)_{\mathbb{R}}$, let $A(Q, u, \alpha)$ denote the linear operator induced on $V/\mathbb{R}u$ by $DQ(u) - \alpha I_{\mathbf{V}}$.

As a routine application of the inverse function theorem, we have

LEMMA 4.6.10. *Let $Q \in P^d(\mathbf{V}, \mathbf{V})$ and $(u, \alpha) \in \mathbf{Y}(Q)$. If $\mathbf{A}(Q, u, \alpha)$ is non-singular, then (u, α) is an isolated point of $\mathbf{Y}(Q)$.*

DEFINITION 4.6.11. Let $Q \in P^d(V,V)$. (a) Q is of *simple type* (respectively, *relatively simple type*) if for all $(u, \alpha) \in \mathbf{Y}(Q)$, $DQ(u) - \alpha I_{\mathbf{V}}$ (respectively, $A(Q, u, \alpha)$) is non-singular. (b) Q is of *hyperbolic type* (respectively, *relatively hyperbolic type*) if Q is of simple type (respectively, relatively simple type) and if, for all $(u, \alpha) \in \mathbf{Y}(Q)$, the characteristic polynomial of $DQ(u) - \alpha I_{\mathbf{V}}$ (respectively, $\mathbf{A}(Q, u, \alpha)$) does not have any factor of the form $T^2 + a$, $a \in \mathbb{C}$.

REMARKS 4.6.12. Let $Q \in P^d(V,V)$. (1) If Q is of simple type, all solution branches of J^Q are curves of simple zeros and J^Q satisfies the branching condition B4 (and hence also satisfies B1 and B3). If Q is of relatively simple type, then J^Q satisfies the branching condition B1.
(2) If Q is of relatively hyperbolic type, then all zeros of the phase vector field $\mathcal{P}_Q$ are hyperbolic and so $Q \in P^d(V,V)_h$.
(3) If Q is of hyperbolic type (or even, less restrictively, if Q is of simple type and of relatively hyperbolic type), then all solution branches of J^Q are branches of hyperbolic zeros, so that J^Q satisfies the branching conditions B1-B3.

LEMMA 4.6.13. *The subspaces of $P^d(\mathbf{V},\mathbf{V})$ consisting of all maps of hyperbolic type, simple type, relatively hyperbolic type or relatively simple type are open subspaces of $P^d(\mathbf{V},\mathbf{V})$.*

The proof is elementary.

EXAMPLE 4.6.14. Let $V = \mathbb{R}^{n+1}$ and define $C \in P^d(V,V)$ by $C(x) = (x_1^d,\ldots,x_{k+1}^d)$. If $d=3$, C is of hyperbolic type and the subspaces of lemma 4.6.13 are all non-empty. If $d=2$, C is of simple type and of relatively hyperbolic type. If we take $V=\mathbb{R}^3$ and define $Q \in P^2(V,V)$ by $Q(x,y,z) = (yz,xz,xy)$, then Q is of relatively hyperbolic type but is not of simple type.

THEOREM 4.6.15. *Let E be a linear subspace of the real linear space $P^d(V,V)$. If E contains an element Q which is of hyperbolic type, then the set of elements of E which are of hyperbolic type is an open dense subset of E. Similar results hold if E contains an element of relatively hyperbolic type, an element of simple type, or an element of relatively simple type.*

The proof of this theorem uses nontrivial results from complex algebraic geometry and is given in an appendix at the end of the chapter.

The next result will be important in the sequel.

THEOREM 4.6.16. (1) *If $P^d_G(V,V)$ contains an element of hyperbolic type, $P^d_G(V,V)_h$ is a dense open subset of $P^d_G(V,V)$.*
(2) *If $P^3_G(V,V)$ contains an element of relatively hyperbolic type, $P^3_G(V,V)_H$ is a dense open subset of $P^3_G(V,V)$.*
(3) *If $P^2_G(V,V)$ contains an element of hyperbolic type, $P^2_G(V,V)_H$ is a dense open subset of $P^2_G(V,V)$.*

PROOF. Set $E = P^d_G(V,V)$ and apply theorem 4.6.15. Parts (1) and (3) are immediate. It remains to prove (2). Assume that $P^3_G(V,V)_h$ is dense in $P^3_G(V,V)$. Let $Q \in P^3_G(V,V)$ and N be a neighbourhood of Q in $P^3_G(V,V)$. It suffices to prove $N \cap P^3_G(V,V)_H \neq \emptyset$. Since $P^3_G(V,V)_h$ is dense, there exists $P \in N \cap P^3_G(V,V)_h$. Let N_1 be a neighbourhood of P such that $N_1 \subset N$. For $a \in \mathbb{R}$, define $P_a \in P^3_G(V,V)$ by $P_a(x) = a\|x\|^2x + P(x)$. Note that $\mathcal{P}_P = \mathcal{P}_{P_a}$, for all $a \in \mathbb{R}$. Choose $\delta > 0$ such that $P_a \in N_1 \cap P^3_G(V,V)_h$ for every a such that $|a| \leq \delta$. If $z \in \mathbf{Z}(\mathcal{P}_P)$, then

$\alpha(P_a, z) = \alpha(P, z) + a$. Thus if $a \in [-\delta, \delta]$ and a does not belong to the finite set $\{-\alpha(P, z) \mid z \in \mathbf{Z}(\mathcal{P}_P)\}$, we have $P_a \in N_1 \cap P_G^3(V, V)_H$. $\square$

4.6.5. The branching pattern of J^Q. It follows from our earlier results that if $Q \in P_G^d(V, V)_H$, then J^Q satisfies the branching conditions B1-B3.

If $Q \in P_G^d(V, V)_H$, then sgn_Q and ind_Q are G-invariant functions defined on $\mathbf{Z}(\mathcal{P}_Q)$, so that $(\mathbf{Z}(\mathcal{P}_Q), \mathrm{sgn}_Q, \mathrm{ind}_Q)$ is a signed, indexed G-set.

LEMMA 4.6.17. *(Notation as above.) If $Q \in P_G^d(V, V)_H$, then the signed indexed branching pattern $\Sigma^\star(J^Q)$ is isomorphic (as a signed, indexed G-set) to $(\mathbf{Z}(\mathcal{P}_Q), sgn_Q, \mathrm{ind}_Q)$.*

The proof is immediate. The isomorphism maps $[\gamma_z]$ to z.

4.7. Transforming to generalized spherical polar coordinates

In this section, we will transform the problem of solving $X = 0$ on $V \times \mathbb{R}$ to a somewhat simpler problem on $(S^n \times \mathbb{R}) \times \mathbb{R}$. The process we use depends on (generalized) spherical polar coordinates or (polar) blowing-up. We continue to assume that $d = 2$ or 3.

4.7.1. Preliminaries. Let V be a finite dimensional real vector space with inner product $(\ ,\ )$ and norm $\|\ \|$. Suppose $\dim(V) = n+1$ and denote the unit sphere $S(V)$ of V by S^n. For the moment we do not assume the presence of a G-action. However, we do restrict attention to the space $C^\infty(V \times \mathbb{R}, V)_0$ of smooth families X which are in the 'normal' form

$$X(x, \lambda) = \lambda x + F(x, \lambda), \quad ((x, \lambda) \in V \times \mathbb{R}), \tag{4.5}$$

where $F(x, \lambda) = O(\|x\|^2)$ ($F_\lambda(0) = 0$ and $DF_\lambda(0) = 0$),

The derivative of X_λ in V-variables at $x = 0$ is λI_V, Obviously $x = 0$ is an equilibrium of (4.5) for all values of λ.

We make a transformation of (4.5) that effectively eliminates the parameter λ and reduces the dimension of the space we have to look at by one. In addition, we will be able to disregard certain terms in the transformed vector field $X(x, \lambda)$. In essence, we replace the study of the family (4.5) by that of a single vector field on a lower dimensional space. Of course, there is a penalty to be paid for this: the new space will no longer be linear. It turns out, this is a small penalty especially in the cases where the dynamics of (4.5) are dominated by the quadratic or cubic homogeneous part of $X(x, \lambda)$. Our transformations are based on rescalings (of space and time) and blowing-up. For our purposes, we shall mainly use *polar blowing-up*. In case $\dim(V) = 2$, the transformation is just polar coordinates. In higher dimensions, polar blowing-up is a generalization of polar coordinates 'without the coordinates'. We use the term 'blowing-up' because of the close relationship with the classical blowing-up construction of algebraic geometry.

4.7.2. Polar blowing-up. We define the polar blowing-up transformation $P : S^n \times \mathbb{R} \to V$ by

$$P(u, R) = Ru, \quad (u \in S^n, R \in \mathbb{R}).$$

We remark the following properties of the transformation P.

(a) Provided $x \neq 0$, $P^{-1}(x)$ consists of precisely two points.
(b) $P^{-1}(0) = S^n \times \{0\}$.

EXAMPLE 4.7.1. If $V = \mathbb{R}^2$, $P : S^1 \times \mathbb{R} \to \mathbb{R}^2$ is polar coordinates. That is, if $u = \theta \in S^1$, then $P(\theta, R) = (R\cos\theta, R\sin\theta) \in \mathbb{R}^2$. Viewed in this way, the natural domain of definition of the polar coordinate transformation is the cylinder $S^1 \times \mathbb{R}$ (rather than the (r, θ)-plane). Note our convention that we allow R to be negative: the points (θ, R), $(\theta + \pi, -R)$ are mapped to the same point of $\mathbb{R}^2$ by P. In fact we have a free action of $\mathbb{Z}_2$ on the cylinder $S^1 \times \mathbb{R}$ generated by the involution $\rho(\theta, R) = (\theta + \pi, -R)$. P is invariant with respect to ρ: $P \circ \rho = P$. Of course, it is natural to require that P is 1:1 off $P^{-1}(0)$ and one way of achieving this is to restrict to the half cylinder defined by $R \geq 0$. Another approach, that avoids the introduction of manifolds with boundary, is to define P on the quotient of $S^1 \times \mathbb{R}$ by the free $\mathbb{Z}_2$-action and so make P 1:1 off $P^{-1}(0)$. We explore this and related questions in some of the exercises.

PROPOSITION 4.7.2. *Under polar blowing-up, the family of vector fields (4.5) transforms to the family (R_λ, S_λ) of vector fields on $S^n \times \mathbb{R}$ defined by*

$$S_\lambda(u, R) = R^{-1}(F(Ru, \lambda) - (F(Ru, \lambda), u)u), \tag{4.6}$$

$$R_\lambda(u, R) = \lambda R + (F(Ru, \lambda), u). \tag{4.7}$$

PROOF. If we write a solution $x(t)$ of (4.5) in the form $x = Ru$, then $x' = R'u + Ru'$. Substituting in (4.5), we obtain $R'u + Ru' = \lambda Ru + F(Ru, \lambda)$. Since $u \in S^n$, $(u, u) = 1$. Differentiating, we have $(u, u') = 0$. The result follows by taking the inner product of our equation for R', u' with u. $\square$

REMARK 4.7.3. If a curve $u = u(t)$ is constrained to lie on S^n, then $u'(t)$ is always tangent to S^n. In particular, the equation $u' = R^{-1}(F(Ru, \lambda) - (F(Ru, \lambda), u)u)$ defines a vector field on S^n. Of course, one can see this directly. If $x \neq 0$, the components of the vector field $\lambda x + F(x, \lambda)$ along and perpendicular to the radial direction through x are $\lambda\|x\| + (F(x, \lambda), x/\|x\|)$ and $F(x, \lambda) - (F(x, \lambda), x/\|x\|)x/\|x\|$ respectively. The equations we obtain in proposition 4.7.2 reflect this orthogonal decomposition of the vector field into radial and tangential components. In the sequel, we sometimes call $S(u, R)$ the *spherical vector field* and $R(u, R)$ the *radial vector field*.

EXAMPLE 4.7.4. Suppose $X(x, \lambda) = \lambda x + Q(x)$, where $Q \in P^d(V, V)$. Then X transforms to the family

$$S_\lambda(u, R) = R^{d-1}\mathcal{P}_Q(u), \tag{4.8}$$

$$R_\lambda(u, R) = \lambda R + R^d(Q(u), u). \tag{4.9}$$

Observe that (4.8) is independent of λ.

EXERCISE 4.7.5. Let $X(x,\lambda) = \lambda x + g(x,\lambda)x$, where $g : V \times \mathbb{R} \to \mathbb{R}$. Show that the associated spherical vector field vanishes identically. What is the radial vector field?

EXERCISE 4.7.6. Suppose that $(u(t), R(t))$ is the solution curve of (4.6,4.7) with initial condition (u_0, R_0). Verify that $(-u(t), -R(t))$ is also a solution curve, now with initial condition $(-u_0, -R_0)$. Verify that the system is equivariant with respect to the $\mathbb{Z}_2$-action on $S^n \times \mathbb{R}$ defined by $(u, R) \mapsto (-u, -R)$.

EXERCISE 4.7.7. Let $p_1 : \mathbb{R}^n \times S^{n-1} \to \mathbb{R}^n$ denote the projection on the first factor. Let $\Sigma \subset \mathbb{R}^n \times S^{n-1}$ be the solution set of the equations

$$x_i u_j - x_j u_i = 0, \quad 1 \leq i, j \leq n, \tag{4.10}$$

where $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $(u_1, \ldots, u_n) \in S^{n-1}$. Set $\Pi = p_1|\Sigma$. Verify that $\Pi : \Sigma \to \mathbb{R}^n$ is the polar blowing-up of $\mathbb{R}^n$. That is, find a diffeomorphism $h : S^{n-1} \times \mathbb{R} \to \Sigma$ such that $P = \Pi \circ h$. If we replace S^{n-1} by $\mathbb{P}^{n-1}(\mathbb{R})$ – the projective space of lines through the origin of $\mathbb{R}^n$ – show that the construction still works but with

$$\Sigma = \{(x, \ell) \in \mathbb{R}^n \times \mathbb{P}^{n-1}(\mathbb{R}) \mid x \in \ell\}.$$

The map $\Pi : \Sigma \to \mathbb{R}^n$ is the classical blowing-up of $\mathbb{R}^n$ at the origin (see also section 5.6.7). Verify that Σ is homeomorphic to the Mobius band if $n = 2$.

4.8. Representations which are $d(V,G)$-determined, $d(V,G) = 2,3$

Suppose that (V, G) is an orthogonal representation of the finite group G and $d = d(V,G) = 2,3$. It follows from Taylor's theorem that if $X \in \mathcal{V}_0$ we may write

$$X(x,\lambda) = \lambda x + Q(x) + F_1(x) + \lambda F_2(x,\lambda),$$

where $Q = J^d(X)$, $F_1(x) = X(x,0) - Q(x) = O(\|x\|^{d+1})$ and $F_2(x,\lambda) = X - Q - F_1 = O(\|x\|^d)$. Transforming using spherical polar coordinates we obtain the following expressions for the spherical and radial terms.

$$S_\lambda(u,R) = R^{d-1}[\mathcal{P}_Q(u) + RA_1(u,R,\lambda) + \lambda A_2(u,R,\lambda)], \tag{4.11}$$

$$R_\lambda(u,R) = \lambda R + R^d[(Q(u),u) + RB_1(u,R,\lambda) + \lambda B_2(u,R,\lambda)]. \tag{4.12}$$

The terms A_1, A_2 are smooth vector fields on S^n that may be given explicitly in terms of F_1 and F_2 respectively. Similarly, B_1 and B_2 are smooth functions that depend on F_1 and F_2 respectively. Let $T(u,R,\lambda)$ be the smooth (R,λ)-parameterized family of vector fields on S^n defined by

$$T(u,R,\lambda) = \mathcal{P}_Q(u) + RA_1(u,R,\lambda) + \lambda A_2(u,R,\lambda).$$

LEMMA 4.8.1. *(Notation as above.) Suppose that $z \in \mathbf{Z}(\mathcal{P}_Q)$ is hyperbolic and $\alpha(Q,z) \neq 0$. There exists a (unique) smooth branch $\gamma_z : [0, \delta_z] \to V \times \mathbb{R}$ of hyperbolic zeros for $X = 0$ with direction of branching $\mathbf{d}(\gamma_z) = z$. Moreover,* *sgn*$([\gamma_z])$ *and* $\operatorname{ind}([\gamma_z])$ *take the same values as for the corresponding branch for* J^Q. *A similar result holds if* z *is a simple zero of* $\mathcal{P}_Q$.

PROOF. Since z is a simple zero for $\mathcal{P}_Q$, we may apply the implicit function theorem and choose a closed neighborhood U of z in S^n, $R_z, \rho_z > 0$ and smooth map $z : [0, R_0] \times [-\rho_z, \rho_z] \to S^n$ such that

(1) $T(z(R, \lambda), R, \lambda) = 0$, all $(R, \lambda) \in [0, R_0] \times [-\rho_z, \rho_z]$.
(2) $z(0, 0) = z$.
(3) The only zeros of T in U for $(R, \lambda) \in [0, R_0] \times [-\rho_z, \rho_z]$ are those given by (1).
(4) $DT_{(R,\lambda)}(z(R, \lambda))$ is hyperbolic, for all $(R, \lambda) \in [0, R_0] \times [-\rho_z, \rho_z]$.

Substitute $z(R, \lambda)$ in the radial equation. Cancelling the factor R (corresponding to the trivial solution), we must solve

$$\lambda + R^{d-1}[\alpha(Q, z(R, \lambda)) + RB_1(R, \lambda) + \lambda B_2(R, \lambda)] = 0,$$

where we have written $B_i(R, \lambda) = B_i(z(R, \lambda), R, \lambda)$, $i = 1, 2$. Choosing $R_z, \rho_z > 0$ smaller if necessary, we may assume that $\operatorname{sgn}(\alpha(Q, z(R, \lambda)) + RB_1(R, \lambda) + \lambda B_2(R, \lambda)) = \operatorname{sgn}(\alpha(Q, z)$, for all $(R, \lambda) \in [0, R_0] \times [-\rho_z, \rho_z]$. Another application of the implicit function theorem shows that we may choose $\delta_z \in (0, R_z]$ and a smooth map $\lambda : [0, \delta_z] \to \mathbb{R}$ such that $\lambda(0) = 0$ and for $R \in [0, \delta_z]$ we have

$$\lambda(R) + R^{d-1}[\alpha(Q, z(R, \lambda(R))) + RB_1(R, \lambda(R)) + \lambda(R)B_2(R, \lambda(R))] = 0,$$

We define the required smooth branch of solutions $\gamma_z : [0, \delta_z] \to V \times \mathbb{R}$ by

$$\gamma_z(s) = (sz(s, \lambda(s)), \lambda(s)), \quad s \in [0, \delta_z].$$

It follows from the construction that $\mathbf{d}(\gamma_z) = z$. Our choices also assure that $\operatorname{sgn}(\gamma_z)$ is the same as that for the corresponding branch of J^Q. It remains to show that we can choose $R_z > 0$ sufficiently small so that γ_z is a branch of hyperbolic zeros. This follows by computing the linearization of $(S_\lambda(u, R), R_\lambda(u, R))$ along the solution curve. We find that

$$D(S_\lambda, R_\lambda)(R) = \begin{pmatrix} R^{d-1}T_z\mathcal{P}_Q + O(R^d) & O(R^d) \\ O(R^d) & -2R^{d-1}\alpha(Q, z) + O(R^d) \end{pmatrix}$$

That we can choose $R_z > 0$ so that γ_z is a branch of hyperbolic zeros and $\operatorname{ind}(\gamma_z)$ is same as that for the corresponding branch of J^Q follows by dividing the matrix by R^{d-1} and using a simple continuity argument. □

THEOREM 4.8.2. *If* $d = 2$, *assume* $P^2_G(V, V)_H \neq \emptyset$, *and if* $d = 3$, *assume* $P^2_G(V, V)_h \neq \emptyset$. *Then*

(1) G *equivariant bifurcation problems on* V *are* d*-determined.*
(2) $X \in \mathcal{V}_0(V, G)$ *is stable if* $j^d_1X(0) \in P^d_G(V, V)_H$.

PROOF. It follows from theorem 4.6.16 that $P^d_G(V, V)_H$ is open and dense in $P^d_G(V, V)$. Let $X \in \mathcal{V}_0(\mathbb{R}^k, G)$. If $j^dX_0(0) \in P^d_G(\mathbb{R}^k, \mathbb{R}^k)_H$, then lemma 4.8.1 implies that X has a well-defined signed indexed branching pattern $\Sigma^\star(X)$. Since

$P^d_G(V,V)_H$ is open and dense it follows that G-equivariant bifurcation problems are d-determined and that X is stable whenever $j^d X_0(0) \in P^d_G(\mathbb{R}^k, \mathbb{R}^k)_H$. $\square$

Proof of Theorem 4.5.11. Suppose that $G \subset H_k$ and $(\mathbb{R}^k, G)$ satisfies conditions (IR,C). Since the basic cubic equivariant $C \in P^3_G(\mathbb{R}^k, \mathbb{R}^k)_H$, it follows from theorem 4.8.2 that G-equivariant bifurcation theorems on $\mathbb{R}^k$ are 3-determined. Since $P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k)_H \subset P^3_G(\mathbb{R}^k, \mathbb{R}^k)_H$, we see that if $X \in \mathcal{V}_0(\mathbb{R}^k, H_k)$ and $j^3_1 X(0) \in P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k)_H$, then $X \in \mathcal{S}(\mathbb{R}^k, G)$. Hence (G_ε) is a symmetry breaking isotropy type for every $\varepsilon \in \mathcal{E}$, Finally if H is subgroup of G such that H satisfies conditions (IR,C) then, by exactly the argument we used above, every G-stable family is, a fortiori, H-stable. Hence $\mathcal{S}(\mathbb{R}^k, G) \subset \mathcal{S}(\mathbb{R}^k, H)$. $\square$

4.8.1. The case when $d(V,G) = 2$ but (V,G) is not 2-determined. Throughout this section we assume that (V,G) is an absolutely irreducible orthogonal representation of critical degree two and that $\dim(P^2_G(V,V)) = 1$. Fix $Q \in P^2_G(V,V)$, $Q \neq 0$. We assume that all the zeros of $\mathcal{P}_Q$ are hyperbolic – so that $Q \in P^2_G(V,V)_h$.

Let $\mathbf{Z}(Q)$ denote the zero set of $Q|S^n$. Since $\mathbf{Z}(Q) \subset \mathbf{Z}(\mathcal{P}_Q)$, if $\mathbf{Z}(Q) \neq \emptyset$, then (V,G) cannot be 2-determined (if $z \in \mathbf{Z}(Q) \cap \mathbf{Z}(\mathcal{P}_Q)$ then the branch γ_z of zeros of $J^Q(x,\lambda) = \lambda x + Q(x)$ will not be a branch of simple zeros). When this situation holds we shall prove that (V,G) is 3-determined.

THEOREM 4.8.3. *If $P^2_G(V,V)) = \mathbb{R}Q$, $Q \in P^2_G(V,V)_h$, and $\mathbf{Z}(Q) \neq \emptyset$, then (V,G) is 3-determined.*

We need some preliminary definitions. If $P \in P^3_G(V,V)$ and $z \in \mathbf{Z}(Q)$, define

$$\begin{aligned}\nu_z(P) &= DQ(z)((T_z\mathcal{P}_Q)^{-1}(\mathcal{P}_P(z))) \in V,\\ \rho_z(P) &= (z, \nu_z(P) - P(z)).\end{aligned}$$

For the definition of $\nu_z(P)$, we regard $T_z\mathcal{P}_Q : T_zS^n \to T_zS^n \subset V$. Let

$$\mathcal{A}^3(V,G) = \{P \in P^3_G(V,V) \mid \rho_z(P) \neq 0,\ \forall z \in \mathbf{Z}(Q)\}.$$

LEMMA 4.8.4. *$\mathcal{A}^3(V,G)$ is an open and dense subset of $P^3_G(V,V)$.*

PROOF. If $P_a(x) = a\|x\|^2 x$, then $\rho_z(P_a) = -a$ and so $P_a \in \mathcal{A}^3(V,G)$, $a \neq 0$. Since $\rho_z(P)$ is a polynomial in the coefficients of P and the zero set of $\mathcal{P}_Q$ is finite, we see easily that $\mathcal{A}^3(V,G)$ is an open and dense subset of $P^3_G(V,V)$. $\square$

Let $\mathcal{A}^3_0(V,G) \subset P^{(3)}_G(V,V)_0$ be the set of polynomials $cQ + P$, where $c \neq 0$ and $P \in \mathcal{A}^3(V,G)$. It follows from the previous lemma that $\mathcal{A}^3_0(V,G)$ is an open and dense subset of $P^{(3)}_G(V,V)_0$.

The next lemma completes the proof of theorem 4.8.3.

LEMMA 4.8.5. *Let $X \in \mathcal{V}_0(V,G)$. If $J^3X(0) \in \mathcal{A}^3_0(V,G)$, then X is stable. In particular, (V,G) is 3-determined.*

PROOF. If $J^3X(0) = (cQ, P) \in \mathcal{A}_0^3(V, G)$, then $c \neq 0$ and $J^2X(0) \in P_G^2(V, V)_h$. If $z \in \mathbf{Z}(\mathcal{P}_Q)$ and $Q(z) \neq 0$, then we have previously shown that we have a hyperbolic branch of solutions γ_z for $X = 0$ (lemma 4.8.1). Hence it suffices to consider the case $z \in \mathbf{Z}(Q)$. Let $(S_\lambda(u, R), R_\lambda(u, R))$ be the transformation of X to spherical polar coordinates, We may write

$$\begin{aligned} S_\lambda(u, R, \lambda) &= R[c(\lambda)\mathcal{P}_Q(u) + Rd(\lambda)\mathcal{P}_P(u)] + O(R^3) + O(R^2\lambda), \\ R_\lambda(u, R, \lambda) &= \lambda R + c(\lambda)(P(u), u)R^2 + d(\lambda)(Q(u), u)R^3 + O(R^4), \end{aligned}$$

where $c(0) = c \neq 0$ and $d(0) = 1$. Dividing through by R, we see that the zeros of the spherical component S_λ are given by

$$\Gamma(u, R, \lambda) = c(\lambda)\mathcal{P}_Q(u) + Rd(\lambda)\mathcal{P}_P(u) + O(R^2) + O(R\lambda) = 0.$$

We have $\Gamma(z, 0, 0) = 0$ and $T_u\Gamma(z, 0, 0) = H(\mathcal{P}_Q, z)$ is invertible. Hence it follows from the implicit function theorem that there is a smooth curve $u(R, \lambda)$ of solutions satisfying $u(0, 0) = 0$. We may write $u(R, \lambda) = z + R\mathbf{a} + \lambda\mathbf{b} + g(R, \lambda)$, where $\mathbf{a}, \mathbf{b} \in T_zS(V)$ and the R- λ-derivatives of g vanish at $R = 0, \lambda = 0$, Substituting in the equation for Γ, we find that $\mathbf{b} = 0$ and $\mathbf{a} = (T_z\mathcal{P}_Q)^{-1}(\mathcal{P}_P(z))$.

Next we substitute for u in the radial equation. We find that

$$R_\lambda(u, R) = \lambda R + R^3\rho_z + O(R^4) + O(\lambda R^3).$$

Now we follow the proof of lemma 4.8.1 and show, using $\rho_z \neq 0$, that there is a smooth branch $\gamma_z : [0, \delta_z] \to V \times \mathbb{R}$ of solutions to to $X = 0$ with $\mathbf{d}(\gamma_z) = z$. That the branch is hyperbolic follows by explicit computation of the linearization of $(S_\lambda(u, R), R_\lambda(u, R))$ along the solution curve (see the proof of lemma 4.8.1). Just as in lemma 4.8.1, the off-diagonal blocks are both $O(R^2)$ and dominated by the two diagonal blocks (of order $O(R)$ and $O(R^2)$). □

4.9. Counting branches and finding their location

We indicate how we can estimate the number of solution branches. We restrict attention to representations $(\mathbb{R}^k, G)$ which satisfy conditions (C,IR) (though the methods apply more generally).

It is easy to verify that generic H_k-equivariant bifurcation problems on $\mathbb{R}^k$ have precisely $3^k - 1$ branches of non-trivial solutions. (H_k has $(3^k - 1)/2$ axes of symmetry and each axis of symmetry is associated to two branches of solutions). Noting the trivial solution, $x = 0$, this gives an upper bound of 3^k on the number of solutions to $\lambda x + Q(x) = 0$, $\lambda \neq 0$ fixed, and Q of simple type (generic). More generally, let $Q \in P^3(\mathbb{R}^k, \mathbb{R}^k)$ and define the homogeneous (complex) cubic $H : \mathbb{C}^{k+1} \to \mathbb{C}^k$ by $H(x, z) = xz^2 + Q(x)$, $(x, z) \in \mathbb{C}^{k+1}$. Each component H_i of H determines a hypersurface Σ_i in $\mathbb{P}^k(\mathbb{C})$. For generic Q, Bézout's theorem implies that the intersection $\cap_{i=1}^k \Sigma_i$ consists of exactly 3^k points (for Bézout's theorem, see[**76**, Chapter 8] or, for a topological proof, [**19**]. For the case $k = 2$, see [**132, 91**]). Each of the real points (x, z) with $x \neq 0$ determines a branch of solutions to $\lambda x + Q(x) = 0$. Conversely, if $Q \in P^3(\mathbb{R}^k, \mathbb{R}^k)$ is of simple type, then the conditions of Bézout's theorem hold for H. Thus there can be at most 3^k real points and so a maximum of $3^k - 1$ real solution branches (the trivial solution branch is always real).

All this is quite trivial for $(\mathbb{R}^k, H_k)$ but becomes much more interesting when G is a proper subgroup of H_k. Suppose that we have a smooth family $Q_s \in P^3_G(\mathbb{R}^k, \mathbb{R}^k)$ and $Q_0 \in P^3_{H_k}(\mathbb{R}^k, \mathbb{R}^k)$. Define $X^s(x, \lambda) = \lambda x + Q_s(x)$. If Q_0 is of simple type, then X^0 has exactly $3^k - 1$ solution branches. As we vary s, it is possible that some of the solutions to $H^s(x, z) = xz^2 + Q^s(x)$ become complex. Hence the number of solution branches for X^s may decrease (we give examples of this phenomenon in chapter 5). Even though some solutions may become complex they are constrained to lie on the *same* fixed point space as the original real solution (strictly speaking, the complexification of the fixed point space). Further variation of the parameter s may result in complex solutions becoming real again but they appear on the original fixed point space. Roughly speaking, if $E \subset \mathbb{R}^k$ is a p-dimensional fixed point space for a subgroup of G, then E 'owns' exactly 3^p solutions of $H|E$. These solutions may be real or complex but under G-equivariant variation of $Q \in P^3_G(\mathbb{R}^k, \mathbb{R}^k)$ they can never leave $E \otimes_{\mathbb{R}} \mathbb{C}$.

We sum up this discussion in the following theorem.

THEOREM 4.9.1. *Let $(\mathbb{R}^k, G)$ satisfy conditions (C,IR). Let J be a subgroup of G and suppose that $E = (\mathbb{R}^k)^J$ is of dimension p. Let $X \in \mathcal{S}_w(\mathbb{R}^k, G)$. Then $X|E$ has at most $3^p - 1$ branches of solutions. Furthermore, generically there will be exactly $3^k - 1$ branches of complex solutions and we can count $3^p - 1$ complex solutions for each p-dimensional fixed point subspace of $(\mathbb{R}^k, G)$.*

PROOF. Since we assume $(\mathbb{R}^k, G)$ satisfies conditions (C,IR), it is no loss of generality by lemma 4.4.6 and theorem 4.8.2 to assume that $J^3(X) \in P^3_G(\mathbb{R}^k, \mathbb{R}^k)$ is of hyperbolic type. The result follows from the previous discussion. $\square$

4.10. The symmetric and alternating groups

In this section, we consider the symmetric group S_{k+1}, $k \geq 2$, acting by its standard representation on $V = \{(x_1, \ldots, x_{k+1}) \in \mathbb{R}^{k+1} \mid \Sigma_i x_i = 0\}$. We start by proving that (S_{k+1}, V) satisfies the MISC. This result was originally proved in [**70**] but we follow the alternative proof given in [**73**]. Let A_{k+1} denote the alternating subgroup of S_{k+1}. We show that (A_{k+1}, V) satisfies the MISC if $k \geq 3$. In addition, we obtain detailed information on the branching patterns for (S_{k+1}, V) and (A_{k+1}, V). If $k+1$ is odd, then (S_{k+1}, V) and (A_{k+1}, V) are 2-determined and our results follow from the results of section 4.8. If $k+1$ is even, then (S_{k+1}, V) and (A_{k+1}, V) are 3-determined. In this case, we use results from section 4.8.1 to prove a stability of branching theorem. Our methods also allow us to get information on symmetry breaking and branching patterns for many other subgroups of the symmetric group.

4.10.1. Preliminaries on S_{k+1}. Let $\mathbf{e}_0 = (1,\dots,1) \in \mathbb{R}^{k+1}$. Then $S_{k+1}\mathbf{e}_0 = \{\mathbf{e}_0\}$ and so $V = \{x \in \mathbb{R}^{k+1} \mid (x, \mathbf{e}_0) = 0\}$ is S_{k+1}-invariant. Moreover (V, S_{k+1}) is absolutely irreducible. Let $S(V) = S^{k-1}$ denote the unit sphere of V.

We need some classical results on invariants, equivariants and axes of symmetry for (V, S_{k+1}).

Let $\pi : \mathbb{R}^{k+1} \to V$ denote the orthogonal projection onto V. We have

$$\pi(x) = x - (k+1)^{-1}\,(x, \mathbf{e}_0)\,\mathbf{e}_0, \ \ (x \in \mathbb{R}^{k+1}).$$

We have $d(V, S_{k+1}) = 2$. Indeed, the space $P^2_{S_{k+1}}(V,V)$ of quadratic equivariants is one-dimensional and generated by

$$Q(x_1,\dots,x_{k+1}) = \pi(x_1^2,\dots,x_{k+1}^2), \ \ ((x_1,\dots,x_{k+1}) \in V). \tag{4.13}$$

We call Q is the *basic quadratic equivariant* for (V, S_{k+1}). Since $V^{S_{k+1}} = \{0\}$, $R^2_{S_{k+1}}(V,V) = \{0\}$. Let $h : V \to \mathbb{R}$ be defined by

$$h(x_1,\dots,x_{k+1}) = \Sigma\, x_i^3, \ \ ((x_1,\dots,x_{k+1}) \in V).$$

Then $Q = \frac{1}{3}\mathrm{grad}(h)$. Hence the zeros of $\mathcal{P}_Q$ are the critical points of $h|S^{k-1}$.

4.10.2. Zeroes of the phase vector field. As in section 4.5, it is convenient to introduce a finite S_{k+1}-invariant subset $\mathcal{E}$ of V which parameterizes the zeros of $\mathcal{P}_Q$. Let $p \in \mathbf{k} = \{1,\dots,k\}$ and set $q = k+1-p$. We define

$$\boldsymbol{\varepsilon}_p = (1/p,\dots,1/p,-1/q,\dots,-1/q) \in V \subset \mathbb{R}^{k+1}$$

(with p coordinates $= 1/p$ and q coordinates $= -1/q$). Set $\mathcal{E}_p = S_{k+1}\boldsymbol{\varepsilon}_p$. Then $\mathcal{E}_p$ consists of all points of V with p coordinates equal to $1/p$ and q coordinates equal to $-1/q$. Let $\mathcal{E} = \cup_{p\in\mathbf{k}}\mathcal{E}_p$. If $p+q = k+1$, then $\mathcal{E}_p = -\mathcal{E}_q$. For $\boldsymbol{\varepsilon} \in \mathcal{E}$, we let L_ε denote the line $\mathbb{R}\boldsymbol{\varepsilon}$. The axes of symmetry for (V, S_{k+1}) are the lines L_ε, $\boldsymbol{\varepsilon} \in \mathcal{E}$. If $\boldsymbol{\varepsilon} \in \mathcal{E}$, then $\boldsymbol{\varepsilon}/\|\boldsymbol{\varepsilon}\|$ is a zero of $\mathcal{P}_Q$ and every zero of $\mathcal{P}_Q$ is of this form. Thus the map $\mu : \mathcal{E} \to \mathbf{Z}(\mathcal{P}_Q)$ defined by $\mu(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}/\|\boldsymbol{\varepsilon}\|$ is an isomorphism of S_{k+1}-sets. Every $z \in \mathbf{Z}(\mathcal{P}_Q)$ is a hyperbolic zero of $\mathcal{P}_Q$.

Let $p \in \mathbf{k}$ and $q = k+1-p$. Let $\boldsymbol{\varepsilon} \in \mathcal{E}_p$. Then $Q(\boldsymbol{\varepsilon}) = \alpha\boldsymbol{\varepsilon}$, where $\alpha = 1/p - 1/q$. If $p \neq q$, then $\alpha \neq 0$. Hence if $k+1$ is odd, then Q is non-degenerate and it follows that Q is of hyperbolic type. If $p = q$, then $\alpha = 0$. Consequently if $k+1$ is even, then Q is degenerate and is of relatively hyperbolic type.

PROPOSITION 4.10.1. *Let $k+1$ be odd. Then*

$$P^2_{S_{k+1}}(V,V)_h = P^2_{S_{k+1}}(V,V)_H = \{aQ \mid a \neq 0\}.$$

In particular, $P^2_{S_{k+1}}(V,V)_h$ is a dense open subset of $P^2_{S_{k+1}}(V,V)$.

PROOF. Since $P^2_{S_{k+1}}(V,V) = \mathbb{R}Q$, the proof is immediate. ◻

4.10.3. Subgroups of S_{k+1}. A proof of the following well-known result may be found in [**25**, page 230] or [**157**, 12.7.1].

LEMMA 4.10.2. *Let G be a subgroup of S_{k+1}. Then (V, G) is absolutely irreducible if and only if G is a doubly transitive subgroup of S_{k+1}.*

LEMMA 4.10.3. *Let G be a subgroup of S_{k+1}. If G satisfies either (a) G is a triply transitive subgroup of S_{k+1}, or (b) $k = 4$ and G is the alternating group A_4, then $P^2_G(V, V) = P^2_{S_{k+1}}(V, V)$.*

PROOF. We leave the proof, which is a direct computation, as an exercise. □

4.10.4. The sign and index functions. Let $\mathbf{Z}(Q) = \{x \in S(V) \mid Q(x) = 0\}$ and $\mathbf{Z}(\mathcal{P}_Q)_1 = \mathbf{Z}(\mathcal{P}_Q) \setminus \mathbf{Z}(Q)$. Set $\mathcal{E}^{(1)} = \mu^{-1}(Z(\mathcal{P}_Q)_1)$. It is easy to check that $\mathcal{E}^{(1)} = \cup_{p\in\mathbf{k},\, 2p\neq k+1}\, \mathcal{E}_p$. If $k+1$ is odd, then $\mathcal{E}^{(1)} = \mathcal{E}$. Clearly $\mathbf{Z}(\mathcal{P}_Q)_1$ (respectively, $\mathcal{E}^{(1)}$) is a S_{k+1}-invariant subset of $\mathbf{Z}(\mathcal{P}_Q)$ (respectively, $\mathcal{E}$). Let $P = aQ$, with $a \neq 0$, and $J^P = \lambda x + P(x)$. For each $z \in \mathbf{Z}(\mathcal{P}_Q)_1$, let the solution curve $\gamma^{(z)}$ for J^P be defined as in section 4.6.3. Let $\gamma_z = \gamma^{(z)}|[0, 1]$. The sign and index of γ_z (or $[\gamma_z]$) are given by:

LEMMA 4.10.4. *Let $p \in \mathbf{k}$ be such that $2p \neq k+1$ and let $q = k+1-p$. Let $\boldsymbol{\varepsilon} \in \mathcal{E}_p$ and let $z = \mu(\boldsymbol{\varepsilon})$, so that $z \in \mathbf{Z}(\mathcal{P}_Q)_1$. Let P, J^P and γ_z be as above.*

(1) *Assume that $a > 0$. Then $sgn(\gamma_z) = sgn(p-q)$. If $p < q$, then* $\mathrm{ind}(\gamma_z) = q-1$*, and if $p > q$, then* $\mathrm{ind}(\gamma_z) = q$.
(2) *Assume that $a < 0$. Then $sgn(\gamma_z) = sgn(q-p)$. If $q < p$, then* $\mathrm{ind}(\gamma_z) = p-1$*, and if $q > p$, then* $\mathrm{ind}(\gamma_z) = p$.

PROOF. The result follows from proposition 4.6.8. □

REMARK 4.10.5. All the branches γ_z given by lemma 4.10.4 satisfy $1 \leq \mathrm{ind}(\gamma_z) < k$. Thus γ_z is never a solution branch of sources or sinks.

It is convenient to define two distinct structures of an abstract signed indexed branching pattern on the S_{k+1}-set $\mathcal{E}^{(1)}$. We define sign functions sgn_+, and sgn_- and index functions ind_+ and ind_- on $\mathcal{E}^{(1)}$ as follows:

We have $\mathcal{E}^{(1)} = \cup_{p\in\mathbf{k},\, 2p\neq k+1}\mathcal{E}_p$. Let $p \in \mathbf{k}$ with $2p \neq k+1$, $q = k+1-p$ and $\boldsymbol{\varepsilon} \in \mathcal{E}_p$. We set $\mathrm{sgn}_+(\boldsymbol{\varepsilon}) = \mathrm{sgn}(p-q)$ and $\mathrm{sgn}_-(\boldsymbol{\varepsilon}) = \mathrm{sgn}(q-p) = -sgn_+(\boldsymbol{\varepsilon})$. If $p < q$, we set $\mathrm{ind}_+(\boldsymbol{\varepsilon}) = q-1$ and $\mathrm{ind}_-(\boldsymbol{\varepsilon}) = p$. If $p > q$, then we set $\mathrm{ind}_+(\boldsymbol{\varepsilon}) = q$ and $\mathrm{ind}_-(\boldsymbol{\varepsilon}) = p-1$. Thus $(\mathcal{E}^{(1)}, \mathrm{sgn}_+, \mathrm{ind}_+)$ and $(\mathcal{E}^{(1)}, \mathrm{sgn}_-, \mathrm{ind}_-)$ are abstract signed indexed branching patterns.

LEMMA 4.10.6. *Let $\boldsymbol{\varepsilon} \in \mathcal{E}^{(1)}$ and set $z = \mu(\boldsymbol{\varepsilon})$. Let $P = aQ$, J^P and γ_z be as above. If $a > 0$, then $sgn(\gamma_z) = sgn_+(\boldsymbol{\varepsilon})$ and* $\mathrm{ind}(\gamma_z) = \mathrm{ind}_+(\boldsymbol{\varepsilon})$. *If $a < 0$, then $sgn(\gamma_z) = sgn_-(\boldsymbol{\varepsilon})$ and* $\mathrm{ind}(\gamma_z) = \mathrm{ind}_-(\boldsymbol{\varepsilon})$.

PROOF. The result follows from the definitions and lemma 4.10.4. □

4.10.5. The case $k+1$ odd.

PROPOSITION 4.10.7. *Let $k+1$ be odd and G be a doubly transitive subgroup of S_{k+1}. Then (V,G) is 2-determined.*

PROOF. Since $k+1$ is odd, Q is a hyperbolic element. The proof now follows that of theorem 4.5.11. □

COROLLARY 4.10.8. *Let $k+1$ be odd. Then*

(1) *(V, S_{k+1}) is 2-determined.*
(2) *(V, A_{k+1}) is 2-determined, $k \geq 4$.*
(3) *Let q be a prime power. If we regard $PGL_2(\mathbb{F}_q)$ as a (triply transitive) subgroup of S_{q+1}, then $(V, PGL_2(\mathbb{F}_q))$ is 2-determined.*

The following theorem, which follows from the previous results and discussion, summarizes the results in case $k+1$ is odd.

THEOREM 4.10.9. *Let $k+1$ be odd and G be a doubly transitive subgroup of S_{k+1}.*

(1) *Let $X \in \mathcal{V}_0(V,G)$ satisfy $J^2(X) = aQ$, with $a \neq 0$. Then $X \in \mathcal{S}(V,G)$. If $a > 0$ (respectively, $a < 0$), then the signed indexed branching pattern $\Sigma^\star(X)$ is isomorphic to $(\mathcal{E}, sgn_+, \mathrm{ind}_+)$ (respectively, $(\mathcal{E}, sgn_-, \mathrm{ind}_-)$).*
(2) *$\{(G_\varepsilon) \mid \boldsymbol{\varepsilon} \in \mathcal{E}\}$ is an admissible family of symmetry breaking isotropy types for (V,G).*
(3) *Suppose that $P^2_G(V,V) = P^2_{S_{k+1}}(V,V)$. If $X \in \mathcal{S}(V,G)$, then the signed, indexed branching pattern $\Sigma^\star(X)$ is isomorphic to either $(\mathcal{E}, sgn_+, \mathrm{ind}_+)$ or $(\mathcal{E}, sgn_-, \mathrm{ind}_-)$. In particular $\{(G_\varepsilon) \mid \boldsymbol{\varepsilon} \in \mathcal{E}\}$ is the unique admissible family of symmetry breaking isotropy types for (V,G) and (G_ε) is generically symmetry breaking for every $\varepsilon \in \mathcal{E}$.*

EXAMPLE 4.10.10. Let $G = \mathrm{Aff}_1(\mathbb{F}_5) \subset S_5$. Then the isotropy type (G_{ε_2}) is a submaximal symmetry breaking isotropy type. Thus (V,G) does not satisfy the MISC. (But the results are quite different for $(V, G \times \langle -I_V \rangle)$, see section5.4.)

4.10.6. The case $k+1$ even. Let $k+1 = 2\ell$ be even and G be a subgroup of S_{k+1} such that

(a) (V,G) is absolutely irreducible,
(b) $P^2_G(V,V) = P^2_{S_{k+1}}(V,V)$.

It follows from our previous results that $\mathbf{Z}(Q) = \mu(\mathcal{E}_\ell)$ and that each $z \in \mathbf{Z}(Q)$ is a hyperbolic zero of $\mathcal{P}_Q$. However, if $z \in \mathbf{Z}(Q)$, then $\alpha(Q,z) = 0$, so that theorem 4.8.2 does not apply. Instead we use the results of section 4.8.1.

Following the notation of section 4.8.1, let $T = cQ + P \in \mathcal{A}_0(V,G)$. We define a sign function sgn_T and an index function ind_T on the G-set $\mathcal{E}$. If $\boldsymbol{\varepsilon} \in \mathcal{E}^{(1)}$ we

define

$$\begin{aligned}
\mathrm{sgn}_T(\boldsymbol{\varepsilon}) &= \mathrm{sgn}_+(\boldsymbol{\varepsilon}), \text{ if } c > 0,\\
&= \mathrm{sgn}_-(\boldsymbol{\varepsilon}), \text{ if } c < 0,\\
\mathrm{ind}_T(\boldsymbol{\varepsilon}) &= \mathrm{ind}_+(\boldsymbol{\varepsilon}), \text{ if } c > 0,\\
&= \mathrm{ind}_-(\boldsymbol{\varepsilon}), \text{ if } c < 0.
\end{aligned}$$

Given $\boldsymbol{\varepsilon} \in \mathcal{E}_\ell = \mu^{-1}(\mathbf{Z}(Q))$, set $z = \mu(\boldsymbol{\varepsilon})$. We define $\mathrm{sgn}_T(\boldsymbol{\varepsilon}) = -\mathrm{sgn}(\rho_z(P))$. If $\mathrm{sgn}_T(\boldsymbol{\varepsilon}) = +1$, then set $\mathrm{ind}_T(\boldsymbol{\varepsilon}) = \ell$ and if $\mathrm{sgn}_T(\boldsymbol{\varepsilon}) = -1$, set $\mathrm{ind}_T(\boldsymbol{\varepsilon}) = \ell - 1$.

THEOREM 4.10.11. *Assume that $k+1 = 2\ell$ is even. Let G be a subgroup of S_{k+1} such that (V,G) is absolutely irreducible and $P^2_G(V,V) = P^2_{S_{k+1}}(V,V)$. Let $X \in \mathcal{V}_0(V,G)$ be such that $J^3(f) \in \mathcal{A}_0(V,G)$ and set $J^3(f) = cQ + P = T$ as above.*

(1) *For each $z \in \mathbf{Z}(\mathcal{P}_Q)$ there exists a smooth curve $\gamma^{(z)} : [-\delta_z, \delta_z] \to V \times \mathbb{R}$ of solutions to $X = 0$ with $\mathbf{d}(\gamma_z) = z$.*
(2) *$X \in \mathcal{S}(V,G)$.*
(3) *$\Sigma^\star(X)$ is isomorphic to $(\mathbf{Z}(\mathcal{P}_Q), \mathrm{sgn}_T, \mathrm{ind}_T)$.*

In particular G-equivariant bifurcation problems on (V,G) are 3-determined.

PROOF. Statements (1,2) and the 3-determinacy follow from lemma 4.8.5, theorem 4.8.3 and lemma 4.8.1. Statement (3) follows from the definition of the index and sgn functions together with the earlier results holding for the 2-determined case. ☐

EXAMPLE 4.10.12. Let $k+1$ be even. It follows from theorem 4.10.11 that (V, S_{k+1}) is 3-determined and, if $k \geq 3$, that (V, A_{k+1}) is 3-determined.

COROLLARY 4.10.13. *Assume $k+1$ is even and let G be as in theorem 4.10.11. The family $\{(G_\varepsilon) \mid \boldsymbol{\varepsilon} \in \mathcal{E}\}$ is the unique admissible family of symmetry breaking isotropy types for (V,G). Consequently the isotropy type (G_ε) is generically symmetry breaking for every $\boldsymbol{\varepsilon} \in \mathcal{E}$.*

PROPOSITION 4.10.14. *($k+1$ arbitrary). (V, S_{k+1}) satisfies the MISC. If $k \geq 3$, then (V, A_{k+1}) satisfies the MISC.*

PROOF. Set $W = S_{k+1}$ and, if $k \geq 3$, $G = A_{k+1}$. It follows from corollary 4.10.13 that $\{(W_\varepsilon) \mid \boldsymbol{\varepsilon} \in \mathcal{E}\}$ (respectively, $\{(G_\varepsilon) \mid \boldsymbol{\varepsilon} \in \mathcal{E}\}$) is the unique admissible family of symmetry breaking isotropy types for (V,W) (respectively, (V,G)). But now it is easy to show that for every $\boldsymbol{\varepsilon} \in \mathcal{E}$, the line $\mathbb{R}\boldsymbol{\varepsilon}$ is an axis of symmetry for (V,W) (respectively, (V,G)). Hence, for every $\boldsymbol{\varepsilon} \in \mathcal{E}$, the isotropy type (W_ε) (respectively, (G_ε)) is a maximal isotropy type. ☐

EXAMPLE 4.10.15. Let q be a prime power ≥ 7 and assume that $q \neq 8$. Let $G = \mathrm{PGL}_2(\mathbb{F}_q) \subset S_{q+1}$. Then an easy combinatorial argument shows that the isotropy type (G_{ε_3}) is submaximal. Thus it follows from theorem 4.10.11 and proposition 4.10.14 that (V,G) does not satisfy the MISC.

4.11. The groups $S_{k+1} \times \mathbb{Z}_2$ and $A_{k+1} \times \mathbb{Z}_2$

Let $G = S_{k+1} \times \mathbb{Z}_2$ and $H = A_{k+1} \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = \langle -I_{\mathbb{R}^k} \rangle$. In this section we examine the equivariant bifurcation theory of (V, G) and (V, H), where $V \cong \mathbb{R}^k$ is as in the previous section. All of the results follow quickly from direct calculations on equivariants and zeros of phase vector fields. In most cases, we simply state results without giving proofs. The equivariant bifurcation theory of these representations is of interest in the study of period doubling bifurcations of representations with S_{k+1} or A_{k+1} symmetry (see [**5**] and chapter 10).

Let $k \geq 3$. The action of $\mathbb{Z}_2 = \langle -I_{\mathbb{R}^k} \rangle$ commutes with the action of S_{k+1} on V and so we get an absolutely irreducible representation of the product group $G = S_{k+1} \times \mathbb{Z}_2$ on V. If $k+1 = 4$, then $(V, G) \cong (\mathbb{R}^3, H_3)$. Since $k \geq 3$, (V, H) is also absolutely irreducible. We list below a number of computational results on equivariants and zeros of phase vector fields for (V, G) and (V, H). These all follow trivially from the corresponding standard results for (V, S_{k+1}).

(a) $P^3_G(V, V) = P^3_{S_{k+1}}(V, V) = P^3_H(V, V)$.
(b) If m is even, then $P^m_H(V, V) = P^m_G(V, V) = \{0\}$.
(c) $P^3_G(V, V) = P^3_H(V, V)$ is 2-dimensional. It has a basis $\{R, C\}$, where R and C are defined as follows:

$$R(x) = \|x\|^2 x, \ \ (x \in V).$$

If $\pi : \mathbb{R}^{k+1} \to V$ denotes the orthogonal projection on V then

$$C(x) = \pi(x_1^3, \ldots, x_{k+1}^3), \ \ (x \in V).$$

We refer to C as the basic cubic equivariant for (V, G) and (V, H).

It follows from (b,c) that $d(V, G) = d(V, H) = 3$.

4.11.1. The zeros of $\mathcal{P}_C$. We find the zeros of the phase vector field $\mathcal{P}_C$. First, some notation. Let $\Gamma(k+1)$ be the set of all triples $(p, q, r) \in \mathbb{N}^3$ satisfying

$$p + q + r = k + 1, \ \ p \leq q \leq r, \ \ q > 0.$$

We define the plane $D \subset \mathbb{R}^3$ by

$$D = \{(a, b, c) \in \mathbb{R}^3 \mid a + b + c = 0\}.$$

Given $(p, q, r) \in \Gamma(k+1)$, set

$$D(p, q, r) = \{(a, b, c) \in D \mid pa + qb + rc = 0\}.$$

If $p = q = r$, then $D(p, q, r) = D$. If $p < r$, then $D(p, q, r)$ is a line through 0 in D. In the this case we set $y(p, q, r) = (r - q, p - r, q - p)$ and $D(p, q, r) = \mathbb{R}y(p, q, r)$.

For each $(p, q, r) \in \Gamma(k+1)$, define the linear map $\tau = (\tau_1, \ldots \tau_{k+1}) : D \to \mathbb{R}^{k+1}$ by

$$\begin{aligned} \tau_i(a, b, c) &= a, \text{ if } 1 \leq i \leq p, \\ &= b, \text{ if } p < i \leq p + q, \\ &= c, \text{ if } p + q < i \leq k + 1. \end{aligned}$$

The map τ is injective and $\tau(D) \cap V = \tau(D(p,q,r))$. Set $C(p,q,r) = \tau(D(p,q,r))$ and $E(p,q,r) = S_{k+1}(C(p,q,r))$. If $p < r$, then $E(p,q,r)$ is a finite union of lines (through 0) in V. If $p = q = r$, then $E(p,q,r) = E(p,p,p)$ is a finite union of 2-planes.

Set $\mathbf{Z}(p,q,r) = E(p,q,r) \cap S(V)$. If $p < r$, set $z(p,q,r) = y(p,q,r)/\|y(p,q,r)\|$. In this case, $\mathbf{Z}(p,q,r)$ is equal to $Gz(p,q,r)$.

We have the following description of $\mathbf{Z}(\mathcal{P}_C)$.

PROPOSITION 4.11.1. *$\mathbf{Z}(\mathcal{P}_C) = \cup_{(p,q,r)\in\Gamma(k+1)}\mathbf{Z}(p,q,r)$.*

REMARK 4.11.2. Assume that $k+1$ is divisible by 3, say $k+1 = 3p$. Then $\mathbf{Z}(p,p,p)$ is a finite union of great circles on $S(V)$. Thus each $z \in \mathbf{Z}(p,p,p)$ is a non-simple zero of $\mathcal{P}_C$.

We define a polynomial function $M : \mathbb{R}^3 \to \mathbb{R}$ by

$$M(x,y,z) = 2(x^3 + y^3 + z^3) + 6xyz - 3(xy(x+y) + yz(y+z) + zx(z+x)).$$

PROPOSITION 4.11.3. *Let $(p,q,r) \in \Gamma(k+1)$ with $p < r$.*

(a) *Assume that $p = 0$. Then $z(p,q,r)$ is a hyperbolic zero of $\mathcal{P}_C$ if $r \neq 2q$ and $z(p,q,r)$ is a non-simple zero of $\mathcal{P}_C$ if $r = 2q$.*
(b) *Assume that $p > 0$ and $(p-q)(q-r) = 0$. Then $z(p,q,r)$ is a hyperbolic zero of $\mathcal{P}_C$.*
(c) *Assume that $r > q > p > 0$. Then $z(p,q,r)$ is a hyperbolic zero of $\mathcal{P}_C$ if $M(p,q,r) \neq 0$ and $z(p,q,r)$ is a non-simple zero of $\mathcal{P}_C$ if $M(p,q,r) = 0$.*

The proof depends on the explicit calculation of the linearization $T_{z(p,q,r)}\mathcal{P}_C$.

LEMMA 4.11.4. *Let $(p,q,r) \in \Gamma(k+1)$ with $0 < p < q < r$. If 3 does not divide $k+1$, then $M(p,q,r) \neq 0$.*

COROLLARY 4.11.5. *If 3 does not divide $k+1$, then every zero of the phase vector field $\mathcal{P}_C$ is hyperbolic.*

4.11.2. Applications.

PROPOSITION 4.11.6. *Let $(p,q,r) \in \Gamma(k+1)$ with $p < r$. Assume that $z(p,q,r)$ is a hyperbolic zero of $\mathcal{P}_C$. Then the isotropy type $(G_{z(p,q,r)})$ is a generically symmetry breaking isotropy type for (V,G).*

The same result holds for (V,H).

PROPOSITION 4.11.7. *Let $(p,q,r) \in \Gamma(k+1)$ with $p < r$.*

(1) *If $0 < p < q < r$, then $(G_{z(p,q,r)})$ is a submaximal isotropy type for (V,G).*
(2) *If either $p = 0$ or $p > 0$ and $(p-q)(q-r) = 0$, then $\mathbb{R}z(p,q,r)$ is an axis of symmetry for (V,G). In this case the isotropy type $(G_{z(p,q,r)})$ is a maximal isotropy type which is generically symmetry breaking for (V,G).*

The same results holds for (V,H).

REMARK 4.11.8. Let $k+1$ be divisible by 3, say $k+1=3\ell$. Then $(0,\ell,2\ell)\in\Gamma(k+1)$. We see from proposition 4.11.3 that $z(0,\ell,2\ell)$ is not a simple zero of $\mathcal{P}_C$, so that proposition 4.11.6 does *not* apply to show that $(G_{z(0,\ell,2\ell)})$ is a symmetry breaking isotropy type. However, it *does* follow from proposition 4.11.7 that the isotropy type $(G_{z(0,\ell,2\ell)})$ is generically symmetry breaking. Note also that $z(0,\ell,2\ell)$ lies on the great circle $\mathbf{Z}(\ell,\ell,\ell)\subset\mathbf{Z}(\mathcal{P}_C)$.

PROPOSITION 4.11.9. *Assume that* 3 *does not divide* $k+1$*. Then the following results hold*

(1) $P^3_G(V,V)_h = P^3_H(V,V)_h$ *is a dense open subset of* $P^3_G(V,V)=P^3_H(V,V)$*. Hence* (V,G) *and* (V,G) *are 3-determined.*
(2) $\{(G_{z(p,q,r)}) \mid (p,q,r)\in\Gamma(k+1)\}$ *is the unique admissible family of symmetry breaking isotropy types for* (V,G)*. Thus, for each* $(p,q,r)\in\Gamma(k+1)$*, the isotropy type* $G_{z(p,q,r)}$ *is generically symmetry breaking. The same result holds for* (V,H)*.*
(3) *Let* $f\in\mathcal{V}_0(V,G)$ *be stable. Then the branching pattern* $\Sigma(f)$ *is isomorphic as a* G*-set to* $\mathbf{Z}(\mathcal{P}_C)=\cup_{(p,q,r)\in\Gamma(k+1)}\mathbf{Z}(\mathcal{P}_C)$*.*

PROPOSITION 4.11.10. *If* $k+1=4,5$*, then* (V,G) *and* (V,H) *satisfy the MISC. If* $k+1\geq 6$*, then there exist submaximal symmetry breaking isotropy types for* (V,G) *and for* (V,H)*.*

4.12. Appendix: Proof of theorem on hyperbolic elements

We assume standard results on semialgebraic sets (basic definitions, results and references on semialgebraic sets may be found in section 6.8). In order to avoid a lengthy exposition of elementary algebraic geometry, we need to assume some basic properties of constructible and algebraic subsets of a complex algebraic variety (we refer the reader to [**132**, Chapter 2] for the theory of constructible sets).

A subset $\mathbf{X}$ of a complex vector space *constructible* if it can be written as a finite union of sets defined by complex polynomial equalities and inequalities (that is, by conditions of the form $p(x)=0$ and $q(x)\neq 0$). More generally, we can define constructible subsets of complex algebraic varieties.

Let $\mathbf{E}$ and $\mathbf{W}$ be (finite dimensional) complex vector spaces. We list some basic properties of constructible sets (see [**132**] for proofs).

A1: If $\mathbf{X}\subset\mathbf{E}$ is an algebraic set and $p:\mathbf{E}\to\mathbf{W}$ is a complex polynomial map, then $p(\mathbf{X})$ is a constructible subset of $\mathbf{W}$.

A2: If $\mathbf{X}\subset\mathbf{E}$ is a constructible set containing an interior point (usual topology), then $\mathbf{X}$ is dense in $\mathbf{E}$ (usual topology).

A3: If $\mathbf{X} \subset \mathbf{E}$ is constructible, then $\mathbf{X}$ is contained in an algebraic subset of $\mathbf{E}$ of the same dimension.

From now on we follow the notation of section 4.6.4. If E is a linear subspace of $P^d(V,V)$, $d \geq 2$, let $\mathbf{E} \subset P^d(\mathbf{V},\mathbf{V})$ denote the complexification of E. Let $\mathbf{Y}$ be the complex algebraic subvariety of $\mathbf{E} \times \Sigma^n \times \mathbb{C}$ defined by

$$\mathbf{Y} = \{(Q, u, \lambda) \mid Q(u) = \lambda u\},$$

and $\mathbf{Y}_\mathbb{R}$ denote the set of real points of $\mathbf{Y}$. That is, $\mathbf{Y}_\mathbb{R} = \mathbf{Y} \cap (E \times S^n \times \mathbb{R})$. We identify $\mathbb{R}^{n+1}$ (respectively, $\mathbb{C}^{n+1}$) with the space of all real (respectively, complex) polynomials in one variable which are of degree $n+1$ and have leading coefficient 1. Thus $\mathbf{a} = (a_1, \dots, a_{n+1}) \in \mathbb{R}^{n+1}$ will correspond to the polynomial

$$p_\mathbf{a}(t) = t^{n+1} + a_1 t^n + \ldots + a_{n+1}.$$

It is well-known (and elementary) that there exists an n-dimensional semialgebraic subset X of $\mathbb{R}^{n+1}$ such that $p_\mathbf{a}$ has a root with real part zero if and only if $\mathbf{a} \in X$. Let $Z_1 \subset \mathbb{R}^{n+1}$ be the set of all $\mathbf{a} \in \mathbb{R}^{n+1}$ such that the polynomial $p_\mathbf{a}$ has a factor of the form $t^2 + \alpha$ for some $\alpha \in \mathbb{R}$. Straightforward elimination theory shows that Z_1 is the zero locus of a non-trivial real polynomial map $h : \mathbb{R}^{n+1} \to \mathbb{R}$. Let Z_2 be the hyperplane in $\mathbb{R}^{n+1}$ defined by $a_{n+1} = 0$ and let $Z = Z_1 \cup Z_2$. It is clear that $X \subset Z$. Let $\mathbf{Z}$ denote the complexification of Z. Thus

$$\mathbf{Z} = \{\mathbf{a} = (a_1, \dots, a_{n+1}) \in \mathbb{C}^{n+1} \mid a_{n+1} h(\mathbf{a}) = 0\}.$$

Note that $\mathbf{a} \in \mathbf{Z}$ if and only if either $p_\mathbf{a}(0) = 0$ or $p_\mathbf{a}$ has a factor of the form $t^2 + \alpha$ for some $\alpha \in \mathbb{C}$.

Now let $\pi : \mathbf{Y} \to \mathbb{R}^{n+1}$ be the map which assigns to $(Q, u, \lambda) \in \mathbf{Y}$ the coefficients of the characteristic polynomial of $DQ(u) - \lambda I_V$. Let $\rho : \mathbf{Y} \to \mathbf{E}$ denote the restriction of the projection $\mathbf{E} \times \Sigma^n \times \mathbb{C} \to \mathbf{E}$. Clearly π, ρ are the restrictions of polynomial maps and $\pi(\mathbf{Y}_\mathbb{R}) \subset \mathbb{R}^{n+1}$ and $\rho(\mathbf{Y}_\mathbb{R}) \subset E$.

LEMMA 4.12.1. *Assume that $\mathbf{E} \subset P^d(\mathbf{V},\mathbf{V})$ contains an element Q of hyperbolic type. Then $\rho\pi^{-1}(\mathbf{Z}) \subset \mathbf{E}$ has no interior points.*

PROOF. Since π is the restriction of a polynomial map, $\pi^{-1}(\mathbf{Z})$ is an algebraic subset of $\mathbf{Y}$. Hence $\rho\pi^{-1}(\mathbf{Z})$ is constructible by A1. If $\rho\pi^{-1}(\mathbf{Z})$ contains an interior point, then it is dense in $\mathbf{E}$ by A2. But this contradicts lemma 4.6.13. □

The next lemma is an immediate consequence of A3 and lemma 4.12.1.

LEMMA 4.12.2. *Assume that $\mathbf{E}$ contains a map of hyperbolic type. Then $\rho\pi^{-1}(\mathbf{Z})$ is contained in an algebraic subset $\mathbf{T}$ of $\mathbf{E}$ of codimension ≥ 1.*

LEMMA 4.12.3. *If E contains a map of hyperbolic type, then there exists an open and dense semi-algebraic subset of E consisting of maps of hyperbolic type.*

PROOF. First of all, note that the set N of elements of E which are not of hyperbolic type form a closed semialgebraic set. It suffices to prove that the interior of N is empty. Let U be an open subset of E contained in N. Clearly $N \subset \rho\pi^{-1}(\mathbf{Z})$. By lemma 4.12.2, $\rho\pi^{-1}(\mathbf{Z})$ is contained in an algebraic subset $\mathbf{T}$ of $\mathbf{E}$ which is of codimension ≥ 1. Hence $\mathbf{T} \cap E$ cannot contain a non-empty open subset of E. Thus U must be empty. $\square$

Proof of theorem 4.6.15. Lemma 4.12.3 implies theorem 4.6.15 in case E contains a map of hyperbolic type. The proof of theorem 4.6.15 in case E contains a map of simple type is similar (and much easier). If E contains a map of relatively hyperbolic type (or relatively simple type), the proof is similar, but somewhat more complicated. We leave details to the reader. $\square$

4.13. Notes on chapter 4

Most of the chapter is based on the work of Roger Richardson and the author on steady state equivariant bifurcation [**70, 71, 72, 73, 57**]. The original motivation was the Maximal Isotropy Subgroup Conjecture (MISC) proposed by Golubitsky [**79**] (in the context of bifurcation theory) and by Michel [**126**] (in the context of Higgs-Landau theories and phase transitions) and we refer to the introduction of [**72**] where there is a more extended discussion, with references. (The article [**71**] is devoted to MISC for finite reflection groups and considers examples we do not discuss in this book – including symmetry breaking for the icosahedral group and the Weyl group of type F_4.)

Historically speaking, the first general results on equivariant bifurcation appear in the important 1973 paper of Ruelle [**151**]. By the mid 1980's there was a substantial body of results on equivariant bifurcation theory and its applications. The state of the art at that time is described very well in the book *Singularities and Groups in Bifurcation Theory* by Golubitsky, Stewart and Schaeffer [**84**]. This book contains an extensive set of references as well as many interesting applications (see [**86**] for an updated perspective). However, the methods (and aims) of [**84**] are rather different from those given here. Unlike [**84**], we make no real use of the theory of singularities of differentiable maps. This is partly because we do not investigate equivariant unfolding theory (see [**84**, Chapters XIV,XV]) – our emphasis is strictly on the codimension one theory. More significantly, singularity theory methods seem inappropriate when it comes to proving genericity theorems or considering actions by non-finite groups. For the analysis of genericity, we use methods based on *equivariant transversality* (see chapter 6). However, the results of chapter 4 show that in many cases one can use quite elementary techniques to verify stability, genericity and determinacy. Nowhere, for example, do we make any use of Schwarz's theorem on smooth invariants [**154**]. Finally, we mention the work of Damon [**37, 38, 39**] on equivariant bifurcation and solution branches. This work uses singularity theory methods to obtain general results on equivariant bifurcations.

CHAPTER 5

Equivariant Bifurcation Theory: Dynamics

In this chapter we investigate the dynamics that can be generated in generic equivariant bifurcations. Unlike what happens in the asymmetric case, when a sink loses stability in a generic steady state equivariant bifurcation, we can expect phenomena ranging from long period limit cycles, through heteroclinic networks to chaotic dynamics. We also give some examples of what can happen in generic equivariant Hopf bifurcations.

We start by describing a very useful technical and conceptual tool for the analysis of steady state bifurcations: the *invariant sphere theorem*. In many cases, we can use this result to remove dependence on parameters and reduce the dimension of the phase space by one. Later in the chapter we give a variant of this result that is appropriate for the analysis of the equivariant Hopf bifurcation and is closely related to the classical blowing-up transformation of complex algebraic geometry.

5.1. The invariant sphere theorem

Before we state the invariant sphere theorem, we need some definitions.

DEFINITION 5.1.1. Suppose that X and Y are topological spaces and Φ_t, Ψ_t are flows on X and Y respectively. The flows Φ_t and Ψ_t are *topologically equivalent* if there is a homeomorphism $h : X \to Y$ such that h maps each trajectory of Φ_t onto a (unique) trajectory of Ψ_t. We call h a *topological equivalence* between Φ_t and Ψ_t. If additionally $h\Phi_t = \Psi_t h$, we say h is a *topological conjugacy* and the flows are *topologically conjugate*

REMARK 5.1.2. Let M be compact and X be a smooth vector field on M with flow Φ_t^X. Then X (or Φ_t) is *structurally stable* if for all smooth vector fields Y sufficiently C^1-close to X, Φ_t^Y is topologically equivalent to Φ_t^X. (See Smale's survey [**163**] for more details on structural stability and its characterization.)

Let V be an $(n+1)$-dimensional real vector space with inner product $(\ ,\ )$ and norm $\|\ \|$. We denote the unit sphere of V by S^n.

DEFINITION 5.1.3. A polynomial $Q \in P^{2p+1}(V, V)$, $p \geq 1$, is *contracting* if

$$(Q(u), u) < 0, \quad (u \in S^n).$$

EXERCISE 5.1.4. Show that if (V, G) is an absolutely irreducible representation and there exists an inner product on V relative to which $Q \in P_G^{2p+1}(V, V)$ is contracting, then Q is contracting with respect to every G-invariant inner product on V. In particular, the contractivity of Q is defined independently of the choice of iG-invariant inner product on an absolutely irreducible representation.

THEOREM 5.1.5 ([**57**, Theorem 5.1]). *Let $p \geq 1$ and suppose that $Q \in P^{2p+1}(V, V)$ is contracting. Then, for every $\lambda > 0$, there exists a unique n-dimensional sphere $S(\lambda) \subset V \setminus \{0\}$ which is invariant by the flow of*

$$x' = \lambda x + Q(x). \tag{5.1}$$

Further,

(a) *$S(\lambda)$ is globally attracting in the sense that every trajectory $x(t)$ of (5.1) with nonzero initial condition is asymptotic to $S(\lambda)$ as $t \to +\infty$.*
(b) *$S(\lambda)$ is embedded as a topological submanifold of V and the bounded component of $V \setminus S(\lambda)$ contains the origin.*
(c) *The flow of (5.1) restricted to $S(\lambda)$ is topologically equivalent to the flow of the phase vector field $\mathcal{P}_Q$.*

PROOF. We start by simplifying (5.1) using rescaling and coordinate transformations. First of all, transform (5.1) using generalized spherical polar coordinates $x = Ru$, $(u, R) \in S^n \times \mathbb{R}$ (see section 4.7). We obtain

$$u' = R^{2p}\mathcal{P}_Q(u), \tag{5.2}$$
$$R' = \lambda R + R^{2p+1}\rho(u), \tag{5.3}$$

where $\rho(u) = (Q(u), u)$ and $\mathcal{P}_Q(u) = Q(u) - \rho(u)u$ is the phase vector field defined in section 4.7. Since $x = 0$ is obviously a global sink for $\lambda \leq 0$, it is no loss of generality to assume that $\lambda > 0$ in what follows. We make the scaling transformations $R = \lambda^{\frac{1}{2p}} r$ and $s = \lambda t$. If we let $\dot{r}, \dot{u}$ denote the derivatives of u and r with respect to s, we find that (5.2,5.3) transform to

$$\dot{u} = r^{2p}\mathcal{P}_Q(u), \tag{5.4}$$
$$\dot{r} = r + r^{2p+1}\rho(u), \tag{5.5}$$

Since it is no loss of generality to assume that $r \geq 0$, we may make a further transformation $I = r^{2p}$. The resulting set of equations is given by

$$\dot{u} = I\mathcal{P}_Q(u), \tag{5.6}$$
$$\dot{I} = 2pI(1 + I\rho(u)). \tag{5.7}$$

Since ρ is strictly negative, it follows that solutions to (5.6,5.7) are defined for all positive time. Clearly $I = 0$ (that is $S^n \times \{0\}$) is an invariant repulsive sphere for this system. Let $\rho_Q = \inf_{u \in S^n} \rho(u)$ and set $I_Q = -1/(2\rho_Q)$. The vector field defined by (5.6,5.7) is transverse to $S^n \times \{I_Q\}$ and outward pointing. In particular, if $(u(s), I(s))$ is a trajectory of (5.6,5.7) with $I(0) \geq I_Q$, then

$$I(s) \geq I_Q, \quad \text{all } s \geq 0.$$

Let $C^0(S^n, \mathbb{R})$ denote the Banach space of all continuous $\mathbb{R}$-valued maps on S^n, supremum norm topology. Denote the associated norm by $\| \|_0$. Let $\mathbf{X}$ denote the closed subspace of $C^0(S^n, \mathbb{R})$ defined by

$$\mathbf{X} = \{\xi \in C^0(S^n, \mathbb{R}) \mid \xi(S^n) \subset [I_Q, \infty)\}.$$

Every element $\xi \in \mathbf{X}$ determines a continuous graph map

$$\mathrm{graph}_\xi : S^n \to S^n \times \mathbb{R},$$

defined by $\mathrm{graph}_\xi(u) = (u, \xi(u))$, $u \in S^n$. Let $\mathrm{graph}(\xi)$ denote the image of graph_ξ. Obviously $\mathrm{graph}(\xi)$ is a topologically embedded sphere, homeomorphic to S^n, which separates $S^n \times \mathbb{R}$ into two pieces, one containing $S^n \times \{0\}$.

We construct the invariant spheres $S(\lambda)$ by proving that there exists a unique flow-invariant graph contained in $S^n \times [I_Q, \infty)$. In order to do this we show that the time-1 map of the flow of (5.6,5.7), restricted to $S^n \times [I_Q, \infty)$, induces a contractive 'graph transform' operator on $\mathbf{X}$. We start by observing that on $S^n \times [I_Q, \infty)$, the phase portrait of (5.6,5.7) is identical to that of the system obtained by dividing the right hand side of (5.6,5.7) by I.

$$\dot{u} = \mathcal{P}_Q(u), \tag{5.8}$$

$$\dot{I} = 2p(1 + I\rho(u)). \tag{5.9}$$

This system is a skew product system over $\dot{u} = \mathcal{P}_Q(u)$. If we let $\Phi_t = (\Phi_t^u, \Phi_t^I)$, $t \geq 0$, denote the semiflow of (5.8,5.9), then for initial conditions $A_1 = (u_1, R_1)$, $A_2 = (u_2, R_2)$, with $u_1 = u_2$, we have $\Phi_t^u(A_1) = \Phi_t^u(A_2)$, all $t \geq 0$. Hence Φ_t^u is a function of $u \in S^n$.

We define the graph transform operator $\mathcal{G} : \mathbf{X} \to \mathbf{X}$ by

$$\mathcal{G}(\xi)(u) = \Phi_1^I(\Phi_{-1}^u(u), \xi(\Phi_{-1}^u(u))), \quad (u \in S^n, \xi \in \mathbf{X}). \tag{5.10}$$

We have

$$\mathrm{graph}(\mathcal{G}(\xi)) = \Phi_1(\mathrm{graph}(\xi)).$$

We claim that $\mathcal{G}$ is a *contraction* mapping. For this, fix $u_0 \in S^n$ and choose $I_1, I_2 \in [I_Q, \infty)$. Let $(u_i(s), I_i(s))$ denote the solutions of (5.8,5.9) with $u_i(0) = u_0$, $I_i(0) = I_i$, $i = 1, 2$. Since Φ^u is independent of I, we may set $u_i(s) = u(s)$, $i = 1, 2$, where $u(s) = \Phi_s^I(u_0)$. Set $\Delta(s) = (I_1 - I_2)(s)$. Computing $\dot{\Delta}$ using (5.9), we find that

$$\dot{\Delta} = 2p\rho(u)\Delta.$$

Now for all $u \in S^n$, $2p\rho(u) \leq -K$, where $K = \inf_{u \in S^n} -\rho(u) > 0$. It follows from Gronwall's inequality (or directly) that $|\Delta(1)| \leq e^{-K}|\Delta(0)|$ and so

$$|I_1(1) - I_2(1)| \leq e^{-K}|I_1 - I_2|. \tag{5.11}$$

In terms of the operator $\mathcal{G}$, we have

$$\|\mathcal{G}(\xi) - \mathcal{G}(\eta)\|_0 \leq e^{-K}\|\xi - \eta\|_0, \quad (\xi, \eta \in \mathbf{X}).$$

Since $e^{-K} < 1$, it follows that $\mathcal{G}$ is a contraction mapping and so $\mathcal{G}$ has a unique fixed point $\Xi \in \mathbf{X}$.

As we have already pointed out, graph(Ξ) defines a topologically embedded sphere $\tilde{S} \subset S^n \times [I_Q, \infty)$ which is homeomorphic to S^n and separates $S^n \times [I_Q, \infty)$ into two components. Since every trajectory with initial condition $(I_1, u_1) \in S^n \times (0, \infty)$ eventually exits $S^n \times (0, I_Q)$, it follows that the forward trajectory of every point of $S^n \times (0, \infty)$ is forward asymptotic to $\tilde{S}$. Consequently, every flow-invariant subset of $S^n \times (0, \infty)$ must be contained in $\tilde{S}$. In particular, $\tilde{S}$ is the unique n-dimensional flow-invariant sphere contained in $S^n \times (0, \infty)$.

We construct the required invariant spheres $S(\lambda) \subset V$, $\lambda > 0$, by transforming $\tilde{S}$ back to V by our λ-dependent coordinate and rescaling transformations. The composite of these transformations determines a topological equivalence between the flow of (5.8,5.9) restricted to $\tilde{S}$ and the flow induced by (5.1) on $S(\lambda)$. Finally observe that the flow on $\tilde{S}$ is topologically equivalent to the flow of $\mathcal{P}_Q$. Indeed, the required topological equivalence is obtained by restricting the projection map $S^n \times \mathbb{R} \to S^n$ to $\tilde{S} = \text{graph}(\Xi)$. $\square$

Remarks 5.1.6. (1) Although the family of invariant spheres given by theorem 5.1.5 may have some differentiability properties, they will almost never be *smoothly* (that is C^∞) embedded. It is easy to construct examples where the invariant spheres are not even C^1-embedded (see exercises).
(2) If we assume that V is a G-representation and Q is G-equivariant, then the invariant spheres given by theorem 5.1.5 are G-invariant. This can be seen in two ways. First, we can make all the constructions and definitions used in the proof of the theorem G-equivariant. (For example, work with the closed subspace of continuous G-invariant maps on S^n.) Alternatively, note that it follows by the G-equivariance of the flow and the uniqueness of the invariant spheres that all the spheres are G-invariant.

Exercise 5.1.7. Consider the differential equation $X' = X + Q(X)$, where $Q : \mathbb{R}^2 \to \mathbb{R}^2$ is homogeneous of degree 3 and contracting. Find examples where (a) the differential equation has four or six or eight nontrivial hyperbolic equilibria; (b) The invariant circle is not a C^1 submanifold of $\mathbb{R}^2$; (c) The invariant circle is a periodic orbit. In case (c), how smooth is the invariant circle?

5.1.1. Extensions and generalizations. Theorem 5.1.5 applies to differential equations which are of the form $x' = \lambda x + Q(x)$, where Q is a homogeneous polynomial of odd degree. This result is already very effective for the analysis of a wide range of problems in equivariant bifurcation theory – for example, most of the 3-determined bifurcation problems considered in the previous chapter. However, it is useful to describe what happens when we allow for higher order terms as well as dependence of nonlinear terms on λ. Some of the results we describe depend on the theory of *normal hyperbolicity* [**93**] and we shall omit most technical details of proofs. We shall also restrict attention to the most interesting case where Q is of degree three.

LEMMA 5.1.8. *Let $Q \in P^3(V,V)$, $H \in C^\infty(V,V)$, $J \in C^\infty(V \times \mathbb{R}, V)$ and define*

$$x' = \lambda x + Q(x) + \lambda H(x) + J(x,\lambda). \tag{5.12}$$

Suppose that Q is contracting, $H(x) = O(\|x\|^2)$ and $J(x,\lambda) = O(\|x\|^4)$. We may find $\lambda_0 > 0$, such that if $\lambda \in (0,\lambda_0]$, and we polar blow-up V and rescale by $R^2 = \lambda I$, $s = \lambda t$, then the system (5.12) transforms to

$$\dot{u} = I(\mathcal{P}_Q(u) + \lambda^{\frac{1}{2}} O(1)) \tag{5.13}$$

$$\dot{I} = 2I(1 + I\rho(u) + \lambda^{\frac{1}{2}} O(1)) \tag{5.14}$$

PROOF. A straightforward computation similar to that used in the proof of Theorem 5.1.5. □

COROLLARY 5.1.9. *(Assume the hypotheses of Lemma 5.1.8.) Suppose that $\mathcal{P}_Q$ is a structurally stable vector field on S^n. There exists $\lambda_1 \in (0,\lambda_0]$ and an open neighbourhood U of $0 \in V$ such that for every $\lambda \in (0,\lambda_1]$. the differential equation (5.12) has a flow-invariant sphere $S(\lambda) \subset U$ such that*

1. *For every $x \in U$, $x \neq 0$, the trajectory of (5.12) through x is forward asymptotic to $S(\lambda)$.*
2. *The flow of (5.12) restricted to $S(\lambda)$ is topologically equivalent to that of $\mathcal{P}_Q$ on S^n.*

PROOF. We sketch the main ideas (which depend on general results on structural stability [**163, 148**]). As in the proof of theorem 5.1.5, it suffices to consider

$$\dot{u} = \mathcal{P}_Q(u) + \lambda^{\frac{1}{2}} O(1)) \tag{5.15}$$

$$\dot{I} = 2 + 2I\rho(u) + \lambda^{\frac{1}{2}} O(1)). \tag{5.16}$$

Suppose $\lambda = 0$ and let $\tilde{S} \subset S^n \times [I_Q, \infty)$ be the invariant sphere given by (the proof of) theorem 5.1.5. Choose an open neighbourhood U of $\tilde{S}$ such that ∂U is smooth and the vector field defined by (5.15,5.16) is transverse to ∂U and everywhere inward pointing. Let $\pi_U : U \to S^n$ denote the projection onto S^n. Let $\Phi_t : U \to U$ denoted the associated semiflow. We remark that Φ_t can be extended to a smooth flow on $S^{n+1} \supset U$ with two repelling hyperbolic points in $S^{n+1} \setminus U$ and such that the trajectory through every other point of $S^{n+1} \setminus U$ eventually crosses ∂U. The flow of $\mathcal{P}_Q$ is topologically equivalent to the induced flow $\Phi_t|\tilde{S} : \tilde{S} \to \tilde{S}$. Each basic set Λ in the spectral decomposition [**163**] of $\mathcal{P}_Q$ corresponds to a transitive invariant set $\tilde{\Lambda} = \pi_U^{-1}(\Lambda) \cap \tilde{S}$ for Φ_t. If $W^s(x)$ is the stable manifold through $x \in \Lambda$, then $\pi_U^{-1}(W^s(x))$ is the stable manifold for $\tilde{x} = \pi_U^{-1}(x) \cap \tilde{S} \in \tilde{\Lambda}$. Similarly, we may lift $W^u(x)$ to the unstable manifold $W^u(\tilde{x})$

through $\tilde{x}$ which projects 1:1 onto $W^u(x)$. We show that each set $\tilde{\Lambda}$ is hyperbolic for Φ_t – this is elementary if $\tilde{\Lambda}$ is either an equilibrium point or a limit cycle. The requisite Axiom A and strong transversality conditions characterizing structural stability continue to hold on U. Since the basic sets we add to extend Φ_t to S^{n+1} are both hyperbolic repellors, it follows that $\Phi_t : S^{n+1} \to S^{n+1}$ satisfies Axiom A and the strong transversality condition and so is structurally stable [**148**]. The same is true for (5.15,5.16) for sufficiently small positive values of λ. $\square$

COROLLARY 5.1.10. *(Assume the hypotheses of Lemma 5.1.8.) Let $\lambda = 0$ and suppose that the invariant sphere $\tilde{S}$ for (5.13,5.14) is normally hyperbolic. Then there exists $\lambda_1 \in (0, \lambda_0]$ and an open neighbourhood U of $0 \in V$ such that for every $\lambda \in (0, \lambda_1]$. the differential equation (5.12) has a flow-invariant sphere $S(\lambda) \subset U$ such that for every $x \in U$, $x \neq 0$, the trajectory of (5.12) through x is forward asymptotic to $S(\lambda)$.*

PROOF. This is a standard application of results on the persistence of normally hyperbolic sets [**93**]. $\square$

COROLLARY 5.1.11. *Suppose that $z' = \lambda z + F(z)$ where $J^2F(0) = 0$. Let Z_R be the one parameter family of smooth vector fields on $S(V)$ defined by*

$$Z_R(u) = R^{-2}(F(Ru) - (F(Ru), u)u), \quad (u \in S(V)).$$

If $R \neq 0$, then every zero $u_0 = u(R)$ of Z_R determines a unique zero $z(u_0)$ of $z' = \lambda z + F(z)$, where $\lambda = \lambda_0$ is uniquely determined by u_0.

PROOF. In order that Ru_0 be a zero of $z' = \lambda z + F(z)$, it suffices that $\lambda R + (F(Ru_0), u_0) = 0$. That is, $\lambda_0 = -R^{-1}(F(Ru_0), u_0)$. $\square$

REMARK 5.1.12. Let (V, G) be absolutely irreducible. If $X \in \mathcal{S}(V, G)$, then for the computation of $\Sigma^\star(X)$ we may always assume that higher order terms are independent of λ (this uses equivariant transversality – see chapter 7). Thus, it follows from corollary 5.1.11 that, for stable families, branches correspond to curves $u(R)$ of zeros in $S(V)$ for the one parameter families Z_R.

5.1.2. Applying the invariant sphere theorem. Let (V, G) be absolutely irreducible. Assume that $P^2_G(V, V) = \{0\}$ and G-equivariant bifurcation problems on (V, G) are 3-determined.

Suppose $X \in \mathcal{V}_0$ can be written $X(x, \lambda) = \lambda x + Q(x) = J^Q(x, \lambda)$, where $Q \in P^3_G(V, V)$. For $a \in \mathbb{R}$, define $X_a(x, \lambda) = X(x, \lambda) + a\|x\|^2 x$. If $X \in \mathcal{S}(V, G)$ (X is stable), it follows from results we prove in chapter 7 that

(1) For all but finitely many values of $a \in \mathbb{R}$, $X_a \in \mathcal{S}(V, G)$.
(2) For all $a \in \mathbb{R}$, $\Sigma(X_a) = \Sigma(X)$.

Note that this result is easy to verify for the hyperoctahedral representations $(\mathbb{R}^k, H_k)$ since the branches of solutions are in 1:1 correspondence with zeros of the phase vector field which is independent of the radial term $a\|x\|^2 x$.

Consequently, if we are interested in the branching pattern, rather than the signed indexed branching pattern, it is no loss of generality to add a radial term $a\|x\|^2 x$ and take a sufficiently negative so that $Q(x) + a\|x\|^2 x$ is contracting. There are several advantages to this approach. First of all, the problem of determining the branching pattern is reduced to that of finding the equilibria of the phase vector field $\mathcal{P}_Q$. More interestingly, the dynamics of the phase vector field $\mathcal{P}_Q$ determine the dynamics generated by the associated (supercritical) bifurcation of X. That is, if the conditions of the invariant sphere theorem hold, all branching is supercritical. Hence we can explore the dynamics generated in the bifurcation merely by examining dynamics of $\mathcal{P}_Q$ on S^n. Since S^n is compact, this allows the easy use of topological tools (such as the Euler-Poincaré index) to determine the number of equilibria. Although the phase vector field will typically not be structurally stable (in the class of non-equivariant vector fields) it will often be equivariantly structurally stable. This allows the possibility of proving equivariant generalizations of corollary 5.1.9.

5.2. The examples of dos Reis and Guckenheimer & Holmes

A feature of equivariant dynamics is the presence of robust saddle connections that would be non-generic for asymmetric vector fields. More formally, a saddle connection is a *non-transverse* intersection of stable and unstable manifolds. Nevertheless this type of intersection can sometimes persist under equivariant perturbations of the vector field and provides an example of what we later call *G-transversality* (see chapter 6). In figure 1, we show a robust saddle connection for a $\mathbb{Z}_2$-equivariant vector field on $\mathbb{R}^2$. In this case $(\mathbb{R}^2)^{\mathbb{Z}_2}$ is the x-axis and a, b are hyperbolic saddles on the x-axis. The connection between a and b cannot be broken by $\mathbb{Z}_2$-equivariant perturbations of the vector field.

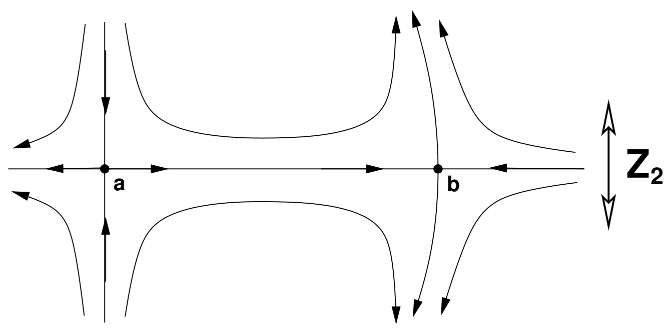


FIGURE 1. Robust non-transverse saddle connection

Dos Reis [**146**], as part of a classification of structurally stable equivariant vector fields on 2-manifolds, observed that it was possible to have a robust *cycle* of non-transverse saddle connections for a $\mathbb{Z}_2^3$-equivariant vector field on the two-sphere. This type of cycle is called a *heteroclinic cycle* (or *homoclinic cycle*, if the connection is between equilibria on the same group orbit). This phenomenon was rediscovered in the context of bifurcation theory by Guckenheimer & Holmes [**87**] (homoclinic cycles are also a well-known phenomenon in the theory of Lotka-Volterra equations and population models [**122, 94, 95**]).

We shall describe the example of Guckenheimer and Holmes as it provides a nice application of the invariant sphere theorem to equivariant dynamics.

With a view to subsequent applications, we first establish some general conventions. Unless stated to the contrary, we restrict to absolutely irreducible representations $(\mathbb{R}^n, G)$ where $n \geq 3$, $G = \Delta_n \rtimes T \subset H_n$ and T is a transitive subgroup of S_n. We define the 'spherical simplex'

$$\Lambda_{n-1} = \{(u_1, \ldots, u_n) \in S^{n-1} \mid u_1, \ldots, u_n \geq 0\}.$$

We remark that Λ_{n-1} is a T-invariant *fundamental domain* for the action of Δ_n on S^{n-1} (see (a,b,c) below). Let Λ°_{n-1} denote the interior of Λ_{n-1}. We record the following properties of Λ_{n-1}.

(a) $\cup_{g\in\Delta_n} g\Lambda_{n-1} = S^{n-1}$.
(b) For all $g \in \Delta_n$, $g\Lambda^\circ_{n-1} \cap \Lambda^\circ_{n-1} \neq \emptyset$ if and only if $g = I$.
(c) Λ_{n-1} is T-invariant.

Every point in $\partial\Lambda_{n-1}$ lies on a coordinate hyperplane. Since a coordinate hyperplane is a fixed point subspace of $\mathbb{R}^n$ disjoint from Λ°_{n-1}, $\partial\Lambda_{n-1}$ is flow invariant for any smooth G-equivariant vector field on S^{n-1}. Consequently, in order to describe the dynamics of a G-equivariant vector field X on S^{n-1}, it suffices to describe the dynamics of $X|\Lambda_{n-1}$.

Suppose that $n = 3$ and $G = \Delta_3 \rtimes \mathbb{Z}_3$. Let $Q = (Q_1, Q_2, Q_3) \in P^3_G(\mathbb{R}^3, \mathbb{R}^3)$. Since Q is Δ_3-equivariant, $Q_i = x_i\tilde{Q}_i(x_1^2, x_2^2, x_3^2)$, where $\tilde{Q}_i$ is linear, $i = 1, 2, 3$. By $\mathbb{Z}_3$-equivariance, once we know $\tilde{Q}_1$ the remaining components can be obtained by cyclic permutation of the coordinates. Since $\tilde{Q}_1$ is linear we therefore have

$$\begin{aligned}
Q_1(x_1, x_2, x_3) &= x_1(a\|x\|^2 + a_2x_2^2 + a_3x_3^2),\\
Q_2(x_1, x_2, x_3) &= x_2(a\|x\|^2 + a_2x_3^2 + a_3x_1^2),\\
Q_3(x_1, x_2, x_3) &= x_3(a\|x\|^2 + a_2x_1^2 + a_3x_2^2).
\end{aligned}$$

where $a, a_2, a_3 \in \mathbb{R}$.

Lemma 5.2.1. *Q is contracting if and only if*

$$a < 0,\ 3a + a_2 + a_3 < 0. \tag{5.17}$$

Proof. We have $(Q(x), x) = a\|x\|^4 + (a_2 + a_3)(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2)$. Taking $x = (1, 0, 0)$ and $x = (1, 1, 1)$, we see that (5.17) is a necessary condition for Q to be contracting. For sufficiency, note that $a\|x\|^4 = \frac{a}{2}\sum_{i<j}(x_i^2 - x_j^2)^2 + 3a(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2)$. □

If (5.17) is satisfied, we may apply the invariant sphere theorem and reduce the analysis of equivariant bifurcations of $x' = J^Q(x, \lambda)$ at zero to the analysis of $\mathcal{P}_Q|\Lambda_2$. In this case, Λ_2 is the spherical triangle with vertices $p_{100} = (1, 0, 0)$,

$p_{010} = (0,1,0)$, and $p_{001} = (0,0,1)$ (see figure 4 and note these vertices have maximal isotropy). The coordinate planes meet S^2 in three invariant circles which determine the boundary of Λ_2. We denote the other point in Λ_2 with maximal isotropy by $p_{111} = (1,1,1)/\sqrt{3}$ ($\mathbb{R}(1,1,1)$ is an axis of symmetry).

Using lemma 4.6.6, it is easy to compute the eigenvalues of the linearization of $\mathcal{P}_Q$ at p_{100} and p_{111}. We find that the eigenvalues of the linearization at p_{100} are a_3 and a_2 with corresponding eigenspaces $\mathbb{R}\mathbf{e}_2$ and $\mathbb{R}\mathbf{e}_3$ respectively. The remaining eigenvalues and eigenspaces at p_{010}, p_{001} are obtained by cyclic permutation.

The complex eigenvalues of the linearization of $\mathcal{P}_Q$ at p_{111} are

$$\frac{1}{3}(-(a_2+a_3) \pm \imath\sqrt{3}(a_2-a_3)).$$

The corresponding eigenspace is the plane $x_1 + x_2 + x_3 = 0$. The eigenvalues at p_{111} are real if and only if $a_2 = a_3$.

We claim that $\mathcal{P}_Q$ has equilibria of submaximal isotropy type $(\mathbb{Z}_2)$ if and only if $a_2 a_3 \geq 0$. Since points with isotropy type $(\mathbb{Z}_2)$ lie on $\partial\Lambda_2$, one coordinate is zero. It suffices to look for equilibria of the form $(u,v,0)$, $uv \neq 0$. Now $(u,v,0)$ is an equilibrium of $\mathcal{P}_Q$ if and only if $J^Q((u,v,0),\lambda) = 0$ for some λ. Computing, the latter condition is equivalent to

$$\begin{aligned} \lambda u + u(a(u^2+v^2) + a_2 v^2) &= 0, \\ \lambda v + v(a(u^2+v^2) + a_3 u^2) &= 0. \end{aligned}$$

Divide the first equation by u, the second by v and subtract the second equation from the first. We obtain

$$a_2 v^2 = a_3 u^2.$$

This equation has non-zero solutions if and only if $a_2 a_3 \geq 0$. If $a_2 a_3 > 0$, we denote the equilibrium point with submaximal isotropy $(\mathbb{Z}_2)$ lying on the edge joining p_{100} and p_{111} by p_{110}. We leave it as an exercise for the reader to show that if $a_2 a_3 > 0$, then p_{110} is a hyperbolic saddle point (the eigenvalues of the linearization of $\mathcal{P}_Q$ at p_{110} are real, non-zero and of opposite sign). Hence all the zeros of $\mathcal{P}_Q$ we have found so far are hyperbolic provided that

$$a_2 a_3, a_2 + a_3 \neq 0.$$

The corresponding branches of $x' = J^Q(x,\lambda)$ will be hyperbolic if, in addition, $a \neq 0$. Granted these conditions on the coefficients, it follows either by theorem 4.9 or direct computation, that we have located all the zeros of $\mathcal{P}_Q$ and corresponding branches of $x' = J^Q(x,\lambda)$.

Referring to figure 2, the stabilities of the equilibria of $\mathcal{P}_Q$ remain constant on the open regions $\pm A_1$, $\pm A_2$, $\pm A_3$ of (a_2,a_3)-parameter space. If we vary (a_2,a_3) counterclockwise round the unit circle in the (a_2,a_3)-plane, we obtain the sequence of phase portraits for $\mathcal{P}_Q$ shown in figure 3. In figure 3(a), we have taken $a_2 = a_3$. In this case $\mathcal{P}_Q$ is H_3-equivariant and a *gradient* vector field. In particular, $\mathcal{P}_Q$ can have no periodic orbits. As we approach the line

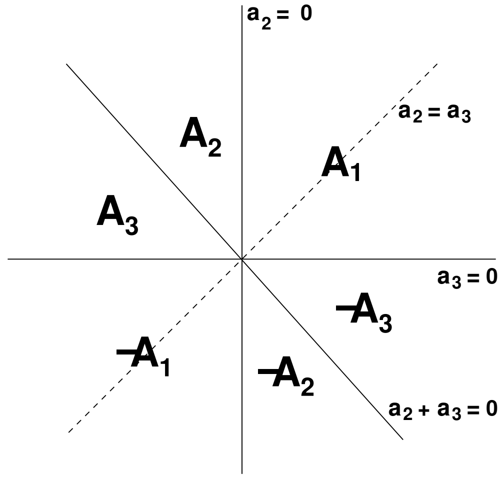


FIGURE 2. Regions of stability for $\mathcal{P}_Q$

$a_2 = 0$, the saddle point p_{110} moves counter-clockwise along the boundary of Λ_2 as shown in figure 3(b). The linearization of $\mathcal{P}_Q$ at p_{111} has complex eigenvalues. In figure 3(c), we have taken $a_2 < 0$, $a_3 > 0$ and $a_2 + a_3 > 0$. The saddle point p_{110} has now disappeared and $\mathcal{P}_Q$ has no sources. Instead, $\mathcal{P}_Q$ has a homoclinic[1] cycle comprised of the stable and unstable manifolds (within Λ_2) of the saddle points p_{100}, p_{010}, p_{001}. The point p_{111} is a sink. As we cross the line $a_2 + a_3 = 0$, $\mathcal{P}_Q$ undergoes a degenerate Hopf bifurcation. We leave it as an exercise – see below – to show that when $a_2 + a_3 = 0$, all trajectories of $\mathcal{P}_Q$ not lying on the homoclinic cycle or equal to p_{111} are periodic (with orbits given by $u_1u_2u_3 = c$, $c \in (0, \sqrt{3}/9)$.)

Passing into the region A_3, p_{111} becomes a source and we continue to have a homoclinic cycle joining the saddle points p_{100}, p_{010}, p_{001} – see figure 3(d).

When we move into $-A_1$, $-A_2$ and $-A_3$, we get the corresponding pictures we saw for A_1, A_2 and A_3 but with arrows reversed.

Let $a_2a_3 < 0$ and denote the corresponding homoclinic cycle of $\mathcal{P}_Q$ by Σ. We say that Σ is an *attracting* homoclinic cycle if we can find an open neighborhood U of Σ such that every trajectory of $\mathcal{P}_Q$ through a point of U is forward asymptotic to Σ. Similarly, we say Σ is *repelling* if we can find U such that backward trajectories through points of U are asymptotic to Σ. It follows from a theorem of dos Reis [**146**], that Σ is attracting (respectively, repelling) if the absolute value of the product of the negative eigenvalues round Σ is greater than (respectively, less than) the product of the positive eigenvalues round Σ. Applying dos Reis' theorem, we see that if $a_2 < 0 < a_3$, then Σ is attracting (respectively, repelling) if $|a_2| > a_3$ (respectively, $|a_2| < a_3$). In our situation, this result may be easily deduced using the Lyapunov function $V(u_1, u_2, u_3) = u_1u_2u_3$

[1]The connections are between equilibria on the same G-orbit

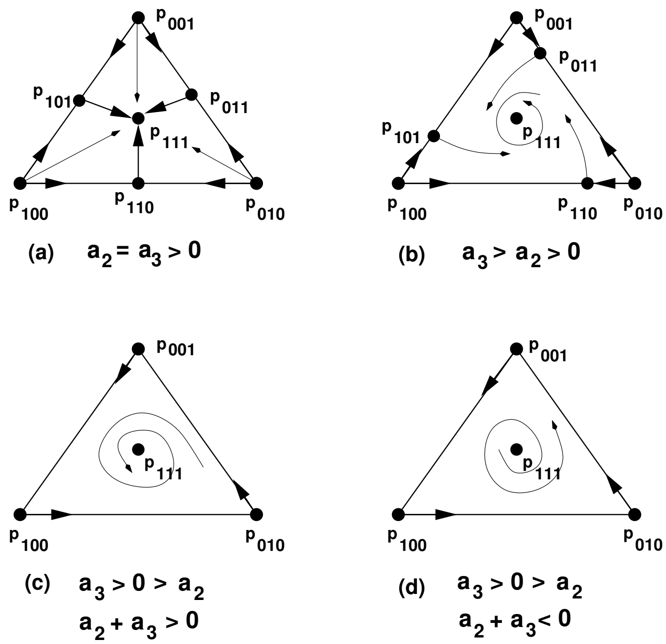


FIGURE 3. Dynamics on Λ_2

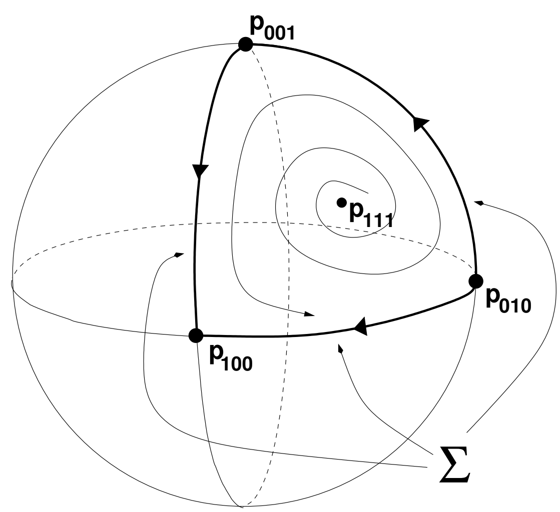


FIGURE 4. An attracting homoclinic cycle on S^2

EXERCISE 5.2.2. Show that if $V(u_1, u_2, u_3) = u_1u_2u_3$, then we have $\dot{V} = (a_2 + a_3)\rho(u_1, u_2, u_3)V$, where $\rho \geq 0$ on Λ_2 and is equal to zero if and only if $u_1 = u_2 = u_3 = 1/\sqrt{3}$. Deduce that Σ is an attracting homoclinic cycle if $a_2 + a_3 < 0$ and repelling if $a_2 + a_3 > 0$. Show also that if $a_2 + a_3 = 0$ then trajectories through points not lying on $\partial\Lambda_2 \cup \{p_{111}\}$ are periodic.

A degenerate Hopf bifurcation occurs at points (a_2, a_3) satisfying $a_2 + a_3 = 0$ (see the previous exercise). If we are prepared to add higher order equivariant terms to Q and consider, via the invariant sphere theorem, the corresponding perturbation of $\mathcal{P}_Q$, it is possible to obtain a non-degenerate Hopf bifurcation – relative to an arc of parameters cutting the line $a_2 + a_3 = 0$ transversally. However, we never obtain branches of limit cycles for generic *primary* G-equivariant bifurcations. Branches of limit cycles only appear stably in *secondary* bifurcations.

We refer the reader to [**87, 74**] for more detailed computations and a discussion of the relevance of this example for applications.

5.3. Steady state bifurcation to limit cycles

Let $G = \Delta_4 \rtimes \mathbb{Z}_4 \subset \mathrm{O}(4)$. In this section, we describe some of the new dynamical phenomena that can occur in G-equivariant bifurcations on $\mathbb{R}^4$ (most of what we say is a summary of results presented in [**74**] and we refer the reader to that article for details of proofs we omit). We also look at the dynamical implications of replacing G by the index two subgroup $G^\star = (\Delta_4 \rtimes \mathbb{Z}_4)'$ ($G^\star$ continues to act absolutely irreducibly on $\mathbb{R}^4$).

5.3.1. Axes of symmetry in Λ_3.

(1) There are four axes $\mathbb{R}\mathbf{e}_1, \ldots \mathbb{R}\mathbf{e}_4$ associated with the maximal isotropy type $(\mathbb{Z}_2^3)$. We let $p_{1000} = (1, 0, 0, 0)$ denote the intersection of $\mathbb{R}\mathbf{e}_1$ with the positive orthant of S^3 and similarly define $p_{0100}, p_{0010}, p_{0001}$.
(2) There is one axis $\mathbb{R}(1, 1, 1, 1)$ with associated maximal isotropy type $(\mathbb{Z}_4)$. We denote the corresponding point on the positive orthant Λ_3 of S^3 by p_{1111}.
(3) There are two axes $\mathbb{R}(1, 0, 1, 0)$, $\mathbb{R}(0, 1, 0, 1)$ with associated maximal isotropy type $(\mathbb{Z}_2^2 \times S_2)$. We denote the corresponding points on Λ_3 by p_{1010} and p_{0101}.

Aside from the points $p_{0100}, \ldots, p_{0101}$ described above, all other points in $\Lambda_3 \subset S^3$ have submaximal isotropy.

5.3.2. Cubic equivariants. Let $Q = (Q_1, \ldots, Q_4) \in P_G^3(\mathbb{R}^4, \mathbb{R}^4)$. Just as in the previous section, it suffices to find Q_1 since the remaining components of Q are obtained by cyclic permutation of the variables in Q_1. We find that

$$Q_1(\mathbf{x}) = x_1(a\|\mathbf{x}\|^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2), \quad (\mathbf{x} = (x_1, \ldots, x_4) \in \mathbb{R}^4),$$

where $a, a_2, a_3, a_4 \in \mathbb{R}$.

LEMMA 5.3.1. *The polynomial Q is contracting if and only if*

$$a,\ 2a + a_3,\ 4a + a_2 + a_4,\ 4a + a_2 + a_3 + a_4 < 0. \tag{5.18}$$

PROOF. We leave this as an exercise for the reader (details are in [**74**]). $\square$

Suppose that Q is contracting. Just as in our analysis of the Guckenheimer-Holmes example, it suffices to consider the dynamics of $\mathcal{P}_Q|\Lambda_3$.

Equilibria of $\mathcal{P}_Q$. We have already identified seven equilibria of $\mathcal{P}_Q|\Lambda_3$ that are forced by symmetry. Since $G \subset H_4$, we can also expect that for certain values of a_2, a_3, a_4 there will be equilibria p_{1100}, p_{0110}, p_{0011} and p_{1001} on the edges of Λ_3 and equilibria p_{1110}, p_{0111}, p_{1011} and p_{1101} on the faces of Λ_3. Since $G \subset H_4$, it follows from the results of chapter 4 that there will be an open and dense subset $\mathcal{A}$ of the space of coefficients a_2, a_3, a_4 such that for all $(a_2, a_3, a_4) \in \mathcal{A}$, $\mathcal{P}_Q$ has at most 15 equilibria and all of these equilibria will be hyperbolic.

LEMMA 5.3.2 ([**74**]). *Define*

$$\kappa_0 = a_2^2 + a_4(a_3 - a_2),\ \kappa_1 = a_3(a_2 - a_3 + a_4),\ \kappa_2 = a_4^2 + a_2(a_3 - a_4),\ \kappa = \kappa_0 + \kappa_1 + \kappa_2.$$

(1) *The eigenvalues of the linearization of $\mathcal{P}_Q$ at p_{1000} are*

$$a_4, a_3, a_2 \tag{5.19}$$

with eigenspaces $\mathbb{R}\mathbf{e}_2$, $\mathbb{R}\mathbf{e}_3$ and $\mathbb{R}\mathbf{e}_4$ respectively.

(2) *The eigenvalues of the linearization of $\mathcal{P}_Q$ at p_{1111} are*

$$(a_3 - (a_2 + a_4))/2,\ -(a_3 \pm \imath(a_2 - a_4))/2. \tag{5.20}$$

(3) *The eigenvalues of the linearization of $\mathcal{P}_Q$ at p_{1010} are*

$$(a_2 - a_3 + a_4)/2 \ (\textit{multiplicity } 2),\ -a_3. \tag{5.21}$$

(4) *There exist zeros with submaximal isotropy type $(\mathbb{Z}_2^2)$ if and only if $a_2a_4 > 0$. If $a_2a_4 > 0$, the eigenvalues of the linearization of $\mathcal{P}_Q$ at p_{1100} are*

$$\frac{-2a_2a_4}{a_2 + a_4},\ \frac{\kappa_2}{a_2 + a_4},\ \frac{\kappa_0}{a_2 + a_4}. \tag{5.22}$$

(5) *There exist equilibria with submaximal isotropy type $(\mathbb{Z}_2)$ if and only if*

$$sgn(\kappa_0) = sgn(\kappa_1) = sgn(\kappa_2) \neq 0 \tag{5.23}$$

If this condition holds, the eigenvalues of the linearization of $\mathcal{P}_Q$ at p_{1110} are

$$\frac{2\sigma_+}{\kappa},\ \frac{2\sigma_-}{\kappa},\ \frac{\kappa_1((a_2 - a_4)^2 + a_3^2)}{\kappa a_3},$$

where $\sigma_\pm$ are the solutions to

$$\sigma^2 + \sigma a_3(a_2^2 + a_4^2) + \kappa_1\kappa_2\kappa_3 = 0.$$

PROOF. We verify the conditions for the existence of equilibria of isotropy type $(\mathbb{Z}_2)$ and leave the remaining straightforward – if tedious – computations to the reader (more details are in [**74**]). Suppose then that there is a solution

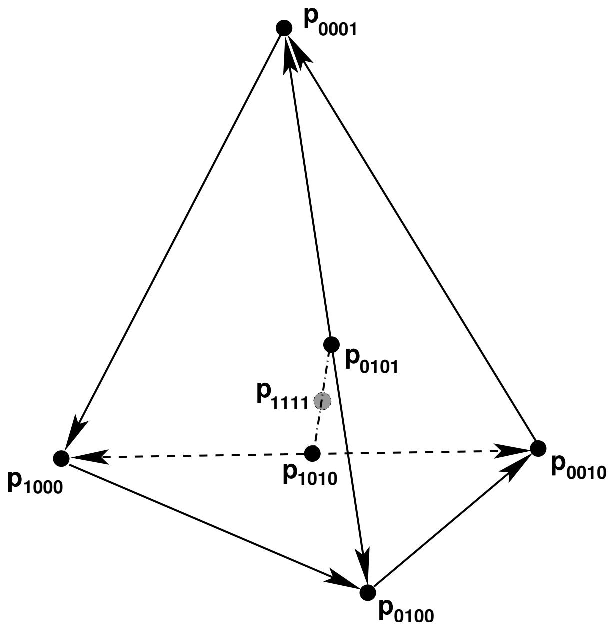


FIGURE 5. Edge cycle on Λ_3

of isotropy type $(\mathbb{Z}_2)$. That is, there exists $(u_1, u_2, u_3, 0) \in S^3$, with $u_1u_2u_3 \neq 0$, and $c \in \mathbb{R}$, such that

$$\begin{aligned} u_1(a_2u_2^2 + a_3u_3^2) &= cu_1, \\ u_2(a_2u_3^2 + a_4u_1^2) &= cu_2, \\ u_3(a_3u_1^2 + a_4u_2^2) &= cu_3. \end{aligned}$$

Since $u_1u_2u_3 \neq 0$, it follows that

$$a_2u_2^2 + a_3u_3^2 = a_2u_3^2 + a_4u_1^2 = a_3u_1^2 + a_4u_2^2.$$

Eliminating u_3^2 from these equations we find that $\kappa_1u_1^2 = \kappa_0u_2^2$ and so $\mathrm{sgn}(\kappa_0) = \mathrm{sgn}(\kappa_1)$. Eliminating u_1^2 we find $\kappa_2u_2^2 = \kappa_1u_3^2$ and so $\mathrm{sgn}(\kappa_1) = \mathrm{sgn}(\kappa_2)$. Hence (5.23) is a necessary condition for the existence of solutions of isotropy type $(\mathbb{Z}_2)$. The proof of the converse is similarly straightforward. $\square$

5.3.3. Homoclinic cycles. It follows from lemma 5.3.2 (1,4) that if $a_2a_4 < 0$, then there is a homoclinic cycle Σ_E connecting the vertices $p_{1110}, \ldots, p_{0001}$. We refer to figure 5 for the case $a_2, a_3 < 0 < a_4$. We refer to Σ_E as an *edge cycle* as it is comprised of edges from the simplex Λ_3.

Let F denote the 2-face of Λ_3 defined by the vertices $p_{1000}, p_{0100}, p_{0001}$. It is possible to choose $a_2, a_3 < 0 < a_4$ such that either p_{0101} is a source in the face F and there is no submaximal equilibrium p_{1101} or $p_{0101} \in F$ is a saddle and there is a source $p_{1101} \in F$.

There are various types of attractivity (and repulsiveness) that are displayed by the edge cycle Σ_E. Before we describe these properties, we need some new notation and definitions.

Let d denote the G-invariant metric on S^3 induced from the Euclidean metric on $\mathbb{R}^4$. Let Φ_t denote the flow of $\mathcal{P}_Q$. Let U be a nonempty open subset of Λ_3 and $\mathbf{u} \in U$. We say that $\mathbf{u}$ *exits* U if there exists $T = T(\mathbf{u}) > 0$ such that

(1) $\Phi_t(\mathbf{u}) \in U$ for all $t \in [0, T)$.
(2) $\Phi_T^Q(\mathbf{u}) \notin U$.

We call $T(\mathbf{u})$ the *exit time* (for $\mathbf{u}$).

The set U is a *transient* set (for the flow Φ_t) if every point of U eventually exits U. We allow the possibility that a trajectory which exits U may, at some later time, re-enter U.

Define $d_\partial : \Lambda_3 \to \mathbb{R}$ by

$$d_\partial(\mathbf{u}) = \inf\{d(\mathbf{u}, \mathbf{v}) \mid \mathbf{v} \in \partial\Lambda_3\}.$$

Let U be a neighbourhood of Σ_E in Λ_3. Define

$$U^\bullet = U \setminus \partial\Lambda_3.$$

Definition 5.3.3 ([**74**]). (1) Σ_E is an *attracting* heteroclinic cycle of $\mathcal{P}_Q$ if for every open neighbourhood*heteroclinic cycle!attracting* V of Σ_E in Λ_3 there exists an open neighbourhood $U \subset V$ of Σ_E such that the forward trajectory of every $\mathbf{u} \in U$ lies in V and is forward asymptotic to Σ_E. We say Σ_E is a *repelling* heteroclinic cycle if Σ_E is attracting for $-\mathcal{P}_Q$.

(2) Σ_E is a *boundary-attracting saddle* if for every open neighbourhood V of Σ_E in Λ_3 there exists an open neighbourhood $U \subset V$ of Σ_E and $\varepsilon > 0$ such that
   (a) $U^\bullet$ is transient.
   (b) The forward trajectory of every $\mathbf{u} \in U \cap \partial\Lambda_3$ lies within $V \cap \partial\Lambda_3$ and is forward asymptotic to Σ_E.
   (c) If $\mathbf{u} \in U^\bullet$ has exit time $T = T(\mathbf{u})$ then $d_\partial(\Phi_T(\mathbf{u})) > \varepsilon$.

   We say that Σ_E is a *boundary-repelling saddle* if Σ_E is a boundary-attracting saddle for $-\mathcal{P}_Q$.

Remark 5.3.4. If Σ_E is a *boundary-attracting saddle*, then the stable set $W^s(\Sigma_E)$ will be a subset of $\partial\Lambda_3$. There will be an 'unstable set' $W^u(\Sigma_E)$ such that $W^u(\Sigma_E)\setminus\Sigma_E$ lies in the interior of Λ_3. In particular, the local unstable set of Σ_E points into the interior of Λ_3 along Σ_E. Viewed from $\partial\Lambda_3$, Σ_E is an attractor but seen from the interior of Λ_3, Σ_E appears to be a repellor. This phenomenon is qualitatively similar to the dynamics of the degenerate saddle $z' = -\overline{z}^3$, where the origin attracts points on the x and y-axis ($z = x + iy$) and repels points along the lines $x \pm y = 0$.

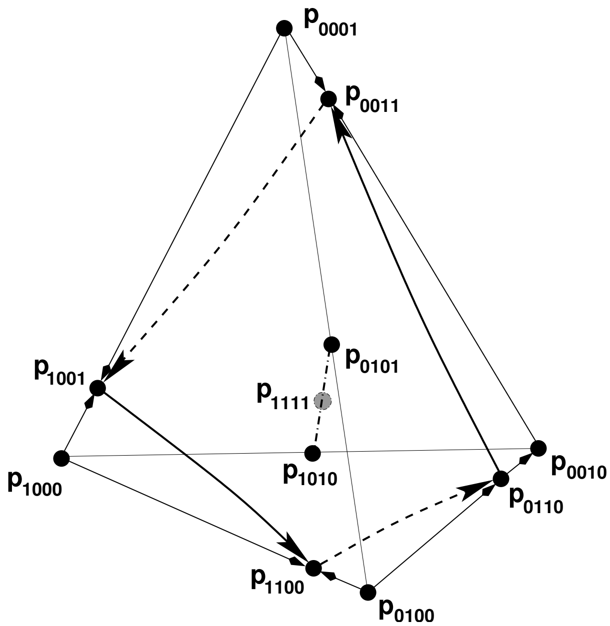


FIGURE 6. Face cycle on Λ_3

PROPOSITION 5.3.5. *Assume that $a_2a_4 < 0$. Let Σ_E denote the homoclinic cycle for the flow of $\mathcal{P}_Q|\Lambda_3$.*

(1) *If $a_2 + a_3 + a_4, a_3 < 0$ (respectively, $a_2 + a_3 + a_4, a_3 > 0$), then Σ_E is an attracting (respectively, repelling) homoclinic cycle.*
(2) *If $a_2 + a_3 + a_4 > 0 > a_3$ (respectively, $a_2 + a_3 + a_4 < 0 < a_3$), Σ_E is a boundary-attracting saddle (respectively, boundary-repelling saddle).*

PROOF. A direct proof parts (1,2) is given in [**74**]. Part (1) of the result also follows from more recent general (and by now standard) results on the stability of heteroclinic cycles (see [**108**] and [**94**]). We give a proof of the more difficult part (2) later in this section (see lemmas 5.3.9, 5.3.11, 5.3.12). □

It is also possible for Λ_3 to support a *face* heteroclinic cycle.

PROPOSITION 5.3.6. *Assume that $a_2a_4 > 0$ and $\kappa_0\kappa_2 < 0$. Then Λ_3 has a face heteroclinic cycle Σ_F joining the equilibria $p_{1100}, p_{0110}, p_{0011}$ and p_{1001} (see figure 6).*

If $\kappa_0 + \kappa_2 < 0 < a_2 + a_4$ (respectively, $\kappa_0 + \kappa_2 > 0 > a_2 + a_4$), then Σ_F is an attracting (respectively, repelling) heteroclinic cycle.

PROOF. We refer the reader to [**74**, Theorem 5.2] (or [**108**]). □

For the remainder of this section we assume that $a_2a_4 < 0$ and set $\Sigma_E = \Sigma$. By lemma 5.3.2, the eigenvalues of the linearization of $\mathcal{P}_Q$ at p_{1111} are $(a_3 - (a_2 + a_4))/2, -(a_3 \pm \imath(a_2 - a_4))/2$. Since $a_2a_4 < 0$ (and so $a_2 \neq a_3$), we always have a complex conjugate pair of eigenvalues. If $a_3 < 0$ and $a_3 < a_2 + a_4$, then p_{1111} is a saddle-point with two-dimensional unstable manifold transverse to the symmetry line through p_{1111} joining p_{0101} to p_{1010} (see figure 5). It follows from proposition 5.3.5 that if $a_2 + a_3 + a_4 > 0 > a_3$, then Σ_E is a boundary-attracting saddle. In this case, the unstable set $W^u(\Sigma_E)$ points into the interior of Λ_3. If $a_2 + a_3 + a_4 > 0 > a_3$, and $a_2 a_4 < 0$ then $a_3 < a_2 + a_4$ and it is easy to check from lemma 5.3.2 that $\mathcal{P}_Q$ has no attracting equilibria in Λ_3. Since Σ_E is not an attractor, this strongly suggests that the interior Λ_3 contains recurrent dynamics.

THEOREM 5.3.7. *Suppose that*

$$a_2 a_4 < 0, \tag{5.24}$$

$$a_3(a_2 + a_3 + a_4) < 0. \tag{5.25}$$

then $\mathcal{P}_Q$ has a periodic orbit contained in the interior of Λ_3.

We need some preliminaries before we give the proof of theorem 5.3.7. We denote the coordinates of points in $\Lambda_3 \subset \mathbb{R}^4$ by $(x_1, \ldots, x_4)$ and regard subscripts as being defined mod 4.

For $i \in \mathbb{Z}$, we define the following subsets of Λ_3 (see figure 7).

$$\begin{aligned}
\mathcal{R}_i &= \{(u_1, \ldots, u_4) \in \Lambda_3 \mid u_i \geq u_{i+2} \text{ and } u_{i+1} = u_{i+3}\} \\
\mathcal{F}_i &= \{(u_1, \ldots, u_4) \in \mathcal{R}_i \mid u_i > u_{i+2} > 0 \text{ and } u_{i+1}, u_{i+3} > 0\} \\
\Omega_i &= \{(u_1, \ldots, u_4) \in \Lambda_3 \mid u_i \geq u_{i+2} \text{ and } u_{i+1} \geq u_{i+3}\} \\
D_i &= \{(u_1, \ldots, u_4) \in \Lambda_3 \mid u_i > u_{i+2} > 0 \text{ and } u_{i+1} > u_{i+3} > 0\}
\end{aligned}$$

Let Λ_3° denote the set of interior points of $\Lambda_3 \subset S^3$. For $i \in \mathbb{Z}$ we have

$$\begin{aligned}
\mathcal{F}_i &= \Lambda_3^\circ \cap \mathcal{R}_i, \ D_i = \Lambda_3^\circ \cap \Omega_i, \\
\partial_{\Lambda_3} \Omega_i &= \partial_{\Lambda_3} D_i = \mathcal{R}_i \cup \mathcal{R}_{i+1}, \\
\mathbb{Z}_4 \Omega_i &= \Delta_3, \\
\Omega_i \cap \Omega_{i+1} &= \mathcal{R}_{i+1}.
\end{aligned}$$

Conditions (5.24,5.25) of the theorem imply that $\kappa_1 = a_3(a_2 + a_4 - a_3) < 0$. Using lemma 5.3.2(5), we may show that $\mathcal{P}_Q$ has no equilibria of isotropy type $(\mathbb{Z}_2)$. Suppose that $a_3 < 0$. Then the points p_{1010}, p_{0101} are sources and the one-dimensional stable manifold of p_{1111} is contained in the arc $u_1 = u_3, u_2 = u_4$ connecting p_{1010} and p_{0101}.

LEMMA 5.3.8. *Suppose that a_2, a_3, a_4 satisfy (5.24,5.25) and $a_4 > 0 > a_2$. Then for $1 \leq i \leq 4$ there exist smooth functions $T_i : \mathcal{F}_i \to \mathbb{R}$ satisfying*

1. *$T_i(\mathbf{u}) > 0$, all $\mathbf{u} \in \mathcal{F}_i$.*
2. *$\Phi(\mathbf{u}, T_i(\mathbf{u})) \in \mathcal{F}_{i+1}$ and $T_i(\mathbf{u})$ is the smallest positive number for which this is true.*
3. *If $g \in \mathbb{Z}_4$ maps $\mathcal{F}_i$ to $\mathcal{F}_j$ then $T_j = T_i g$.*

A similar result holds if $a_4 < 0 < a_2$.

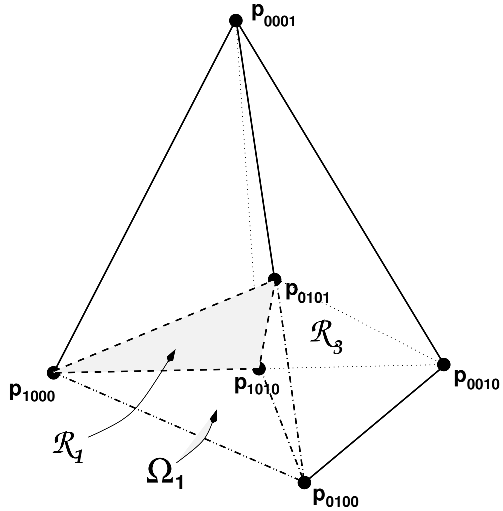


FIGURE 7. The sets $\mathcal{R}_1$, Ω_1

PROOF. We construct $T = T_1 : \mathcal{F}_1 \to \mathbb{R}$. The remaining maps T_i are then defined by $\mathbb{Z}_4$-equivariance. We assume $a_3 < 0$ and so, $a_3(a_2 + a_4 - a_3) < 0$, we have $a_2 + a_4 > a_3$.

For $t \in [0, 1]$, define $L_t : \mathbb{R}^3 \to \mathbb{R}$ by

$$L_t(x_1, \dots, x_4) = (1 - t)(x_1 - x_3) - tx_3.$$

The zero sets $H_t = L_t^{-1}(0)$ define a pencil of planes meeting Λ_3 along the curve $u_1 = u_3 = 0$. Clearly, $H_0 \cap \Omega_1 = \mathcal{R}_2$, $H_1 \cap \Omega_1$ is the face of Ω_1 defined by $u_3 = 0$ and

$$\bigcup_{t\in[0,1]} H_t \cap \Omega_1 = \Omega_1.$$

The normal to H_t, pointing towards $\mathcal{F}_1$ away from $\mathcal{F}_2$ in Ω_1 is given by

$$n_t = (1 - t, 0, -1, 0).$$

Let $\mathbf{u} \in D_1$. Computing, we find that

$$(\mathcal{P}_Q(\mathbf{u}), n_t) = (1 - t)u_1((a_2 - a_4)(u_2^2 - u_4^2) + a_3(u_1^2 - u_3^2)).$$

Since we are assuming $a_3, a_2 < 0 < a_4$, it follows from the definition of D_1 that

$$(\mathcal{P}_Q(\mathbf{u}), n_t) < 0, \quad \text{for all } \mathbf{u} \in D_1,\ t \in [0, 1).$$

Hence, if $\mathbf{u} \in \mathcal{F}_1$, the forward Φ-trajectory through $\mathbf{u}$ meets each H_t transversally, $t \in [0, 1)$. There are two possibilities: Either (1) $\Phi_t(\mathbf{u})$ is asymptotic to a zero of $\mathcal{P}_Q$ in Ω_1 or (2) $\Phi_t(\mathbf{u})$ eventually exits D_1. It follows from our remarks prior to the statement of the lemma that there are no points in D_1 lying in the stable manifold of a zero of $\mathcal{P}_Q$. Hence (1) cannot occur. Consequently, the trajectory $\Phi_t(\mathbf{u})$ eventually exits through a face of Ω_1. Since $\partial\Lambda_3$ is Φ-invariant and $\mathcal{P}_Q$ is inward pointing along $\mathcal{F}_1$, $\Phi_t(\mathbf{u})$ must exit through $\mathcal{F}_2$. Hence there exists a

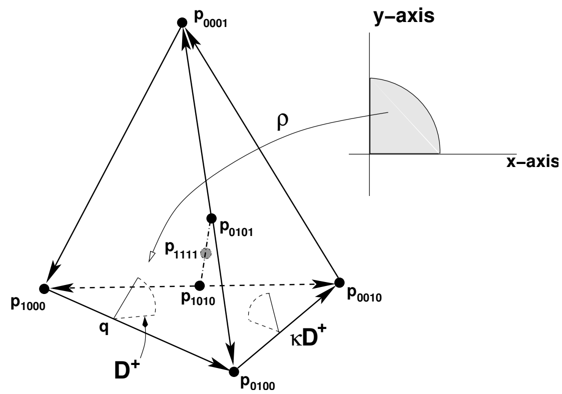


FIGURE 8. The local section D_1^+

(minimal) $T(\mathbf{u}) > 0$ such that $\Phi(\mathbf{u}, T(\mathbf{u})) \in \mathcal{F}_2$. This construction defines the required map $T : \mathcal{F}_1 \to \mathbb{R}$. The smoothness of T is a routine application of the implicit function theorem. □

Let $A_i : \mathcal{F}_i \to \mathcal{F}_{i+1}$ be the diffeomorphism defined by $A_i(\mathbf{u}) = \Phi(\mathbf{u}, T_i(\mathbf{u}))$ (notation of lemma 5.3.8). If we define $S = A_4 \circ A_3 \circ A_2 \circ A_1 : \mathcal{F}_1 \to \mathcal{F}_1$, then S is a smooth $\mathbb{Z}_4$-equivariant diffeomorphism of $\mathcal{F}_1$. We regard S as the Poincaré map of Φ associated to the cross-section $\mathcal{F}_1$.

Fix a point $q \in \Lambda_3$ lying in the interior of the edge E_{12} joining p_{1000} to p_{0100}. Let D_r denote the open r-disk, center zero in $\mathbb{R}^2$. Set $D_1 = D$ and let $D^+ = \{(x, y) \in D \mid x, y \geq 0\}$. We similarly define D_r^+, $0 < r < 1$. Fix an embedding $\rho : D \to S^3$ such that (1) $\rho \pitchfork E_{12}$, (2) $\rho(D) \cap E_{12} = \{q\}$ and (3) $\rho(D) \cap \Lambda_3 = \rho(D^+)$. Give $\rho(D^+) = D_1^+$ the coordinates inherited from D^+. We may assume that $x = 0$ corresponds to the face $x_3 = 0$ and $y = 0$ to the face $x_4 = 0$ (see figure 8). Let κ be the generator of $\mathbb{Z}_4$ satisfying $\kappa p_{1000} = p_{0100}$. We may choose $r \in (0, 1)$ such that Φ_t determines a Poincaré map $P : D_r^+ \to \kappa D^+ \approx D^+$. For $x \neq 0$, we define $P(x, y) = \Phi_{t(x,y)}(x, y)$, where $t(x, y)$ is the smallest strictly positive number such that $\Phi_{t(x,y)}(x, y) \in \kappa D^+$. We define $P(0, y) = 0$ (κq). With this definition of P it is easy to verify that P is continuous.

LEMMA 5.3.9. *Assume that $a_2, a_3 < 0 < a_4$ and $a_3(a_2 + a_3 + a_4) < 0$. Let $\mu = -\frac{a_3}{a_4}$, $\nu = -\frac{a_2}{a_4}$ (so that $\mu + \nu \in (0, 1)$).*

(1) *$\mathcal{P}_Q$ is C^1 linearizable at p_{1000}.*
(2) *We may choose coordinates on D_r^+ and find continuous strictly positive functions $A, B : D_r^+ \to \mathbb{R}$ such that*

$$P(x, y) = (A(x, y)x^\mu y, B(x, y)x^\nu).$$

PROOF. The conditions on the coefficients imply that $|a_2 + a_4| \neq |a_3|$, $|a_2 + a_3| \neq |a_4|$ and $|a_4 + a_3| \neq |a_2|$. Since $\mathcal{P}_Q$ is C^2 (in fact, C^∞), it follows from the Bellickii-Samoval linearization theorems [**10, 11**] that $\mathcal{P}_Q$ is C^1-linearizable at p_{1000}. Taking a C^1-linear coordinate system at p_{1000}, we can explicitly solve

$\mathcal{P}_Q$ on a neighbourhood of p_{1000} and hence estimate the time function $t(x, y)$ used in the definition of $P(x, y)$. We omit the straightforward details (see [**74**] or [**108**]). $\square$

REMARK 5.3.10. If $a_2 + a_3 + a_4, a_2a_4, a_3 < 0$, the second part of lemma 5.3.9 will hold provided that the non-resonance conditions $|a_i + a_j| \neq |a_k|$ are satisfied. In this case $\mu + \nu > 1$ and this gives an easy way to prove that Σ_E is an attracting homoclinic cycle (see [**74**]). It can be shown that this result holds if the non-resonance conditions fail.

Continuing with the notational assumptions of lemma 5.3.9, define

$$\lambda_d = \frac{\mu + \sqrt{\mu^2 + 4\nu}}{2}.$$

If $a_3(a_2 + a_3 + a_4) < 0$, and so $\mu + \nu \in (0, 1)$, it follows that $\lambda_d \in (0, 1)$. The asymptotic behaviour of iterates of the map P defined in lemma 5.3.9 is dominated by λ_d. Specifically, suppose that we are given $c, \alpha > 0$. Let $H(c, \alpha)$ denote the "hyperbola" in D^+ defined by $x^\alpha y = c$ and define

$$E(c, \alpha) = \{(x, y) \in D^+ \mid x^\alpha y \geq c\}.$$

LEMMA 5.3.11. *(Notation as above.) Let $P : D_r^+ \to D^+$ be the map defined by $P(x, y) = (Ax^\mu y, Bx^\nu)$, where A, B are for the present assumed to be strictly positive constants. Set $C = (BA^{\lambda_d})^{1/(1-\lambda_d)}$. Let $S : [0, \infty) \to [0, \infty)$ be the map defined by*

$$S(c) = B(Ac)^{\lambda_d},\ c \in \mathbb{R}.$$

Then

(1) *For all $c \in (0, \infty)$, $P(H(c, \lambda_d)) = H(S(c), \lambda_d)$; $P(E(c, \lambda_d)) = E(S(c), \lambda_d)$.*
(2) *If $\lambda_d < 1$ (equivalently, $\mu + \nu < 1$), then $S(c) > c$, for all $c \in (0, C)$.*
(3) *We may choose a closed neighbourhood U of $0 \in D_r^+$ and find $\varepsilon > 0$ such that if we set $U^\bullet = U \setminus \partial D^+$, then $U^\bullet$ is transient and if $x \in U^\bullet$ exits after N iterations of P, we have $d(P^N(x), \partial D^+) > \varepsilon$.*

PROOF. Fix $c > 0$. The hyperbola $H(c, \lambda_d) \subset D^+$ has parameterization $(t, ct^{-\lambda_d})$, $t > 0$. Computing, we find that $P(t, ct^{-\lambda_d}) = (T, B(Ac)^{\lambda_d}T^{-\lambda_d})$, where $T = Act^{\mu-\lambda_d}$. Hence, we have proved (1). The proof of (2) is a simple computation which depends on the fact that C is the (unique) fixed point of S. It remains to prove (3). Let $x_0 > 0$ and set

$$y_1 = Bx_0^\nu,\ y_0 = B(\frac{x_0}{Ay_1})^{\nu/\mu}.$$

Let U be the open rectangular neighbourhood of $O \in D$ with edges the x- and y-axis together with the lines $y = y_1$, $x = x_0$. Using the conditions $\nu + \mu < 1$, $\mu, \nu > 0$, we may find $\tilde{x}_0 > 0$ such that for all $x_0 \in (0, \tilde{x}_0]$, we have $y_0 < y_1$. Fix $x_0 \in (0, \tilde{x}_0]$. A simple computation shows that if we choose $\varepsilon > 0$ satisfying

$$\varepsilon < \min\{x_0^{(1-\nu)\nu/\mu}, Bx_0^\nu\},$$

then every $X \in U^\bullet$ eventually exits U. If $X \in U^\bullet$ exits after N iterations of P, then $d(P^N(X), \partial D^+) > \varepsilon$. $\square$

A straightforward openness and continuity argument proves the following extension of lemma 5.3.11 which allows for the coefficients of P to be continuous non-constant functions.

LEMMA 5.3.12. *(Notation as above.) Let $P : D_r^+ \to D^+$ be the map defined by $P(x,y) = (Ax^\mu y, Bx^\nu)$, where A, B are strictly positive continuous functions. Assume $\lambda_d < 1$. There exist $\bar{C} > 0$ and a continuous strictly monotone increasing map $\bar{S} : [0, \bar{C}) \to [0, \infty)$ such that*

(1) *For all $c \in (0, \bar{C})$, $P(H(c, \lambda_d)) \subset E(S(c), \lambda_d)$.*
(2) *We may choose a closed neighbourhood U of $0 \in D_r^+$ and find $\varepsilon > 0$ such that if we set $U^\bullet = U \setminus \partial D^+$, then $U^\bullet$ is transient and if $x \in U^\bullet$ exits after N iterations of P, we have $d(P^N(x), \partial D^+) > \varepsilon$.*

REMARK 5.3.13. Proposition 5.3.5(2) follows from lemma 5.3.12.

Proof of theorem 5.3.7 We assume $a_2, a_3 < 0 < a_4$ (the proof is similar, or follows by time-reversal, if $a_3 > 0$ and/or $a_4 < 0 < a_2$). In order to prove that $\mathcal{P}_Q$ has a limit cycle in the interior of Λ_3, it suffices to show that $S : \mathcal{F}_1 \to \mathcal{F}_1$ has a fixed point. For this it suffices to construct a closed disc $D \subset \mathcal{F}_1$ such that $S(D) \subset D$. It then follows from the Brouwer fixed point theorem that S has a fixed point. Our construction of D depends on finding a solid torus

$$\mathbf{T} \subset \Lambda_3^\circ \setminus \overline{W^s(p_{1111})}$$

such that $\mathcal{P}_Q$ is everywhere transverse to $\partial \mathbf{T}$ and inward pointing and $\mathbf{T} \cap \mathcal{F}_1 = D$ is a closed disc. It then follows from the transversality of $\mathcal{P}_Q$ to $\partial \mathbf{T}$ that $S(D) \subset D$.

Let $L \subset \Lambda_3$ denote the arc joining p_{1010}, p_{0101} defined by the intersection of Λ_3 with the plane $x_1 = x_3$, $x_1 = x_3$. Since the equilibria p_{1010}, p_{0101}, p_{1111} are hyperbolic and $\overline{W^u(p_{1111})} = L$, it follows that we may construct a closed neighbourhood V of L in Λ_3 such that

(1) ∂V is smooth and diffeomorphic to a cylinder (with boundary contained in $\partial \Lambda_3$).
(2) $\mathcal{P}_Q$ is transverse to ∂V and outward pointing.
(3) The forward trajectory of every point in $V \setminus L$ eventually exits U.

It follows from lemma 5.3.12 that we can choose a transverse local section D^+ through a point q of the edge joining p_{1000} to p_{0100} and $c, r > 0$ such that $P^4(H(c, \lambda_d) \cap D_r^+) \subset E(c', \lambda_d)$, where $c' > 0$. If $\mathbf{u} \in H(c, \lambda_d) \cap D_r^+$, define $t(\mathbf{u})$ by $P^4(\mathbf{u}) = \Phi_{t(\mathbf{u})}(\mathbf{u})$ and let M be the smooth surface with boundary and corners swept out by Φ_t: $M = \cup_{\mathbf{u} \in H(c, \lambda_d) \cap D_r^+} \Phi_{\mathbf{u}}([0, t(\mathbf{u})])$. Since the boundary component $\cup_{\mathbf{u} \in H(c, \lambda_d) \cap D_r^+} (\mathbf{u}, t(\mathbf{u}))$ is disjoint from $H(c, \lambda_d) \cap D_r^+$, we may deform M to M' so that (a) $\mathcal{P}_Q$ is everywhere transverse to M' and points into the interior of Λ_3, and (b) M' is a smooth surface with boundary consisting of two components, diffeomorphic to circles. We now continue the surface M' to ∂V. Rounding corners where the extension meets ∂V, we obtain a smooth surface $K \subset \Lambda_3^\circ$ which equals M' near Σ_E and ∂V away from the faces of Λ_3. A further deformation of K supported outside V and M' enables us to require $\mathcal{P}_Q \pitchfork K$ and inward pointing (for this we can use transversality estimates used in the proof of lemma 5.3.8). The surface K is a 2-torus and we let $\mathbf{T} \subset \Lambda_3^\circ$ denote the solid torus comprising K together with all the points inside K, The intersection of $\mathbf{T}$ $\mathcal{F}_1$ will be a closed 2-disk such that $S(D) \subset D$. $\square$

5.3.4. Stabilities and a 'hidden Hopf bifurcation'. Theorem 5.3.7 gives no information on the number of limit cycles in Λ_3 or their stabilities. We conjecture that there there is at most one limit cycle in Λ_3 with stability determined in the obvious way by the stabilities of the equilibria on Λ_3. In particular, if $a_3, a_2a_4, a_3(a_2 + a_4 + a_3) < 0$, then the limit cycle should be attracting.

Fix the parameters a_2, a_4 and assume that $a_2 \pm a_4 \neq 0$. Set $a_3 = s$ and for $s \in \mathbb{R}$, consider the family $\mathcal{P}^{Q(s)}$, where

$$Q_1(s)(x_1, \ldots, x_4) = x_1 - x_1(a\|x\|^2 + a_2x_2^2 + sx_3^2 + a_4x_4^2).$$

The eigenvalues of the linearization of $\mathcal{P}^{Q(s)}$ at p_{1111} are given by

$$\mu_s = \frac{s - (a_2 + a_4)}{2},\ \lambda_s = \frac{-s + \imath(a_4 - a_2)}{2},\ \bar{\lambda}_s = \frac{-s - \imath(a_4 - a_2)}{2}.$$

It follows that for all $s \in \mathbb{R}$, the imaginary parts of the complex conjugate pair $\lambda_s, \bar{\lambda}_s$ are non-zero. Further, for s near zero, the real eigenvalue μ_s is non-zero. As s increases through zero, the complex conjugate pair $\lambda_s, \bar{\lambda}_s$ cross the imaginary axis at non-zero speed. The next result, which is proved in [**74**, section 4] using a centre-manifold analysis, shows that there is a *non-degenerate* Hopf bifurcation at $s = 0$.

THEOREM 5.3.14. *Assume $a_2 \pm a_4 \neq 0$ and set $\sigma = -\mathrm{sign}(a_2 + a_4)$.*

(1) *There exists $\alpha = \alpha(a_2, a_4) > 0$, such that $\mathcal{P}^{Q(s)}$ has a branch of limit cycles for $s \in (0, \sigma\alpha]$. As $s \to 0$, the period of the limit cycles is asymptotic to $2/|a_4 - a_2|$ and the diameter to $\sqrt{|s|}$. The limit cycles are attracting if $\sigma = -1$ and repelling if $\sigma = 1$. Furthermore, for $c \in (0, -\sigma\alpha]$, $\mathcal{P}^{Q(s)}$ has no limit cycles in a neighbourhood of p_{1111}.*
(2) *Let $f \in \mathcal{V}_0(\mathbb{R}^4 \times \mathbb{R}, G)$ and set $Q = j^3f_0(0)$. Suppose that the coefficients of Q satisfy $a_2 \pm a_4 \neq 0$, $4a + a_2 + a_4 \neq 0$ and $a_3 \in (0, \sigma\alpha]$ where σ, α are as defined in (1). Then there exists $\lambda_0 \neq 0$, with $\mathrm{sign}(\lambda_0) = -\mathrm{sign}(4a + a_2 + a_4)$, such that there is a branch of limit cycles of $x' = f(x, \lambda)$ for $\lambda \in (0, \lambda_0]$. As $\lambda \to 0$, the period of the limit cycles is asymptotic to $|\lambda|^{-1}/|(a_4 - a_2)|$ and the diameter is asymptotically proportional to $\sqrt{|\lambda a_3|}$. The limit cycles are attracting if $\sigma < 0 < \lambda_0$, repelling if $\lambda_0 < 0 < \sigma$ and saddles if $\lambda_0\sigma > 0$.*

We refer to the phenomenon described by theorem 5.3.14 (2) as a 'hidden Hopf bifurcation'. Using part (1) if the theorem, it is possible to identify non-empty open subsets of the (a_2, a_3, a_4)-parameter space for which there exist exactly one limit cycle in Λ_3.

REMARKS 5.3.15. (1) In this section we have emphasized phenomena associated to the edge cycle Σ_E; notably the co-existence of Σ_E and a limit cycle. It follows from theorem 5.3.14 and proposition 5.3.6, that the face cycle Σ_F can also coexist with a limit cycle.
(2) If instead of the group $\Delta_4 \rtimes \mathbb{Z}_4$, we look at the determinant one subgroup $G^\star = (\Delta_4 \rtimes \mathbb{Z}_4)'$, the regions Λ_3 are no longer flow-invariant for $\mathcal{P}_Q$. The components of a general homogeneous cubic equivariant Q are given by

$$Q_j(x_1, \ldots, x_4) = x_j(a\|x\|^2 + \Sigma_{i=2}^4 a_i x_{i+j-1}^2) + (-1)^j e x_1 \ldots \hat{x}_j \ldots x_4,$$

where $\hat{}$ denotes omission and $a, \ldots, e \in \mathbb{R}$. If $a_2 a_4 < 0$, we continue to have the edge cycle Σ_E but now trajectories can wind around Σ_E, exit Λ_3 and explore other parts of S^3. Chaotic dynamics are seen in numerical explorations of the dynamics of $\mathcal{P}_Q$. If we look at a $G^\star$-equivariant family $x' = X(x, \lambda)$, $X \in \mathcal{V}_0(\mathbb{R}^4 \times \mathbb{R}, \mathbb{R}^4)$, then the bifurcation that occurs as λ passes through zero can result in bifurcation to chaotic dynamics or 'instant chaos'. We refer the reader to the articles by Guckenheimer & Worfolk [**89**], and Worfolk [**182**] for more details about this phenomenom. (We give another example of instant chaos in the next section.)

5.4. Bifurcation to complex dynamics in dimension four

In this section we describe some of the complex dynamics associated to bifurcations on a four dimensional absolutely irreducible representation. This representation also gives the lowest dimensional example where the converse to the MISC fails. We start with a description of the representation which closely follows [**61**, Appendix].

Let $\hat{G} \subset S_5$ be the group generated by $s = (12345)$, $t = (2453) \in S_5$. Straightforward computations verify that $st = ts^2$ and $|\hat{G}| = 20$. It is easy to see that $\hat{G}$ is isomorphic to the group $\mathrm{Aff}_1(\mathbb{F}_5)$ of affine automorphisms of $\mathbb{F}_5$ (see section 4.5.6 and examples 1.2.7(4)). Take the standard irreducible representation of S_5 on $\mathbb{R}^4$ and restrict to $\hat{G}$. Since $\hat{G}$ is a doubly transitive subgroup of S_5, $(\mathbb{R}^4, \hat{G})$ is absolutely irreducible by lemma 4.10.2.

EXERCISE 5.4.1. (1) Verify the relations $ts = s^3 t$, $ts^3 = s^4 t$ and $ts^4 = s^2 t$.
(2) Verify that $\langle s \rangle$ is a normal subgroup of $\hat{G}$ and that $\hat{G} = \langle s \rangle \rtimes \langle t \rangle$.

5.4.1. A basis for the action of $\hat{G}$ on $\mathbb{C}^4$. Complexifying the action of $\hat{G}$, we obtain a complex irreducible action of $\hat{G}$ on $\mathbb{C}^4$. Let S denote the 5×5 permutation matrix corresponding to the generator $s = (12345)$. Set $e^{\frac{2\pi\imath}{5}} = \omega$. The eigenvalues of $S : \mathbb{C}^5 \to \mathbb{C}^5$ are $\{\omega^k \mid k = 0, \ldots, 4\}$. The eigenvectors $\mathbf{e}_i$

corresponding to ω^i, $1 \leq i \leq 4$ are

$$\begin{aligned}
\mathbf{e}_1 &= (1, \omega^4, \omega^3, \omega^2, \omega),\\
\mathbf{e}_2 &= (1, \omega^3, \omega, \omega^4, \omega^2),\\
\mathbf{e}_3 &= (1, \omega^2, \omega^4, \omega, \omega^3),\\
\mathbf{e}_4 &= (1, \omega, \omega^2, \omega^3, \omega^4).
\end{aligned}$$

Since $\sum_{i=0}^4 \omega^i = 0$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is a basis for $\mathbb{C}^4 \cong z_1 + \ldots z_5 = 0$.

LEMMA 5.4.2. *Relative to the basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ *we have*

$$\begin{aligned}
s(z_1, z_2, z_3, z_4) &= (\omega z_1, \omega^2 z_2, \omega^3 z_3, \omega^4 z_4),\\
t(z_1, z_2, z_3, z_4) &= (z_3, z_1, z_4, z_2).
\end{aligned}$$

PROOF. We leave the straightforward verification to the reader. ☐

LEMMA 5.4.3. *Let V be the four dimensional real subspace of $\mathbb{C}^4$ defined by*

$$V = \{(z_1, z_2, \bar{z}_2, \bar{z}_1) \mid z_1, z_2 \in \mathbb{C}\}.$$

Then

(a) *V is a $\hat{G}$-invariant subspace of $\mathbb{C}^4$ and $(V, \hat{G}) \cong (\mathbb{R}^4, \hat{G})$ (as real representations).*
(b) *Relative to the complex coordinates (z_1, z_2) on V, we have*

$$\begin{aligned}
s(z_1, z_2) &= (\omega z_1, \omega^2 z_2),\\
t(z_1, z_2) &= (\bar{z}_2, z_1).
\end{aligned}$$

PROOF. Since $\bar{\omega} = \omega^4$ and $\bar{\omega}^2 = \omega^3$, $\hat{G}$-invariance of V follows from

$$\begin{aligned}
s(z_1, z_2, \bar{z}_2, \bar{z}_1) &= (\omega z_1, \omega^2 z_2, \overline{\omega^2 z_3}, \overline{\omega^4 z_4}),\\
t(z_1, z_2, \bar{z}_2, \bar{z}_1) &= (\bar{z}_2, z_1, \bar{z}_1, z_2).
\end{aligned}$$

Since $(\mathbb{R}^4, \hat{G})$ is absolutely irreducible, $(\mathbb{C}^4, \hat{G})$ is isomorphic as a real representation to two copies of $(\mathbb{R}^4, \hat{G})$. In particular, every nontrivial $\hat{G}$-invariant subspace of $(\mathbb{C}^4, \hat{G})$ is isomorphic as a real representation to $(\mathbb{R}^4, \hat{G})$. ☐

5.4.2. The representation (V, G). Let $G \subset \mathrm{O}(V)$ be the group generated by $\hat{G}$ and $-I_V$. Obviously, $|G| = 40$, (V, G) is absolutely irreducible and $P_G^{2n}(V, V) = \{0\}$, $n \geq 0$. We leave as an exercise the straightforward computation that a vector space basis $\{R, p, q\}$ for $P_G^3(V, V)$ is given by

$$\begin{aligned}
R(z_1, z_2) &= (|z_1|^2 + |z_2|^2)(z_1, z_2),\\
p(z_1, z_2) &= (\bar{z}_1^2 \bar{z}_2, z_1 \bar{z}_2^2),\\
q(z_1, z_2) &= (z_2^3, \bar{z}_1^3).
\end{aligned}$$

EXERCISE 5.4.4. (1) Verify that $\{R, p, q\}$ is a basis for $P^3_G(V, V)$.
(2) Show that $q_1(z_1, z_2) = (\bar{z}_2^2, z_1^2)$, $q_2(z_1, z_2) = (\bar{z}_1 z_2, \bar{z}_1 \bar{z}_2)$ is a basis for $P^2_{\hat{G}}(V, V)$. Hence show that there exists a nontrivial homogeneous quadratic $Q \in P^2_{\hat{G}}(V, V)$ such that $(Q(x), x) = 0$, all $x \in V$.

We will mainly be interested in looking at the dynamics and bifurcation theory of normalized cubic truncations $\lambda z + Q(z) \in P^{(3)}_G(V, V)$. The next lemma gives conditions when the invariant sphere theorem applies.

LEMMA 5.4.5. *Let $Q(z) = \alpha R(z) + \beta p(z) + \gamma q(z)$, where $\alpha, \beta, \gamma \in \mathbb{R}$. Then Q is contracting if and only if*

$$\alpha < 0, \quad \textit{and } |\beta + \gamma| < -2\alpha.$$

PROOF. Setting $(z_1, z_2) = u$ and computing, we find that

$$(Q(u), u) \;=\; \alpha\|u\|^4 + \frac{\beta+\gamma}{2}(\bar{z}_1^3 \bar{z}_2 + z_1^3 z_2 + \bar{z}_1 z_2^3 + z_1 \bar{z}_2^3).$$

Making the polar coordinate substitutions $z_i = r_i e^{\imath\theta_i}$, $i = 1, 2$, we find that

$$\begin{aligned}
(Q(u), u) &= \alpha(r_1^2 + r_2^2)^2 + (\beta+\gamma) r_1 r_2 [r_1^2 \cos(3\theta_1 + \theta_2) + r_2^2 \cos(3\theta_2 - \theta_1)], \\
&\leq \alpha(r_1^2 + r_2^2)^2 + |\beta+\gamma| r_1 r_2 (r_1^2 + r_2^2), \\
&\leq \alpha(r_1^2 + r_2^2)^2 + |\beta+\gamma| \frac{r_1^2 + r_2^2}{2}(r_1^2 + r_2^2), \\
&\leq (\alpha + \frac{|\beta+\gamma|}{2})(r_1^2 + r_2^2)^2.
\end{aligned}$$

The sufficiency of the conditions follows immediately from this estimate. Necessity follows by taking $3\theta_1 \in \{0, \pi\}$, $\theta_2 = 0$ and $r_1 = r_2$. □

5.4.3. Geometry of the representation (V, G)**.** Take complex coordinates (z_1, z_2) on V and real coordinates (x_1, y_1, x_2, y_2), where $z_i = x_i + \imath y_i$, $i = 1, 2$.

LEMMA 5.4.6. *Representative proper fixed point spaces of (V, G) are given by*

$\mathbf{A} = \mathbb{R}(1, 0, 1, 0) = V^{\langle t \rangle}$ *(axis of symmetry, $\langle t \rangle \cong \mathbb{Z}_4$).*
$\mathbf{B} = \mathbb{R}(1, 0, -1, 0) = V^{\langle -t \rangle}$ *(axis of symmetry, $\langle -t \rangle \cong \mathbb{Z}_4$).*
$\mathbf{S} = \{(x_1, 0, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\} = V^{\langle t^2 \rangle}$ *(submaximal stratum, $\langle t^2 \rangle \cong \mathbb{Z}_2$).*
$\mathbf{P} = \{(0, y_1, 0, y_2) \mid y_1, y_2 \in \mathbb{R}\} = V^{\langle -t^2 \rangle}$ (maximal *orbit stratum, $\langle -t^2 \rangle \cong \mathbb{Z}_2$) and $N(\langle -t^2 \rangle)/\langle -t^2 \rangle \cong \mathbb{Z}_4$.*

There are five axes of type (A), five axes of type (B), five planes of type (S) and five planes of type (P). There are no three dimensional fixed point subspaces and all other points in V have trivial isotropy.

The proof is straightforward and omitted.

REMARK 5.4.7. Absolutely irreducible representations on $\mathbb{R}^n$, $n \leq 3$ have one-dimensional maximal orbit strata. Hence lemma 5.4.6 gives the smallest dimension for which one can have a two-dimensional maximal orbit stratum.

5.4.4. Equilibria of a normalized cubic family. We consider the normalized cubic family of 5.4.2 with $\alpha = -1$. In coordinates we have

$$z_1' = \lambda z_1 - (|z_1|^2 + |z_2|^2) z_1 + \beta \bar{z}_1^2 \bar{z}_2 + \gamma z_2^3, \tag{5.26}$$

$$z_2' = \lambda z_2 - (|z_1|^2 + |z_2|^2) z_2 + \beta z_1 \bar{z}_2^2 + \gamma \bar{z}_1^3. \tag{5.27}$$

From now on assume that $|\beta + \gamma| < 2$ and so, by lemma 5.4.5, the conditions of the invariant sphere theorem hold.

For our computations of equilibria and their stabilities, it is useful to rewrite the system (5.26,5.27) in real coordinates.

$$\begin{aligned}
x_1' &= \lambda x_1 - \|x\|^2 x_1 + \beta(x_1^2 x_2 - y_1^2 x_2 - 2x_1 y_1 y_2) + \gamma(x_2^3 - 3x_2 y_2^2),\\
y_1' &= \lambda y_1 - \|x\|^2 y_1 + \beta(-x_1^2 y_2 + y_1^2 y_2 - 2x_1 y_1 x_2) + \gamma(-y_2^3 + 3x_2^2 y_2),\\
x_2' &= \lambda x_2 - \|x\|^2 x_2 + \beta(x_1 x_2^2 - x_1 y_2^2 + 2y_1 x_2 y_2) + \gamma(x_1^3 - 3x_1 y_1^2),\\
y_2' &= \lambda y_2 - \|x\|^2 y_2 + \beta(y_1 x_2^2 - y_1 y_2^2 - 2x_1 x_2 y_2) + \gamma(y_1^3 - 3x_1^2 y_1).
\end{aligned}$$

Equilibria along axes of symmetry. If $\lambda > 0$, there is a pair of equilibria $\pm a(\lambda)$ on the axis $\mathbf{A} = V^{\langle t\rangle} \subset V^{\langle t^2\rangle} \subset S$. Computing, we find that

$$a(\lambda) = \left(\sqrt{\frac{-\lambda}{\beta+\gamma-2}}, 0, \sqrt{\frac{-\lambda}{\beta+\gamma-2}}, 0\right).$$

The eigenvalues of the linearization of the cubic system at $\pm a(\lambda)$ are

$$[-2\lambda : \frac{4\lambda\gamma}{\beta+\gamma-2} : \lambda\frac{(\gamma+3\beta) \pm \imath(3\gamma-\beta)}{\beta+\gamma-2}].$$

We remark that -2λ is the eigenvalue associated to the radial direction and that the eigenspace of the eigenvalue $\frac{4\lambda\gamma}{\beta+\gamma-2}$ lies in the x_1, x_2-plane $\mathbf{S}$. The eigenspace associated to the complex conjugate pair of eigenvalues is the y_1, y_2-plane $\mathbf{P} = V^{(-t^2)}$ which meets $\mathbf{S}$ orthogonally. Note that these eigenvalues are complex unless $3\gamma = \beta$ (and then the cubic equivariant will have S_5-symmetry).

For $\lambda > 0$, there is a pair of equilibria $\pm b(\lambda)$ on the axis $\mathbf{B} = V^{\langle -t\rangle} \subset \mathbf{S}$ given by

$$b(\lambda) = \left(\sqrt{\frac{\lambda}{\beta+\gamma+2}}, 0, -\sqrt{\frac{\lambda}{\beta+\gamma+2}}, 0\right).$$

The eigenvalues of the linearization of the cubic system at $\pm b(\lambda)$ are

$$[-2\lambda : \frac{4\lambda\gamma}{\beta+\gamma+2} : \lambda\frac{(\gamma+3\beta) \pm \imath(3\gamma-\beta)}{\beta+\gamma+2}].$$

The eigenspace of $\frac{4\lambda\gamma}{\beta+\gamma+2}$ lies in the x_1, x_2-plane $\mathbf{S}$. The eigenspace space associated to the complex conjugate pair of eigenvalues is the y_1, y_2-plane $\mathbf{P}$.

Thus far we have shown that for $\lambda > 0$ there are four nontrivial equilibria on the plane $\mathbf{S}$. The equations of the system restricted to $\mathbf{S}$ are

$$x_1' = \lambda x_1 - (x_1^2 + x_2^2)x_1 + \beta x_1^2 x_2 + \gamma x_2^3, \tag{5.28}$$
$$x_2' = \lambda x_2 - (x_1^2 + x_2^2)x_2 + \beta x_1 x_2^2 + \gamma x_1^3. \tag{5.29}$$

LEMMA 5.4.8. *Let $\lambda > 0$. If $\gamma \neq 0$ and $|\beta + \gamma| < 2$, then $\pm a(\lambda)$, $\pm b(\lambda)$ are the only nonzero equilibrium points of (5.28,5.29).*

PROOF. If (x_1, x_2) is an equilibrium of (5.28,5.29) and $x_1 x_2 \neq 0$, it is easy to show that $\gamma(x_1^4 - x_2^4) = 0$. Hence, provided $\gamma \neq 0$, $x_1 = \pm x_2$. $\square$

REMARK 5.4.9. A straightforward computation shows that if $\gamma \neq 0$, $\gamma \neq \beta$, $\gamma \neq 2\beta$ and $\beta \neq 2\gamma$, then (5.28,5.29) has a quadruple of *hyperbolic complex equilibria* on the lines $x_1 = \pm \imath x_2$ (the eigenvalues of the linearization are real). Counting real and complex equilibria and the origin of $\mathbf{S}$, it follows that we have found a total of $4 + 4 + 1 = 3^2$ nonsingular equilibria on $\mathbf{S}$. This is the maximum number we can expect from Bézout's theorem (theorem 4.9.1).

Equilibria on the y_1, y_2-plane $\mathbf{P}$. Dynamics on $\mathbf{P}$ are governed by the system

$$y_1' = \lambda y_1 - (y_1^2 + y_2^2)y_1 + \beta y_1^2 y_2 - \gamma y_2^3, \tag{5.30}$$
$$y_2' = \lambda y_2 - (y_1^2 + y_2^2)y_2 - \beta y_1 y_2^2 + \gamma y_1^3. \tag{5.31}$$

By lemma 5.4.6, $N((-t^2))/(-t^2) \cong \mathbb{Z}_4$ acts freely on $\mathbf{P}$ and so the number of nontrivial equilibria is divisible by four.

PROPOSITION 5.4.10. *Suppose $\lambda > 0$, $\gamma \neq 0$.*

(a) *If $\beta\gamma < 0$ or $\beta^2 < \gamma^2$, then (5.30,5.31) has no nontrivial equilibria.*
(b) *If $\beta^2 > \gamma^2$ and $\beta\gamma > 0$, then there exist two distinct $\mathbb{Z}_4$-orbits of nontrivial equilibria of (5.30,5.31). One $\mathbb{Z}_4$-orbit will consist of hyperbolic sinks, the other of hyperbolic saddles and (5.30,5.31) will have no other nontrivial equilibria.*
(c) *If $\beta = \gamma > 0$, there is one $\mathbb{Z}_4$-orbit of singular nontrivial equilibria.*

For a nonempty open dense subset of values of β, γ satisfying $\beta^2 > \gamma^2$ and $\beta\gamma > 0$, the zeros given by (b) are hyperbolic zeros of (5.26,5.27).

PROOF. If (y_1, y_2) is a nontrivial equilibrium of (5.30,5.31), we may show

$$2\beta y_1^2 y_2^2 = \gamma(y_1^4 + y_2^4). \tag{5.32}$$

Hence, for nontrivial solutions, β and γ have the same sign. Since $\gamma \neq 0$, there are no nontrivial solutions of (5.32) with $y_1 y_2 = 0$. Rewriting (5.32), we see that y_1/y_2 satisfies the equation

$$X^4 - 2\frac{\beta}{\gamma}X^2 + 1 = 0.$$

This equation only has real solutions if $\beta^2 > \gamma^2$. Hence a necessary condition for the existence of nontrivial equilibria is $\beta^2 \geq \gamma^2$ and $\beta\gamma > 0$. In particular, there are no equilibria if either $\beta\gamma < 0$ or $\beta^2 < \gamma^2$. In case $\beta = \gamma > 0$, we find by direct computation that there are four singular zeros on the lines $y_1 = \pm y_2$. Computing, we find that if we regard $\mu = \beta/\gamma$ as a parameter (and fix λ), then (5.30,5.31) has a nondegenerate saddle-node bifurcation at $\mu = 1$. Eigenvalues of the linearization in the direction transverse to $\mathbf{P}$ are easily computed to be real, nonzero and of opposite sign when $\mu = 1$.

As we increase μ through one, we obtain two $\mathbb{Z}_4$-orbits of hyperbolic equilibria (four saddles and four sinks). For $\mu > 1$ close to one, these equilibria will be hyperbolic within V not just the (y_1, y_2)-plane. A further computation verifies that both $\mathbb{Z}_4$-orbits of equilibria will be hyperbolic (in V) for all $\mu > 1$. $\square$

PROPOSITION 5.4.11. *There is an open and dense semialgebraic subset $\mathcal{R}$ of the (β, γ)-plane such that if $(\beta, \gamma) \in \mathcal{R}$ then*

(1) *Every nontrivial zero of (5.26,5.27) is hyperbolic.*
(2) *Every nontrivial zero of (5.26,5.27) lies either on a G-orbit of the axis $\mathbf{A}$, or on a G-orbit of the axis $\mathbf{B}$ or on a G-orbit of the 2-plane $\mathcal{P}$.*

Moreover, there is a nonempty open subset $\mathcal{C}$ of $\mathcal{R}$ such that if $(\beta, \gamma) \in \mathcal{C}$, then (5.26,5.27) has no zeros on the maximal isotropy subspace $\mathbf{P}$. In particular, the converse of the MISC fails for the representation (V, G).

PROOF. By proposition 5.4.10 and the results of section 5.4.4, in particular remark 5.4.9, there is a nonempty open subset U of the (β, γ)-plane such that if $(\beta, \gamma) \in U$, then (5.26,5.27) has a total of at least $1 + 5 \times 8 + 5 \times 8 = 81$ real and complex zeros, all of which are hyperbolic. Hence, by theorem 4.9.1, for $(\beta, \gamma) \in U$ there are exactly 81 zeros. In particular, there can be no equilibria with trivial isotropy. If $(\beta, \gamma) \in U$, then $\alpha p + \beta q \in P^3_G(V, V)$ is of hyperbolic type and so the proposition follows by theorem 4.6.16. $\square$

5.4.5. Dynamics close to the plane S. Dynamics on $\mathbf{S}$ are governed by the system (5.28,5.29).

Since we are assuming that $|\beta + \gamma| < 2$, the conditions of the invariant sphere theorem are satisfied for the original system (5.26,5.27) and hence for (5.28,5.29). Consequently, (5.28,5.29) will have a globally attracting invariant circle $C(\lambda)$, for $\lambda > 0$. Necessarily, $\pm a(\lambda), \pm b(\lambda) \in C(\lambda)$.

LEMMA 5.4.12. *Let $\lambda > 0$. If $\gamma \neq 0$ and $|\beta + \gamma| < 2$, then $\pm a(\lambda)$, $\pm b(\lambda)$ are the only nonzero equilibrium points of (5.28,5.29). Stabilities of $a(\lambda)$, $b(\lambda)$ in the x_1, x_2-plane are as follows.*

(a) *If $\gamma > 0$, $\pm a(\lambda)$ are sinks, $\pm b(\lambda)$ are saddles and*
$$C(\lambda) = W^u(b(\lambda)) \cup W^u(-b(\lambda)) \cup \{a(\lambda), -a(\lambda)\}.$$
(b) *If $\gamma < 0$, $\pm b(\lambda)$ are sinks, $\pm a(\lambda)$ are saddles and*
$$C(\lambda) = W^u(a(\lambda)) \cup W^u(-a(\lambda)) \cup \{b(\lambda), -b(\lambda)\}.$$

(See figure 9 for the case $\gamma < 0$.)

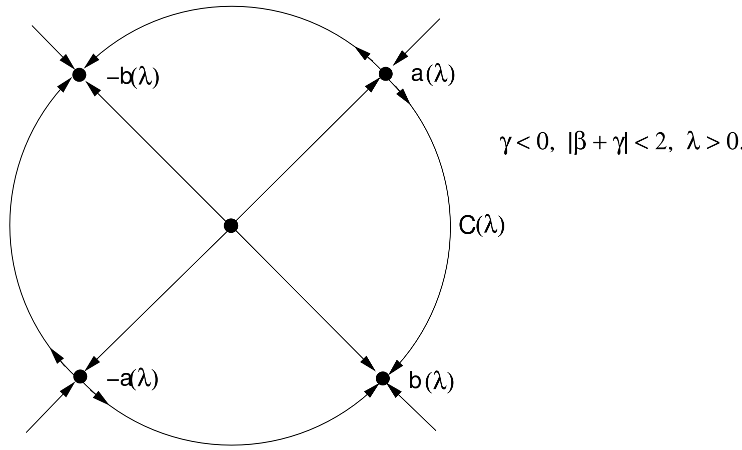


FIGURE 9. Dynamics on $\mathbf{S}$

LEMMA 5.4.13. *Suppose that* $\lambda > 0$ *and and* $|\beta+\gamma| < 2$.

(1) *If* $\gamma > 0$ *and* $\gamma + 3\beta < 0$, *then* $\dim(W^u(a(\lambda))) = 2$, $\dim(W^s(b(\lambda))) = 3$, $W^u(a(\lambda))$ *and* $W^s(b(\lambda))$ *are transverse to* $\mathbf{S}$ *and there is a 1-dimensional connection from* $b(\lambda)$ *to* $a(\lambda)$ *in* $\mathbf{S}$.
(2) *If* $\gamma < 0$ *and* $\gamma + 3\beta > 0$, *then dim*$(W^u(b(\lambda))) = 2$, *dim*$(W^s(a(\lambda))) = 3$, $W^u(b(\lambda))$ *and* $W^s(a(\lambda))$ *are transverse to* $\mathbf{S}$ *and there is a 1-dimensional connection from* $a(\lambda)$ *to* $b(\lambda)$ *in* $\mathbf{S}$.
(3) *If the conditions of (1) and (2) are not satisfied and* $\gamma(\gamma+3\beta) \neq 0$, *then exactly one of* $a(\lambda)$, $b(\lambda)$ *is a sink.*

Lemmas 5.4.12, 5.4.13 follow from our earlier computations.

Since we are assuming the conditions of the invariant sphere hold, it is no loss of generality to fix $\lambda > 0$ in lemma 5.4.13 and restrict dynamics to the associated invariant sphere, $S(\lambda)$. We refer to figure 10 for the dynamics near the intersection of $\mathbf{S}$ with $S(\lambda)$ when $\gamma < 0$ and $\gamma + 3\beta > 0$.

Referring to figure 10, there are one-dimensional connections from $\pm a(\lambda)$ to $\pm b(\lambda)$. These connections are not transverse and are forced by the symmetry. Since $S(\lambda)$ is attracting, $W^u(\pm a(\lambda)) \subset S(\lambda)$. Restricting the flow to $S(\lambda)$, it follows from lemma 5.4.13 that (within $S(\lambda)$) we have $\dim(W^u(b(\lambda))) = 2$, $\dim(W^s(a(\lambda))) = 2$. Since G acts freely on $V \setminus (G(\mathbf{S}) \cup G(\mathbf{P}))$ and the invariant manifolds $W^u(b(\lambda))$, $W^s(a(\lambda))$ intersect $G(\mathbf{S}) \cup G(\mathbf{P})$ only at the equilibrium points $a(\lambda), b(\lambda)$, we can expect that, for all $g \in G$, either $W^u(\pm b(\lambda) \cap W^s(\pm a(g\lambda)) = \emptyset$, or that $W^u(b(\lambda) \cap W^s(a(g\lambda))$ is transverse and consists of a finite set of trajectories. Of course, this is a genericity statement and to achieve

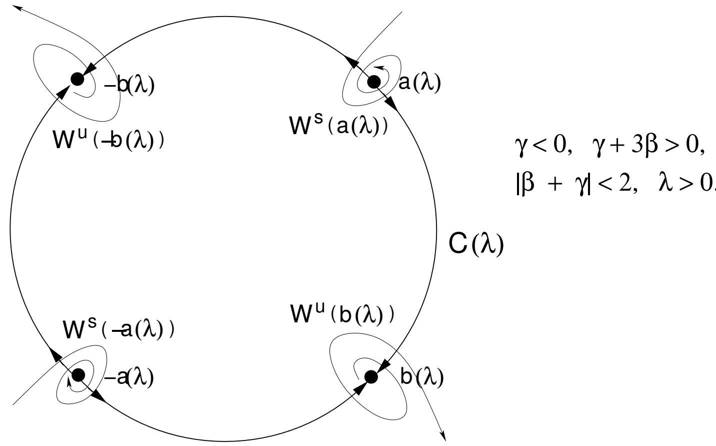


FIGURE 10. Dynamics near $S(1) \cap \mathbf{S} = C(1)$

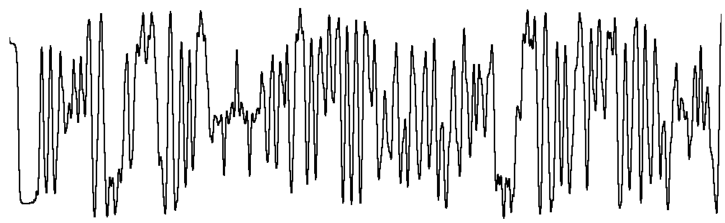
transversality of the invariant manifolds might well involve G-equivariants of degree greater than three.

Suppose that there exists $g \in G$ such that $W^u(b(\lambda)) \cap W^s(ga(\lambda)) \neq \emptyset$ and is transverse. Then, by equivariance, the intersections $W^u(g^j b(\lambda)) \cap W^s(g^{j+1}a(\lambda))$ will also be nonempty and transverse. Since every element of G has order at most 5, the sequence of connections $W^u(g^j a(\lambda)) \cap W^s(g^j b(\lambda))$ and $W^u(g^j b(\lambda) \cap W^s(g^{j+1}a(\lambda))$, $j \geq 0$ will define a cycle which connects back to $b(\lambda)$. In fact each time we connect to $g^j a(\lambda)$, we also have 1-dimensional connections to $\pm g^j b(\lambda)$.

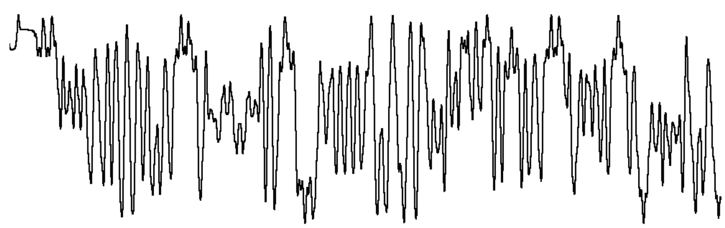
LEMMA 5.4.14. *Suppose that $g \in G$ and $g \notin \langle t, -I \rangle$ (so $g\mathbf{S} \cap \mathbf{S} = \emptyset$). The single connection $b(\lambda) \to ga(\lambda)$ generates a G-invariant connected heteroclinic network consisting of* 20 *equilibria and* 60 *connections. Of these connections, 20 are of the form $ha(\lambda) \to \pm hb(\lambda)$, $h \in G$, and the remaining connections are of the form $hb(\lambda) \to j\sigma a(\lambda)$, where $hj^{-1} \notin \langle t, -I \rangle$ and σ is either* $+1$ *or* -1.

PROOF. We sketch the proof in case $g = s$ when there is a connection $b(\lambda) \to sa(\lambda)$. Since the isotropy group of $b(\lambda)$ is $\langle -t \rangle$, $b(\lambda) \to -tsa(\lambda)$, $b(\lambda) \to t^2 sa(\lambda)$ and $b(\lambda) \to -t^3 sa(\lambda)$ are all connections forced by G-equivariance. From exercise 5.4.1(2), we have $-ts = -s^3 t$, $t^2 s = s^4 t^2$ and $-t^3 s = -s^2 t$. Hence, since $ta(\lambda) = a(\lambda)$, we obtain distinct connections $b(\lambda) \to sa(\lambda)$, $b(\lambda) \to -s^2 a(\lambda)$, $b(\lambda) \to s^3 a(\lambda)$ and $b(\lambda) \to -s^4 a(\lambda)$. Acting by $-I$ gives another four connections $-b(\lambda) \to -sa(\lambda), \ldots, -b(\lambda) \to s^4 a(\lambda)$. Acting by $\langle s \rangle$, we see that there are a total of 40 connections between points on the G-orbits of $b(\lambda)$ and $a(\lambda)$. However, there is no connection from $b(\lambda)$ to $\pm a(\lambda)$ in the G-orbit of $b(\lambda) \to ga(\lambda)$. If the G-orbit of $b(\lambda) \to ga(\lambda)$ contains $hb(\lambda) \to j\sigma a(\lambda)$ then it does not contain $hb(\lambda) \to -j\sigma a(\lambda)$. $\square$

REMARK 5.4.15. If there is a connection $b(\lambda) \to ha(\lambda)$, $h \notin \langle t, -I \rangle$, and $W^u(b(\lambda))$ intersects $W^u(ha(\lambda))$ transversally along the connection, we call the network given by lemma 5.4.14 a *Shilnikov network* on account of the similarity to the setup for the Shilnikov bifurcation (see also [**61**, Appendix]).



Time series for x_1



Times series for x_2

FIGURE 11. Time-series for x_1, x_2: $\gamma = -1.0$, $\beta = 0.5$, $\lambda = 1.0$

If the system (5.26,5.27) satisfies the conditions for the invariant sphere theorem and contains a Shilnikov network, then it reasonable to expect the presence of complex dynamics. In fact simple numerical experiments show highly chaotic behaviour as well as random switching of typical trajectories between the nodes (equilibria) of what appears to be a Shilnikov network.

In figure 11 we show the time series[2] for the variables x_1, x_2 when $\gamma = -1.0$, $\beta = 0.5$ and $\lambda = 1.0$. For these parameter values, the plane **P** contains a limit cycle which is a saddle within S^3. In figure 12, we show the projection into the (y_1, y_2)-plane **P** of a typical trajectory. The outer boundary (envelop) of the trajectories is the limit cycle in **P**. Note the abrupt changes of direction of

[2]Computed using *Dstool*, 4th order Runge-Kutta with a fixed time step of 0.001

trajectories – the expectation is that these correspond to the trajectory passing close to an equilibrium point of the flow.

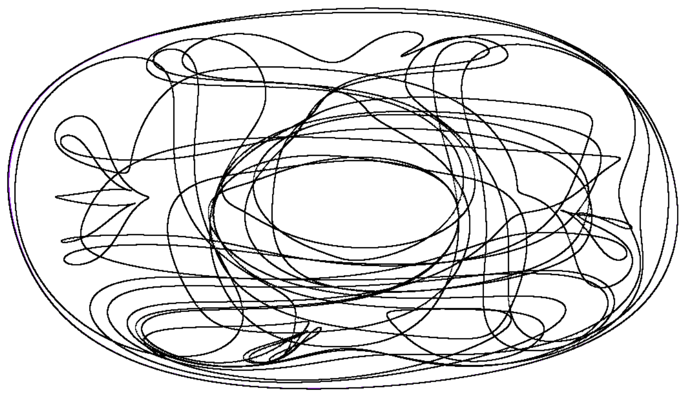


FIGURE 12. Projection into (y_1, y_2)-plane. Same parameter values as before.

In lemma 5.4.14 we showed that a connection between equilibria in distinct planes $\mathbf{S}$, $h\mathbf{S}$ led to a network with some 60 connections. There is also the possibility of connection between limit cycles in the G-orbit of $\mathbf{P}$ as well as connections between equilibria and limit cycles (see [**4**] for a description of the various possibilities). Since the limit cycles are hyperbolic saddles for an non-empty open set of (γ, β), it follows that if their invariant manifolds meet transversally there will be transverse homoclinic points and hence (suspended) horseshoes by Smale's theorem (see [**100**, Theorem 6.5.5]).

5.4.6. Random switching and dynamics near a network. It has been known for some time that equivariant dynamical systems can possess heteroclinic networks[3] and that dynamics can often be complicated near these networks (see for example [**101, 73, 6**]). In her 2003 Porto thesis, Manuela Aguiar made a study of random switching in the system (5.26,5.27). This work is reported on in [**4**]. We describe some of the ideas that relate to the Shilnikov network generated by a connection $b(1) \to \pm sa(1)$ and refer the reader to [**4**] for more complete details. (We assume some background on subshifts of finite type [**139**],[**100**, 1.9].)

Using *Dstool* and GAIO [**40**], Aguiar gave numerical evidence of parameter values γ, β, for which there were coexisting connections $b(\lambda) \to \pm sa(\lambda)$ such that the intersections of invariant manifolds along the connections were transverse.

For the remainder of the section we assume that γ, β are chosen so that $|\gamma+\beta| < 2$, $\gamma < 0$ and $\gamma+3\beta > 0$. We take $\lambda = 1$ and set $a(1) = \mathbf{a}$, $b(1) = \mathbf{b}$. Let

[3]For us a heteroclinic network will consist of a connected set of connections between hyperbolic saddles. The non-transverse connections will be forced by symmetry. The simplest examples are provided by the G-orbits of heteroclinic cycles.

$S(V) \cong S^3$ denote the invariant sphere for the dynamics. Assume that $|\gamma|, |\beta|$ are chosen sufficiently small so that $S(V)$ is a C^2-submanifold of V. Assume that we have a transverse connection $\mathbf{b} \to s\mathbf{a}$ and let $\mathcal{N}$ be the 60 connection Shilnikov network given by lemma 5.4.14. Recall that $G = \langle -s \rangle \rtimes \langle t \rangle$. Set $J = \langle -s \rangle$, $K = \langle t \rangle$. Since J acts *freely* on $S(V)$, the flow determined by (5.26,5.27) induces a C^2 K-equivariant flow Φ_t on the oriented K-manifold $\tilde{S} = S(V)/J$. Let X denote the vector field determined by Φ_t. Let $p : S(V) \to \tilde{S}$ denote the orbit map. Since $-I \in J$, $p(S(V) \cap \mathbf{S})$ is a Φ_t-invariant circle C in $\tilde{S}$ containing two equilibria $\tilde{\mathbf{a}} = p(\pm\mathbf{a})$, $\tilde{\mathbf{b}} = p(\pm\mathbf{b})$. The equilibria are the two fixed points of the K-action on $\tilde{S}$. Both fixed points are of the *same* K-isotropy type. The circle C contains all points in $\tilde{S}$ which are not of principal isotropy type. Set $\gamma = p(\mathbf{b} \to s\mathbf{a})$. The K-orbit of γ consists of four distinct connections $\tilde{\mathbf{a}} \to \tilde{\mathbf{b}}$. If we let $\tilde{\mathcal{N}} = \mathcal{N}/J$, then $\tilde{\mathcal{N}}$ is a heteroclinic network for Φ_t consisting of the two equilibria $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}$, two connections $\tilde{\mathbf{b}} \to \tilde{\mathbf{a}}$, and four connections $\tilde{\mathbf{a}} \to \tilde{\mathbf{b}}$. We refer the reader to figure 13 – we have only shown one of the four connections $\tilde{\mathbf{a}} \to \tilde{\mathbf{b}}$.

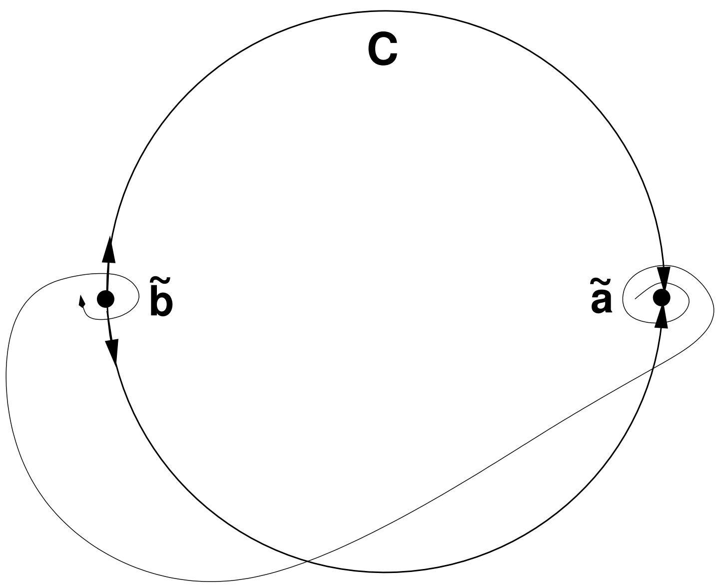


FIGURE 13. The heteroclinic network $\tilde{\mathcal{N}}$

If there are p connections $\mathbf{a} \to s\mathbf{b}$, and q connections $\mathbf{a} \to -s\mathbf{b}$, then there will be a total of $4(p+q)$ connections $\tilde{\mathbf{a}} \to \tilde{\mathbf{b}}$.

We assume that the network $\tilde{\mathcal{N}}$ has transverse connections γ_i, $1 \le i \le m = 4N$, from $\tilde{\mathbf{a}}$ to $\tilde{\mathbf{b}}$, together with the two connections π_α, π_β from $\tilde{\mathbf{b}}$ to $\tilde{\mathbf{a}}$ forced by K-symmetry. Note that K acts freely on $\{\gamma_1, \dots, \gamma_m\}$ and acts with isotropy $\mathbb{Z}_2$ on $\{\pi_\alpha, \pi_\beta\}$. Let $\Sigma = \Sigma_A \subset \{\alpha, \beta, 1, \dots, m\}^{\mathbb{Z}}$ be the subshift of finite type with

01-matrix A specified by

$$\begin{aligned} a_{ij} &= 1, \text{ if } i \in \{\alpha, \beta\},\ j \in \{1, \dots, m\}, \\ &= 1, \text{ if } i \in \{1, \dots, m\},\ j \in \{\alpha, \beta\}, \\ &= 0, \text{ if } i, j \in \{1, \dots, m\}, \text{ or } i, j \in \{\alpha, \beta\}. \end{aligned}$$

The matrix A is irreducible (and so $\sigma : \Sigma \to \Sigma$ is transitive) but A is not aperiodic (σ is not topologically mixing [**139**]). If we identify α with π_α, β with π_β and i with γ_i, $1 \leq i \leq m$, then the action of K on the set of connections induces an action of K on $\{\alpha, \beta, 1, \dots, m\}$ and hence on Σ. The shift map is then K-equivariant (we refer the reader to section 9.3 for basic theory of 'G-subshifts of finite type'). If $r : \Sigma \to \mathbb{R}$ is a G-invariant roof function, then the suspension flow $\sigma_t^r : \Sigma^r \to \Sigma^r$ is K-equivariant (see section 9.4 for details on equivariant suspensions).

Definition 5.4.16. An open neighbourhood W of $\tilde{\mathcal{N}}$ is *admissible* if

(1) W is K-invariant.
(2) W is connected.
(3) There exist disjoint closed neighbourhoods A of $\tilde{\mathbf{a}}$ and B of $\tilde{\mathbf{b}}$ such that $W \setminus A \cup B$ has $m + 2$ connected components, $U_\alpha, U_\beta, U_1, \dots, U_m$.
(4) For $j = \alpha, \beta$, $U_j \cup (A \cup B)$ will be a neighbourhood of π_j and for $1 \leq i \leq m$, $U_i \cup (A \cup B)$ will be a neighbourhood of γ_i.

Remark 5.4.17. The network $\tilde{\mathcal{N}}$ has a base of admissible neighbourhoods.

5.4.7. Random switching theorem. Let W be an admissible neighbourhood of $\tilde{\mathcal{N}}$. It is shown in [**4**, section 6] that there exists a K-invariant roof function $r : \Sigma \to \mathbb{R}$ and a K-equivariant embedding $\chi : \Sigma^r \to W$ such that

(1) $\Phi_t | \chi(\Sigma^r) = \sigma_t^r$.
(2) $\Phi_t | \chi(\Sigma^r)$ is hyperbolic.
(3) If the open sets $A, B, U_\alpha, U_\beta, U_1, \dots, U_m$ satisfy the conditions of definition 5.4.16, then for each $\mathbf{x} = (x_i) \in \Sigma$, there exists a (unique) trajectory $\phi(\mathbb{R}) \subset \chi(\Sigma^r)$ and set $\{I_k \mid -\infty < k < \infty\}$ of disjoint open subintervals of $\mathbb{R}$ such that
   (a) If $t \notin \cup_k I_k$, $\phi(t) \in A \cup B$.
   (b) If $x_k \in \{\alpha, \beta, 1, \dots, k\}$, then $\phi(t) \in U_{x_k}$, $t \in I_k$.

This result – which requires transverse intersection of invariant manifolds along the connections γ_i, together with certain eigenvalue conditions that are satisfied for the system (5.26,5.27) – shows that there is 'random switching' round the nodes of the network $\tilde{\mathcal{N}}$. The random switching is quantified by a subshift of finite type and occurs arbitrarily close to the network. Indeed, there is a countable set of embedded subshifts Σ^{r_j} which satisfy the conditions of the theorem and accumulate on the network $\tilde{\mathcal{N}}$.

Since the orbit map $p : S^3 \to \tilde{S}$ is a 5:1 covering map, everything we have described lifts to the network $\mathcal{N} \subset S^3$. In particular, if $\phi(t)$ is a trajectory in $\tilde{S}$ satisfying (a,b), then there is a lift (in fact 5 lifts) $\psi(t)$ to S^3. The trajectory $\psi(t)$ will randomly switch round the nodes of the network $\mathcal{N}$. Once $\psi(0)$ is chosen (so that $p(\psi(0)) = \phi(0)$, the transitions of $\psi(t)$ between nodes of $\mathcal{N}$ are uniquely determined by the transitions for $\phi(t)$. In particular, we can quantify the random switching round the nodes of the network $\mathcal{N}$ in terms of the subshift Σ on $m+2$ symbols.

Since the embedded suspensions Σ^r have hyperbolic structure, the same is true for the lifts of Σ^r to S^3. Hence the random switching will persist under C^1-small perturbations which break the symmetry (and the $\mathbf{b} \to \mathbf{a}$ connections). The persistence of apparently chaotic dynamics is observed numerically when we make small symmetry breaking perturbations to the system (5.26,5.27).

5.5. The converse to the MISC

Suppose that (V, G) is an absolutely irreducible representation of the compact Lie group G. We recall that the converse of the MISC states that if τ is a maximal isotropy type, then τ is generically symmetry breaking. If $H \in \tau$ and $\dim(V^H)$ is odd, it follows by degree theory that τ is generically symmetry breaking [**34, 173**]. Hence, in order to find counterexamples to the MISC, it suffices to restrict to maximal isotropy types which have even dimensional fixed point spaces. Proposition 5.4.11 gives the lowest dimensional example for which a maximal isotropy type is not generically symmetry breaking. Prior to this example, families of counterexamples were obtained by Melbourne [**124**] who found six dimensional representations of a class of finite groups for which the converse of the MISC failed. In the remainder of this section, we briefly describe Melbourne's approach (the reader should consult [**124**] for more details and examples).

Suppose that $\tau \in \mathcal{O}(V, G)$ is a maximal isotropy type and let $H \in \tau$. For simplicity, assume G is finite. Let $\dim(V^H) = 2$ and set $J = N(H)/H$. The group J acts freely on V^H. Suppose J is nontrivial and $J \not\cong \mathbb{Z}_2$. Since $\dim(V^H) = 2$, $J \cong \mathbb{Z}_p$, where $p > 2$, and (V^H, J) is irreducible of complex type. Of course, if (V^H, J) is irreducible of complex type, the generic J-equivariant bifurcations on V^H have no branches of equilibria. The generic bifurcation will be a Hopf bifurcation to a branch of limit cycles. However, since (V, G) is absolutely irreducible, linear equivariants on V always restrict to *real* multiples of the identity on V^J. Consequently, we cannot generate branches of limit cycles on V^J via the Hopf bifurcation. On the other hand, we can generate a branch of limit cycles, even if the linear term is a real multiple of the identity, provided that the cubic term is a complex multiple of $\|X\|^2 X$. Adding higher order terms will not destroy the branch of limit cycles. This approach is very natural when there are no quadratic equivariants (or the quadratics vanish identically on V^H). In this case, the determinacy theorems (proved in chapter 7) allow us to add a term $-a\|X\|^2 X$, $a \gg a$, so that the conditions of the invariant sphere hold and we do create or destroy any branches of equilibria. On the two-dimensional space V^H we will have an invariant circle γ which must either be a limit cycle for the flow or contain at least two equilibria. If we can show there are no equilibria, we are done (cf the proof of proposition 5.4.11).

Melbourne finds examples where the restriction map $P^3_G(V,V) \to P^3_J(V^H,V^H)$ is onto. Counterexamples can then be constructed along the lines described above. The basic idea is that at homogeneous cubic order, the equivariants should behave like the equivariants of a complex representation. The resulting branches of limit cycles will not be destroyed by the addition of higher order terms and so there will be no branches of equilibria (the isotropy type will not be symmetry braking – unlike what occurs for the representation studied in the previous section). We now describe some of Melbourne's examples in more detail.

EXAMPLE 5.5.1 (Melbourne [**124**, §3]). Let $\rho, \kappa : \mathbb{C}^3 \to \mathbb{C}^3$ be defined by

$$\begin{aligned} \rho(z_1,z_2,z_3) &= (z_2,z_3,z_1) \\ \kappa(z_1,z_2,z_3) &= (\bar z_1,\bar z_3,\bar z_2) \end{aligned}$$

The group $\langle \rho,\kappa\rangle$ is isomorphic to S_3 and acts orthogonally on $\mathbb{C}^3$.

Let $p \geq 3$. Define an action of $\mathbb{Z}_p^3$ on $\mathbb{C}^3$ by

$$(\omega_1,\omega_2,\omega_3)(z_1,z_2,z_3) = (\omega_1 z_1,\omega_2 z_2,\omega_3 z_3),$$

where $(\omega_1,\omega_2,\omega_3) \in \mathbb{Z}_p^3$. If we let $G = \langle S_3, \mathbb{Z}_p^3\rangle$, then $G = \mathbb{Z}_p^3 \rtimes S_3$. Since $p \geq 3$, $\mathbb{Z}_p$ acts irreducibly on $\mathbb{C}$ and so G acts absolutely irreducibly on $\mathbb{C}^3$. If $z \neq 0$, let $H = G_{(0,z,\bar z)}$. It is easy to verify that H is maximal. Clearly, $(\mathbb{C}^3)^H = \{(0,z,\bar z) \mid z \in \mathbb{C}\}$ is a two dimensional real subspace of $\mathbb{C}^3$. Let $X = (X_1,X_2,X_3) \in \mathcal{V}_0(G,\mathbb{C}^3)$. Since $\rho \in G$, we have

$$X_2(z) = X_1(\rho z), \;\; X_3(z) = X_1(\rho^2 z), \;\; (z \in \mathbb{C}^3).$$

It suffices to describe X_1. To simplify computations, we assume $p \geq 5$ and give only the cubic truncation of X_1. Using $\mathbb{Z}_p^3$-equivariance, we have

$$X_1(z) = \lambda z_1 + z_1(\alpha|z_1|^2 + \beta|z_2|^2 + \gamma|z_3|^2),$$

where $\alpha,\beta,\gamma \in \mathbb{C}$. Using equivariance with respect to $\kappa \in S_3$, we see that $\alpha \in \mathbb{R}$, $\beta = \bar\gamma$. Consequently,

$$X_1(z) = \lambda z_1 + z_1(a|z_1|^2 + (b+\imath c)|z_2|^2 + (b-\imath c)|z_3|^2),$$

where $a,b,c \in \mathbb{R}$. If we identify $(\mathbb{C}^3)^H$ with $\mathbb{C}$ via the map $(0,z,\bar z) \mapsto z$, the truncated family $z' = X(z)$ restricts to

$$z' = \lambda z + z|z|^2(a+b+\imath c). \tag{5.33}$$

Transforming to polar coordinates $z = re^{\imath\theta}$, we obtain the system

$$\begin{aligned} r' &= \lambda r + (a+b)r^3 \\ \theta' &= cr^2 \end{aligned}$$

If $a+b, c \neq 0$, then (5.33) has no branches of equilibria but does have a branch of limit cycles of (constant) radius $r_\lambda = \sqrt{-\lambda}/(a+b)$. The corresponding periods are given by $\omega_\lambda = -\frac{2\pi(a+b)}{c\lambda}$. Using rescaling arguments similar to those used in the proof of the invariant sphere theorem, we may show that these results continue to hold when we take account of higher order terms. In particular, we have obtained a counterexample to the converse of the MISC.

REMARK 5.5.2. The period ω_λ of the limit cycles constructed above goes to infinity like λ^{-1}, $\lambda \to 0$. The asymptotics are the same as those that occur in the branch of limit cycles for the $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$-equivariant steady state bifurcations considered in section 5.3. However, the mechanism for generating the limit cycles is different. While the cycles that appeared in section 5.3 were generated by a 'hidden' Hopf bifurcation of the phase vector field, the limit cycles in example 5.5.1 appear because the third order truncation restricted to $(\mathbb{C}^3)^H$ is a general cubic SO(2)-equivariant with respect to the natural SO(2)-action on $(\mathbb{C}^3)^H$.

5.6. Hopf bifurcation and the invariant sphere theorem

Let V be a finite dimensional complex vector space with (Hermitian) inner product $\langle\ ,\ \rangle$. Suppose that we are given a non-trivial irreducible unitary representation of the compact Lie group G on V. For example, if $V = \mathbb{C}^n$, $(\mathbb{C}^n, G)$ might be the complexification of an absolutely irreducible representation of G on $\mathbb{R}^n$. In this section we study the bifurcation theory of families

$$z' = X(z, \lambda), \quad (z, \lambda) \in V \times \mathbb{R},$$

where X is a smooth family of G-equivariant vector fields on V. It follows from the equivariance of X and the irreducibility of (V, G) that $X(0, \lambda) \equiv 0$ and we may assume the linearization $DX_\lambda(0)$ of X_λ at $z = 0$ is equal to $\sigma_X(\lambda) I_V$, where $\sigma_X : \mathbb{R} \to \mathbb{C}$ is smooth[4].

LEMMA 5.6.1. *Let $X \in C_G^\infty(V \times \mathbb{R}, V)$ and suppose that*

(a) $\mathrm{Re}(\sigma_X(0)) = 0$, $\mathrm{Im}(\sigma_X(0)) \neq 0$.
(b) $\mathrm{Re}(\sigma_X'(0)) \neq 0$.

We may rescale time and the parameter so that in a neighbourhood of the origin of $V \times \mathbb{R}$, $z' = X_\lambda(z)$ transforms to $\dot{z} = \bar{X}_\mu(z)$, where $\dot{z}$ denotes differentiation with respect to rescaled time and $\sigma_{\bar{X}}(\mu) = \mu + \imath$.

PROOF. Let $T(\lambda) = \mathrm{Im}(\sigma_X(\lambda))$ and choose $\varepsilon_0 > 0$ so that $T \neq 0$ on $[-\varepsilon_0, \varepsilon_0]$. If we rescale time by $s = T(\lambda)t$, X transforms to $\bar{X}$ where

$$\sigma_{\bar{X}}(\lambda) = \mathrm{Re}(\sigma_X(\lambda))/T(\lambda) + \imath, \quad \lambda \in [-\varepsilon_0, \varepsilon_0].$$

Set $\mu(\lambda) = \mathrm{Re}(\sigma_X(\lambda))/T(\lambda)$. Since $\mathrm{Re}(\sigma_X(0)) = 0$ and $\mathrm{Re}(\sigma_X'(0)) \neq 0$, $\mu'(0) \neq 0$. Hence, for possibly smaller $\varepsilon_0 > 0$, μ is a smooth embedding of $(-\varepsilon_0, \varepsilon_0)$ onto an

[4]We refer to chapter 10, section 10.1, or [**84**, Chapter XVI] for justification that we can always assume $DX_\lambda(0)$ is $\mathbb{C}$-linear.

open interval I containing the origin. If we define the new parameter $\mu = \mu(\lambda)$, then $\sigma_{\bar{X}}(\mu) = \mu + \imath$, $\mu \in I$. $\square$

Following our earlier definitions for steady state bifurcations, we say that $X \in C^\infty_G(V \times \mathbb{R}, V)$ is a *normalized family* if

$$\sigma_X(\lambda) = \lambda + \imath,\ (\lambda \in \mathbb{R}).$$

If X is normalized then there is a non-degenerate change of stability of the trivial solution $z = 0$ at $\lambda = 0$. Let $\mathcal{V}_0 = \mathcal{V}_0(V, G)$ denote the space of normalized families and give $\mathcal{V}_0$ the induced C^∞-topology.

If $X \in \mathcal{V}_0$, then we expect the appearance of branches of limit cycles for $z' = X(z, \lambda)$ as λ passes through zero. We refer to this phenomenon as an *equivariant Hopf bifurcation*. The period of the limit cycles spawned in the bifurcation will be asymptotic to 2π as $\lambda \to 0$.

A number of techniques have been developed for the study of the equivariant Hopf bifurcation. A particularly successful method, developed by Golubitsky and Stewart [**85, 84**], depends on a reduction of X to (truncated) normal form followed by an analysis based on the determination of the one-complex dimensional fixed point spaces in V. A detailed description of this technique is given in the book by Golubitsky, Stewart and Schaeffer [**84**] and we only give a brief review.

Suppose $X \in \mathcal{V}_0$. It follows from the theory of equivariant normal forms [**84**, chapter XVI, §5] that if $1 \le d \le \infty$, we may make a smooth G-equivariant λ-dependent change of coordinates on a neighbourhood of the origin of $V \times \mathbb{R}$ so that $j^d X_\lambda(0) \in P^{(d)}_G(V, V)$ is $G \times S^1$ equivariant. Here S^1 acts as complex scalar multiplication on V by complex numbers of unit modulus. In particular, we can remove all terms of even order from the Taylor series of X_λ. In general, it is not possible to make a local change of coordinates so that X is $G \times S^1$-equivariant on a neighbourhood of the origin in $V \times \mathbb{R}$ (that is, we cannot normalize the tail or flat terms). Since (V, G) is a complex representation, all fixed point spaces V^H will be complex linear subspaces of V. The next step in the analysis is to determine all isotropy groups of the action of $G \times S^1$ which have fixed point spaces of complex dimension one. The corresponding isotropy types will be maximal – for the same reason that isotropy types of one-dimensional fixed point spaces in the steady state theory are maximal. It is then easy to show that there will be a branch of limit cycles associated to each one-dimensional fixed point space. After an analysis of stabilities, the final step is to show that branches persist when we take account of the tail which is only G-equivariant. This is typically not hard for finite groups but can be technically quite demanding when G is not finite. Notwithstanding the success of methods based on one-dimensional fixed point subspaces, there are some limitations. Even for low dimensional representations there may be submaximal branches of limit cycles and the presence of such branches often has a major impact on the dynamics (see Swift [**166**] and example 5.6.27 for the case $(\mathbb{C}^2, \mathbf{D}_4)$). Submaximal branches cannot be found by an analysis of the 1-dimensional fixed point spaces. On the other hand, the normal form analysis works quite generally and in [**60, 62**] it is proved that (generically) branches and their stabilities persist when we take account of the tail. In the case when the $G \times S^1$-equivariant problem is 3-determined, results of this type were first proved by Ruelle [**151**] who also showed that cubic truncations sufficed to determine the stabilities in the original unnormalized problem (see also [**60**, Theorem 5.3.1]).

In the remainder of this section we describe one approach to the equivariant Hopf bifurcation that parallels the use of the invariant sphere theorem in steady state theory. Just as in chapter 4, we continue to emphasize the case when G is finite. We also assume that vector fields are in normal form.

5.6.1. $G \times S^1$-equivariant families. We start by considering the space $\mathcal{V}_0(V, G \times S^1)$ of normalized families on $(V, G \times S^1)$. Let $\dim_{\mathbb{C}}(V) = n$. So as to avoid trivial cases, we always assume that (V, G) is a non-trivial representation and that $n \geq 2$. Let $(\ ,\ )$ denote the Euclidean inner product on V associated to $\langle\ ,\ \rangle$ and $S(V) = S^{2n-1}$ denote the unit sphere of V.

EXERCISE 5.6.2. Show that $\langle z, w\rangle = (z, w) + \imath(z, \imath w)$, $z, w \in V$, and deduce that $(z, \imath z) = 0$, for all $z \in V$.

LEMMA 5.6.3. *Let $Q \in P^d(V, V)$, $d \geq 1$. Then Q is S^1-equivariant if and only if*

$$Q(cz) = c|c|^{d-1}Q(z), \ \ (c \in \mathbb{C}, z \in V).$$

In particular, if Q is S^1-equivariant then d is odd.

PROOF. Suppose Q is S^1-equivariant. Then $Q(uz) = uQ(z)$ for all $u \in S^1$, $z \in V$. Every nonzero complex number c may be written uniquely as $c = |c|u$, $u \in S^1$. Hence $Q(cz) = Q(|c|uz) = |c|^d uQ(z) = c|c|^{d-1}Q(z)$. The converse is trivial (take $|c| = 1$). ☐

The next lemma, which can viewed as a generalization of lemma 5.6.3, allows us to apply blowing-up techniques to S^1-equivariant families of smooth maps.

LEMMA 5.6.4. *Let $H : V \times \mathbb{R} \to V$ be a smooth $G \times S^1$-equivariant family of maps. Suppose $DH_\lambda(0) = 0$, $\lambda \in \mathbb{R}$. Then we may write*

$$H(cz, \lambda) = c|c|^2\tilde{H}(z, |c|^2, \lambda), \ \ ((z, c, \lambda) \in V \times \mathbb{C} \times \mathbb{R}),$$

where $\tilde{H} : V \times \mathbb{R} \times \mathbb{R} \to V$ is smooth and G-equivariant.

PROOF. We define the smooth S^1-equivariant family $J_{z,\lambda} : \mathbb{C} \to V$ by $J_{z,\lambda}(c) = H(cz, \lambda)$, $c \in \mathbb{C}$, $(z, \lambda) \in V \times \mathbb{R}$. As an S^1-representation, V is isomorphic to n-copies of the standard representation of S^1 on $\mathbb{C}$. Identify V with $\mathbb{C}^n$ and let $j_{z,\lambda}$ be any component of $J_{z,\lambda}$. The $\mathbb{R}$-algebra of polynomial invariants $P(\mathbb{C})^{S^1}$ is generated by $|z|^2$ and a basis for the $P(\mathbb{C})^{S^1}$-module $P_{S^1}(\mathbb{C}, \mathbb{C})$ of S^1-equivariants is given by the identity map. It follows from Schwarz's smooth invariant theorem [**154**] (see chapter 6) that we may write

$$j_{z,\lambda}(c) = \bar{j}(z, |c|^2, \lambda)c,$$

where $\bar{j}$ is a smooth function on $\mathbb{C}^n \times \mathbb{R} \times \mathbb{R}$. Since $DH_\lambda(0) = 0$, $\bar{j}(z, 0, \lambda) = 0$ and so we may write $\bar{j}(z, |c|^2, \lambda) = |c|^2 \tilde{h}(z, |c|^2, \lambda)$, where $\tilde{h}$ is smooth. Applying this result to each of the components of $J_{z,\lambda}$, we obtain a smooth function $\tilde{H} : V \times \mathbb{R} \times \mathbb{R} \to V$ such that

$$H(cz, \lambda) = c|c|^2 \tilde{H}(z, |c|^2, \lambda), \ \ ((z, c, \lambda) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{R}).$$

Averaging both sides of this equation over G, the left hand side is unchanged and so we may assume $\tilde{H}$ is G-equivariant. □

EXAMPLE 5.6.5. The standard complex irreducible action of $\mathbf{D}_n$ $(n \geq 3)$ on $\mathbb{C}^2$ is generated by

$$\rho = \begin{pmatrix} \exp(2\pi\imath/n) & 0 \\ 0 & \exp(-2\pi\imath/n) \end{pmatrix} \quad \text{and} \quad \kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Define $R, Q, S \in P^3_{\mathbf{D}_n \times S^1}(\mathbb{C}^2, \mathbb{C}^2)$ by

$$\begin{aligned} R(z_1, z_2) &= \|(z_1, z_2)\|^2 (z_1, z_2), \\ Q(z_1, z_2) &= (|z_1|^2 z_1, |z_2|^2 z_2), \\ S(z_1, z_2) &= (\bar{z}_1 z_2^2, \bar{z}_2 z_1^2). \end{aligned}$$

It is easy to verify that $\{R, Q, S\}$ is a basis of $P^3_{\mathbf{D}_4 \times S^1}(\mathbb{C}^2, \mathbb{C}^2)$ and that, if $n \neq 4$, $\{R, Q\}$ is a basis of $P^3_{\mathbf{D}_n \times S^1}(\mathbb{C}^2, \mathbb{C}^2)$. (A full description of the polynomial invariants and equivariants for $(\mathbb{C}^2, \mathbf{D}_n \times S^1)$ is given in [**84**, chapter XVIII,§2].)

EXAMPLE 5.6.6. Let $(\mathbb{C}^2, \mathbf{D}_3)$ be the standard complex irreducible representation of $\mathbf{D}_3$ on $\mathbb{C}^2$. If we define $H(z_1, z_2) = (z_1|z_1|^2 + \bar{z}_1^2 z_2^3, z_2|z_2|^2 + \bar{z}_2^2 z_1^3)$, then H is $\mathbf{D}_3 \times S^1$-equivariant. We see that $\tilde{H}(c, z_1, z_2) = (z_1|z_1|^2, z_2|z_2|^2) + |c|^2(\bar{z}_1^2 z_2^3, \bar{z}_2^2 z_1^3)$.

EXAMPLE 5.6.7. Take the standard absolutely irreducible action of $\mathrm{O}(n)$ on $\mathbb{R}^n$ and complexify to obtain a complex irreducible representation $(\mathbb{C}^n, \mathrm{O}(n))$. There are two independent invariants of the $G = \mathrm{O}(n) \times S^1$-action: R^2 and $|S|^2$ where

$$R^2 = \|z\|^2, \ \ S = \sum_{i=1}^n z_i^2.$$

Note that S is $\mathrm{O}(n)$-invariant but not S^1-invariant. The general smooth G-equivariant map may be written $zA(R^2, |S|^2) + \bar{z} S B(R^2, |S|^2)$ where $A, B : \mathbb{C}^2 \to \mathbb{C}$ are smooth. If we truncate to third order, we obtain the cubic normal form

$$J(z, \lambda) = (\lambda + \imath) z + a(\lambda)\|z\|^2 z + b(\lambda) S \bar{z},$$

where $a, b : \mathbb{R} \to \mathbb{C}$ are smooth. (See [**84**, chapter XVII,§4] for a different treatment.)

DEFINITION 5.6.8. A smooth map $X : V \to V$ is *radial* if there is a smooth map $p : V \to \mathbb{C}$ such that $X(z) = p(z)z$, for all $z \in V$.

REMARK 5.6.9. The definition of radial we give here is a complex version of that given in chapter 4. For example, $X(z) = \imath z$ is radial according to definition 5.6.8 but most definitely not radial according to the definition given in chapter 4.

EXAMPLE 5.6.10. Let $p : \mathbb{R} \to \mathbb{C}$ be smooth. If we define $X : V \to V$ by $X(z) = p(\|z\|^2)z$ then X is $G \times S^1$-equivariant and radial.

DEFINITION 5.6.11. We say (V, G) has critical degree $d_{\mathbb{C}}(V, G) = d_{\mathbb{C}}$ if $d_{\mathbb{C}}$ is the smallest positive integer such that $P_G^{d_{\mathbb{C}}}(V, V)$ contains non-radial terms.

If G is finite then $\infty > d_{\mathbb{C}}(V, G) \geq 2$. If (V, G) is irreducible of complex type and G acts transitively on $S(V)$ (for example, $G = \mathrm{U}(n)$, $V = \mathbb{C}^n$), then all equivariants are radial and $d_{\mathbb{C}}(V, G) = \infty$.

EXERCISE 5.6.12. Show that if we take the standard representation of SU(2) on $\mathbb{C}^2$ then the SU(2)-equivariants are not all radial in the sense of 5.6.8 even though SU(2) acts transitively on S^3. (Note that $(\mathbb{C}^2, \mathrm{SU}(2))$ is *not* irreducible of complex type.) What happens if $G = \mathrm{SU}(2) \times S^1$?

5.6.2. (Complex) phase vector field.

DEFINITION 5.6.13. Let $Q \in P_{G\times S^1}^{2d+1}(V, V)$, $d \geq 1$. The (complex) phase vector field $\mathcal{P}^Q$ of Q is the vector field on $S(V)$ defined for $u \in S(V)$ by

$$\mathcal{P}^Q(u) = Q(u) - \langle Q(u), u\rangle u, \tag{5.34}$$

$$= Q(u) - (Q(u), u)u - (Q(u), \imath u)\imath u \tag{5.35}$$

REMARKS 5.6.14. (1) Exercise 5.6.2 shows that the two expressions for $\mathcal{P}^Q(u)$ are the same.
(2) If Q is radial then $\mathcal{P}^Q \equiv 0$.

LEMMA 5.6.15. *Let $Q \in P_{G\times S^1}^{2d+1}(V, V)$, $d \geq 1$. For all $u \in S(V)$, $\mathcal{P}^Q(u)$ is orthogonal to the S^1-orbit through u.*

PROOF. Since $(z, \imath z) = 0$ for all $z \in V$, the result follows from the second expression (5.35) for $\mathcal{P}^Q(u)$. □

5.6.3. Projective space and the Hopf fibration. Let $P(V)$ denote the complex projective space of V – that is the set of $\mathbb{C}$-lines through the origin of V. Since every $\mathbb{C}$-line meets $S(V)$ in an S^1-orbit and S^1-acts freely on $S(V)$, we may represent $P(V)$ as the orbit space $S(V)/S^1$. In particular, $P(V)$ has the natural structure of a compact manifold with respect to which the projection $\nu : S(V) \to P(V)$ has the structure of an S^1-principal bundle. If we identify V with $\mathbb{C}^n$, $S(V)$ with S^{2n-1}, and $P(V)$ with $\mathbb{P}^{n-1}(\mathbb{C})$ ($n-1$-dimensional complex projective space), then $\nu : S^{2n-1} \to \mathbb{P}^{n-1}(\mathbb{C})$ is the Hopf fibration (see chapter 2, examples 2.3.16, for a discussion of the case $n = 2$ when $\mathbb{P}^{n-1}(\mathbb{C})$ can be identified with the Riemann sphere S^2).

Since the action of G on V is $\mathbb{C}$-linear, G maps $\mathbb{C}$-lines to $\mathbb{C}$-lines and so there is an induced smooth action of G on $P(V)$. Every smooth $G \times S^1$-equivariant vector field on $S(V)$ drops down to a smooth G-equivariant vector field on $P(V)$. In particular, if $Q \in P^{2d+1}_{G\times S^1}(V,V)$, then the complex phase vector field $\mathcal{P}^Q \in C^\infty_{G\times S^1}(S(V))$ induces a smooth G-equivariant vector field on $P(V)$ which we shall denote by $\mathcal{P}_Q$. Since $\mathcal{P}^Q$ has no component along S^1-orbits, we may think of $\mathcal{P}^Q$ as the unique 'drift-free' lift of $\mathcal{P}_Q$ to $S(V)$. We refer to [**166**] for a method of writing $\mathcal{P}_Q$ in spherical polar coordinates when $n = 2$.

EXAMPLE 5.6.16. Take the standard complex irreducible representation of $\mathbf{D}_n$ on $\mathbb{C}^2$, $n \geq 3$ (see example 5.6.5) and consider the induced smooth action of $\mathbf{D}_n$ on $\mathbb{P}^1(\mathbb{C})$ (the Riemann sphere S^2). We describe the orbit structure of the action on $\mathbf{D}_n$. For subsequent reference, we distinguish three cases: n odd, $n \equiv 2$, mod 4, and $n \equiv 0$, mod 4.

In all cases the subset of $\mathbb{P}^1(\mathbb{C})$ with non-trivial isotropy is finite. In the table below we give representative lines $(\alpha, \beta) \in \mathbb{P}^1(\mathbb{C})$ and corresponding isotropy groups for each $\mathbf{D}_n$-orbit of points with non-trivial isotropy. We also give the corresponding representative isotropy groups for the action of $\mathbf{D}_n \times S^1$ on $\mathbb{C}^2$ (the table is based on [**84**, Table 1.1, page 368], but the notation is a little different).

n odd	Representative line	$\mathbf{D}_n$-isotropy	$\mathbf{D}_n \times S^1$-isotropy
	$\mathbb{C}(1,0)$	$\mathbb{Z}_n$	$\tilde{\mathbb{Z}}_n = \{(\gamma,\gamma^{-1}) \mid \gamma \in \mathbb{Z}_n\}$
	$\mathbb{C}(1,1)$	$\langle\kappa\rangle$	$\langle\kappa\rangle$
	$\mathbb{C}(1,-1)$	$\langle\kappa\rangle$	$\langle-\kappa\rangle$
$n \equiv 2, \mathrm{mod} 4$			
	$\mathbb{C}(1,0)$	$\mathbb{Z}_n$	$\tilde{\mathbb{Z}}_n = \{(\gamma,\gamma^{-1}) \mid \gamma \in \mathbb{Z}_n\}$
	$\mathbb{C}(1,1)$	$\langle\kappa\rangle$	$\langle\kappa,(e^{\imath\pi},e^{\imath\pi})\rangle$
	$\mathbb{C}(1,-1)$	$\langle\kappa\rangle$	$\langle-\kappa,(e^{\imath\pi},e^{\imath\pi})\rangle$
$n \equiv 0, \mathrm{mod} 4$			
	$\mathbb{C}(1,0)$	$\mathbb{Z}_n$	$\tilde{\mathbb{Z}}_n = \{(\gamma,\gamma^{-1}) \mid \gamma \in \mathbb{Z}_n\}$
	$\mathbb{C}(1,1)$	$\langle\kappa\rangle$	$\langle\kappa,(e^{\imath\pi},e^{\imath\pi})\rangle$
	$\mathbb{C}(1,e^{2\pi\imath/n})$	$\langle\rho\kappa\rangle$	$\langle\rho\kappa,(e^{\imath\pi},e^{\imath\pi})\rangle$

TABLE 1. Maximal isotropy groups for the actions of $\mathbf{D}_n$ on $\mathbb{P}^1(\mathbb{C})$ and $\mathbf{D}_n \times S^1$ on $\mathbb{C}^2$

Every smooth $\mathbf{D}_n$-equivariant vector field on $\mathbb{P}^1(\mathbb{C})$ will have a zero at each point of $\mathbb{P}^1(\mathbb{C})$ which has non-trivial isotropy. For example, a $\mathbf{D}_5$-equivariant vector field on $\mathbb{P}^1(\mathbb{C})$ has exactly 12 zeros forced by symmetry.

5.6.4. Phase blowing-up. Let $X \in \mathcal{V}_0(V, G \times S^1)$. We transform $z' = X(z,\lambda)$ into a set of three equations that determine the radial, drift and tangential behaviour of the system. For this, we use a variation of polar blowing-up.

Let S^1 act smoothly on $S(V) \times S^1 \times \mathbb{R}$ by

$$e^{\imath\theta}(u, e^{\imath\phi}, R) = (e^{\imath\theta}u, e^{\imath\phi}, R).$$

DEFINITION 5.6.17. The *phase blowing-up* of V at the origin is the smooth $G \times S^1$-equivariant map $\pi : S(V) \times S^1 \times \mathbb{R} \to V$ defined by

$$\pi(u, \theta, R) = R\exp(\imath\theta)u, \quad ((u, \theta, R) \in S(V) \times S^1 \times \mathbb{R}).$$

In order to obtain a *unique* lift of a vector field on V to $S(V) \times S^1 \times \mathbb{R}$ using phase blowing up, we make the transformation $z = \pi(u, \theta, R)$ subject to the constraint $(u', \imath u) = 0$ – that is, the $S(V)$-component of the lifted vector field will be orthogonal to S^1-orbits. Since $u \in S(V)$, we may write the constraint equivalently as

$$\langle u', u\rangle = 0, \ (u \in S(V)).$$

In the next lemma we give the general formulas for the transformation of $G \times S^1$-equivariant vector fields on V under phase blowing-up.

LEMMA 5.6.18. *Let $X \in \mathcal{V}_0(V, G \times S^1)$ and write $X(z, \lambda) = (\lambda + \imath)z + F(z, \lambda)$. Suppose $\hat{X}$ is a smooth $G \times S^1$-equivariant family of vector fields on $S(V) \times S^1 \times \mathbb{R}$ such that $T\pi\hat{X} = X\pi$. Then the components of $\hat{X}$ are given by*

$$\begin{aligned} u' &= \imath(1 + R^2\Theta(u, R^2, \lambda) - D(u, R^2, \lambda))u + R^2U(u, R^2, \lambda), \\ \theta' &= D(u, R^2, \lambda), \\ R' &= \lambda R + R^3P(u, R^2, \lambda), \end{aligned}$$

where D is smooth and G-invariant, and U, Θ and P are smooth functions given explicitly by

$$\begin{aligned} U(u, R^2, \lambda) &= R^{-3}(F(Ru, \lambda) - \langle F(Ru), \lambda), u\rangle u), \\ \Theta(u, R^2, \lambda) &= R^{-3}(F(Ru, \lambda), \imath u), \\ P(u, R^2, \lambda) &= R^{-3}(F(Ru, \lambda), u). \end{aligned}$$

(1) *If $D = 1 + R^2\Theta$, we set $\hat{X} = X^{\star\star}$ and refer to $X^{\star\star}$ as* the phase blowing-up *of X. The $S(V)$-component of $X^{\star\star}$ is everywhere orthogonal to the action of S^1 on $S(V)$.*
(2) *If $D = 0$, so $\theta' = 0$, we set $\hat{X} = X^\star$. The vector field $X^\star$ induces a $G \times S^1$-equivariant vector field on $S(V) \times \mathbb{R}$ which is precisely the transform of X under the polar blowing-up transformation $z = Ru$.*

PROOF. The proof is similar to that of proposition 4.7.2. We make the substitution $z = Re^{\imath\theta}u$. Since F is S^1-equivariant, it follows from lemma 5.6.4 that $F(Re^{\imath\theta}u) = e^{\imath\theta}F(Ru)$. Differentiating, substituting and using $(u, u') = (u, \imath u) = 0$, we find that

$$R' = \lambda R + (F(Ru, \lambda), u)u = \lambda R + R^3P(u, R^2, \lambda),$$

where $P(u, R^2, \lambda) = R^{-3}(F(Ru, \lambda), u)$. Using the expression for R', and the identity $\langle w, u\rangle = (w, u) + \imath(w, \imath u)$, we find the following expression relating u' and θ'

$$u' = \imath(1 + (F(Ru), \imath u) - \theta')u + R^{-1}(F(Ru) - \langle F(Ru), u\rangle u). \tag{5.36}$$

Noting our requirement that the transformed vector field be $G \times S^1$-equivariant, we may write

$$\theta' = D(u, R^2, \lambda),$$

where D is a smooth real-valued and G-invariant function on $S(V) \times \mathbb{R} \times \mathbb{R}$. It follows that $u' = \imath(1 + R^2\Theta(u, R^2, \lambda) - D(u, R^2, \lambda))u + R^2U(u, R^2, \lambda)$. Granted our definitions of U, Θ, P, choosing any smooth G-invariant function $D(u, R^2, \lambda)$, gives a smooth family $\hat{X}$ of vector fields on $S(V) \times S^1 \times \mathbb{R}$ satisfying $T\pi\hat{X} = X\pi$.

If we add the constraint $(u', \imath u) = 0$, then, by (5.36), we have $1+(F(Ru), \imath u) = \theta' = D(u, R^2, \lambda)$, proving (1). On the other hand, if we set $\theta' = D \equiv 0$, then we have

$$\begin{aligned} u' &= \imath u(1 + R^2\Theta(u, R^2, \lambda)) + R^{-1}(F(Ru, \lambda) - \langle F(Ru), \lambda), u\rangle u), \\ &= \imath u + R^{-1}(F(Ru, \lambda) - (F(Ru, \lambda), u)u), \end{aligned}$$

which is exactly what we get when we transform X under $z = Ru$. □

COROLLARY 5.6.19. *(Notation of lemma 5.6.18) Let $X \in \mathcal{V}_0(V, G \times S^1)$ have phase blowing-up $X^{\star\star}$. Suppose that the G-equivariant vector field $P_{R,\lambda}$ induced on $\mathbb{P}^{n-1}(\mathbb{C})$ by $U(u, R^2, \lambda)$ has a zero u_0 such that $P_{R,\lambda}(u_0) = 0$, for all (R, λ) in some open open neighbourhood of $(0, 0) \in \mathbb{R}^2$. Then $z' = X(z, \lambda)$ has a branch of limit cycles which is tangent to the line u_0 at $\lambda = 0$.*

PROOF. The equations for the phase blowing-up $X^{\star\star}$ on $S(V) \times S^1 \times \mathbb{R}$ are given by

$$\begin{aligned} u' &= R^2U(u, R^2, \lambda), \\ \theta' &= 1 + R^2\Theta(u, R^2, \lambda), \\ R' &= \lambda R + R^3P(u, R^2, \lambda). \end{aligned}$$

Let $\tilde{u}_0 \in S(V)$ be any point such that $\nu(\tilde{u}_0) = u_0$. Taking $u = \tilde{u}_0$, $U(\tilde{u}_0, R^2, \lambda) = 0$ and so we are reduced to solving

$$\begin{aligned} \theta' &= 1 + R^2\Theta(\tilde{u}_0, R^2, \lambda), \\ R' &= \lambda R + R^3P(\tilde{u}_0, R^2, \lambda). \end{aligned}$$

By the implicit function theorem, the R-equation has a smooth curve $\lambda(R) = R^2k(R^2), R \in [0, R_0]$ of zeros. We have the corresponding branch of limit cycles in V defined for each $R \in [0, R_0]$ by

$$\gamma_R(t) = Re^{\imath(1+R^2\Theta(\tilde{u}_0,R^2,\lambda(R))t}u_0.$$

Obviously the branch is tangent to the line $\tilde{u}_0 = u + 0$ and indeed is contained in the linear subspace $\mathbb{C}\tilde{u}_0 \times \mathbb{R} \subset V \times \mathbb{R}$. □

EXAMPLE 5.6.20. Take the standard action of $\mathbf{D}_n$ on $\mathbb{C}^2$ (example 5.6.5). We showed in example 5.6.16 that certain zeros of smooth $\mathbf{D}_n$-equivariant vector fields on $\mathbb{P}^1(\mathbb{C})$ were forced by symmetry. We now interpret these results in the light of corollary 5.6.19. For $n \geq 3$, every smooth $\mathbf{D}_n$-equivariant vector fields on $\mathbb{P}^1(\mathbb{C})$ has a zero $(1, 0)$ of isotropy type $\mathbb{Z}_n$. Points on the branch $\gamma_R(t)$ of limit cycles tangent to $\mathbb{C}(1, 0)$ given by corollary 5.6.19 have isotropy $\tilde{\mathbb{Z}}_n = \{(\gamma, -\gamma) \mid \gamma \in \mathbb{Z}_n\}$. As is explained in [**84**], we may interpret $\tilde{\mathbb{Z}}_n$ as a *spatiotemporal symmetry*. Specifically, a spatial rotation of $2\pi/n$ has the same effect as a phase shift by one nth of a period:

$$\rho\gamma_R(t) = \gamma_R(t + \frac{T}{n}),$$

where $T \approx 2\pi$ is the period of γ_R[5]. Limit cycles with this type of symmetry are called *rotating waves*. Branches of limit cycles with the spatiotemporal symmetry $\tilde{\mathbb{Z}}_n$ occur in all three cases described in example 5.6.16. Now suppose that n is odd. The branch of limit cycles tangent to the line $\mathbb{C}(1, 1)$ has the spatial symmetry $\langle\kappa\rangle \approx \mathbb{Z}_2$. On the other hand, the branch γ_R of limit cycles tangent to the line $\mathbb{C}(1, -1)$ has the half-period symmetry

$$\kappa\gamma_R(t) = \gamma_R(t + \frac{T}{2}),$$

and so the isotropy group of points on the cycle is $\langle e^{\imath\pi}\kappa\rangle$. That is $\kappa(1, -1) = (-1, 1) = e^{\imath\pi}(1, -1)$.

Next we consider the case n even. We have $-I_{\mathbb{C}^2} \in \mathbf{D}_n \cap S^1$ and we set $\mathbb{Z}_2^c = \langle -I_{\mathbb{C}^2}\rangle$. Branches tangent to $\mathbb{C}(1, 1)$ have symmetry $\langle\kappa, -I_{\mathbb{C}^2}\rangle = \langle\kappa\rangle \oplus \mathbb{Z}_2^c$ (where we have followed the notation of [**84**, chapter VIII]). Next we consider branches tangent to $\mathbb{C}(1, -1)$ in the case $n \equiv 2$, mod 4. We find that branches have symmetry $\langle e^{\imath\pi}\kappa, -I_{\mathbb{C}^2}\rangle = \langle e^{\imath\pi}\kappa\rangle \oplus \mathbb{Z}_2^c$. Finally, suppose $n \equiv 0$, mod 4. Branches tangent to $\mathbb{C}(1, e^{2\pi\imath/n})$ have symmetry $\langle\kappa\rho\rangle \oplus \mathbb{Z}_2^c$.

5.6.5. Transforming a cubic normal form. Since $P^{2d}_{G\times S^1}(V, V) = \{0\}$, $d \geq 1$, it follows that $d_{\mathbb{C}}(V, G \times S^1) \geq 3$ for all complex irreducible representations (V, G). Aside from the cases when all equivariants are radial, we know of no examples where all the cubic equivariants are radial. In particular, if $V = W \otimes_{\mathbb{R}} \mathbb{C}$ and (W, G) is an absolutely irreducible orthogonal representation, then $d_{\mathbb{C}}(V, G \times S^1) = 3$. Indeed, if we let $z \mapsto z^2$ denote the complexification of $x \mapsto \|x\|^2$, then $z \mapsto z^2\bar{z}$ is always a non-radial cubic $G \times S^1$-equivariant. For this reason we will focus on cubic truncations and, just as in chapter 4, start by assuming that $X \in \mathcal{V}_0(V, G \times S^1)$ is in the normalized form

$$X(z, \lambda) = (\lambda + \imath)z + Q_\lambda(z),$$

[5]We can always rescale time so that, for the given branch of limit cycles, $T \equiv 2\pi$. In this way time advance by a fraction of the period will be equal to the corresponding rotation by an element of S^1.

where $Q_\lambda \in P^3_{G\times S^1}(V,V)$.

Applying lemma 5.6.18 to $z' = X(z,\lambda)$, we obtain the following set of transformed equations

$$u' = R^2\mathcal{P}^{Q_\lambda}(u), \tag{5.37}$$
$$\theta' = 1 + R^2(Q_\lambda(u), \imath u), \tag{5.38}$$
$$R' = \lambda R + R^3(Q_\lambda(u), u), \tag{5.39}$$

EXAMPLE 5.6.21. Suppose that $X(z,\lambda) = (\lambda+\imath)z + Q(z)$. Assume that G is finite and the zeros of $\mathcal{P}_Q$ are all simple. Since $\chi(P(V)) = n$, it follows from the Poincaré-Hopf theorem that $\mathcal{P}_Q$ has at least n zeros. Let $u_0 \in \mathbf{Z}(\mathcal{P}_Q)$. Suppose that $(Q(u_0), u_0) \neq 0$. Let (R,λ) be a point on the curve $\lambda + R^2(Q(u_0), u_0) = 0$. The S^1-orbit through $(u_0, 0, R)$ is then a limit cycle of (5.38,5.39,5.39) with period $2\pi/(1 + R^2(Q(u_0), \imath u_0))$. Transforming back to V, we obtain a Hopf bifurcation which is supercritical if $(Q(u_0), u_0) < 0$ and subcritical if $(Q(u_0), u_0) > 0$. All this is parallel to the arguments of chapter 4: every zero of $\mathcal{P}_Q$ determines a unique branch of limit cycles to the original equation. Although we suppressed the dependence of Q on λ, everything we do continues to work if Q depends on λ provided the zeros of $\mathcal{P}_Q$ at $\lambda = 0$ are non-singular.

5.6.6. The invariant sphere theorem for the Hopf bifurcation. In this section we give a version of the invariant sphere theorem that applies to the Hopf bifurcation. We start with a new definition.

DEFINITION 5.6.22 (cf [**52**]). Let M, N be $G\times S^1$-spaces and suppose that Φ_t, Ψ_t are continuous $G\times S^1$-equivariant flows on M, N respectively. We say that Φ_t and Ψ_t are *drift conjugate* if there exists a $G\times S^1$-equivariant homeomorphism $h : M \to N$ and continuous $G\times S^1$-invariant map $k : M\times\mathbb{R} \to S^1$ such that for all $x \in M$, $t \in \mathbb{R}$ we have

$$h(\Phi_t(x)) = k(z,t)\Phi_t(h(z)).$$

REMARK 5.6.23. If Φ_t and Ψ_t are drift conjugate then Φ_t and Ψ_t induce topologically conjugate flows on M/S^1 and N/S^1.

THEOREM 5.6.24. *Let $z' = X(z,\lambda) = (\lambda+\imath)z+Q_\lambda(z)$, where $Q_\lambda \in P^3_{G\times S^1}(V,V)$ and Q_0 is contracting. We may choose $\lambda_0 > 0$ such that*

(1) *For every $\lambda \in (0,\lambda_0]$, there exists a $(2n-1)$-dimensional G-invariant topological sphere $S(\lambda) \subset V\setminus\{0\}$ which is invariant under the flow of $z' = X_\lambda(z)$.*
(2) *$S(\lambda)$ is embedded as a topological submanifold of V and the bounded component of $V\setminus S(\lambda)$ contains the origin of V.*
(3) *Every $z \in V\setminus\{0\}$ is forward asymptotic under the flow of $z' = X_\lambda(z)$ to $S(\lambda)$.*
(4) *The flow of $z' = X_\lambda(z)$ restricted to $S(\lambda)$ is drift conjugate to the flow of the complex phase vector field $\mathcal{P}^{Q_\lambda}$.*

(5) *The Hopf fibration* $\pi : S(\lambda) \to P(V)$ *maps the trajectories of* $z' = X_\lambda(z)$ *onto those of* $\mathcal{P}_{Q_\lambda}$.

PROOF. Our proof is similar to that of theorem 5.1.5 with an additional twist caused by the introduction of a phase. We start by making the coordinate transformation $z = R\exp(\imath\theta)u$, $(u, \exp(\imath\theta), R) \in S(V) \times S^1 \times \mathbb{R}$, subject to $(u', \imath u) = 0$, to obtain the following system of vector fields on $S(V) \times S^1 \times \mathbb{R}$

$$u' = R^2\mathcal{P}^{Q_\lambda}(u), \tag{5.40}$$

$$\theta' = 1 + R^2(Q_\lambda(u), \imath u), \tag{5.41}$$

$$R' = \lambda R + R^3((Q_\lambda(u), u). \tag{5.42}$$

Consider the system of vector fields on $S(V) \times \mathbb{R}$ obtained by setting $\theta' = 0$.

$$u' = R^2\mathcal{P}^{Q_\lambda}(u), \tag{5.43}$$

$$R' = \lambda R + R^3((Q_\lambda(u), u). \tag{5.44}$$

It follows from the proof of theorem 5.1.5 that we can choose $\lambda_0 > 0$ so that for each $\lambda \in [0, \lambda_0]$ there exists a $G \times S^1$-invariant topologically embedded sphere $\Sigma(\lambda) \subset S(V) \times \mathbb{R}$ satisfying

(a) $\Sigma(\lambda)$ is invariant by the flow of (5.44,5.44).
(b) The flow of (5.44,5.44) restricted to $\Sigma(\lambda)$ is topologically conjugate to the flow of $\mathcal{P}^{Q_\lambda}$.
(c) $\Sigma(\lambda)$ is globally attracting for the flow of (5.44,5.44) in the sense that every trajectory of (5.44,5.44) with initial condition in $S(V) \times \mathbb{R}^+$ is forward asymptotic to $\Sigma(\lambda)$.

Let $\lambda \in [0, \lambda_0]$ and $(u, R) \in \Sigma(\lambda)$. Let $(u(t), R(t)) \in \Sigma(\lambda)$ denote the trajectory of (5.44,5.44) with initial condition (u, R). Since $\Sigma(\lambda)$ is a compact flow-invariant set, $(u(t), R(t))$ is defined for all $t \in \mathbb{R}$. Substitute for u, R in the equation $\theta' = 1 + R^2(Q_\lambda(u), \imath u)$. Solving for θ, we obtain a continuous map

$$\theta_\lambda : \Sigma(\lambda) \times \mathbb{R} \to \mathbb{R}$$

such that for each $(u, R) \in \Sigma(\lambda)$, $(u(t), R(t), \theta_\lambda(u, R, t))$ is the trajectory through $(u, R, 0)$ for (5.41,5.42,5.42). Define $k_\lambda : \Sigma(\lambda) \to S^1$ by

$$k_\lambda(t) = \exp(\imath\theta_\lambda(u, R, t)).$$

Let Φ_t^λ and $\hat{\Phi}_t^\lambda$ respectively denote the flows induced by (5.41,5.42,5.42) and (5.44,5.44) on $\Sigma(\lambda)$. It follows from our construction of k_λ that for all $\lambda \in [0, \lambda_0]$, $(u, R) \in \Sigma(\lambda)$ we have

$$\Phi_t^\lambda(u, R) = k_\lambda(u, R, t)\hat{\Phi}_t^\lambda(u, R)$$

and so the flows Φ_t^λ and $\hat{\Phi}_t^\lambda$ are drift conjugate. Transforming back to V the theorem follows. $\square$

EXERCISE 5.6.25. Let $X \in \mathcal{V}_0(V, G \times S^1)$. Let $\hat{X}^1$, $\hat{X}^2$ be any two transformations of f to $S(V) \times S^1 \times \mathbb{R}$ given by lemma 5.6.18. Show that the flows of $\hat{X}^1$, $\hat{X}^2$ are drift conjugate. In particular, verify that the flow of $X^\star$ is drift conjugate to the flow of $X^{\star\star}$.

5.6.7. The algebraic Hopf theorem. We have seen both for steady state and now Hopf bifurcations how the use of polar blowing-up can be a very effective tool for the analysis of equivariant bifurcation problems. In this section we show how we can use the classical blowing-up or quadratic transformations from complex algebraic geometry in the analysis of the equivariant Hopf bifurcation. Even though the equations we consider are not holomorphic (complex analytic), the S^1-equivariance allows us to apply these complex algebraic techniques.

As we shall soon need to define local coordinate structures on $P(V)$, we shall for the remainder of this section identify V with $\mathbb{C}^n$ and, where necessary, write $z \in \mathbb{C}^n$ in the coordinate form $(z_1, \ldots, z_n)$. As usual, we denote the complex projective space of $\mathbb{C}^n$ by $\mathbb{P}^{n-1}(\mathbb{C})$ and recall that we always assume $n \geq 2$.

We define an equivalence relation $\sim$ on $\mathbb{C}^n \setminus \{0\}$ by $z \sim z'$ if and only if there exists $\lambda \in \mathbb{C}^\star$ such that $z' = \lambda z$. We may identify $\mathbb{C}^n \setminus \{0\}/\sim$ with $\mathbb{P}^{n-1}(\mathbb{C})$ by mapping z to the $\mathbb{C}$-line through z. We regard nonzero n-tuples $(z_1, \ldots, z_n)$ as defining *homogeneous coordinates* on $\mathbb{P}^{n-1}(\mathbb{C})$. Two nonzero n-tuples $(z_1, \ldots, z_n)$, $(z_1, \ldots, z_n)$ define the same point of $\mathbb{P}^{n-1}(\mathbb{C})$ if and only if there exists $\lambda \in \mathbb{C}^\star$ such that $z_i = \lambda z_i'$, $1 \leq i \leq n$.

We define an atlas $\mathcal{A} = \{(U_i, \phi_i) \mid 1 \leq i \leq n\}$ of charts for $\mathbb{P}^{n-1}(\mathbb{C})$ by

$$\begin{aligned} U_i &= \{(z_1, \ldots, z_n) \mid z_i \neq 0\}, \\ \phi_i(z_1, \ldots, z_n) &= (z_1/z_i, \ldots, z_{i-1}/z_i, z_{i+1}/z_i, \ldots, z_n/z_i),\ z \in U_i. \end{aligned}$$

The atlas $\mathcal{A}$ gives $\mathbb{P}^{n-1}(\mathbb{C})$ the structure of a complex manifold (in fact an algebraic variety – the maps $\phi_i \phi_j^{-1}$ are rational functions). For future reference note that each chart map ϕ_i is a diffeomorphism of U_i onto $\mathbb{C}^{n-1}$ with inverse $\psi_i = \phi_i^{-1}$ given by

$$\psi_i(Z_1, \ldots, Z_{n-1}) = (Z_1, \ldots, Z_{i-1}, 1, Z_i, \ldots, Z_{n-1}). \tag{5.45}$$

We define a new complex $G \times S^1$-manifold $\tilde{\mathbb{C}}^n$ and projection map $B : \tilde{\mathbb{C}}^n \to \mathbb{C}^n$ such that

(a) B is a complex analytic map.
(b) $B^{-1}(0) \approx \mathbb{P}^{n-1}(\mathbb{C})$.
(c) B restricts to a complex analytic diffeomorphism of $\tilde{\mathbb{C}}^n \setminus \mathbb{P}^{n-1}(\mathbb{C})$ onto $\mathbb{C}^n \setminus \{0\}$.
(d) $(\tilde{\mathbb{C}}^n)^{S^1} = B^{-1}(0)$ and $B^{-1}(0)$ is G-invariant.

We call $B : \tilde{\mathbb{C}}^n \to \mathbb{C}^n$ (or $\tilde{\mathbb{C}}^n$) the *blowing-up* of $\mathbb{C}^n$ at the origin.

Since $\mathbb{P}^{n-1}(\mathbb{C})$ is the set of $\mathbb{C}$-lines through the origin of $\mathbb{C}^n$, we may define

$$\tilde{\mathbb{C}}^n = \{(z, u) \in \mathbb{C}^n \times \mathbb{P}^{n-1}(\mathbb{C}) \mid z \in u\}.$$

Ignoring for a moment the smoothness of $\tilde{\mathbb{C}}^n$, we define the map B to be the restriction to $\tilde{\mathbb{C}}^n$ of the projection $\pi_1 : \mathbb{C}^n \times \mathbb{P}^{n-1}(\mathbb{C}) \to \mathbb{C}^n$ onto the first factor. With these definitions, it is immediate that $B^{-1}(0) \approx \mathbb{P}^{n-1}(\mathbb{C})$ and B maps $\tilde{\mathbb{C}}^n \setminus \mathbb{P}^{n-1}(\mathbb{C})$ 1:1 onto $\mathbb{C}^n \setminus \{0\}$.

Next we define an atlas of charts for $\tilde{\mathbb{C}}^n$. In order to do this we define for $1 \leq i \leq n$ maps $\Psi_i = (\psi_i^C, \psi_i^P) : \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{P}^{n-1}(\mathbb{C})$ where

$$\begin{aligned}
\psi_i^C(Z_1, \dots, Z_n) &= (Z_1 Z_i, \dots, Z_{i-1} Z_i, Z_i, Z_{i+1} Z_i, \dots, Z_n Z_i), \\
\psi_i^P(Z_1, \dots, Z_n) &= (Z_1, \dots, Z_{i-1}, 1, Z_{i+1}, \dots, Z_n).
\end{aligned}$$

Each Ψ_i is a smooth embedding of $\mathbb{C}^n$ onto an open and dense subset $\tilde{U}_i$ of $\tilde{\mathbb{C}}^n$. Since $\cup_i \tilde{U}_i = \tilde{\mathbb{C}}^n$, $\tilde{\mathcal{A}} = \{(\tilde{U}_i, \Psi_i^{-1}) \mid 1 \leq i \leq n\}$ is an atlas of charts for $\tilde{\mathbb{C}}^n$.

If we set $B_i = B \circ \Psi_i$, then

$$B_i(Z_1, \dots, Z_n) = (Z_1 Z_i, Z_2 Z_i, \dots, Z_i, \dots, Z_n Z_i), \quad ((Z_1, \dots, Z_n) \in \mathbb{C}^n).$$

This is the formula that enables is to compute, in Ψ_i-coordinates, the B-transform of functions or vector fields defined on $\mathbb{C}^n$.

Finally, take the $G \times S^1$-action on $\mathbb{C}^n \times \mathbb{P}^{n-1}(\mathbb{C})$ defined as the product of the given $G \times S^1$-action on $\mathbb{C}^n$ with the induced $G = G \times S^1$-action on $\mathbb{P}^{n-1}(\mathbb{C})$. Since $\tilde{\mathbb{C}}^n$ is a $G \times S^1$-invariant submanifold of $\mathbb{C}^n \times \mathbb{P}^{n-1}(\mathbb{C})$, $\tilde{\mathbb{C}}^n$ inherits the structure of a smooth $G \times S^1$-manifold. Clearly condition (d) is satisfied.

Example 5.6.26. We compute the blowing-up of the holomorphic vector field defined on $\mathbb{C}^2$ by

$$\begin{aligned}
z_1' &= z_1 z_2^2, \\
z_2' &= z_1^3 + z_2^3.
\end{aligned}$$

We work out the transform of the vector field in Ψ_1-coordinates. To do this, we make the substitutions $z_1 = Z_1$, $z_2 = Z_2 Z_1$. Differentiating we obtain $z_1' = Z_1'$, $z_2' = Z_2' Z_1 + Z_2 Z_1'$. After substitution and simplification we obtain

$$\begin{aligned}
Z_1' &= Z_1^3 Z_2^2, \\
Z_1 Z_2' &= Z_1^3.
\end{aligned}$$

The set $Z_1 = 0$ lies in $B^{-1}(0)$ – the *exceptional variety* of the blowing-up. Cancelling the factor Z_1, gives the transformed vector field $\tilde{X}(Z_1, Z_2) = (Z_1^3 Z_2^2, Z_1^2)$. Up to a scale – and off the exceptional variety – we may reduce to

$$\begin{aligned}
Z_1' &= Z_1 Z_2^2, \\
Z_2' &= 1.
\end{aligned}$$

These equations may be solved explicitly and, projecting back to $\mathbb{C}$, we find the phase portrait of the original system (except on $z_1 = 0$). We may similarly look at the system in Ψ_2-coordinates.

The blowing-up technique described in the previous example will not generally work for smooth families of vector fields $z' = X(z,\lambda)$ on $\mathbb{C}^n$, $X(0,\lambda) \equiv 0$. The reason is that if the map X is not holomorphic, then we get conjugate complex factors $\bar{Z}_i$ in the blown-up vector field which we cannot cancel. However, if we assume that the family is S^1-equivariant, we find that we can again cancel the Z_i factors associated to the exceptional variety. We illustrate with an example from equivariant bifurcation theory.

EXAMPLE 5.6.27. We consider the cubic truncation of a $\mathbf{D}_4 \times S^1$-equivariant system on $\mathbb{C}^2$.

$$\begin{aligned} z_1' &= (\lambda + \imath)z_1 + (a(|z_1|^2 + |z_2|^2) + b|z_1|^2)z_1 + c\bar{z}_1 z_2^2, \\ z_2' &= (\lambda + \imath)z_2 + (a(|z_1|^2 + |z_2|^2) + b|z_2|^2)z_2 + c\bar{z}_2 z_1^2, \end{aligned}$$

where $a, b, c \in \mathbb{C}$. Transforming using $z_1 = Z_1$, $z_2 = Z_1 Z_2$, we find that

$$\begin{aligned} Z_1' &= (\lambda + \imath)Z_1 + (a(|Z_1|^2 + |Z_1|^2|Z_2|^2) + b|Z_1|^2|Z_2|^2)Z_1 + c|Z_1|^2 Z_2^2, \\ Z_1 Z_2' &= Z_1|Z_1|^2\left(bZ_2(|Z_2|^2 - 1) + c(\bar{Z}_2 - Z_2^3)\right). \end{aligned}$$

Dividing the second equation by Z_1 and then by the scale $|Z_1|^2$, we obtain

$$Z_2' = bZ_2(|Z_2|^2 - 1) + c(\bar{Z}_2 - Z_2^3).$$

This equation is independent of both Z_1 and $|Z_1|^2$ and corresponds to the phase vector field (defined on $\mathbb{P}^1(\mathbb{C})$). The Z_1'-equation can be thought of as a complex version of the radial equation. Notice that every zero of the Z_2'-equation gives rise to a branch of limit cycles for the original system. This follows trivially by substitution for Z_2 in the first equation, applying the (standard) Hopf bifurcation and projecting back to $\mathbb{C}^2$. It was shown by Swift [**166**] that for this truncation there could exist branches of limit cycles of submaximal isotropy type. Swift's calculations depended on the use of spherical polar coordinates. We briefly sketch how we calculate the zeros of the equation $P(z) = bz(|z|^2 - 1) + c(\bar{z} - z^3) = 0$. First of all observe that $z = 0, \pm 1, \pm\imath$ are all zeros (these solutions are all forced by symmetry and may be deduced by an analysis based on 1-complex dimension fixed point spaces). We investigate the possibility of new solutions. First of all we may assume $b, c \neq 0$. If $b = 0$, $c \neq 0$, then we just get the solutions $z = 0, \pm 1, \pm\imath$. If $c = 0, b \neq 0$, then the problem degenerates and every z with $|z| = 1, 0$ is a solution. Dividing $P(z)$ by b, we reduce to considering the equation $Q(z) = z(|z|^2 - 1) + d(\bar{z} - z^3) = 0$, where d is a non-zero complex number. Multiplying Q by $\bar{z}$, we consider the equation $\bar{z}Q(z) = |z|^2(1 - |z|^2) + d(\bar{z}^2 - |z|^2 z^2) = 0$. Write $d = \alpha + \imath\beta$, $z = x + \imath y$ and take real and imaginary parts of $\bar{z}Q$. Setting $r^2 = x^2 + y^2$, we derive the equations

$$\begin{aligned} r^2(1 - r^2) + \alpha(x^2 - y^2)(1 - r^2) + 2\beta xy(1 + r^2) &= 0, \\ \beta(x^2 - y^2)(1 - r^2) - 2\alpha xy(1 + r^2) &= 0. \end{aligned}$$

Now add α times the first equation to β times the second equation. We obtain

$$\alpha r^2(1-r^2)+\alpha^2(x^2-y^2)(1-r^2))+\beta^2(x^2-y^2)(1-r^2)=0.$$

Cancelling the term $1-r^2$ (which generates no new solutions as $d\neq 0$), and after some rearrangement, we obtain

$$x^2(\alpha^2+\beta^2+\alpha)=y^2(\alpha^2+\beta^2-\alpha).$$

It follows that a necessary condition for the existence of new solutions is that $|d|^2>|\mathrm{Re}(d)|$. We use this relation between x^2 and y^2 to eliminate x from the equation for the imaginary part of $\bar{z}Q$. After a little work, we find the following equation for y^2.

$$2y^2(\alpha^2+\beta^2)(\sqrt{(\alpha^2+\beta^2)^2-\alpha^2}\pm\beta)=(\alpha^2+\beta^2+\alpha)(\beta\mp\sqrt{(\alpha^2+\beta^2)^2-\alpha^2}).$$

The terms involving square roots have to have the same sign in order that there be real solutions. That is $|\beta|>\sqrt{(\alpha^2+\beta^2)^2-\alpha^2}$. Squaring and simplifying, we find that $\alpha^2+\beta^2<1$. Consequently,

$$1>|d|^2>|\mathrm{Re}(d)|$$

are necessary and sufficient conditions for the existence of submaximal solution branches. In fact, as was shown by Swift, if the conditions of the invariant sphere theorem hold then, by applying the Poincaré-Bendixon theorem to the flow on $\mathbb{P}^1(\mathbb{C})=S^2$, there exists an open subspace of parameters b,c where there are limit cycles for the induced flow on S^2. These limit cycles lift to give branches of 2-tori with quasi-periodic flow for the original system on $\mathbb{C}^2$ [**166**, §3.9].

LEMMA 5.6.28. *Let $X\in\mathcal{V}_0(\mathbb{C}^n,G\times S^1)$ and suppose $X(z,\lambda)=(\lambda+\imath)z+F(z,\lambda)$. Under blowing-up, X transforms to a smooth $G\times S^1$-equivariant family $\tilde{X}$ of vector fields on $\tilde{\mathbb{C}}^n$. The equations for $\tilde{X}$ in Ψ_i-coordinates are given by*

$$\begin{aligned}
Z_1' &= (F_1(Z_1Z_i,\dots,Z_i,\dots,Z_nZ_i,\lambda)-Z_1F_i(Z_1Z_i,,\dots,Z_i,\dots,Z_nZ_i,\lambda))/Z_i,\\
\dots &= \dots\\
Z_i' &= (1+\imath)Z_i+F_i(Z_1Z_i,\dots,Z_i,\dots,Z_nZ_i,\lambda),\\
\dots &= \dots\\
Z_n' &= (F_n(Z_1Z_i,\dots,Z_i,\dots,Z_nZ_i,\lambda)-Z_nF_i(Z_1Z_i,,\dots,Z_i,\dots,Z_nZ_i,\lambda))/Z_i.
\end{aligned}$$

PROOF. In order to compute $\tilde{X}$ in Ψ_i-coordinates, we make the substitutions $z_i=Z_i$, and $z_j=Z_jZ_i$, $j\neq i$. Noting that $z_i'=Z_i'$, $z_j'=Z_j'Z_i+Z_jZ_i'$ and substituting in the equations $z_\ell'=(\lambda+\imath)z_\ell+F_\ell(z_1,\dots,z_n,\lambda)$, $1\leq\ell\leq n$, leads to equations for $Z_1',\dots,Z_n'$. The smoothness of the expressions $(F_j-Z_jF_i)/Z_i$ follows from lemma 5.6.4. ☐

REMARK 5.6.29. If the vector field F in lemma 5.6.28 is a homogenous cubic, then

$$(F_j-Z_jF_i)(Z_1Z_i,\dots,Z_i,\dots,Z_nZ_i)/Z_i=|Z_i|^2(F_j-Z_jF_i)(Z_1,\dots,1,Z_n).$$

This implies that, up to a scale, the equations for Z_j', $j \neq i$, are independent of Z_i.

Recall the phase blowing-up map $\pi : S^{2n-1} \times S^1 \times \mathbb{R} \to \mathbb{C}^n$ and the Hopf fibration $\nu : S^{2n-1} \to \mathbb{P}^{n-1}(\mathbb{C})$. Noting that both π and ν are $G \times S^1$-equivariant (S^1 acts trivially on $\mathbb{P}^{n-1}(\mathbb{C})$), we define the smooth $G \times S^1$-equivariant map $\chi : S^{2n-1} \times S^1 \times \mathbb{R} \to \tilde{\mathbb{C}}^n$ by

$$\chi(u, e^{\imath\theta}, R) = (\pi(u, e^{\imath\theta}, R), \nu(u)).$$

LEMMA 5.6.30. (1) $\pi = B \circ \chi$.

(2) *If $X \in \mathcal{V}_0(\mathbb{C}^n, G \times S^1)$ has B-transform $\tilde{X}$ and $\hat{X}$ is any π-transform of X – for example, $X^{\star\star}$ or $X^\star$ – then $\tilde{X}$ is related to $\hat{X}$ by*

$$(T_w\chi)\hat{X}(w) = \tilde{X}(\chi(w)).$$

PROOF. The equality $\pi = B \circ \chi$ is immediate from the definitions of χ and B. The second statement follows since $\pi = B \circ \chi$ and $\tilde{X}$ is uniquely determined by the relation $T_{\tilde{x}}\tilde{X}(\tilde{x}) = X(B(\tilde{x}))$, $\tilde{x} \in \tilde{\mathbb{C}}^n$. $\square$

THEOREM 5.6.31. *Let $X \in \mathcal{V}_0(\mathbb{C}^n, G \times S^1)$ and write*

$$X(x, \lambda) = (\lambda + \imath)z + F(z).$$

There exist smooth maps $P^j : \mathbb{C}^{n-1} \times \mathbb{R} \to \mathbb{C}^{n-1}$, $1 \leq j \leq n$, such that each zero $(z.\rho)$, $\rho > 0$, of P^j determines a unique limit cycle of X. Conversely, to every S^1-invariant limit cycle of $z' = X(z, \lambda)$, there is associated a zero of some P^j.

PROOF. We construct the map P^1. By lemma 5.6.28, we may write $\tilde{X}$ in Ψ_1-coordinates as

$$\begin{aligned} Z_1' &= (1 + \imath)Z_1 + F_1(Z_1, \ldots, \ldots, Z_iZ_1, \ldots, Z_nZ_1), \\ Z_j' &= (F_j(Z_1, \ldots, Z_iZ_1, \ldots, Z_nZ_1) - Z_jF_1(Z_1, \ldots, Z_iZ_1, \ldots, Z_nZ_1))/Z_1,\ j > 1. \end{aligned}$$

Applying lemma 5.6.4, there are smooth functions $\tilde{F}_i^1 : \mathbb{C}^{n-1} \times \mathbb{R} \to \mathbb{C}$ such that $F_i(Z_1, \ldots, Z_iZ_1, \ldots, Z_nZ_1) = Z_1|Z_1|^2\tilde{F}_i^1(Z_2, \ldots, Z_n, |Z_1|^2)$, $1 \leq i \leq n$. Define

$$P_j^1(Z_2, \ldots, Z_n, |Z_1|^2) = \tilde{F}_j^1(Z_2, \ldots, Z_n, |Z_1|^2) - Z_j\tilde{F}_1^1(Z_2, \ldots, Z_n, |Z_1|^2),\ j \geq 2.$$

We may write the equation for Z_j' in the form

$$Z_j' = |Z_1|^2P_j^1(Z_2, \ldots, Z_n, |Z_1|^2),\ j \geq 2.$$

Define $P^1 = (P_2^1, \ldots, P_n^1) : \mathbb{C}^{n-1} \times \mathbb{R} \to \mathbb{C}^{n-1}$. We similarly define P^j, $2 \leq j \leq n$.

Suppose that $P^1(Z_2^0, \ldots, Z_n^0, R_0^2) = 0$, where we have set $|Z_1|^2 = R_0^2$ and we assume $R_0 \neq 0$. If we make the substitution $Z_1 = R\exp(\imath\theta)$ in the equation for Z_1' and fix $(Z_2, \ldots, Z_n) = (Z_2^0, \ldots, Z_n^0)$, we find that

$$\begin{aligned} R' &= \lambda R + R^3\alpha(Z_2^0, \ldots, Z_n^0, R^2), \\ \theta' &= 1 + R^2\beta(Z_2^0, \ldots, Z_n^0, R^2), \end{aligned}$$

where $\tilde{F}_1^1 = \alpha + \imath\beta$. In particular, if $\lambda = -R_0^2\alpha(Z_2^0, \dots, Z_n^0, R_0^2)$, then $R = R_0$ will be an equilibrium of the radial equation. If we set $p = 1 + R_0^2\beta(Z_2^0, \dots, Z_n^0, R_0^2)$, then there is a limit cycle solution of the original system with with period $2\pi/p$ (we allow the possibility of the degeneracy $p = 0$ in which case we obtain a circle of equilibria).

Conversely, if γ is an S^1-invariant limit cycle of $z' = X_\lambda(z)$, then γ will lift to an S^1-invariant limit cycle of $\tilde{X}_\lambda$. For some j, we may choose local Ψ_j coordinates so that Z_j is non-vanishing on $\tilde{\gamma}$. Just as we did above, it is then easy to verify that $\tilde{\gamma}$ determines a zero of P^j. $\square$

The maps P^j given by theorem 5.6.31 are, up to a scale, local representatives of the vector fields $U(u, R)$ induced on $P^{n-1}(\mathbb{C})$ via phase blowing-up. In particular, if $F = Q$ is a homogeneous cubic, then the P^j are local representatives of the phase vector field $\mathcal{P}_Q$ (up to a scale). We indicate why this is so for P^1.

We start by defining some maps – as usual $\nu : S^{2n-1} \to \mathbb{P}^{n-1}(\mathbb{C})$ denotes the Hopf fibration. Let $\boldsymbol{\nu} : S^{2n-1} \times S^1 \times \mathbb{R} \to \mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{R}$ be the map defined by $\boldsymbol{\nu}(u, e^{\imath\theta}, R) = (\nu(u), R^2)$. Define $b : \tilde{\mathbb{C}}^n \to \mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{R}$ by $b(z, u) = (u, \|z\|^2)$. Let $\Psi_1 : \mathbb{C}^n \to \tilde{\mathbb{C}}^n$ be the chart map defined by

$$\Psi_1(Z_1, \dots, Z_n) = ((Z_1, Z_1Z_2, \dots, Z_1Z_n), (1, Z_2, \dots, Z_n)).$$

Observe that $b \circ \Psi_1 : \mathbb{C}^n \to \mathbb{P}^{n-1}(\mathbb{C}) \times \mathbb{R}$ is the map

$$(b \circ \Psi_1)(Z_1, \dots, Z_n) = ((1, Z_2, \dots, Z_n), (|Z_1|^2\|(1, Z_2, \dots, z_n)\|^2).$$

In particular, if we identify $\mathbb{C}^{n-1} \subset \mathbb{C}^n$ with $Z_1 = 0$, the $\mathbb{P}^{n-1}(\mathbb{C})$-component of $(b \circ \Psi_1)|\mathbb{C}^{n-1}$ is the standard chart map $\phi_1^{-1} : \mathbb{C}^{n-1} \to \mathbb{P}^{n-1}(\mathbb{C})$ for $\mathbb{P}^{n-1}(\mathbb{C})$. But ϕ_1^{-1} maps P^1 to a $|Z_1|^2$-dependent vector field $\bar{P}^1$ on $U_1 \subset \mathbb{P}^{n-1}(\mathbb{C})$. Since $\boldsymbol{\nu} = b \circ \chi$, it follows that, up to a strictly positive scale, $\bar{P}^1$ is equal to $U(u, R)|U_1$.

Remark 5.6.32. We assumed in the statement of theorem 5.6.31 that F was independent of λ. We explain now why this is not a serious restriction. For simplicity, we continue to assume G is finite. Later, in chapter 10, we prove that there is an open and dense subset $\mathcal{S}(\mathbb{C}^n, G \times S^1)$ of $\mathcal{V}_0(\mathbb{C}^n, G \times S^1)$ consisting of stable families. Each $X \in \mathcal{S}$ has a finite number of branches of hyperbolic limit cycles and the stability is determined by a finite jet $j^pX_0(0)$. For stable families, the signed indexed branching pattern of $z' = (\lambda + \imath)z + F(z, \lambda)$ will be the the same as that of $z' = (\lambda + \imath)z + F(z, 0)$.

Typically the zero $u(R) = (Z_2^0, \dots, Z_n^0)$ of P^j described in the proof of theorem 5.6.31 will depend smoothly on R. For $R > 0$, u will be a hyperbolic zero of P^j provided that the corresponding branch of limit cycles is hyperbolic. Of course, with appropriate variation, all of these remarks apply to steady state bifurcations (see remark 5.1.12).

Example 5.6.33. We investigate an example where the cubic truncation does not determine the dynamics. We consider the standard irreducible representation of $\mathbf{D}_3$ on $\mathbb{C}^2$. A set of generators for the module of $(\mathbb{C}^2)^{\mathbf{D}_3\times S^1}$-equivariants is given by

$$\{(z_1, z_2),\ (|z_1|^2 z_1, |z_2|^2 x_2),\ (\bar{z}_1^2 z_2^3, \bar{z}_2^2 z_1^3),\ (z_1^4 \bar{z}_2^3, z_2^4 \bar{z}_1^3)\}.$$

(See [**84**, chapter XVIII,§2] for more details.) We consider $X \in \mathcal{V}_0(\mathbb{C}^2, \mathbf{D}_3 \times S^1)$ of the form $X(z, \lambda) = (\lambda + \imath)z + F(z)$ where

$$\begin{aligned} F_1(z_1, z_2) &= (a\|z\|^2 + b|z_1|^2)z_1 + c\bar{z}_1^2 z_2^3 + dz_1^4 \bar{z}_2^3, \\ F_2(z_1, z_2) &= (a\|z\|^2 + b|z_2|^2)z_2 + c\bar{z}_2^2 z_1^3 + dz_2^4 \bar{z}_1^3, \end{aligned}$$

where $a, b, c, d \in \mathbb{C}$. If $d = 0$, this is the normal form for the $\mathbf{D}_3$-equivariant Hopf bifurcation (see [**84**, chapter XVIII, §§1–4] and note that we will not assume any results from [**84**] in what follows). Computing we find that

$$P^1(Z_2, |Z_1|^2) = bZ_2(|Z_2|^2 - 1) + c|Z_1|^2(\bar{Z}_2^2 - Z_2^4) + d|Z_1|^4(|Z_2|^2 \bar{Z}_2^2 - Z_2^4). \tag{5.46}$$

We have seven Z_1-independent solutions $Z_2 = 0$ and $Z_2 = \exp(2p\pi\imath/6)$, $0 \le p \le 5$. These determine branches of limit cycles along the lines $z_2 = 0$ and $z_2 = \exp(2p\pi\imath/6)z_1$. Examination of P^2 yields the additional solution $Z_1 = 0$ and corresponding branch of limit cycles along $z_1 = 0$ (these are the solutions forced by symmetry - see examples 5.6.16, 5.6.20). There remains the possibility of solutions of (5.46) which depend on $|Z_1|^2$. We shall assume that $b, c \neq 0$ and $\mathrm{Re}(b\bar{c}) \neq 0$. Note that if $c = 0$ then $Z_2 = w$ is a solution of (5.46) whenever $|w| = 1$. On the other hand if $b = 0$, then we get the seven solutions described above. Dividing by b, we reduce to solving the equation

$$z(|z|^2 - 1) + er^2(\bar{z}^2 - z^4) + fr^4(|z|^2 \bar{z}^2 - z^4) = 0, \tag{5.47}$$

where $e = c\bar{b}/|b|^2 \neq 0$, $f = d/b$, we have set $Z_2 = z$, $|Z_1|^2 = r^2$ and we assume $\mathrm{Re}(e) \neq 0$. It follows from the general theory that we develop in chapters 6, 10 that if $X \in \mathcal{S}(\mathbb{C}^2, \mathbf{D}_3 \times S^1)$ is a stable family then we may find a real analytic solution to (5.47) of the form

$$z(t) = \sum_{i=0}^{\infty} z_i t^i,\ \ r(t) = \sum_{i=0}^{\infty} r_i t^i,\ \ (t \in [0, \delta]),$$

where $r(0) = 0$ and so $r_0 = 0$. Substituting in (5.47), and comparing terms of the same order we find that provided $\mathrm{Re}(e) \neq 0$, we must have $z_0 = \exp(2p\pi\imath/6)$, where $p \in \{0, \dots, 5\}$. Without loss of generality, assume $p = 0$ and $z_0 = 1$ (the analysis is similar if $z_0 = \exp(\pi\imath/3)$). Write $z = 1 + w$ and substitute in (5.47). Taking real and imaginary parts, we may use the implicit function theorem to solve for the real part of $w = x + \imath y$ and thereby obtain $x = x(y, r)$ as a smooth function of y, r, $0 \le |y|, r < \delta$. Differentiating implicitly, we find that $x(y, r) = yr\beta(y, r)$, where β is smooth. Substituting in the equation for the imaginary part of (5.47), we obtain an equation of the form $yrA(y, r) = 0$ where $A(0, 0) \neq 0$. It follows that $y = x = 0$ for r sufficiently close to zero and so $z(t) \equiv 1$ for sufficiently small $|Z_1|$. The argument we have given is valid provided that X is stable. As we shall see later, the stability of X depends only on the coefficients b, c, d (evaluated at zero if they depend on λ). More precisely, the computations we have given are valid provided b, c, d lie outside of a closed semialgebraic subset A of $\mathbb{C}^3$ of real codimension at least 1. In fact, the coefficient d plays no role (consistent with the normal form given in [**84**, chapter XVIII, §§1–4]) and we may take $\mathbb{C}^3 \setminus A = \{(b, c, d) \in \mathbb{C}^3 \mid \mathrm{Re}(b\bar{c}) \neq 0, bc \neq 0\}$. The radial coefficient a is needed for computations of the stability of branches of limit cycles. In this way we are able to determine all generic branches of limit cycles for the $\mathbf{D}_3 \times S^1$-equivariant problem. When we break symmetry to $\mathbf{D}_3$ – and thereby allow for flat $\mathbf{D}_3$-equivariant terms – all the branches we have found persist. However, there still remains the tricky problem of showing that no *new* branches of limit cycles appear when we allow for flat $\mathbf{D}_3$-equivariant terms.

5.7. Notes on chapter 5

The first part of the chapter is largely based on the treatment of the invariant sphere theorem given in [**57**] together with the applications described in [**57, 73**] and the paper with Jim Swift [**74**]. More applications and examples of homoclinic cycles and networks are in [**61**]. There is now a quite extensive literature on homoclinic and heteroclinic cycles in equivariant bifurcation theory. In particular, we mention the survey article by Krupa [**106**], and the papers by Krupa and Melbourne [**107, 108**] on asymptotic stability and bifurcation of cycles. For general results on heteroclinic cycles in systems with wreath product symmetries there is the article by Dias, Dionne and Stewart [**42**]. Heteroclinic networks arising from symmetry have been studied by Kirk and Silber [**101**]. In a slightly different direction, we mention the work of Ashwin and Field [**6**]. There is also an extensive literature on heteroclinic cycles in population models that goes back to May and Leonard's paper [**122**].

The remainder of the chapter is about the equivariant Hopf bifurcation and based on [**75, 59**] (and inspired by the earlier works [**74, 166**]). Methods used are based on a complex version of the invariant sphere theorem as well as blowing-up techniques from complex algebraic geometry. Overall, the aim is to combine geometric, analytical and algebraic techniques to discover information about existence of branches of limit cycles (with no maximal isotropy constraints) and dynamics. As indicated in the text, there is an extensive set of examples and applications on the equivariant Hopf bifurcation in Golubitsky, Stewart and Schaeffer [**84**]. The authors use methods based on the complex version of the equivariant branching lemma which yield branches of limit cycles which have necessarily have maximal spatiotemporal normal form symmetry.

CHAPTER 6

Equivariant transversality

6.1. Introduction

The topic of this chapter is the theory of G-*transversality* or *equivariant general position*. After preliminaries on C^∞-topologies, jet bundles and transversality theory we start work on the theory of general position for equivariant maps. Unlike what happens in standard transversality theory, we do not have a simple geometric definition of what it means for a map to be G-transversal to a G-invariant submanifold. It is, however, possible to show that there is an intrinsic definition of equivariant transversality that is independent of all choices. It takes some work to get to this definition. The first step is straightforward. We reduce the problem of defining equivariant transversality to a local problem about solving equivariant equations $f(x) = 0$, where $f : V \to W$ and (V, G) and (W, G) are G-representations. We reformulate the problem of finding 'generic' solutions to $f(x) = 0$ in terms of transversality to an algebraic variety Σ. In order to do this we need some serious preliminaries involving spaces of polynomial and smooth equivariant maps ('smooth invariant theory'), the theory of stratifying (partitioning) semialgebraic sets into smooth pieces that fit together well and the theory of transversality of maps to stratified sets. The key step towards obtaining an intrinsic definition of G-transversality comes next. We prove that we can formulate the equivariant transversality of a map $f : V \to W$ to $0 \in W$ at $0 \in V$ in terms of transversality to a stratification $\mathcal{A}$ of an invariantly defined vector space $\mathbb{U}$, where $\dim(\mathbb{U})$ depends only the representations V, W. We prove that the stratification $\mathcal{A}$ of $\mathbb{U}$ depends only on the representations V and W. In addition, we show that $\mathcal{A}$ admits natural symmetries related to coordinate invariance. It is then fairly routine to show that equivariant transversality satisfies the characteristic properties of transversality including openness, density and a variant of the transversality isotopy theorem. We conclude with a section showing how the assumption of G-transversality imposes constraints on the symmetries of solutions.

We have tried to present the local theory in as simple a way as possible. Thus we generally work on representations (rather than proper G-invariant open subsets) and use generating sets of *homogeneous* polynomials. (This will be the setting for our subsequent applications of G-transversality to equivariant bifurcation theory.) Only when we come to proving the openness of equivariant transversality do we consider the more complicated issue of sets of inhomogeneous polynomial generators. Readers mainly interested in applications of equivariant transversality to bifurcation theory are advised to omit 6.7.1 and 6.11 – 6.14 at first reading.

6.2. C^∞-topologies on function spaces

We have already given the definition of the C^∞-topology for smooth vector valued maps on open subsets of $\mathbb{R}^n$ (chapter 4). There are several equivalent ways to proceed when we come to consider the space $C^\infty(M, N)$ of smooth maps between the differential manifolds M and N. These approaches are covered in introductory texts on differential manifolds and all we do here is we review the definitions and main ideas. We start by giving a conceptually easy approach which builds on our definition for $C^\infty(\mathbb{R}^p, \mathbb{R}^q)$ (this needs the tubular neighbourhood theorem), then we give a definition based on local coordinates. Finally, after a short digression on jet spaces, we give a coordinate-free definition of the C^∞-topologies (although we do not need jet spaces in this chapter, they will play an important role in chapter 7). We emphasize C^∞-topologies; the definitions for C^r-topologies are similar (and simpler).

Our first method uses Whitney's embedding theorem [**92**, chapter 1,§3] to represent M and N as closed submanifolds of $\mathbb{R}^p$ and $\mathbb{R}^q$. We may then topologize $C^\infty(M, N)$ as a (closed) subspace of $C^\infty(\mathbb{R}^p, \mathbb{R}^q)$. We call the resulting topology on $C^\infty(M, N)$ the C^∞-topology. For this approach, we need the tubular neighbourhood theorem (proposition 3.4.1, remarks 3.4.2) to construct an extension operator $L : C^\infty(M, N) \to C^\infty(\mathbb{R}^p, \mathbb{R}^q)$ and so embed $C^\infty(M, N)$ in $C^\infty(\mathbb{R}^p, \mathbb{R}^q)$ (L composed with restriction to M is the identity map of $C^\infty(M, N)$). Our second approach uses local constructions. Let $f \in C^\infty(M, N)$. Suppose that (U, ϕ) is an M-chart, K is a compact subset of U, and (V, ψ) is an N-chart such that $f(K) \subset V$. Working in the local coordinates given by (U, ϕ) and (V, ψ), we define for every $r \in \mathbb{N}$ and $\varepsilon > 0$, the set $W(f, K, r, \varepsilon) \subset C^\infty(M, N)$ to consist of all maps g such that $g(K) \subset V$ and $\|f - g\|_r^K < \varepsilon$ (see chapter 4 for the definition of $\| \|_r^K$). The collection of sets $W(f, K, r, \varepsilon)$ over all (U, ϕ), (V, ψ), K, r, ε then defines a basis of open sets for the C^∞-topology on $C^\infty(M, N)$.

If M is not compact, the C^∞-topology gives no control over the behavior of maps at infinity. To remedy this problem, Whitney introduced a finer topology, now known as the Whitney C^∞-topology. We describe the Whitney topology, first for $C^\infty(\mathbb{R}^p, \mathbb{R}^q)$. For each $f \in C^\infty(\mathbb{R}^p, \mathbb{R}^q)$, we construct a neighbourhood base of f for the Whitney topology. Given $r \in \mathbb{N}$, $x \in \mathbb{R}^p$, let $\rho(f, r)(x) = \|f\|_r^x$. Let $\delta : \mathbb{R}^p \to \mathbb{R}$ be a smooth (continuous will do) strictly positive function. We define

$$W(f, r, \delta) = \{g \in C^\infty(\mathbb{R}^p, \mathbb{R}^q) \mid \rho(f - g, r)(x) < \delta(x),\ x \in \mathbb{R}^p\}.$$

The collection of all $W(f, r, \delta)$, $r \in \mathbb{N}$, $\delta : \mathbb{R}^p \to \mathbb{R}^+$ defines a neighbourhood base for f. Varying over all $f \in C^\infty(\mathbb{R}^p, \mathbb{R}^q)$, we define a base for the Whitney C^∞-topology on $C^\infty(\mathbb{R}^p, \mathbb{R}^q)$.

Once we have the Whitney C^∞-topology on $C^\infty(\mathbb{R}^p, \mathbb{R}^q)$, we can define the Whitney C^∞-topology on $C^\infty(M, N)$ by embedding M and N as closed submanifolds of $\mathbb{R}^p$, $\mathbb{R}^q$ just as we did for the C^∞-topology. Alternatively, we can follow the local construction but now allowing for arbitrary families of compact subsets $K_i \in M$ (see [**92**, chapter 2] for details).

For future reference, we highlight two properties of the C^∞ and Whitney C^∞-topologies (we refer the reader to [**92**, chapter 2], [**81**, chapter II] or [**1**] for more details and proofs.

LEMMA 6.2.1. (1) *If M is compact, the Whitney C^∞ and C^∞-topologies coincide. If M is not compact, the topologies differ.*
(2) *$C^\infty(M, N)$ is a Baire space*[1] *in either the Whitney C^∞- or C^∞-topology.*
(3) *If $(f_n) \subset C^\infty(M, N)$ converges to $F \in C^\infty(M, N)$ in the Whitney C^∞-topology, then there exists a compact subset K of M such that for all sufficiently large n, $f_n = F$ on $M \setminus K$.*

REMARK 6.2.2. Lemma 6.2.1 continues to hold when M, N are G-manifolds and we replace $C^\infty(M, N)$ by $C^\infty_G(M, N)$ – the space of smooth equivariant maps from M to N.

EXERCISE 6.2.3. (1) Construct the extension operator $L : C^\infty(M, N) \to C^\infty(\mathbb{R}^p, \mathbb{R}^q)$ required for the first construction of the (Whitney) C^∞-topology.
(2) Verify that the two definitions we have given of the C^∞-topology are equivalent. Similarly for the Whitney C^∞-topology.
(3) Show that in the Whitney topology, $C^\infty(\mathbb{R}^p, \mathbb{R}^q)$ is not first countable and is therefore not metrizable. Show that $C^\infty(M, N)$ is metrizable in the C^∞-topology even if M is not compact.
(4) Show that the Weierstrass approximation theorem is not true for $C^\infty(\mathbb{R}^p, \mathbb{R}^q)$ if we take the Whitney topology.
(5) Let M, N be smooth G-manifolds. Show that $C^\infty_G(M, N)$ is a closed subset of $C^\infty(M, N)$ in either the C^∞- or Whitney C^∞-topology.

6.2.1. Jet bundles. For $d \geq 0$, let $L^d_s(\mathbb{R}^p, \mathbb{R}^q)$ denotes the vector space of symmetric d-linear maps from $\mathbb{R}^p$ to $\mathbb{R}^q$ ($L^0_s(\mathbb{R}^p, \mathbb{R}^q) = \mathbb{R}^q$). We recall the natural isomorphism $L^d_s(\mathbb{R}^p, \mathbb{R}^q) \approx P^d(\mathbb{R}^p, \mathbb{R}^q)$ defined by mapping $A \in L^d_s(\mathbb{R}^p, \mathbb{R}^q)$ to $P_A \in P^d(\mathbb{R}^p, \mathbb{R}^q)$, where $P_A(x) = \frac{1}{d!}A(x^d)$. In particular, for $r \geq 0$,

$$\prod_{j=0}^{r} L^j_s(\mathbb{R}^p, \mathbb{R}^q) \approx P^{(r)}(\mathbb{R}^p, \mathbb{R}^q).$$

Let U be an open subset of $\mathbb{R}^p$. For $r \geq 0$, define the space $J^r(U, \mathbb{R}^q)$ of r-jets from U to $\mathbb{R}^q$ by

$$J^r(U, \mathbb{R}^q) = U \times \prod_{j=0}^{r} L^j_s(\mathbb{R}^p, \mathbb{R}^q) \approx U \times P^{(r)}(\mathbb{R}^p, \mathbb{R}^q).$$

[1] The countable intersection of open and dense subsets is dense.

There are natural projections $\pi_p : J^r(U, \mathbb{R}^q) \to U$ and $\pi_q : J^r(U, \mathbb{R}^q) \to \mathbb{R}^q$ defined by mapping $(x, (A_0, A_1, \ldots, A_r))$ to x and A_0 respectively.

We define the *r-jet extension map* $j^r : C^\infty(U, \mathbb{R}^q) \to C^\infty(U, J^r(\mathbb{R}^p, \mathbb{R}^q))$ by

$$j^r f(x) = (x, (f(x), Df(x), \ldots, D^r f(x)), \ (x \in U).$$

We refer to $j^r f(x)$ as the r-jet of f at x. If we use the identification with polynomial maps, then the r-jet of f at x may be identified with the degree r Taylor polynomial $T_r f(x)$ of f at x and conversely.

Suppose that M, N are smooth manifolds and $x \in M$, $y \in N$. Let f, g be smooth N-valued maps defined on an open neighbourhood of $x \in M$. We write $f \sim^r_{x,y} g$ if $f(x) = g(x)$ and we can find coordinate charts (U, ϕ), $x \in U$, and (V, ψ), $y \in V$, such that

$$j^r(\psi f \phi^{-1})(\phi(x)) = j^r(\psi g \phi^{-1})(\phi(x)).$$

This definition is independent of the choice of charts and $\sim^r_{x,y}$ is an equivalence relation. Let $J^r_{x,y}(M, N)$ denote the set of equivalence classes of $\sim^r_{x,y}$. It is straightforward to check that $J^r_{x,y}(M, N) \cong \prod_{j=1}^r L^j_s(\mathbb{R}^p, \mathbb{R}^q)$. We define the bundle of r-jets of smooth maps from M to N by

$$J^r(M, N) = \cup_{(x,y) \in M \times N} J^r_{x,y}(M, N) \ \ \text{(disjoint union)}.$$

We have natural projections $\pi_M : J^r(M, N) \to M$, $\pi_N : J^r(M, N) \to N$. For each $f \in C^\infty(M, N)$, we define the r-jet extension map $j^r f : M \to J^r(M, N)$ by requiring $j^r f(x)$ to be the $\sim^r_{x,f(x)}$ equivalence class of f.

Suppose that (U, ϕ) is an M-chart, (V, ψ) is an N-chart. If we let $J^r(U, V) = \cup_{(x,y) \in U \times V} J^r_{x,y}(M, N) \subset J^r(M, N)$ then there is a natural bijection

$$\nu : J^r(U, V) \to J^r(\phi(U), \psi(V)) \subset J^r(\mathbb{R}^m, \mathbb{R}^n).$$

In order to construct ν, suppose that $x \in U$, $y \in V$ and α is the $\sim^r_{x,y}$-equivalence class of a smooth map f defined on some open neighbourhood of x in U and such that $f(x) = y$. We define $\nu(\alpha)$ to be the $\sim^r_{\phi(x),\psi(y)}$-equivalence class of $\psi f \phi^{-1}$ in $J^r(\phi(U), \psi(V))$. Obviously, ν is a bijection onto $J^r(\phi(U), \psi(V))$. Since $J^r(\phi(U), \psi(V))$ is an open subset of $J^r(\mathbb{R}^m, \mathbb{R}^n)$, we may use $(J^r(U, V), \nu)$ as a chart for $J^r(M, N)$. With some straightforward, if tedious, computations, we may show that the collection of all such charts gives $J^r(M, N)$ the structure of a differential manifold (we use the charts to define a topology on $J^r(M, N)$).

THEOREM 6.2.4. *Let M, N be smooth manifolds.*

(1) *For $r \geq 0$, $J^r(M, N)$ has the natural structure of a smooth manifold and $\dim(J^r(M, N)) = \dim(J^r(\mathbb{R}^m, \mathbb{R}^n))$, where $m = \dim(M)$, $n = \dim(N)$.*
(2) *The maps $\pi_M : J^r(M, N) \to M$, $\pi_N : J^r(M, N) \to N$ are smooth submersions.*
(3) *For each $f \in C^\infty(M, N)$, the r-jet extension $j^r f : M \to J^r(M, N)$ is smooth.*

PROOF. For the details we refer to [**92**] or [**81**]. □

6.2.2. Natural constructions of C^∞-topologies. We may use the jet bundles $J^r(M,N)$ to give definitions of the C^∞- and Whitney C^∞-topologies that avoid the Whitney embedding theorem or the use of charts. First, the C^∞-topology. The r-jet extension defines a map $j^r : C^\infty(M,N) \to C^0(M, J^r(M,N))$. Take the compact open topology on $C^0(M, J^r(M,N))$. We take the weakest topology on $C^\infty(M,N)$ such that $j^r : C^\infty(M,N) \to C^0(M, J^r(M,N))$ is continuous for all $r \geq 0$. The resulting topology is the C^∞-topology on $C^\infty(M,N)$.

In order to define the Whitney C^∞-topology, let $r \geq 0$ and U be any open subset of $J^r(M,N)$. Let

$$W_k(U) = \{f \in C^\infty(M,N) \mid j^k f(M) \subset U\}$$

The family of all sets $\{W_k(U) \mid k \geq 0, U \subset J^r(M,N)\}$ is a basis for the Whitney C^∞-topology on $C^\infty(M,N)$.

PROPOSITION 6.2.5. *Let $r \geq 0$. The map $j^r : C^\infty(M,N) \to C^\infty(M, J^r(M,N))$ is continuous in the Whitney C^∞-topology.*

PROOF. Details may be found in [**81**]. □

6.3. Transversality

In this section we review parts of the theory of transversality for smooth maps. Suppose that M, N are connected differential manifolds, $\dim(M) = m$, $\dim(N) = n$. Let P be a p-dimensional submanifold of N. If $f : M \to N$ is a smooth map, f is *transverse* to P at $x \in M$ if either $f(x) \notin P$ or $f(x) \in P$ and

$$T_x f(T_x M) + T_{f(x)} W = T_{f(x)} N.$$

We write this symbolically as $f \pitchfork_x P$. If f is transverse to P at all points of M, we say f *is transverse to* P and write this $f \pitchfork P$.

6.3.1. Basic theorems on transversality.

(a) (Openness) If $f \pitchfork_x P$, then there exists an open neighbourhood U of x in M such that $f \pitchfork_y P$, for all $y \in U$.
(b) (Dimension) If $f \pitchfork P$ and $f(M) \cap P \neq \emptyset$ (so $m \geq n - p$), then $f^{-1}(P)$ is an $m - n + p$ dimensional submanifold of M.
(c) (Stability) If M is compact, P is a closed submanifold of N and $f \pitchfork P$, then there exists a C^1-open neighbourhood $\mathcal{U}$ of f in $C^\infty(M,N)$ (C^∞-topology) such that for all $g \in \mathcal{U}$, $g \pitchfork P$. If M is not compact, openness holds in the Whitney C^∞-topology.
(d) (Isotopy theorem) If M is compact, P is a closed submanifold of N, $f : M \times [0,1] \to N$ is smooth and $f_t \pitchfork P$, $t \in [0,1]$, then there is a smooth isotopy $K_t : M \to M$ such that $K_t(f_t^{-1}(P)) = f_0^{-1}(P)$ and $K_0 = I_M$.

(e) (Density) There is a residual subset $\mathcal{T}$ of $C^\infty(M,N)$ such that for all $g \in \mathcal{T}$, $g \pitchfork P$. In case P is closed, $\mathcal{T}$ is C^1-open and C^∞-dense in $C^\infty(M,N)$ (if M is not compact, we take the Whitney topology).

Remarks 6.3.1. (1) Proofs of these results may be found in the books by Abraham and Robbin [**1**], Hirsch [**92**] or Golubitsky and Guillemin [**81**] (for the isotopy theorem, see [**1**]).
(2) It is obvious that (c,d) fail is P is not closed,
(3) If M is not compact, we require the Whitney C^∞-topology in order to establish openness results. In fact, as our focus is mainly local, it is safe for the most part to think in terms of the C^∞-topology.

The local theory of transversality is easily reformulated in terms of solutions to equations locally defined on vector spaces. Specifically, suppose that $f \in C^\infty(M,N)$ and $f(x) \in P$. We may choose smooth charts (U,ϕ) for M at x and (V,ψ) for N at x such that $f(U) \subset V$ and

(1) ϕ is a diffeomorphism of U onto an open neighbourhood U_1 of $0 \in \mathbb{R}^m$.
(2) ψ is a diffeomorphism of V onto an open neighbourhood $V_1 \times V_2$ of $\psi f(x) = (0,0) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$.
(3) $\psi(V \cap P) = V_2$ (an open neighbourhood of $\psi(f(x)) = 0 \in \mathbb{R}^p$).

Let $\pi : \mathbb{R}^n = \mathbb{R}^{n-p} \times \mathbb{R}^p \to \mathbb{R}^{n-p}$ denote the projection on the first factor. Then it follows from the definition of transversality that $f \pitchfork_x P$ if and only if $0 \in U_1$ is not a critical point of $\tilde{f} = \pi\psi f \phi^{-1} : U_1 \to \mathbb{R}^{n-p}$. If $0 \in U_1$ is not a critical point, we can always choose U, V sufficiently small so that $0 \in V_1$ is a regular value of $\tilde{f} : U_1 \to \mathbb{R}^{n-p}$. In particular, $f \pitchfork_y P$ for all $y \in U$, proving property (a). Property (b) follows by applying the implicit function theorem to the equation $\tilde{f}(x) = 0$. In other words by reformulating transversality as a statement about the solution to an equation, we can easily derive the local properties of transversality. Global results, such as the isotopy theorem, require more work (for the density theorem, we use Sard's theorem).

Our investigation of equivariant transversality will emphasize local properties. However, for our general theorems, we will use results about transversality (no symmetry); in particular results on maps which are transverse to Whitney regular *stratified sets* (as described in chapter 3, section 3.9). We will also need results on the existence of Whitney stratifications of semialgebraic sets.

6.4. Stratumwise transversality and stability

Suppose that M, N are G-manifolds, P is a G-invariant submanifold of N and $f : M \to N$ is a smooth equivariant map. As a first attempt at formulating a theory of equivariant transversality it is natural to require stratumwise transversality. More precisely, $f : M \to N$ is *stratumwise transverse* to P if for every isotropy group $H \in \tau \in \mathcal{O}(M,G)$ for the action of G on M, we have $f^H \pitchfork P^H$, where $f^H = f|M_\tau^H : M_\tau^H \to N^H$. If f is an embedding, stratumwise transversality is equivalent $f_\tau \pitchfork P_\tau$, for all $\tau \in \mathcal{O}(M,G)$, where $f_\tau = f|M_\tau : M_\tau \to N_\tau$. It is not hard to check that if we want a stability or isotopy result for equivariant transversality then stratumwise transversality is necessary – else we can change the local topological type of the intersection by a small perturbation. However, as the following example shows, stratumwise transversality is not sufficient for the local stability of intersections.

EXAMPLES 6.4.1. (1) Let $\mathbb{Z}_2$ act on $\mathbb{R}$ as multiplication by ± 1. Set $P = \{0\} \in \mathbb{R}$. Clearly P is a $\mathbb{Z}_2$-invariant submanifold of $\mathbb{R}$. Define the $\mathbb{Z}_2$-equivariant map $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^5$. Since $f(x) = 0$ if and only if $x = 0$ and $f'(x) \neq 0$, $x \neq 0$, f is stratumwise transverse to P (and $f^{-1}(P) = \{0\}$). For $a \geq 0$, define $f_a(x) = x^5 - 2ax^3 + a^2x$. Then f_a is $\mathbb{Z}_2$-equivariant, $f_a(\pm\sqrt{a}) = 0$ and $f_a'(\pm\sqrt{a}) = 0$. Hence f_a is *not* stratumwise transverse to P, for all $a > 0$. Further, the topological type of $f_a^{-1}(P)$, $a > 0$, is different from that of $f^{-1}(P)$.
(2) The pathology described in (1) can be removed by perturbing f to $f_\varepsilon(x) = x^5 + \varepsilon x$, $\varepsilon \neq 0$. We then have $f_\varepsilon \pitchfork_0 P$, $\varepsilon \neq 0$. However, this approach fails if we work with families of $\mathbb{Z}_2$-equivariant maps. If $F_t : \mathbb{R} \to \mathbb{R}$ is a smooth family of $\mathbb{Z}_2$-equivariant maps, $t \in \mathbb{R}$, we may write $F_t(x) = xg(x,t)$, where g is smooth and an even function of x. We have $F \pitchfork_{(0,t)} P$ if and only if $g(0,t) \neq 0$. If we take $g(x,t) = t$, then however we perturb g (or F), there will always be a value of t near zero for which $g(0,t) = 0$. Hence there is no way we can equivariantly perturb away the point of non-transverse intersection of F with P.

A characteristic feature of transversality theory is that transverse intersections $f^{-1}(P)$ are submanifolds of the ambient manifold. Moreover, the intersections vary smoothly as we vary f. Both properties fail for equivariant maps.

EXAMPLES 6.4.2. (1) Continuing with the notation of the previous example, let $F_t(x) = xt$. The zero set of F is the union of the t and x axes and so has a 'crossing' type singularity at the origin. If G_t is a smooth $\mathbb{Z}_2$-equivariant family C^2-close to F_t, we have $G_t(x) = xh(x,t)$, where $|h(x,t) - t|$, $|h_t(x,t) - 1|$ are small near $(0,0)$. Applying the implicit function theorem, we see that there exists t_0 close to 0 and a smooth map $t(x)$, $t(x_0) = t_0$, such that $h(x,t(x))) = 0$, x near zero. Since $x = 0$ lies in the zero set, we see again that there is a crossing singularity at $(0,t_0)$ (see figure 1),
(2) We cannot, in general, require differential stability of solutions sets for equivariant mappings. Let SO(2) act on $\mathbb{C}^2$ as multiplication by $e^{\imath\theta}$ and on $\mathbb{C}$ as multiplication by $e^{4\imath\theta}$. Every SO(2)-equivariant homogeneous polynomial map $P : \mathbb{C}^2 \to \mathbb{C}$ of degree less than four is identically zero. The complex vector space $P^4_{\mathrm{SO}(2)}(\mathbb{C}^2, \mathbb{C})$ has basis $\{z_1^4, z_1^3 z_2, \ldots, z_2^4\}$ and so the general SO(2)-equivariant polynomial of degree four may be written

$$P_c(z_1, z_2) = \sum_{j=0}^{4} c_j z_1^j z_2^{4-j},$$

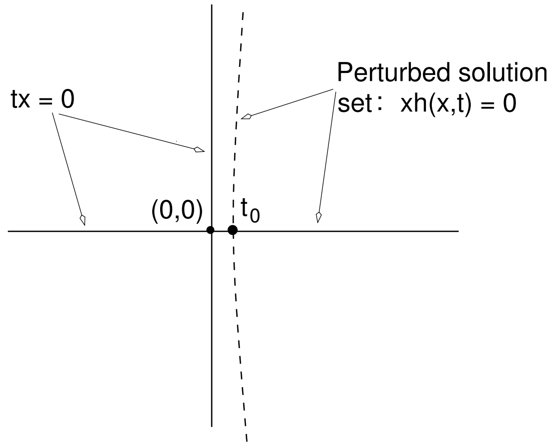


FIGURE 1. Stable crossing singularity

where $c = (c_0, \ldots, c_4) \in \mathbb{C}^5$. In this case a natural (necessary) condition to require for the SO(2)-equivariant transversality of P_c to $0 \in \mathbb{C}$ is that c does not lie on the discriminant locus $\mathcal{D} \subset \mathbb{C}^5$ associated to homogeneous polynomials of degree 4. Indeed, if $c \notin \mathcal{D}$, then $P_c^{-1}(0)$ will consist of four distinct complex lines through the origin of $\mathbb{C}^2$. On the other hand, if $c \in \mathcal{D}$, then the topological type of $P_c^{-1}(0)$ can be changed by perturbing c off $\mathcal{D}$. But now suppose $c, c^\star \notin \mathcal{D}$. If $T : \mathbb{C}^2 \to \mathbb{C}^2$ is a smooth SO(2)-equivariant diffeomorphism mapping $P_c^{-1}(0)$ to $P_{c^\star}^{-1}(0)$, then the same must be true of the derivative of T at the origin. But this can only happen if the two sets of four $\mathbb{C}$-lines have the same cross-ratio [**177**, section 16], [**119**, section 8] and so differentiable stability fails.

6.5. Reduction to a problem about solving equations

Suppose that M, N are Riemannian G-manifolds and P is a G-invariant submanifold of N. Let $f : M \to N$ be a smooth G-equivariant map. Suppose that $x \in M$ and $f(x) \in P$. Since f is G-equivariant $G_x \subset G_{f(x)}$ and so $T_{f(x)}N$ has the structure of an orthogonal G_x-representation. Let $W_1 = T_{f(x)}P$ and W_2 denote the orthogonal complement of W_1 in $T_{f(x)}N$. As G_x-representations, $T_{f(x)}N = W_1 \oplus W_2$. By Bochner's theorem 3.4.9, we may choose a G_x-equivariant diffeomorphism ψ of a G_x-invariant open neighbourhood D of $f(x) \in N$ onto an open neighbourhood $B \times C$ of $(0, 0) \in W_1 \oplus W_2$ so that $\psi(f(x)) = (0, 0)$ and $\psi(D \cap P) = B \times \{0\}$. Let $\pi : W_1 \oplus W_2 \to W_2$ denote the projection on W_2.

Turning now to M, choose a slice S_x for the action of G at x so that $f(S_x) \subset D$. Let (V, G_x) be the G_x-representation induced on the normal space $T_xG(x)^\perp$. Choose a G_x-equivariant diffeomorphism ϕ of S_x onto an open neighbourhood A of $0 \in V$. We may assume $\phi(x) = 0$. Define $\tilde{f} : A \subset V \to B \times C \subset W_1 \oplus W_2$ by

$$\tilde{f} = \psi f \phi^{-1}.$$

Clearly $\tilde{f}$ is a smooth G_x-equivariant map satisfying $\tilde{f}(0) = 0$. Moreover, given any G_x-equivariant map $h : A \subset V \to B \times C \subset W_1 \oplus W_2$, we may define a G-equivariant map $h^\star : G(S_x) \to N$ by $h^\star(\gamma y) = \gamma\psi^{-1}h(\phi y)$, all $y \in S_x$ and $\gamma \in G$ (we leave it to the reader to check that $h^\star$ is a well-defined G-equivariant map and that $\tilde{f}^\star = f|G(S_x)$).

Set $\pi\tilde{f} = F$. Thus $F : A \subset V \to C \subset W_2$ is a smooth G_x-equivariant map and $F(0) = 0$. Observe that if $y \in S_x$ then $f(y) \in P$ if and only if

$$F(\phi(y)) = 0.$$

In particular, every solution z of $F = 0$ determines a unique G-orbit $G(\phi^{-1}(z)) \subset G(S_z) \subset M$ mapped by f to P. Conversely every G-orbit $\alpha \subset G(S_x)$ which is mapped to P by f determines a unique solution of $F = 0$. We refer the reader to figure 2.

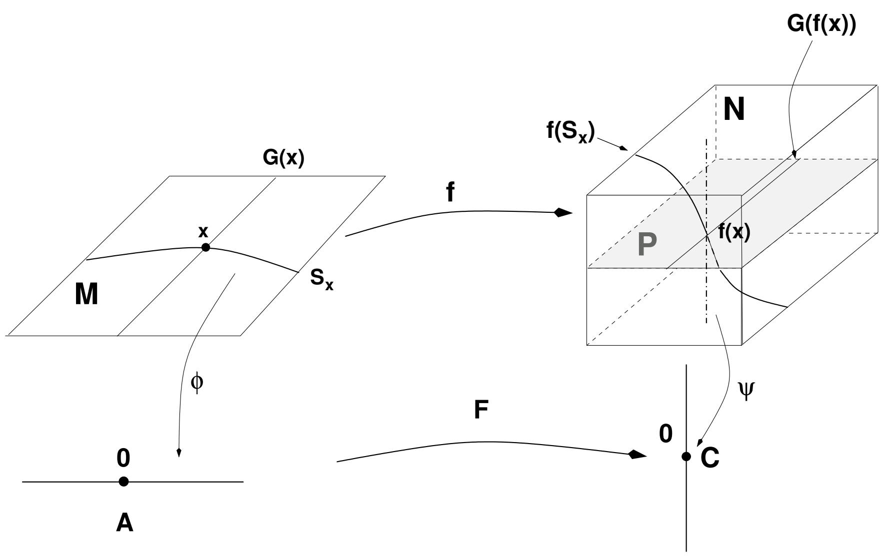


FIGURE 2. Reduction of transversality to an equation

The preceding discussion shows that the general problem of defining the G-transversality of a map to a G-invariant submanifold can be reduced to a 'local' problem about solving equivariant equations defined on a representation. In general, the group changes from G to an isotropy group for the action of G. This motivates our strategy for defining G-transversality. We study smooth G-equivariant maps $f : V \to W$ where V, W are G-representations and find conditions for the solution set $f^{-1}(0)$ to be "generic". We also verify basic properties such as invariance under coordinate changes and the openness of G-transversality. Using the local theory, we may then define what it means for a G-equivariant map $f : M \to N$ to be G-transverse to a G-invariant submanifold of N. Here it is important to check that our definitions are independent of all choices (for example, of slice or local coordinates). Genericity, openness, stability and isotopy theorems follow relatively straightforwardly from the corresponding local results for maps transverse to stratified sets.

The theory we present in the following sections is very much a C^∞-theory. This is on account of our use of smooth invariant theory. We discuss this issue further at appropriate places in the text. Our approach is a mix of those originally used in Bierstone [**14**] and Field [**50**] (see also the notes at the end of the chapter). As a simplification we assume as far as possible that functions are defined on representations (linear G-spaces) rather than on G-invariant open neighbourhoods of the origin of a representation (as in [**14**]) – this is a harmless restriction, we indicate why in various exercises. Almost all of the time we assume that polynomial generators are *homogeneous*. In fact, the only place where we have to allow for inhomogeneous generators is when we prove the openness of G-transversality. Following [**50**], we frame our definition of G-transversality in terms of a natural stratification of a parameter space rather than in terms of transversality to an algebraic variety (which depends on choices). This approach works well for bifurcation theory (see [**70, 57, 60**]). It also gives a simple proof of the density theorem.

6.6. Invariants and equivariants

Let V, W be finite dimensional real representations of the compact Lie group G. The vector space $P_G(V,W)$ has the structure of a $P(V)^G$-module. Using Haar measure (averaging polynomials over G) together with the Hilbert basis theorem, it may be shown that $P(V)^G$ is a finitely generated $\mathbb{R}$-algebra (in particular, a Noetherian ring) and that $P_G(V,W)$ is a finitely generated $P(V)^G$-module (see Poenaru [**143**] for details).

6.6.1. Smooth equivariants. As usual we let $C^\infty(V)^G$ denote the $\mathbb{R}$-algebra of smooth G-invariant real valued functions on V and $C^\infty_G(V,W)$ denote the $C^\infty(V)^G$-module of smooth G-equivariant maps from V to W. Clearly, $P(V)^G \subset C^\infty(V)^G$ and $P_G(V,W) \subset C^\infty_G(V,W)$.

LEMMA 6.6.1. *If $\mathcal{F} = \{F_1, \ldots, F_k\}$ is a set of generators for the $P(V)^G$-module $P_G(V,W)$, then $\mathcal{F}$ generates the $C^\infty(V)^G$-module $C^\infty_G(V,W)$.*
More generally, if U is a G-invariant open subset of V, then $\mathcal{F}$ generates the $C^\infty(U)^G$-module $C^\infty_G(U,W)$.

PROOF. Method I. The equivariant Stone-Weierstrass approximation theorem, theorem 2.10.8, implies that $P_G(V,W)$ is a dense subset of $C^\infty_G(V,W)$. It suffices to show that the $C^\infty(V)^G$-submodule of $C^\infty_G(V,W)$ generated by $\mathcal{F}$ is closed. This follows from the Malgrange Division theorem which implies that if U is a non-empty subset of V, then every finite subset of $P(V,W)$ generates a *closed* $C^\infty(U)$-submodule of $C^\infty(U,W)$ (see [**171**, Chapter VI, Corollary 1.5]). In particular, taking $U = V$, $\mathcal{F}$ generates a closed submodule of $C^\infty(V,W)$. The result follows by averaging over G. This argument works for G-invariant open

subsets U of V since $P_G(V,W)$ is a dense subset of $C^\infty_G(U,W)$ in the C^∞-topology for all non-empty open subsets U of V (chapter 3 exercise 2.10.9).
Method II. Using an approach due to Malgrange, the result may also be deduced from Schwarz' theorem on smooth invariants [**154**]. We sketch the proof at the end of the section (see also Poenaru [**143**] or Golubitsky et al. [**84**, chapter XII, §§5,6] and remark 6.6.14). □

REMARKS 6.6.2. (1) If U is a G-invariant open subset of V and $\mathcal{F}$ is a set of polynomial generators for the $C^\infty(U)^G$-module $C^\infty_G(U,W)$, it does not follow that $\mathcal{F}$ generates the $P(V)^G$-module $P_G(V,W)$ unless the elements of $\mathcal{F}$ are homogeneous.
(2) Lemma 6.6.1 *fails* for C^r equivariants, $r<\infty$. In fact it is possible – with some difficulty – to show that if r is sufficiently large then a C^r-equivariant $f:V\to W$ may be written as $\sum f_jF_j$. However, the invariants f_j will typically only be C^s, where $s\sim[\frac{r}{d}]$, where $d>1$ (see [**147, 153**]). In spite of this, it is typically the case in bifurcation problems that it is possible to prove *strong determinacy* results that imply higher order terms do not matter (even if they are not equivariant, see [**60, 62**] and chapter 10). Examples of this phenomenon are given by the theory we presented for $(\mathbb{R}^k,H_k)$ where for the codimension 1 theory, it is enough to assume vector fields are C^3 and we do not need to be concerned with equivariants of degree greater than 3.

EXERCISE 6.6.3. Let U be a non-empty G-invariant open subset of V. Show that if K is a compact G-invariant subset of U, there exists a continuous (C^∞- or Whitney C^∞-topology) linear map $e_K: C^\infty_G(U,W)\to C^\infty_G(V,W)$ such that for all $f\in C^\infty_G(U,W)$, $e_K(f)|K=f$.

6.6.2. Generators for equivariants. For the remainder of the chapter, $\mathcal{F}=\{F_1,\dots,F_k\}$ will always denote a *minimal set of homogenous generators* for the $P(V)^G$-module $P_G(V,W)$. By minimal, we mean that no proper subset of $\mathcal{F}$ generates $P_G(V,W)$. We set degree$(F_j)=d_j$ and label the generators so that $0\le d_1\le d_2\le\ldots\le d_k$.

LEMMA 6.6.4. *(Notation as above.) If $p_1,\dots,p_k\in P(V)^G$ and*

$$\sum_{j=1}^k p_jF_j=0,$$

then $p_j(0)=0$, $1\le j\le k$. The same result holds if we allow $p_j\in C^\infty(V)^G$.

PROOF. If $p\in P(V)^G$, let p^ℓ denote the homogeneous part of p of degree ℓ. For $1\le i\le k$ we have

$$-p_iF_i=\sum_{j\ne i}p_jF_j.$$

Taking the homogeneous parts of degree d_i we see that

$$-p_i(0)F_i = \sum_{j\neq i} p_j^{d_i-d_j} F_j.$$

Hence $p_i(0) = 0$ by the minimality of $\mathcal{F}$. If we allow the coefficients p_i to be smooth invariants, the result follows by taking the d_k-jet of $\sum_{j=1}^k p_j F_j$ at the origin and applying the result for polynomials. $\square$

Let $\mathfrak{M} = \{p \in P(V)^G \mid p(0) = 0\}$ and $\mathfrak{M}_\infty = \{f \in C^\infty(V)^G \mid f(0) = 0\}$.

LEMMA 6.6.5. (1) $C^\infty_G(V,W)/\mathfrak{M}_\infty C^\infty_G(V,W) \approx P_G(V,W)/\mathfrak{M}P_G(V,W)$ *(as vector spaces).*

(2) *Any minimal set of homogeneous generators for $P_G(V,W)$ maps to a vector space basis of $P_G(V,W)/\mathfrak{M}P_G(V,W)$.*

PROOF. (1) follows from lemma 6.6.1; (2) from lemma 6.6.4. $\square$Set $\mathbb{U} = P_G(V,W)/\mathfrak{M}P_G(V,W)$ and let $\Pi : C^\infty_G(V,W) \to \mathbb{U}$ be the projection given by lemma 6.6.5.

REMARKS 6.6.6. (1) Lemma 6.6.5 implies that the number of polynomials in a minimal set of homogeneous generators for $P_G(V,W)$ depends only on the isomorphism classes of the representations V and W. We let $k = k(V,W)$ denote the cardinality of a minimal homogeneous generating set.
(2) If $\mathcal{F}$ is a minimal set of homogeneous generators for $P_G(V,W)$, then the set of degrees (counting multiplicities) $\{d_1,\dots,d_k\}$ depends only on isomorphism class of the representations V and W.
(3) If the generators are not all homogeneous, the second part of lemma 6.6.5 may fail. As a simple example, $\{x^2+x^4, x^4\}$ generate $\mathbb{R}[x^2]$ but neither x^4 nor x^2+x^4 generate $\mathbb{R}[x^2]$. Note that x^2+x^4 (but not x^4) defines a basis of $\mathbb{R}[x^2]/\mathfrak{M}\mathbb{R}[x^2]$.

Although most of the time we work exclusively with homogeneous generators, we need to consider sets of inhomogeneous generators when we come to prove openness of G-transversality. The next result provides a version of lemma 6.6.5 that applies to sets of polynomials that define a spanning set of $P_G(V,W)/\mathfrak{M}P_G(V,W)$.

LEMMA 6.6.7. *Let $\mathcal{G} = \{G_1,\dots,G_r\}$ be a finite subset of $P_G(V,W)$ and suppose that $\Pi(\mathcal{G}) = \{\Pi(G_1),\dots,\Pi(G_r)\}$ spans $\mathbb{U}$. If the subset $\mathcal{G}'$ of $\mathcal{G}$ defines a basis of $\mathbb{U}$, then there are exactly $k = k(V,W)$ elements in $\mathcal{G}'$. If we label $\mathcal{G}$ so that $\mathcal{G}' = \{G_1,\dots,G_k\}$, then we may write*

$$\begin{aligned} G_i(x) &= \sum_{j=1}^k \alpha_{ij}(x) G_j(x), \quad i > k, \\ F_i(x) &= \sum_{j=1}^k \beta_{ij}(x) G_j(x), \end{aligned}$$

where the coefficients α_{ij}, β_{ij} are rational functions which are real analytic on some open G-invariant neighbourhood U of $0 \in V$.

PROOF. Since $\mathcal{F}$ generates $P_G(V,W)$, there exist $\gamma_{ij} \in P(V)^G$ such that

$$G_i(x) = \sum_{j=1}^{k} \gamma_{ij}(x) F_j(x),\ 1 \le i \le r.$$

Since $\Pi(\mathcal{G}') = \{\Pi(G_1), \ldots, \Pi(G_k)\}$ is a basis of $\mathbb{U}$, the $k \times k$-matrix $[\gamma_{ij}(0)]$ is non-singular. Hence, by Cramer's rule,

$$F_i(x) = \sum_{j=1}^{k} \beta_{ij}(x) G_j(x),\ \ 1 \le i \le k,$$

where the β_{ij} are rational functions which are real analytic on some open G-invariant neighbourhood U of $0 \in V$. If $j > k$, we have $G_j = \sum_{j=1}^{k} \gamma_{ij} F_j$, and so, on substituting for F_j, we have $G_i = \sum_{j=1}^{k} \alpha_{ij} G_j$, $i > k$, where the α_{ij} are rational functions which are real analytic on U. □

COROLLARY 6.6.8. *Let $\mathcal{G} \subset P_G(V,W)$. If there exists a G-invariant open neighbourhood U of $0 \in V$ such that $\mathcal{G}$ generates the $C^\infty(U)^G$-module $C^\infty_G(U,V)$, then $\Pi(\mathcal{G})$ spans $\mathbb{U}$. Conversely, if $\Pi(\mathcal{G})$ spans $\mathbb{U}$ and we choose a subset $\mathcal{G}' \subset \mathcal{G}$ such that $\Pi(\mathcal{G}')$ is a basis for $\mathbb{U}$, then there exists a base $\mathcal{U}$ of G-invariant open neighbourhoods of $0 \in V$ such that for all $U \in \mathcal{U}$, $\mathcal{G}'$ generates the $C^\infty(U)^G$-module $C^\infty_G(U,W)$.*

PROOF. Use lemma 6.6.7. □

It follows from lemma 6.6.5 that a choice of minimal homogeneous generating set $\mathcal{F}$ determines a vector space isomorphism $I_{\mathcal{F}} : \mathbb{U} \to \mathbb{R}^k$. Let $\gamma^{\mathcal{F}} = I_{\mathcal{F}}\Pi : C^\infty_G(V,W) \to \mathbb{R}^k$ be the projection map given by lemma 6.6.5. Providing the set $\mathcal{F}$ is clear from the context, we often drop the subscript from $\gamma^{\mathcal{F}}$ and just write γ. If $\mathcal{F}'$ is another minimal set of homogeneous generators for $P_G(V,W)$, then a change from $\mathcal{F}$ to $\mathcal{F}'$ induces a linear isomorphism $A_{\mathcal{F},\mathcal{F}'} : \mathbb{R}^k \to \mathbb{R}^k$ such that

$$\gamma^{\mathcal{F}}(f) = A_{\mathcal{F},\mathcal{F}'}(\gamma^{\mathcal{F}'}(f)),\ \ (f \in C^\infty_G(V,W)).$$

Let $d = d_k$ be the maximal degree of the polynomials in a minimal set of homogeneous of generators. Given $f \in C^\infty_G(V,W)$, let $J^d(f)$ denote the d-jet (Taylor polynomial degree d) of f at the origin. If $J^d(f) = 0$ then $\gamma(f) = 0$. Hence γ factorizes as

$$C^\infty_G(V,W) \xrightarrow{J^d} P^{(d)}_G(V,W) \xrightarrow{\bar{\gamma}} \mathbb{R}^k,$$

where $\bar{\gamma} = \gamma | P^{(d)}_G(V,W)$. Clearly γ is continuous if we give $C^\infty_G(V,W)$ the C^r-topology, $\infty \ge r \ge d$ (Whitney or uniform convergence on compact sets).

LEMMA 6.6.9. *Suppose V, W_1, W_2 are G-representations and that $\mathcal{F}^1$, $\mathcal{F}^2$ are minimal sets of homogeneous generators for $P_G(V, W_1)$ and $P_G(V, W_2)$ respectively. Then $\mathcal{F} = \mathcal{F}^1 \cup \mathcal{F}^2$ is a minimal set of homogeneous generators for $P_G(V, W_1 \oplus W_2)$. (Here $F^1 \in \mathcal{F}^1$ is identified with $(F^1, 0) \in \mathcal{F}$, and $F^2 \in \mathcal{F}^2$ with $(0, F^2) \in \mathcal{F}$.)*

PROOF. The result follows easily using lemma 6.6.4. □

LEMMA 6.6.10. *Suppose that V, W are G-representations and that $\mathbb{R}^s$ is a trivial G-representation. Every minimal set of homogeneous generators $\mathcal{F}$ for $P_G(V, W)$ defines a minimal set of homogeneous generators for $P_G(V \times \mathbb{R}^s, W)$. (Each $F \in \mathcal{F}$ determines a map $F : V \times \mathbb{R}^s \to W$ by $F(x, t) = F(x)$.)*

PROOF. We leave this as an easy exercise for the reader. □

Suppose that V, W are G-representations, $\mathbb{R}^s$ is a trivial G-representation and $\mathcal{F}$ is a minimal set of homogeneous generators for $P_G(V, W)$. It follows from lemmas 6.6.5, 6.6.10 that we have a linear map $\Pi^s : C^\infty_G(V \times \mathbb{R}^s, W) \to C^\infty(\mathbb{R}^s, \mathbb{U})$ defined by $\Pi^s(f)(t) = \Pi(f_t) = I_{\mathcal{F}}^{-1}\gamma^{\mathcal{F}}(f_t)$, for all $t \in \mathbb{R}^s$. In terms of $\mathcal{F}$, suppose $f \in C^\infty_G(V \times \mathbb{R}^s, W)$. We may write

$$f(x, t) = \sum_{j=1}^{k} f_j(x, t) F_j(x),$$

where $f_j \in C^\infty(V \times \mathbb{R}^s)^G$, $1 \leq j \leq k$. Define $\gamma^{\mathcal{F}} = I_{\mathcal{F}}\Pi^s : C^\infty_G(V \times \mathbb{R}^s, W) \to C^\infty(\mathbb{R}^s, \mathbb{R}^k)$ by

$$\gamma^{\mathcal{F}}(f)(t) = (f_1(0, t), \dots, f_k(0, t)), \quad (t \in \mathbb{R}^s, f \in C^\infty_G(V \times \mathbb{R}^s, W)).$$

LEMMA 6.6.11. *The map $\gamma^{\mathcal{F}} : C^\infty_G(V \times \mathbb{R}^s, W) \to C^\infty(\mathbb{R}^s, \mathbb{R}^k)$ is continuous with respect to the C^∞-topologies on $C^\infty(\mathbb{R}^s, \mathbb{R}^k)$ and $C^\infty_G(V \times \mathbb{R}^s, W)$.*

PROOF. ([**14**]) Let $\alpha : (C^\infty(V \times \mathbb{R}^s)^G)^k \to C^\infty_G(V \times \mathbb{R}^s, W)$ and $\beta : (C^\infty(V \times \mathbb{R}^s)^G)^k \to C^\infty(\mathbb{R}^s, \mathbb{R}^k)$ be defined by

$$\alpha(f_1, \dots, f_k) = \sum_{j=1}^{k} f_j F_j, \text{ and } \beta(f_1, \dots, f_k)(t) = (f_1(0, t), \dots, f_k(0, t)),\ t \in \mathbb{R}^s.$$

Both α and β are continuous (with respect to the C^∞-topologies on function spaces). Since α is a continuous linear surjective map between Fréchet spaces, it follows by the Open Mapping Theorem that α is an open map. Since $\gamma^{\mathcal{F}}\alpha = \beta$ it follows that for all open subsets V in $C^\infty(\mathbb{R}^s, \mathbb{R}^k)$, $\alpha(\beta^{-1}(V)) = (\gamma^{\mathcal{F}})^{-1}(V)$ is open and so $\gamma^{\mathcal{F}}$ is continuous. □

EXERCISE 6.6.12. Using the continuity with respect to the C^∞-topology show that $\gamma^{\mathcal{F}} : C^\infty_G(V \times \mathbb{R}^s, W) \to C^\infty(\mathbb{R}^s, \mathbb{R}^k)$ is continuous with respect to the Whitney C^∞-topology.

6.6.3. Smooth invariant theory. Let $p_1, \dots, p_\ell$ be a minimal set of homogeneous generators for the $\mathbb{R}$-algebra $P(V)^G$. We recall Schwarz's theorem on smooth invariants.

THEOREM 6.6.13 ([**154**]). *(Notation as above.) Let $f \in C^\infty(V)^G$. Then there exists a smooth function $F : \mathbb{R}^\ell \to \mathbb{R}$ such that*

$$f(x) = F(p_1(x), \dots, p_\ell(x)), \quad (x \in V).$$

The same result holds if we replace V by a G-invariant open subset of V.

REMARK 6.6.14. Let $P = (p_1, \dots, p_\ell) : V \to \mathbb{R}^\ell$. Since we can always take $p_1 = \| \ \|^2$, P is a *proper* mapping (inverse images of compact sets are compact). The map P defines the pullback map $P^\star : C^\infty(\mathbb{R}^\ell) \to C^\infty(V)^G$ by $P^\star(F)(x) = F(p_1(x), \dots, p_\ell(x))$. Working with the C^∞-topology, it is easy to show that $P^\star(C^\infty(\mathbb{R}^\ell))$ is dense in $C^\infty(V)^G$ (because $P^\star(P(\mathbb{R}^\ell)) = P(V)^G$). The difficulty is in showing that $P^\star(C^\infty(\mathbb{R}^\ell))$ is a closed subspace of $C^\infty(V)^G$. Schwarz's proof of theorem 6.6.13 used the fact that $p_1, \dots, p_\ell$ was a set of generators for the invariants. Subsequently, far more general results have become available (see for example [**17, 16, 170**]). In particular, if $Q : \mathbb{R}^p \to \mathbb{R}^q$ is any proper polynomial map then (a) $Q^\star(C^\infty(\mathbb{R}^q))$ is a closed subspace of $C^\infty(\mathbb{R}^p)$, and (b) the sequence $C^\infty(\mathbb{R}^q) \xrightarrow{Q^\star} Q^\star(C^\infty(\mathbb{R}^q)) \to 0$ splits. That is, there exists a continuous linear section $\sigma : Q^\star(C^\infty(\mathbb{R}^q)) \to C^\infty(\mathbb{R}^q)$ of the map $Q^\star$. The special case of the splitting result for smooth invariants was first obtained by Mather [**120**]. Subsequent work of Vogt and Wagner (see [**175**]) leads to simple proofs of the splitting theorem based on general results from functional analysis. We may also use methods based on the splitting theorem to show that there exists a continuous linear section $\rho : C^\infty_G(V, W) \to (C^\infty(V)^G)^k$ (see [**14**]). However, we will not need these splitting results in our development of equivariant transversality. Note, however, that they can be used to give a simple alternative proof of lemma 6.6.11.

Proof of Lemma 6.6.1 using smooth invariant theory Let V, W be G-representations. We show how to prove lemma 6.6.1 using Schwarz's theorem 6.6.13 on smooth invariants. The method we describe is based on using a specific set of generators for $P_G(V, W)$ constructed by Malgrange.

Let V, W be G-representations. The action of G on W induces an action on the dual space $W^\star$ defined by $(g\phi)(v) = \phi(g^{-1}v)$, $g \in G$, $v \in V$, $\phi \in W^\star$. Let $p_1, \dots, p_k$ be a set of generators for the $\mathbb{R}$-algebra $P(V \times W^\star)^G$. Each p_j determines a polynomial $P_j \in P_G(V, W)$ by $\langle P_j(v), \phi \rangle = p_j(v, \phi)$, $v \in V, \phi \in W^\star$. Conversely, if $P \in P_G(V, W)$, we may define $p \in P(V \times W^\star)^G$ by $p(v, \phi) = \langle P(v), \phi \rangle$. Noting that $P(V \times W^\star)^G$ has the structure of a $P(V)^G$-module, it may be verified that this correspondence defines a $P_G(V)$-module isomorphism between $P_G(V, W)$ and $P(V \times W^\star)^G$. Applying Schwarz's theorem, we deduce that $\{P_1, \dots, P_k\}$ is a set of generators for the $C^\infty(V)^G$-module $C^\infty_G(V, W)$. ☐

6.6.4. Smooth invariants and the orbit space. Let $\{p_1, \ldots, p_\ell\}$ be a minimal set of homogeneous generators for the $\mathbb{R}$-algebra $P(V)^G$. Let $P = (p_1, \ldots, p_\ell) : V \to \mathbb{R}^\ell$ and $p : V \to V/G$ denote the orbit map. Since P is a proper map, $P(\mathbb{R}^\ell)$ is a closed subset of $\mathbb{R}^\ell$. Since P separates G-orbits (examples 2.10.13(2)), it follows that there is a natural homeomorphism $\eta : V/G \to P(V)$ such that $\eta \circ p = P$. We may define a smooth structure on V/G by requiring $f : V/G \to \mathbb{R}$ to be smooth if and only if $f \circ p : V \to \mathbb{R}$ is smooth (in the usual sense). We refer to Bredon [**26**, chapter VI] for generalities on smooth structures on orbit spaces. If we restrict to an orbit stratum V_τ, $\tau \in \mathcal{O}(V, G)$, then V_τ/G is a smooth manifold (proposition 3.7.2) and $p : V_\tau \to V_\tau/G$ will be a smooth (in the usual sense). Every smooth function on V/G obviously restricts to a smooth function on V_τ/G. A map $g : P(V) \subset \mathbb{R}^\ell \to \mathbb{R}$ is smooth if it is the restriction of a smooth function defined on $\mathbb{R}^\ell$ (it follows from Whitney's extension theorem that the smoothness of g is a property that can be defined intrinsically on $P(V)$). Observe that if $g : P(V) \to \mathbb{R}$ is smooth, then $g \circ p : V \to \mathbb{R}$ is a smooth invariant and so g induces (via η) a smooth function on V/G. Conversely, if $f : V/G \to \mathbb{R}$ is smooth is follows from Schwarz's theorem that $f \circ \eta^{-1} : P(V) \to \mathbb{R}$ is a smooth function on $P(V)$. Consequently, the smoothness structures on V/G and $P(V)$ coincide. Hence we may and shall identify V/G with $P(V) \subset \mathbb{R}^\ell$.

With this identification, each set $P(V_\tau) \approx V_\tau/G$ is a smooth submanifold of $\mathbb{R}^\ell$. Bierstone [**13**] proved that $\{P(V_\tau) \mid \tau \in \mathcal{O}(V, G)\}$ is a Whitney regular stratification of $P(V)$ (in fact the *canonical* or *minimal* stratification of $P(V)$). Bierstone's important result is the starting point for orbit space computations in equivariant dynamics (for example, see [**104**]). We shall not use Bierstone's result in this work as we avoid orbit space computations and prefer to work at the level of the G-space. We discuss these matters more in the next chapter.

6.7. The universal variety

We continue to assume that $\mathcal{F}$ is a minimal set of homogeneous generators for the $P(V)^G$-module $P_G(V, W)$. Define the polynomial map $\boldsymbol{\vartheta} \in P_G(V \times \mathbb{R}^k, W)$ by

$$\boldsymbol{\vartheta}(x, t) = \sum_{j=1}^{k} t_j F_j(x), \quad ((x, t) \in V \times \mathbb{R}^k).$$

Let $\Sigma^{\mathcal{F}} = \Sigma = \boldsymbol{\vartheta}^{-1}(0) \subset V \times \mathbb{R}^k$. The set Σ is a G-invariant algebraic subset of $V \times \mathbb{R}^k$. We sometimes refer to Σ as the *universal variety*, and $\boldsymbol{\vartheta}$ as the *universal polynomial* (for the pair (V, W)). For $s \geq 1$, we regard V and $\mathbb{R}^s$ as embedded in $V \times \mathbb{R}^s$ as $V \times \{0\}$ and $\{0\} \times \mathbb{R}^s$ respectively. For future reference note that

$$\Sigma \supset V \quad \subset \quad V \times \mathbb{R}^k, \tag{6.1}$$

$$\Sigma \supset \mathbb{R}^k \quad \subset \quad V \times \mathbb{R}^k, \quad \text{if } W^G = \{0\}. \tag{6.2}$$

EXAMPLE 6.7.1. Let $\mathbb{Z}_2$ act on $V = \mathbb{R}^2$ as $(x, y) \mapsto (-x, -y)$ and on $W = \mathbb{R}^2$ as $(x, y) \mapsto (-x, y)$. As minimal set of homogeneous generators we may take

$$F_1(x, y) = (x, 0);\ F_2(x, y) = (y, 0);\ F_3(x, y) = (0, 1).$$

We have $\boldsymbol{\vartheta}(x, y; t_1, t_2, t_3) = (t_1 x + t_2 y, t_3)$. In order to describe $\boldsymbol{\vartheta}^{-1}(0)$, observe that since $\boldsymbol{\vartheta}(x, y; t_1, t_2, t_3) = 0$ only if $t_3 = 0$, it suffices to describe the zero set $\overline{\Sigma}$ of the map $\bar{\boldsymbol{\vartheta}} : \mathbb{R}^4 \to \mathbb{R}$ defined by $\bar{\boldsymbol{\vartheta}}(x, y; t_1, t_2) = t_1 x + t_2 y$. Let $\pi : \mathbb{R}^4 \to \mathbb{R}^2$ denote the projection $\pi(x, y; t_1, t_2) = (t_1, t_2)$. For all $\mathbf{t} = (t_1, t_2) \neq (0, 0)$, $\pi^{-1}(\mathbf{t})$ is a line $L_{\mathbf{t}} \subset \mathbb{R}^4$ through $\mathbf{t}$. It is straightforward to show that $\overline{\Sigma} \setminus \pi^{-1}(\mathbf{0})$ is a ruled three-manifold. On the other hand $\pi^{-1}(\mathbf{0})$ is a singular fibre for $\pi : \overline{\Sigma} \to \mathbb{R}^2$. Computing we find that $(0, 0, 0, 0)$ is the unique singular point of $\overline{\Sigma}$.

The importance of Σ and $\boldsymbol{\vartheta}$ lies in the fact that every $f \in C^\infty_G(V, W)$ factorizes through $\boldsymbol{\vartheta}$. Specifically, if $f \in C^\infty_G(V, W)$, then we may write $f(x) = \sum_{j=1}^k f_j(x) F_j(x)$, where $f_j \in C^\infty(V)^G$. Define $\Gamma^{\mathcal{F}}_f = \Gamma_f : V \to V \times \mathbb{R}^k$ by $\Gamma_f(x) = (x, f_1(x), \ldots, f_k(x))$. Then

$$\begin{aligned} f &= \boldsymbol{\vartheta} \circ \Gamma_f, \\ f^{-1}(0) &= \Gamma_f^{-1}(\Sigma). \end{aligned}$$

Subsequently, we will define f to be G-transverse to $0 \in W$ at $0 \in V$, if $\Gamma^{\mathcal{F}}_f$ is transverse to Σ at $x = 0$. However, since Σ will typically have singularities (see example 6.7.1), we first need give Σ a Whitney stratification so that we can apply the Thom-Mather theory of maps tranverse to stratified sets. We also need to show that the proposed definition is independent of choices, in particular of $\mathcal{F}$ and the coefficient functions $f_j(x)$.

EXAMPLE 6.7.2. We continue with the notation and assumptions of example 6.7.1. If $f \in C^\infty_{\mathbb{Z}_2}(\mathbb{R}^2, \mathbb{R}^2)$, then $f(x, y) = (f_1(x, y)x + f_2(x, y)y, f_3(x, y))$, where $f_1, f_2, f_3 \in C^\infty(\mathbb{R}^2)^{\mathbb{Z}_2}$. We can always require $\Gamma_f(0, 0) \notin \Sigma$ simply by requiring $f_3(0, 0) \neq 0$. It is also easy to perturb f so that $\Gamma_f(x, y) \neq \mathbf{0} = (0, 0, 0, 0, 0)$ for all $(x, y) \in \mathbb{R}^2$. We may then further perturb f so that $\Gamma_f(x, y) \neq \mathbf{0}$ and Γ_f is transverse to $\Sigma \setminus \{\mathbf{0}\}$, all $(x, y) \in \mathbb{R}$. The zeros of f will then be isolated. Note that $f_3^{-1}(0)$ will typically be a curve in $\mathbb{R}^2$. Now add parameters to the problem and consider families $f \in C^\infty_{\mathbb{Z}_2}(\mathbb{R}^2 \times \mathbb{R}^s, \mathbb{R}^2)$, $s \geq 1$. If $s \leq 2$, we can always perturb f so that $\Gamma_f \pitchfork (\Sigma \setminus \{\mathbf{0}\})$ and $\Gamma_f(x, y, t) \neq \mathbf{0}$, all $((x, y), t) \in \mathbb{R}^2 \times \mathbb{R}^s$. The zero set of f will then typically be a curve ($s = 1$) or surface ($s = 2$) without singularities. However, if $s = 3$, we can expect that there will be isolated points (x, y, t) for which $\Gamma_f(x, y, t) = \mathbf{0}$. In this case, as long as we require that $\Gamma_f \pitchfork \mathbf{0}$ at (x, y, t), the local zero set of f near (x, y, t) will be diffeomorphic to a neighbourhood of $\mathbf{0}$ in Σ. Indeed, the transversality of Γ_f to $\mathbf{0}$ at (x, y, t) implies that $\Gamma_f(x, y, t)$ is a diffeomorphism of an open neighbourhood A of (x, y, t) onto an open neighbourhood U of $\mathbf{0}$ in $\mathbb{R}^2 \times \mathbb{R}^3$. Then $f^{-1}(0) \cap A = (\Gamma_f | A)^{-1}(\Sigma \cap U)$.

If $\mathcal{F}$ is a minimal set of homogeneous generators of $P_G(V,W)$, then $\mathcal{F}$ is a minimal set of generators for the $P(V\times\mathbb{R}^s)^G$-module $P_G(V\times\mathbb{R}^s,W)$. Consequently, $(V\times\mathbb{R}^s,W)$ and (V,W) have the same universal varieties. If $f\in C^\infty_G(V\times\mathbb{R}^s,W)$, we may write $f(x,t)=\sum_{j=1}^k f_j(x,t)F_j(x)$, $f_j\in C^\infty(V\times\mathbb{R}^k)^G$, and define $\Gamma_f: V\times\mathbb{R}^s\to V\times\mathbb{R}^k$ by

$$\Gamma_f(x,t)=(x,f_1(x,t),\dots,f_k(x,t)).$$

We have $f=F\circ\Gamma_f$ and $f^{-1}(0)=\Gamma_f^{-1}(\Sigma)$.

It follows from this discussion that it is no loss of generality to assume that V contains no trivial subrepresentations (equivalently, $V^G=\{0\}$). With this assumption,

$$\Sigma_{(G)}=\Sigma^G\subset\mathbb{R}^k\subset V\times\mathbb{R}^k.$$

(If V and W contain no trivial subrepresentations then $\Sigma_{(G)}=\mathbb{R}^k\subset V\times\mathbb{R}^k$.)

Remark 6.7.3. Of course, the universal variety Σ depends on the choice of generating set $\mathcal{F}$. We address this issue shortly. Roughly speaking the germ of Σ along $\{0\}\times\mathbb{R}^k$ is independent of choice of generating set. More precisely, we shall show that there is a natural stratification $\mathcal{A}$ of the space $\mathbb{U}$ (*independent* of choice of generating set) that allows us to express the G-transversality of $f\in C^\infty_G(V\times\mathbb{R}^s,W)$ to $0\in W$ along a subset K of $\mathbb{R}^s$ in terms of the transversality of the map $\Pi^s(f):\mathbb{R}^s\to\mathbb{U}$ to $\mathcal{A}$ along K. This formulation of equivariant transversality turns out to be very useful in applications to equivariant bifurcation theory; it is also independent of all choices.

6.7.1. Changing generators. We start by looking at the easiest case when both generator sets are minimal and homogeneous.

Lemma 6.7.4. *Suppose that $\mathcal{F}=\{F_1,\dots,F_k\}$, $\mathcal{G}=\{G_1,\dots,G_k\}$ are minimal sets of homogeneous generators for $P_G(V,W)$ (we allow $\mathcal{F}=\mathcal{G}$). We may find homogeneous $p_{ij}\in P(V)^G$ such that*

$$F_i(x)=\sum_{i=1}^k p_{ij}(x)G_j(x),\ i=1,\dots,k,$$

and if we define $P=P_{\mathcal{G},\mathcal{F}}:V\times\mathbb{R}^k\to V\times\mathbb{R}^k$ by

$$P_{\mathcal{G},\mathcal{F}}(x,t)=(x,(\sum_{i=1}^k t_ip_{i1}(x),\dots,\sum_{i=1}^k t_ip_{ik}(x))).$$

then

(1) *$P_{\mathcal{G},\mathcal{F}}$ is a G-equivariant polynomial automorphism of $V\times\mathbb{R}^k$.*
(2) *$P_{\mathcal{G},\mathcal{F}}(\Sigma^{\mathcal{F}})=\Sigma^{\mathcal{G}}$.*
(3) *If we write $f\in C^\infty_G(V,W)$ as $f(x)=\sum_{i=1}^k f_i(x)F_i(x)$, then $f(x)=\sum_{j=1}^k(\sum_{i=1}^k f_i(x)p_{ij}(x))G_j(x)$ and $\Gamma^{\mathcal{G}}_f=P_{\mathcal{G},\mathcal{F}}\Gamma^{\mathcal{F}}_f$.*

PROOF. Since $\deg(F_i) = \deg(G_i)$, $1 \leq i \leq k$, we may choose $p_{ij} \in P^{d_i - d_j}(V)^G$ so that $F_i = \sum_{i=1}^k p_{ij} G_j$. If $d_j > d_i$, $p_{ij} = 0$, and if $d_j = d_i$, p_{ij} is constant. We may write the matrix $\mathbf{P}(x) = [p_{ij}(x)]$ in block lower-triangular form and the diagonal blocks will be constant non-singular matrices. Hence $\mathbf{P}(x)$ will is invertible for all $x \in V$ and $\mathbf{P}(x)^{-1}$ has polynomial entries. Let $\mathbf{P}(x)^t$ denote the transpose of $\mathbf{P}(x)$. We define $P_{\mathcal{G},\mathcal{F}}(x, \mathbf{t}) = (x, \mathbf{P}(x)^t(\mathbf{t}))$, $(x, \mathbf{t}) \in V \times \mathbb{R}^k$. Since $\mathbf{P}(x)$ is invertible (in the class of polynomial matrices), $P_{\mathcal{G},\mathcal{F}}$ is a polynomial automorphism of $V \times \mathbb{R}^k$, proving (1). Statements (2) and (3) are trivial computations. $\square$

When we come to verify openness of G-transversality we need to allow for sets of generators which are neither minimal nor homogeneous.

LEMMA 6.7.5. *Suppose that $\mathcal{F}$ be a minimal set of homogeneous generators for $P_G(V, W)$. Let $\mathcal{G} = \{G_1, \ldots, G_r\} \subset P_G(V, W)$ and $\{\Pi(G_1), \ldots \Pi(G_k)\}$ be a basis for $\mathbb{U}$. There exists an open G-invariant neighbourhood $U_1 = U \times \mathbb{R}^k$ of $\{0\} \times \mathbb{R}^k \subset V \times \mathbb{R}^k$ and an analytic G-equivariant diffeomorphism $R = R_{\mathcal{G},\mathcal{F}}$ of $U_1 \times \mathbb{R}^{r-k}$ onto an open neighbourhood U_2 of $\mathbb{R}^r$ in $V \times \mathbb{R}^r$ such that*

(1) *$R_{\mathcal{G},\mathcal{F}}(U_1 \cap \Sigma^{\mathcal{F}}) = U_2 \cap \Sigma^{\mathcal{G}}$ (we regard $\Sigma^{\mathcal{F}}$ as embedded in $V \times \mathbb{R}^k \times \mathbb{R}^{r-k}$ as $\Sigma^{\mathcal{F}} \times \mathbb{R}^{r-k}$).*
(2) *Given a representation of $f \in C^\infty_G(V, W)$ as $f = \sum_{i=1}^k f_i F_i$, we may choose a representation $f = \sum_{i=1}^r g_i G_i$ on U so that $\Gamma^{\mathcal{G}}_f = R_{\mathcal{G},\mathcal{F}} \Gamma^{\mathcal{F}}_f$, where $\Gamma^{\mathcal{F}}_f(x) = (x, (f_1(x), \ldots, f_k(x)), 0, \ldots, 0)$. Conversely, if we are given a representation $f = \sum_{i=1}^r g_i G_i$, we can choose a representation $f = \sum_{i=1}^k f_i F_i$ so that $\Gamma^{\mathcal{F}}_f = R_{\mathcal{F},\mathcal{G}} \Gamma^{\mathcal{G}}_f$.*

PROOF. There exist $q_{ij} \in P(V)^G$ such that $G_i = \sum_{j=1}^k q_{ij} F_j$, $i = 1, \ldots, r$. By lemma 6.6.7 there is an open G-invariant neighbourhood U of $0 \in V$ such that the $k \times k$ matrix $[q_{ij}]_{1 \leq i,j \leq k}$ is invertible on U. Denote the inverse matrix of $[q_{ij}(x)]$ by $[p_{ij}(x)]$ where the components p_{ij} will be real analytic rational functions on U. We have

$$\begin{aligned} F_i &= \sum_{j=1}^k p_{ij} G_j, \ 1 \leq i \leq k, \\ G_i &= \sum_{j=1}^k \alpha_{ij} G_j, \ k+1 \leq i \leq r, \end{aligned}$$

where $\alpha_{ij} = \sum_{\ell=1}^k q_{i\ell} p_{\ell j}$, $k+1 \leq i \leq r$, $1 \leq j \leq k$.

Take coordinates $t = (t_1, \ldots, t_k, t_{k+1}, \ldots t_r)$ on $\mathbb{R}^k \times \mathbb{R}^{r-k}$. Set $U_1 = U \times \mathbb{R}^r$ and define the G-equivariant map $R_{\mathcal{G},\mathcal{F}} : U_1 \to V \times \mathbb{R}^r$ by

$$R_{\mathcal{G},\mathcal{F}}(x, t) = (x, (\sum_{i=1}^k t_i p_{ij}(x) - \sum_{i=k+1}^r t_i \alpha_{ij}(x)), t_{k+1}, \ldots, t_r).$$

Since $[p_{ij}(x)]$ is invertible on U, it follows that $R_{\mathcal{G},\mathcal{F}}$ is 1:1 and $DR_{\mathcal{G},\mathcal{F}}(x,t)$ is non-singular for all $(x,t) \in U_1$. Hence $R_{\mathcal{G},\mathcal{F}}$ is an equivariant diffeomorphism onto a G-invariant open neighbourhood U_2 of $\mathbb{R}^r$ in $V \times \mathbb{R}^r$. It follows from the construction that $R_{\mathcal{G},\mathcal{F}}(U_1 \cap \Sigma^{\mathcal{F}}) = U_2 \cap \Sigma^{\mathcal{G}}$.

If $f = \sum_j f_j F_j$, define $g_j = \sum_{i=1}^k f_i p_{ij}$, if $1 \leq j \leq k$, and set $g_j = 0$, if $j > k$. We have $\sum_j g_j G_j = f$ and so $\Gamma_f^{\mathcal{G}} = R_{\mathcal{G},\mathcal{F}} \Gamma_f^{\mathcal{F}}$. The proof of the converse is similar. ☐

Remark 6.7.6. In the case where both sets of generators are minimal and homogeneous, lemma 6.7.5 provides a version of lemma 6.7.4 that applies when we do not 'optimize' the choice of the coefficients p_{ij}.

6.8. Stratifications and semialgebraic sets

In chapter 3, section 3.9, we introduced some of the basic definitions on stratified sets and, in particular, the Whitney regularity conditions and the Thom-Mather transversality theorem. In this section we develop these ideas with particular reference to the class of semialgebraic sets.

6.8.1. Semialgebraic sets. We start by recalling the definition and some of the basic properties of semialgebraic sets. We refer the reader to Costi [**36**], Gibson et al. [**77**], and Risler [**12**] for proofs and the general theory of semialgebraic sets and to Mather [**119**] for the relevant stratification theory. We remark that there is a corresponding theory for analytic and sub-analytic sets.

Definition 6.8.1. A semialgebraic subset X of $\mathbb{R}^n$ is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid p_i(x) = 0,\ q_j(x) > 0\},$$

where $p_i, q_j : \mathbb{R}^n \to \mathbb{R}$ are a finite set of polynomials.

Exercise 6.8.2. Show that the collection of semialgebraic subsets of $\mathbb{R}^n$ is closed under finite union, intersection and complementation.

(P1) *The closure, interior and frontier of a semialgebraic set $X \subset \mathbb{R}^n$ are semialgebraic. The frontier ∂X of X is of dimension strictly less than that of X.*
(P2) *A semialgebraic subset has finitely many connected components.*
(P3) (Tarski-Seidenberg theorem) *If $p : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial map and $X \subset \mathbb{R}^m$ is semialgebraic, then $p(X)$ is a semialgebraic subset of $\mathbb{R}^n$.*

Lemma 6.8.3. *Let (V, G) be a G-representation. Each orbit stratum V_τ, $\tau \in \mathcal{O}(V, G)$, is a semialgebraic subset of V.*

Proof. Let $p_1, \ldots, p_\ell$ be a minimal set of homogeneous generators for $P(V)^G$ and $P : V \to \mathbb{R}^\ell$ denote the associated orbit map. Let $H \in \tau \in \mathcal{O}(V, G)$ and $\{\nu_1, \ldots, \nu_p\} \subset \mathcal{O}$ be the set of all isotropy types ν such that $\nu > \tau$. For each ν_j, choose $K_j \in \nu_j$. We have

$$V_\tau = G(V_\tau^H) = G\left(V^H \setminus \cup_{j=1}^p G(V^{K_j})\right).$$

Since $G(V^{K_j}) = P^{-1}(P(V^{K_j}))$, it follows from (P3) that $G(V^{K_j})$ is a semialgebraic subset of V. Hence $V_\tau^H = V^H \setminus \cup_{j=1}^p G(V^{K_j})$ is a semialgebraic subset of V and so $V_\tau = P^{-1}(P(V_\tau^H))$ is semialgebraic. □

REMARK 6.8.4. For all $\tau \in \mathcal{O}(V, G)$, $\overline{V}_\tau$ is a real algebraic subset of V. We refer to [**60**, Lemma 9.6.1] for a proof and additional properties of orbit strata.

EXAMPLE 6.8.5. Let $p_1, \dots, p_\ell$ be a minimal set of homogeneous generators for $P(V)^G$ and set $P = (p_1, \dots, p_\ell) : V \to \mathbb{R}^\ell$. Since P is proper, it follows from (P3) that $P(V)$ is a closed semialgebraic subset of $\mathbb{R}^\ell$. The set $\{P(V_\tau) \mid \tau \in \mathcal{O}V, G)\}$ is a stratification of $P(V) \approx V/G$ by semialgebraic submanifolds. By Bierstone's theorem [**13**], this stratification is Whitney regular (see section 6.6.4). We refer the reader to Procesi & Schwarz [**145**] for explicit results on inequalities defining the orbit space $P(V)$.

6.8.2. Semialgebraic stratifications. Let $X \subset \mathbb{R}^n$ be a semialgebraic set. A stratification $\mathcal{S}$ of X is a *semialgebraic stratification* if each stratum is semialgebraic.

REMARK 6.8.6. If $\mathcal{S}$ is a semialgebraic stratification of X, then each stratum of $\mathcal{S}$ has the structure of a *real-analytic* manifold.

EXAMPLE 6.8.7. If (V, G) is a G-representation then the stratification of V by (normal) isotropy type is a Whitney semialgebraic stratification into smooth G-invariant submanifolds (for Whitney regularity see proposition 3.9.13)

EXAMPLE 6.8.8. The Whitney regular stratification of the universal variety described in example 6.7.1 is given by $\{\Sigma \setminus \{\mathbf{0}\}, \{\mathbf{0}\}\}$.

6.8.3. The canonical stratification. Let $\mathcal{S}$ be a stratification of $X \subset \mathbb{R}^n$. We define the associated filtration of X by dimension to be the filtration (X^i) of X obtained by taking X^i to be the union of all strata of dimension $\leq i$. If $\mathcal{S}$ is a Whitney stratification, we say that $\mathcal{S}$ is *canonical* if, for each i, $\mathcal{S}_i$ is the largest smooth submanifold of X^i for which Whitney regularity holds at all points of $\mathcal{S}_j$ for the pair $(\mathcal{S}_j, \mathcal{S}_i)$, $j < i$. If X has a canonical stratification, it is unique and minimal relative to the order on stratifications defined via filtration.

(P4) [**119**] *Every semialgebraic subset X of $\mathbb{R}^n$ has a canonical stratification and the corresponding strata are semialgebraic.*

EXAMPLE 6.8.9. Let $\{V_\tau \mid \tau \in \mathcal{O}(V, G)\}$ denote the Whitney regular stratification of V by isotropy type. Let $P : V \to \mathbb{R}^\ell$ be the polynomial map defined by a minimal set of homogeneous generators for $P(V)^G$ (see example 6.8.5). The semialgebraic stratification $\{P(V_\tau) \mid \tau \in \mathcal{O}(V, G)\}$ of $P(V)$ coincides with the canonical semialgebraic stratification of $P(V)$ [**13**].

6.9. Canonical stratification of the universal variety Σ

We continue to assume V, W are real G-representations and $\mathcal{F}$ is a minimal set of homogeneous generators for $P_G(V, W)$.

6.9.1. Partition of Σ by isotropy type. Since G acts trivially on $\mathbb{R}^k$, $\mathcal{O}(V \times \mathbb{R}^k, G) = \mathcal{O}(V, G)$. Let $\{\Sigma_\tau \mid \tau \in \mathcal{O}(V, G)\}$ be the partition of Σ into points of the same isotropy type.

LEMMA 6.9.1. *If $\tau, \mu \in \mathcal{O}(V, G)$ then $\overline{\Sigma}_\tau \cap \Sigma_\mu \neq \emptyset$ if and only if $\tau \leq \mu$.*

PROOF. Since $\Sigma \supset V \times \{0\}$ (6.1), $\tau \leq \mu$ implies $\overline{\Sigma}_\tau \cap \Sigma_\mu \neq \emptyset$ by the result for representations. Lemma 3.7.5 gives the converse. □

REMARK 6.9.2. In general it is *not* true that if $\overline{\Sigma}_\tau \cap \Sigma_\mu \neq \emptyset$, then $\partial \Sigma_\tau \supset \Sigma_\mu$.

Let $H \in \tau \in \mathcal{O}(V, G)$. We define

(a) $g_\tau = g_\tau(G) = \dim(G/H)$.
(b) $n_\tau = n_\tau(G) = \dim(N(H)/H)$.
(c) $d_\tau = d_\tau(V) = \dim(V_\tau^H) = \dim(V^H)$.
(d) $e_\tau = e_\tau(W) = \dim(W^H)$.
(e) $i_\tau = i_\tau(V, W) = d_\tau - e_\tau$.

REMARK 6.9.3. If $V = W$, then $d_\tau = e_\tau$. Depending on the representations V, W, e_τ may be less than d_τ, in particular zero, or greater than d_τ.

LEMMA 6.9.4 (cf [**70**, Proposition 3.6],[**31**]). *Let $\tau \in \mathcal{O}(V, G)$. Then Σ_τ is a G-invariant submanifold of $V \times \mathbb{R}^k$ and*

$$\dim(\Sigma_\tau) = k + g_\tau - n_\tau + i_\tau.$$

PROOF. The proof follows that given in [**70**] in case $V = W$. Let $H \in \tau$ and choose $x \in V_\tau^H$, $X \in W^H$. By theorem 3.5.2(5), there exists $f \in C_G^\infty(V, W)$ such that $f(x) = X$. Since $\mathcal{F}$ generates $G^\infty(V, W)$ over $C^\infty(V)^G$, $\{F_1(x), \ldots, F_k(x)\}$ span W^H. Hence the rank of the linear map $[F_1(x), \ldots, F_k(x)] : \mathbb{R}^k \to W^H$ on V_τ^H is constant equal to e_τ. Consequently, for fixed $x \in V_\tau^H$, the kernel of $\boldsymbol{\vartheta}$ has dimension $k - e_\tau$. Varying x over V_τ^H, we see that $\Sigma_\tau \cap (V_\tau^H \times \mathbb{R}^k)$ is a submanifold of dimension $k + i_\tau$. Hence $\dim(\Sigma_\tau) = \dim(G(\Sigma_\tau \cap (V_\tau^H \times \mathbb{R}^k))) = k + g_\tau - n_\tau + i_\tau$. □

REMARKS 6.9.5. (1) The manifold Σ_τ may be represented as an *algebraic subset* of $(V \times \mathbb{R}) \times \mathbb{R}^k$, see [**60**, Lemma 10.6.1].
(2) If $W = V$, then $\dim(\Sigma_\tau) = k + g_\tau - n_\tau$. If G is finite or Abelian, then all the submanifolds have the same dimension k. If G is not finite it is more interesting to look for relative equilibria – invariant G-orbits – and this what is done in [**57, 60**]. We return to this question in chapter 10.

LEMMA 6.9.6 (cf [**60**, Lemma 4.3.4]). *Let $\mu > \tau \in \mathcal{O}(V, G)$. Then*

$$\dim(\Sigma_\mu \cap \overline{\Sigma}_\tau) < \dim(\Sigma_\mu), \quad \textit{if } i_\tau - n_\tau \leq i_\mu - n_\mu.$$

PROOF. Let $P = (p_1, \ldots, p_m) : V \times \mathbb{R}^k \to \mathbb{R}^m$ where $p_1, \ldots, p_m$ is a minimal set of homogeneous generators for $P(V \times \mathbb{R}^k)^G$. By example 6.8.9 and lemma 6.9.4, $P(\Sigma_\gamma)$ is a smooth $k + i_\gamma$-dimensional semialgebraic submanifold of $\mathbb{R}^m$. Since P is proper, $\overline{P(\Sigma_\gamma)} = P(\overline{\Sigma}_\gamma)$. Hence, if if $\mu > \tau \in \mathcal{O}(V, G)$ we have

$$\begin{aligned} P(\Sigma_\mu \cap \overline{\Sigma}_\tau) &= P(\Sigma_\mu) \cap \overline{P(\Sigma_\tau)}, \\ &= P(\Sigma_\mu) \cap \partial P(\Sigma_\tau). \end{aligned}$$

By (P1), $\dim(\partial P(\Sigma_\tau)) < \dim(P(\Sigma_\tau)) = k - n_\tau + i_\tau$. Hence

$$\begin{aligned} \dim(\Sigma_\mu \cap \overline{\Sigma}_\tau) &= \dim(P^{-1}(P(\Sigma_\mu) \cap \partial P(\Sigma_\tau))), \\ &< k - n_\tau + i_\tau + g_\mu, \\ &\leq \dim(\Sigma_\mu), \end{aligned}$$

provided $i_\tau - n_\tau \leq i_\mu - n_\mu$. □

REMARK 6.9.7. If $V = W$, then we have $\dim(\Sigma_\mu \cap \overline{\Sigma}_\tau) < \dim(\Sigma_\mu)$ whenever $\mu > \tau$. This will be the situation of greatest interest in our applications to equivariant vector fields. However, if we impose additional structure, such as reversibility, then the indices i_τ may be non-zero. We refer the reader to the survey by Lamb and Roberts [**114**] for background on reversible systems and to [**31**] for an approach to the bifurcation theory of equivariant reversible systems based on ideas from equivariant transversality. Later we include one or two examples of reversible equivariant vector fields.

6.9.2. The definition of G-transversality. Let $\mathcal{S}$ denote the canonical stratification of Σ. Each stratum of $\mathcal{S}$ is a semialgebraic subset of $V \times \mathbb{R}^k$. Since the stratification is canonical and Σ is G-invariant, G permutes strata. In particular, group orbits of connected strata are G-manifolds. Our convention will be that if $S \in \mathcal{S}$ is a stratum then S is a G-manifold and S/G (rather than S) is connected. We say a smooth map is transverse to Σ if the map is transverse to each stratum of $\mathcal{S}$ (we might as well have said 'f transverse to $\mathcal{S}$').

DEFINITION 6.9.8. Let $f \in C^\infty_G(V, W)$. The map f is *G-transverse to* $0 \in W$ *at* $0 \in V$ if $\Gamma_f : V \to V \times \mathbb{R}^k$ is transverse to Σ at $0 \in V$. More generally, if $s \geq 0$ and K is a subset of $\mathbb{R}^s \subset V \times \mathbb{R}^s$, then f is G-transverse to $0 \in W$ along K if f is G-transverse to $0 \in W$ at $(0, k)$, all $k \in K$. That is, f is G-transverse to $0 \in W$ along K if $\Gamma_f : V \times \mathbb{R}^s \to V \times \mathbb{R}^k$ is transverse to Σ on K.

REMARKS 6.9.9. (1) Openness of transversality implies that if f is G-transverse to $0 \in W$ at $0 \in V$, then $\Gamma_f \pitchfork \Sigma$ on a G-invariant neighbourhood U of $0 \in V$. As we see later, if $\Gamma_f \pitchfork \Sigma$ on U then f will be G-transverse to $0 \in W$ on U – openness of G-transversality.
(2) For the definition of G-transversality along $K \subset \mathbb{R}^s$, we are thinking of subsequent applications to families. In fact, since every G-representation V is isomorphic to $\bar{V} \times \mathbb{R}^u$, where $u = \dim(V^G)$ and $\bar{V}^G = \{0\}$, we could have equivalently given the definition of G-transversality to $0 \in W$ at $0 \in V$ under the

assumption that $V^G = \{0\}$ (we can take the same generating sets for $P_G(V,W)$ and $P_G(\bar{V},W)$ by lemma 6.6.10).

LEMMA 6.9.10. *The definition of G-transversality is independent of choice of minimal set of homogeneous generators for $P_G(V,W)$.*

PROOF. Lemma 6.7.4 and the definition of the canonical stratification. $\square$

REMARKS 6.9.11. (1) Lemma 6.9.10 is to be understood in the sense that if we have transversality of Γ_f to $\Sigma^{\mathcal{F}}$ with respect to one choice of coefficient functions f_i, then we will have transversality of Γ_f to $\Sigma^{\mathcal{G}}$ for at least one set of coefficient functions g_j. The relation between f_i and g_j is given by lemma 6.7.4.
(2) While we have framed the definition of G-transversality in terms of a minimal set of homogeneous generators of $P_G(V,W)$, it follows easily from lemma 6.7.5 that if U is a G-invariant open neighbourhood of $0 \in V$, we can give a definition G-transversality for maps in $C^\infty_G(U,W)$ in terms of transversality to $\Sigma^{\mathcal{F}}$ where $\mathcal{F}$ is any set of polynomial generators for the $C^\infty(U)^G$-module $C^\infty_G(U,W)$.
(3) Although lemma 6.9.10 shows that the definition of G-transversality is independent of the choice of generators, we still need to show that the definition is independent of choice of coefficient functions f_j in our representation of f as $\sum_j f_j F_j$. While this can be done directly (see [**14**]), we proceed by proving that transversality to Σ at $0 \in V$ is determined by the values of $f_1(0), \dots, f_k(0)$. It follows from lemmas 6.6.4 and 6.6.11 that these values are uniquely determined by f, granted a choice of minimal set of homogeneous generators.

EXAMPLES 6.9.12. (1) Let $\mathbb{Z}_2$ act on $V = \mathbb{R}^2$ as multiplication by ± 1. As minimal set of homogeneous generators for $P_{\mathbb{Z}_2}(V,V)$ we take $F_1(x,y) = (x,0)$, $F_2(x,y) = (y,0)$, $F_3(x,y) = (0,x)$, $F_4(x,y) = (0,y)$. The universal variety is the subset of $V \times \mathbb{R}^4$ defined by $t_1x + t_2y = s_1x + s_2y = 0$. If we define $\Delta_3 = \{(x,t,s) \in \Sigma \mid t_1s_2 = t_2s_1, x = y = 0\}$ and $\Delta_0 = \{(0,\dots,0)\}$, then $\{\Sigma \setminus \Delta_3, \Delta_3 \setminus \Delta_0, \Delta_0\}$ is the canonical stratification of Σ. In particular $f \in C^\infty_{\mathbb{Z}_2}(V,V)$ is $\mathbb{Z}_2$-transverse to $0 \in V$ at $0 \in V$ if and only if $\gamma(f) \notin \Delta_3$. If $\gamma(f) \notin \Delta_3$ then $Df(0)$ is non-singular and $(x,y) = (0,0)$ is an isolated zero of f.
(2) Same assumptions as for (1) but now consider $f \in C^\infty_{\mathbb{Z}_2}(V \times \mathbb{R}, V)$. Then f is $\mathbb{Z}_2$-transverse to $0 \in V$ along $\mathbb{R} \subset V \times \mathbb{R}$ if $\gamma(f) : \mathbb{R} \to \mathbb{R}^4$ is transverse to Δ_3 (that is to the Whitney stratification $\{\Delta_3 \setminus \Delta_0, \Delta_0\}$ of Δ_3). In this case, if $\gamma(f)(\lambda^0) \in \Delta_3$ then $\gamma(f)(\lambda^0) \notin \Delta_0$. The germ of $f^{-1}(0,0)$ will be singular at $(0,\lambda^0)$ and locally homeomorphic to

$$\{(0,0,\lambda_0 + t) \mid t \in \mathbb{R}\} \cup \{(x,y,\lambda_0) \mid ax + by = 0\},$$

where $ax + by = 0$ is the kernel of the 2×2-matrix defined by $\gamma(f)(\lambda^0)$.

6.10. Stratifying Σ_τ and $\mathbb{U}$

For the rest of the chapter we assume that V contains no proper trivial G-representations and so $V^G = \{0\}$ (this is no loss of generality by remarks 6.9.9(2)).

Let $\tau \in \mathcal{O}(V,G)$ and $S \in \mathcal{S}$. Then $S_\tau = \Sigma_\tau \cap S$ is G-invariant semialgebraic submanifold of $V \times \mathbb{R}^k$. Since S_τ is semialgebraic, it follows by property (P2) that S_τ has finitely many connected components and so S_τ is a finite union of G-invariant strata. Hence the intersection of $\mathcal{S}$ with Σ_τ determines uniquely a stratification $\mathcal{S}_\tau$ of Σ_τ into G-invariant strata.

THEOREM 6.10.1 (cf [**60**, Theorem 4.3.7], [**70**, Theorem 5.10]). *For all $\tau \in \mathcal{O}(V,G)$, $\mathcal{S}_\tau$ is a Whitney regular stratification of Σ_τ.*

(1) *$\cup_\tau \mathcal{S}_\tau$ is a Whitney stratification of Σ.*
(2) *If $S \in \mathcal{S}_\tau$, then ∂S is a union of strata from $\cup_{\mu\geq\tau}\mathcal{S}_\mu$.*
(3) *If $i_\tau \geq i_\mu$ for all $\tau > \mu$, then $\mathcal{S}_\tau$ is a* union *of $\mathcal{S}$ strata.*

REMARKS 6.10.2. (1) Suppose $V = W$. Theorem 6.10.1 implies that each stratification $\mathcal{S}_\tau$ is a *union* of $\mathcal{S}$-strata. In particular, $\mathcal{S}$ induces a Whitney regular stratification of $\Sigma_{(G)} \subset \mathbb{R}^k$.
(2) If $W^G = \{0\}$, then $\Sigma_{(G)}$ is a stratification of $\mathbb{R}^k$. If not, we add the stratum $\mathbb{R}^k \setminus \Sigma$ to $\Sigma_{(G)}$ and thereby obtain a Whitney regular stratification of $\mathbb{R}^k$.
(3) The proof that we give for theorem 6.10.1 applies in case Σ is defined by any finite set of generators, not necessarily minimal nor homogeneous.

Before we prove theorem 6.10.1, we need a preliminary result. Let $\pi_s : V \times \mathbb{R}^s \to \mathbb{R}^s$ denote the projection on $\mathbb{R}^s$.

LEMMA 6.10.3. *Let E be a G-invariant linear subspace of $V \times \mathbb{R}^s$. Then*
$$E = (V \cap U) \oplus (\mathbb{R}^s \cap V).$$
In particular, $E \cap \mathbb{R}^s = \pi_s(E)$.

PROOF. The subrepresentation E of $V \times \mathbb{R}^s$ may be written uniquely as $A \oplus B$ where $A = B^\perp$ and $B = E^G$ is a trivial G-representation. Since V does not contain any trivial factors, $A \subset V$ and $B \subset \mathbb{R}^s$. □

Proof of Theorem 6.10.1: We start by proving that $\mathcal{S}_{(G)}$ is Whitney regular. Suppose that $S, T \in \mathcal{S}_{(G)}$ are such that $\partial S \cap T \neq \emptyset$. It suffices to prove that (T,S) satisfies Whitney (b)-regularity at points of $\partial S \cap T$. Suppose $x \in \partial S \cap T$ and $(p_n) \subset S$, $(q_n) \subset T$ are sequences converging to x such that $T_{p_n}S$ converges to a $\dim(S)$-linear subspace E and $\mathbb{R}(p_n - q_n)$ converges to the line ℓ. We have to prove $E \supset \ell$. Suppose that S is a connected component of $\hat{S} \cap \mathbb{R}^s$, T is a connected component of $\hat{T} \cap \mathbb{R}^s$, $\hat{S}, \hat{T} \in \mathcal{S}$. By choosing subsequences we may suppose that $T_{p_n}\hat{S}$ converges to a $\dim(\hat{S})$-linear subspace $\hat{E}$ of $V \times \mathbb{R}^k$. Since $(\hat{T}, \hat{S})$ satisfies Whitney (b)-regularity, $\hat{E} \supset \ell$. Applying lemma 6.10.3, $E = \mathbb{R}^s \cap \hat{E}$. Hence, since $\ell \subset \mathbb{R}^s$, $E \supset \ell$, proving (b)-regularity. The result for $\mathcal{S}_\tau$, $\tau \neq (G)$, follows similarly using slices.

It is known (see [**119**, proposition 8.7]) that a Whitney stratification satisfies the frontier condition and so (2) follows from (1). Suppose $S, T \in \cup_\tau \mathcal{S}_\tau$ are such that $\partial S \cap T \neq \emptyset$. We verify Whitney regularity in the special case $T \in \mathcal{S}_{(G)}$

and leave the general case to the reader. By the first part of the theorem, we may assume $S \in \mathcal{S}_\tau$, $\tau \neq (G)$. (Note that we cannot have $S \in \mathcal{S}_{(G)}$ and $T \in \mathcal{S}_\tau$ since $\Sigma_{(G)}$ is closed.) Let $x \in \partial S \cap T$. Choose sequences $(p_n) \subset S$, $(q_n) \subset T$ converging to x such that $T_{p_n}S$ converges to a $\dim(S)$-linear subspace E and $\mathbb{R}(p_n - q_n)$ converges to the line ℓ. Using the compactness of G, we may choose a subsequence (r_n) of $(g_n p_n)$, $g_n \in G$, such that G_{r_n} is constant, equal to $H \in \tau$, we have convergence of $T_{r_n}S$ to a linear subspace E_1 and $\mathbb{R}(r_n - q_n)$ convergent to a line ℓ_1. We have $E_1, \ell_1 \subset (V \times \mathbb{R}^k)^H$. As in the first part of the proof, we may write $S = \Sigma_\tau \cap \hat{S}$, $T = \mathbb{R}^k \cap \hat{T}$, where $(\hat{T}, \hat{S})$ satisfy (b)-regularity. By choosing a subsequence, we may assume $T_{r_n}\hat{S}$ is convergent to a linear subspace which contains ℓ_1 by (b)-regularity. But now $T_{r_n}\hat{S} \cap (V \times \mathbb{R}^k)^H = T_{t_n}S$ and so, just as before, we must have $E_1 \supset \ell_1$.

The final statement of the theorem follows by noting that if $i_\tau \geq i_\mu$ for all $\tau > \mu$, then $\overline{\Sigma}_\mu \cap \Sigma_\tau$ is a proper subset of Σ_τ with no interior points (in Σ_τ). Hence, by the definition of the canonical stratification, $\Sigma_\tau \setminus \cup_\mu \overline{\Sigma}_\mu$ must be contained in a union of $\mathcal{S}$-strata. But now the boundary of $\mathcal{S}$-strata is a union of strata in $\mathcal{S}$. Hence Σ_τ must be a union of $\mathcal{S}$-strata. □

EXAMPLE 6.10.4. We assume the hypotheses of example 6.7.1. We have $k = 3$ and $\mathcal{S}_{(\mathbb{Z}_2)} = \{\mathbb{R}^3 \setminus \{(0,0,0)\}, \{(0,0,0)\}\}$, $\mathcal{S}_{(e)} = \{(x,y,t,u,v) \in \mathbb{R}^2 \times \mathbb{R}^3 \mid tx = uy, v = 0,\ (x,y,u,v) \neq (0,0,0,0)\}$.

EXAMPLE 6.10.5. Consider the absolutely irreducible representation $(\mathbb{R}^k, H_k)$, $k \geq 2$. The stratification $\mathcal{S}_{(H_k)}$ induced on $\mathbb{R}^k$ by the canonical stratification of $\Sigma \subset \mathbb{R}^k \times \mathbb{R}^k$ has associated filtration defined by the flag $\mathbb{R}^k \supset \mathbb{R}^{k-1} \supset \ldots \supset \mathbb{R} \supset \{0\}$ where $\mathbb{R}^{k-j} = \{(t_1, \ldots, t_n) \mid t_1 = \ldots t_j = 0\}$. We refer the reader to [**70**] for the proof and results for most of the remaining finite reflection groups.

THEOREM 6.10.6. *Let $\mathcal{F}$ be a minimal set of homogeneous generators for $P_G(V,W)$ and let $\mathcal{A}_\mathcal{F}$ be the Whitney stratification of $\mathbb{R}^k \subset V \times \mathbb{R}^k$ given by theorem 6.10.1. If we let $\mathcal{A} = I_\mathcal{F}^{-1}(\mathcal{A}_\mathcal{F})$ denote the Whitney stratification of $\mathbb{U}$ induced from $\mathcal{A}_\mathcal{F}$ then*

(1) *$\mathcal{A}$ is independent of the choice of minimal set of generators $\mathcal{F}$.*
(2) *If $f \in C_G^\infty(V \times \mathbb{R}^s, W)$ and $K \subset \mathbb{R}^s$, then f is G-transverse to $0 \in W$ along K if and only if $\Pi^s(f) : \mathbb{R}^s \to \mathbb{U}$ is transverse to $\mathcal{A}$ along K.*

PROOF. Suppose that $\mathcal{G}$ is a minimal set of homogeneous generators for $P_G(V,W)$. By lemma 6.7.4(2), the linear isomorphism $A_{\mathcal{G},\mathcal{F}} : \mathbb{R}^k \to \mathbb{R}^k$ is the restriction of the polynomial automorphism $P_{\mathcal{G},\mathcal{F}}$ of $V \times \mathbb{R}^k$ to $\mathbb{R}^k$. Since $P_{\mathcal{G},\mathcal{F}}(\Sigma^\mathcal{F}) = \Sigma^\mathcal{G}$, $P_{\mathcal{G},\mathcal{F}}$ maps the canonical stratification of $\Sigma^\mathcal{F}$ to the canonical stratification of $\Sigma^\mathcal{G}$. Hence $A_{\mathcal{G},\mathcal{F}}(\mathcal{A}_\mathcal{F}) = \mathcal{A}_\mathcal{G}$, proving (1). For (2) we use theorem 6.10.1 and the observation that $\Gamma_f^\mathcal{F}$ is transverse to $S \in \mathcal{S}$ along K if and only if $\gamma_\mathcal{F}(f)$ is transverse to S^G along K. □

COROLLARY 6.10.7. *Let $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ and K be a subset of $\mathbb{R}^s$. The definition of the G-transversality of f to $0 \in W$ along K is well-defined, independent of choice of minimal set of homogeneous generators for $P_G(V,W)$ and coefficient functions for f.*

PROOF. Theorem 6.10.6 and lemmas 6.9.10, 6.6.11. □

EXAMPLE 6.10.8. Let $(V, \mathbf{D}_3)$ denote the standard absolutely irreducible representation of $\mathbf{D}_3$, where $V = \mathbb{R}^2 \approx \mathbb{C}$. Regarding $\mathbf{D}_3 \subset \mathrm{O}(2)$, we may write $\mathbf{D}_3 = \langle \rho, \kappa \rangle$ where $\rho(z) = e^{2\pi\imath/3} z$ and $\kappa(z) = \bar{z}$. Let $(W, \mathbf{D}_3)$ denote the orthogonal representation of $\mathbf{D}_3$ on $W = \mathbb{R}^2 \approx \mathbb{C}$ defined by $\rho(z) = e^{2\pi\imath/3} z$ and $\kappa(z) = -\bar{z}$. The representations $(V, \mathbf{D}_3)$ and $(W, \mathbf{D}_3)$ are isomorphic (multiplication by $\imath$ defines an intertwining operator). A minimal set of homogeneous generators for $P_{\mathbf{D}_3}(V, W)$ is given by $\{\imath z, \imath \bar{z}^2\}$. The associated stratification of $\mathbb{U} \cong \mathbb{R}^2$ is given by $t_1 = 0$ and $t_1 = t_2 = 0$. If $f \in C^\infty_{\mathbf{D}_3}(V \times \mathbb{R}, W)$, then f is $\mathbf{D}_3$-transverse to $0 \in W$ along $K \subset \mathbb{R}$ if and only if $\gamma(f) : \mathbb{R} \to \mathbb{R}^2$ is transverse to $t_1 = 0$ along K. Points of intersection of $\gamma(f)$ with $t_1 = 0$ correspond to curves of solutions of $f = 0$ along axes of symmetry for $(V, \mathbf{D}_3)$. For this example, $f \in C^\infty_{\mathbf{D}_3}(V \times \mathbb{R}, W)$ may be viewed as a family of *time-reversible* equivariant vector fields on $(V, \mathbf{D}_3)$. That is, $f : V \times \mathbb{R} \to V$ is $\mathbb{Z}_3 \subset \mathbf{D}_3$-equivariant and if $g \in \mathbf{D}_3 \setminus \mathbb{Z}_3$, then $f(gz, t) = -gf(z, t)$. It is a straightforward exercise to check that the linearization of f at any zero on an axis of symmetry always has eigenvalues on the imaginary axis.

Using lemma 6.7.5, rather than lemma 6.7.4, we may extend corollary 6.10.7 to allow for general sets of polynomial generators.

PROPOSITION 6.10.9. *Let $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ and K be a subset of $\mathbb{R}^s$. If we choose a finite set $\mathcal{G}$ of polynomial generators for $P_G(V,W)$ then $\Gamma^{\mathcal{G}}_f$ is transverse to $\Sigma^{\mathcal{G}}$ along K if and only if f is G-transverse to $0 \in W$ along K.*

We note the following special case of stratumwise transversality.

LEMMA 6.10.10. *Let $f \in C^\infty_G(V \times \mathbb{R}^s, W)$. If f is G-transverse to $0 \in W$ along $K \subset \mathbb{R}^s$ then $f|(V \times \mathbb{R}^s)^G$ is transverse to $0 \in W^G$ along K.*

PROOF. Let $W = W_1 \oplus W_2$, where $W^G = W_2$. We may write $f(x, s) = \sum_{j=1}^{k_1} f_j F_j + \sum_{\ell=1}^{k_2} c_\ell \mathbf{e}_\ell$ where $F_1, \dots, F_{k_1}$ is a minimal set of homogeneous generators for $P_G(V, W_1)$ and $\mathbf{e}_1, \dots, \mathbf{e}_{k_2}$ is a basis for W_2. If we let $\Sigma_1 \subset V \times \mathbb{R}^{k_1}$ be the zero set of $\sum t_j F_j$, then the universal variety for maps from V to W is $\Sigma_1 \times \{0\} \subset (V \times \mathbb{R}^{k_1}) \times \mathbb{R}^{k_2}$. Since $V^G = \{0\}$, we have $(V \times \mathbb{R}^s)^G = \mathbb{R}^s$. If we set $f^G = f|\mathbb{R}^s$, then $f^G(s) = \sum_{j=1}^{k_2} c_j(s) \mathbf{e}_j$. If f is G-transverse to $0 \in W$ along K, then $\gamma(f) : \mathbb{R}^s \to \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ is transverse to $\mathcal{A}$ along K. Identifying $\mathbb{R}^{k_2}$ with W^G, we see that the composition of $\gamma(f)$ with the projection onto $\mathbb{R}^{k_2}$ is equal f^G. Since the strata of $\mathcal{A}$ are all of the form $S \times \{0\} \subset \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$, if $\gamma(f)$ is transverse to $\mathcal{A}$, then f^G is transverse to $0 \in W$. □

6.11. Equivariant coordinate changes on $V \times \mathbb{R}^s$ and W

In this section we show that the definition of equivariant transversality is invariant under equivariant coordinate changes on source and target.

If M is a G-manifold (in particular, a representation), let $\mathrm{Diff}_G(M)$ denote the group of smooth G-equivariant diffeomorphisms of M. Give $\mathrm{Diff}_G(M)$ the topology induced from $C^\infty_G(M,M)$.

PROPOSITION 6.11.1. *Let $s \geq 0$, K be a subset of $\mathbb{R}^s$ and $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ be G-transverse to $0 \in W$ along K. If $\alpha \in \mathrm{Diff}_G(V \times \mathbb{R}^s)$ then $f \circ \alpha$ is G-transverse to $0 \in W$ along $\alpha^{-1}(K)$.*

PROOF. Write $\alpha(x,s) = (\delta_s(x), \kappa_s(x))$, $(x,s) \in V \times \mathbb{R}^s$. Since $D\delta_s(0) \in L_G(V,V)$ is non-singular along $\mathbb{R}^s$, we may choose a G-invariant open neighbourhoods A, B of $\mathbb{R}^s \subset V \times \mathbb{R}^s$ such that if we set $A_s = (V \times \{s\}) \cap A$, then $\delta_s : A_s \subset V \to V$ is a G-equivariant embedding onto the open neighbourhood $B_s = (V \times \{s\}) \cap B$ of $0 \in V$, for all $s \in \mathbb{R}^s$. Let $\{F_1, \ldots, F_k\}$ be a minimal set of homogeneous generators for $P_G(V,W)$. Since $F_i \circ \delta \in C^\infty_G(V \times \mathbb{R}^s, W)$, there exist $\gamma_{ij} \in C^\infty(V \times \mathbb{R}^s)^G$ such that

$$F_i(\delta_s(x)) = \sum_{j=1}^k \gamma_{ji}(x,s) F_j(x), \quad 1 \leq i \leq k.$$

For all $s \in \mathbb{R}^s$, the matrix $[\gamma_{ij}(0,s)]$ is invertible. Since $F_i(x) = F_i(\delta_s(\delta_s^{-1}(x)))$, $x \in B_s$, we have

$$F_i(x) = \sum_{j=1}^k \gamma_{ji}(\delta_s^{-1}(x), s) F_j(\delta_s^{-1}(x)), \quad (s \in \mathbb{R}^s, x \in B_s). \tag{6.3}$$

If $f(x,s) = \sum_{i=1}^k f_i(x,s) F_i(s)$, then for $s \in \mathbb{R}^s$, $x \in V$ we have

$$\begin{aligned} f \circ \alpha(x,s) &= \sum_{i=1}^k f_i(\delta_s(x), \kappa_s(x)) F_i(\delta_s(x)), \\ &= \sum_{i=1}^k f_i(\delta_s(x), \kappa_s(x)) \gamma_{ji}(x,s) F_j(x). \end{aligned}$$

For $s \in \mathbb{R}^s$, define the G-equivariant diffeomorphism $T_s : B_s \times \mathbb{R}^k \to V \times \mathbb{R}^k$ by

$$T_s(x,t) = (\delta_s^{-1}(x), (\sum_{j=1}^k \gamma_{ij}(\delta_s^{-1}(x), s)) t_j)).$$

Computing we find, using (6.3), that $\boldsymbol{\vartheta}(T_s(x,t)) = \sum_{j=1}^k t_j F_j(x)$, $(x,s) \in B$. Hence for $s \in \mathbb{R}^s$, T_s defines a diffeomorphism of an open neighbourhood U_1 of $\mathbb{R}^k \subset V \times \mathbb{R}^k$ onto an open neighbourhood U_2 of $\mathbb{R}^k \subset V \times \mathbb{R}^k$ such that $T_s(\Sigma \cap U_1) = \Sigma \cap U_2$. In particular T_s maps strata of $\Sigma \cap U_1$ diffeomorphically onto strata of $\Sigma \cap U_2$ (this uses the fact that the stratification of Σ is canonical). But now $\Gamma_{f\circ\alpha}(x,s) = T_s(\Gamma_f \circ \alpha(x,s))$ and so $\Gamma_{f\circ\alpha}$ is transverse to Σ along $\alpha^{-1}(K)$ if (and only if) Γ_f is transverse to Σ along K. □

PROPOSITION 6.11.2. *Let $\beta \in \mathrm{Diff}_G(W)$ satisfy $\beta(0) = 0$. If $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ is G-transverse to $0 \in W$ along $K \subset \mathbb{R}^s$, then $\beta \circ f$ is G-transverse to $0 \in W$ along $K \subset \mathbb{R}^s$.*

PROOF. Write $f(x,s) = \sum_{i=1}^k f_j(x,s)F_j(x)$, where $\{F_1,\dots,F_k\}$ is a minimal set of homogeneous generators for $P_G(V,W)$. We may easily show that

$$\beta(\sum_{i=1}^k t_i F_i(x)) = \sum_{i=1}^k \tau_i(x,t)F_i(x),$$

where $J(x,t) = (x,(\tau_i(x,t))$ restricts to a G-equivariant diffeomorphism on an open neighbourhood of $\mathbb{R}^k \subset V \times \mathbb{R}^k$. Since $\beta(0) = 0$, $J(\Sigma) \subset \Sigma$, and so if $\Gamma_f \pitchfork \Sigma$ along K then $J \circ \Gamma_f = \Gamma_{\beta\circ f}$ is transverse to Σ along K. Hence $\beta \circ f$ is G-transverse to $0 \in W$ along K. □

In order to verify that G-transversality is invariantly defined on manifolds we need a slight strengthening of proposition 6.11.2.

PROPOSITION 6.11.3. *Suppose that $W = W_1 \oplus W_2$ (as G-representations) and let $\pi : W \to W_2$ denote the projection map. Let $\beta = (\beta_1, \beta_2) \in \mathrm{Diff}_G(W)$ satisfy $\beta_2(w_1, 0) = 0$, for all $w_1 \in W_1$ (that is, $\beta(W_1) \subset W_1$). Let $f \in C^\infty_G(V \times \mathbb{R}^s, W)$. If $\pi \circ f$ is G-transverse to $0 \in W_2$ along $K \subset \mathbb{R}^s$ then $\pi \circ (\beta \circ f)$ is G-transverse to $0 \in W_2$ along $K \subset \mathbb{R}^s$.*

PROOF. Let $\mathcal{G} = \{G_1,\dots,G_{\bar{k}}\}$ and $\mathcal{F} = \{F_1,\dots,F_k\}$ be minimal sets of homogeneous generators for $P_G(V,W_1)$ and $P_G(V,W_2)$ respectively. Let $\Sigma \subset V \times \mathbb{R}^k$ be the universal variety associated to $\mathcal{F}$ and define

$$\overline{\Sigma} = \{(x,\bar{\mathbf{t}},\mathbf{t}) \in V \times \mathbb{R}^{\bar{k}} \times \mathbb{R}^k \mid (x,\mathbf{t}) \in \Sigma\}.$$

Obviously $\overline{\Sigma} \approx \Sigma \times \mathbb{R}^{\bar{k}}$. As in the proof of proposition 6.11.2, there exist smooth G-invariant functions $\bar{\tau}_j$, τ_i such that for all $(x,\bar{\mathbf{t}},\mathbf{t}) \in V \times \mathbb{R}^{\bar{k}} \times \mathbb{R}^k$ we have

$$\beta(\sum_{j=1}^{\bar{k}} \bar{t}_j G_j(x), \sum_{i=1}^k \bar{t}_i F_i(x)) = (\sum_{j=1}^{\bar{k}} \bar{\tau}_j(x,\bar{\mathbf{t}},\mathbf{t})G_j(x), \sum_{i=1}^k \tau_i(x,\bar{\mathbf{t}},\mathbf{t})F_i(x)).$$

Define $J : V \times \mathbb{R}^{\bar{k}} \times \mathbb{R}^k \to V \times \mathbb{R}^{\bar{k}} \times \mathbb{R}^k$ by

$$J(x,\bar{\mathbf{t}},\mathbf{t}) = (x,\bar{\tau}(x,\bar{\mathbf{t}},\mathbf{t}),\tau(x,\bar{\mathbf{t}},\mathbf{t})).$$

Then J is a smooth G-equivariant map which restricts to a G-equivariant diffeomorphism on some open neighbourhood of $\{0\} \times \mathbb{R}^{\bar{k}} \times \mathbb{R}^k$ in $V \times \mathbb{R}^{\bar{k}} \times \mathbb{R}^k$. Since $\beta_2(w_1,0) = 0$, $w_1 \in W_1$, $J(\overline{\Sigma}) \subset \overline{\Sigma}$. Let $\Gamma_f : V \times \mathbb{R}^s \to V \times \mathbb{R}^{\bar{k}} \times \mathbb{R}^k$ be the graph of the coefficient map for f determined by $\mathcal{G}$, $\mathcal{F}$, and $\Gamma_{\pi\circ} f : V \times \mathbb{R}^s \to V \times \mathbb{R}^k$ be the corresponding map for $\pi \circ f$. We have $\Gamma_{\pi\circ f} = \tilde{\pi} \circ \Gamma_f$, where $\tilde{\pi}$ is the projection of $V \times \mathbb{R}^{\bar{k}} \times \mathbb{R}^k$ on $V \times \mathbb{R}^k$. Observe that $\Gamma_{\pi\circ f}$ is transverse to Σ along K if and only if Γ_f is transverse to $\overline{\Sigma}$ along K. Since J s a local isomorphism of $\overline{\Sigma}$ near $x = 0$, $\Gamma_f \pitchfork \overline{\Sigma}$ if and only if $J \circ \Gamma_f = \Gamma_{\beta\circ f}$ is transverse to $\overline{\Sigma}$ along K. In turn this condition is equivalent to $\Gamma_{\beta\circ f} \pitchfork \Sigma$ along K. $\square$

6.12. Symmetries of the stratification $\mathcal{A}$

We start by remarking that since $V^G = \{0\}$, we have $h(0) = 0$ for all $h \in \mathrm{Diff}_G(V)$.

Given $t \in \mathbb{U}$, choose $f \in C^\infty_G(V, W)$ such that $\Pi(f) = t$. If $h \in \mathrm{Diff}_G(V)$, then $fh \in C^\infty_G(V, W)$ and we may define $\omega(h)(t) = \Pi(fh) \in \mathbb{U}$.

Lemma 6.12.1. (1) *For $h \in \mathrm{Diff}_G(V)$, $t \in \mathbb{U}$, $\omega(h)(t)$ is well-defined, independent of the choice of f such that $\Pi(f) = t$.*

(2) *ω defines a continuous map $\omega : \mathrm{Diff}_G(V) \to \mathrm{GL}(\mathbb{U}) \subset \mathrm{Diff}(\mathbb{U})$.*

Proof. Choose a minimal set $\mathcal{F}$ of homogeneous generators for $P_G(V, W)$. If $t = \Pi(f)$, we may write $f(x) = \sum_{j=1}^k f_j(x)F_j(x)$, where $f_j(0) = t_j$, $1 \le j \le k$. If $h \in \mathrm{Diff}_G(V)$, we may write $F_j \circ h = \sum_{i=1}^k q_{ji}F_i$, $q_{ji} \in C^\infty(V)^G$. We have

$$\begin{aligned} f(h(x)) &= \sum_{j=1}^k t_j F_j(hx), \\ &= \sum_{j=1}^k t_j \left(\sum_{i=1}^k q_{ji}(x) F_i(x) \right), \\ &= \sum_{i=1}^k \left(\sum_{j=1}^k t_j q_{ji}(x) \right) F_i(x). \end{aligned}$$

and so $\Pi(fh) = Mt$, where the components $M = [m_{ij}] \in \mathrm{GL}(k, \mathbb{R}^k)$ are given by $m_{ij} = q_{ji}(0)$. This shows that $\omega(h)(t)$ is well-defined and that $\omega(h) \in \mathrm{GL}(\mathbb{U})$. Continuity of M follows from our explicit computation of ω. $\square$

Remark 6.12.2. If (V, G) is absolutely irreducible, it is straightforward to show that the group $\omega(\mathrm{Diff}_G(V))$ consists of invertible lower triangular matrices.

Given $t \in \mathbb{U}$, choose $f \in C^\infty_G(V, W)$ such that $\Pi(f) = t$. If $h \in \mathrm{Diff}_G(W)$, then $hf \in C^\infty_G(V, W)$ and we may define $\sigma(h)(t) = \Pi(hf) \in \mathbb{U}$.

Lemma 6.12.3. *For $h \in \mathrm{Diff}_G(W)$, $t \in \mathbb{U}$, $\sigma(h)(t)$ is well-defined, independent of the choice of f such that $\Pi(f) = t$.*

(1) *If $W^G = \{0\}$, then σ defines a continuous map $\sigma : \mathrm{Diff}_G(W) \to \mathcal{P}^{(d)}(\mathbb{U}) \subset \mathrm{Diff}(\mathbb{U})$, where $\mathcal{P}^{(d)}(\mathbb{U})$ denotes the group of polynomial diffeomorphisms of $\mathbb{U}$ of degree less than or equal to d, $d = [d_k/d_1]$ and $\sigma(h)(0) = 0$, for all $h \in \mathrm{Diff}_G(W)$.*

(2) *If* $W^G = W$ *(trivial representation), then* $\sigma : \mathrm{Diff}_G(W) \to \mathrm{Diff}(\mathbb{U})$ *and is a group isomorphism.*

(3) *Suppose that* $W = W_1 \oplus W_2$ *where* $W_1^G = \{0\}$ *and* W_2 *is a trivial* G*-representation. If* $k_1 = k(V, W_1)$, $k_2 = k(V, W_2) = \dim(W_2)$, *and we denote coordinates on* $\mathbb{R}^k = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ *by* (t, s), *then* $\sigma(h)(t, s) = (p(s;t), q(s))$, *where* $p(s;t)$ *is a polynomial of degree less than or equal to* $d = [d_{k_1}/d_1]$ *in* t, *coefficients smoothly depending on* s, *and* $q \in \mathrm{Diff}(W_2)$.

Proof. Suppose $h \in \mathrm{Diff}_G(W)$. We may write $h(w) = \sum_{i=1}^m h_i(w)G_i(w)$, where $\mathcal{G} = \{G_1, \ldots, G_m\}$ is a minimal set of homogeneous generators for the $P(W)^G$-module $P_G(W, W)$. For $(x, t) \in V \times \mathbb{R}^k$ we have

$$h(\sum_{j=1}^k t_j F_j(x)) = \sum_{i=1}^m h_i(\sum_{j=1}^k t_j F_j(x)) G_i(\sum_{j=1}^k t_j F_j(x)). \tag{6.4}$$

We may write $G_i(w) = A_i(w, w, \ldots, w)$, where $A : W^{g_i} \to W$ is a symmetric multilinear map and $\deg(G_i) = g_i$. Hence

$$G_i(\sum_{j=1}^k t_j F_j(x)) = \sum_{j_1, j_2, \ldots, j_{g_i}} t_{j_1} \ldots t_{j_{g_i}} A_i(F_{j_1}(x), \ldots, F_{j_{d_i}}(x)).$$

Since $A_i(F_{j_1}(x), \ldots, F_{j_{d_i}}(x))$ is a G-equivariant polynomial, we may write

$$A_i(F_{j_1}(x), \ldots, F_{j_{d_i}}(x)) = \sum_{n=1}^k q^n_{i;j_1,\ldots,j_{g_i}}(x) F_n(x),$$

where $q^n_{i;j_1,\ldots,j_{g_i}} \in P(V)^G$. Thus

$$G_i(\sum_{j=1}^k t_j F_j(x)) = \sum_{j_1, j_2, \ldots, j_{g_i}, n=1}^k t_{j_1} \ldots t_{j_{g_i}} q^n_{i;j_1,\ldots,j_{g_i}}(x) F_n(x).$$

Substituting in (6.4), we see that $\sigma(h)(t) = \Pi(hf)$ is well-defined, independent of our choice of f such that $\Pi(f) = (t_1, \ldots, t_k)$.

If $W^G = \{0\}$, then $d_i \geq 1$, $1 \leq i \leq k$, and so $h_i(\sum_{j=1}^k t_j F_j(0)) = h_i(0)$ and is independent of t. Since $q^n_{i;j_1,\ldots,j_{g_i}}(0) = 0$ whenever $d_{j_1} + \ldots d_{j_{d_i}} > d_k$, $\deg(\sigma(h)) \leq [d_k/d_1]$, proving (1).

For (2), observe that if W is a trivial G-representation, we may take $F_j(w) = \mathbf{e}_j$, where $1 \leq j \leq k = \dim(W)$, and $\mathbf{e}_1, \ldots, \mathbf{e}_k$ is a basis for W. Equation (6.4) reduces to

$$h(\sum_{j=1}^k t_j F_j(x)) = h(t_1, \ldots, t_k),$$

and so $\sigma(h)(t) = h(t)$.

We leave the verification of (3) to the reader. $\square$

DEFINITION 6.12.4. Let $\mathcal{G} = \mathcal{G}(V, W) \subset \mathrm{Diff}(\mathbb{U})$ denote the group of transformations of $\mathbb{U}$ generated by $\omega(\mathrm{Diff}_G(V))$ and $\sigma(\mathrm{Diff}_G(W))$.

EXAMPLE 6.12.5. Suppose that V, W are 2-dimensional representations of $\mathbb{Z}_2$ containing no trivial subrepresentations. Choose coordinates (x_1, x_2) on V, (y_1, y_2) on W. If we define $F_1(x_1, x_2) = (x_1, 0)$, $F_2(x_1, x_2) = (x_2, 0)$, $F_3(x_1, x_2) = (0, x_1)$ and $F_4(x_1, x_2) = (0, x_2)$, then $\{F_1, \ldots, F_4\}$ is a minimal set of homogeneous generators for $P_{\mathbb{Z}_2}(V, W)$ and so $\mathbb{U} \cong \mathbb{R}^4$. The group $\omega(\mathrm{Diff}_{\mathbb{Z}_2}(V))$ is the subgroup of $\mathrm{GL}(4, \mathbb{R})$ consisting of all transformations of the form $(t_1, \ldots, t_4) \mapsto (A(t_1, t_2), A(t_3, t_4))$, $A \in \mathrm{GL}(2, \mathbb{R})$. The group $\sigma(\mathrm{Diff}_{\mathbb{Z}_2}(W))$ is also isomorphic to $\mathrm{GL}(2, \mathbb{R})$. In this case elements $A \in \mathrm{GL}(2, \mathbb{R})$ act diagonally on the odd and even coordinates. It can be shown that $\mathcal{G} \subset \mathrm{GL}(4, \mathbb{R})$ is the semi-direct product of $\omega(\mathrm{Diff}_{\mathbb{Z}_2}(V))$ and $\sigma(\mathrm{Diff}_{\mathbb{Z}_2}(W))$ (both groups are normal subgroups of $\mathcal{G}$). The group $\mathcal{G}$ leaves invariant the cone $t_1 t_4 = t_2 t_3$ corresponding to the singular linear maps from V to W.

THEOREM 6.12.6. *The stratification $\mathcal{A}$ is invariant by the group $\mathcal{G} \subset \mathrm{Diff}(\mathbb{U})$ defined in Definition 6.12.4.*

PROOF. The result follows from propositions 6.11.1, 6.11.2. □

EXERCISE 6.12.7. Compute $\mathcal{G}(V, V)$ when V is $(\mathbb{R}^n, H_n)$, $n = 2, 3$. To what extent is $\mathcal{A}$ determined by $\mathcal{G}$ for these representations?

6.13. Openness of equivariant transversality

Suppose (V, G) is an orthogonal representation relative to a G-invariant inner product on V. We continue to assume $V^G = \{0\}$. Given $\bar{x} \in V$, let $V_{\bar{x}}$ be the orthogonal complement of $T_{\bar{x}} G\bar{x}$. Note that $\bar{x} \in V_{\bar{x}}$ (since $G\bar{x}$ lies in the sphere of radius $\|\bar{x}\|$, centre the origin) and $V_{\bar{x}}$ has the structure of a $G_{\bar{x}}$-orthogonal representation. If $\bar{z} = (\bar{x}, s) \in V \times \mathbb{R}^s$, then the orthogonal complement of $T_{\bar{z}} G\bar{z}$ in $V \times \mathbb{R}^s$ is $V_{\bar{x}} \times \mathbb{R}^s$. Let $S^r_{\bar{z}}$ denote the r-disk centre $\bar{z}$ in $V_{\bar{z}} \times \mathbb{R}^s$, For r sufficiently small, $S^r_{\bar{z}}$ is a slice for the action of G at $\bar{z}$.

LEMMA 6.13.1. *Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a minimal set of homogeneous generators for $P_G(V, W)$. Let $\bar{z} = (\bar{x}, \bar{s}) \in V \times \mathbb{R}^s$, $G_{\bar{z}} \neq G$, and choose $r > 0$ such that $S^r_{\bar{z}} \subset V_{\bar{x}} \times \mathbb{R}^s$ is a slice for the action of G at $\bar{z}$. Let $\overline{\mathcal{F}} = \{\overline{F_1}, \ldots, \overline{F}_k\}$ be the set of $G_{\bar{x}}$-equivariant polynomials on $V_{\bar{x}} \times \mathbb{R}^s$ defined by $\overline{F}_j(x, s) = F_j(x + \bar{x}, s + \bar{s})$. Then $\overline{\mathcal{F}}$ is a set of generators for $C^\infty(S^r_{\bar{z}})^{G_{\bar{x}}}$-module $C^\infty_{G_{\bar{x}}}(S^r_{\bar{z}}, W)$.*

PROOF. Let $f \in C^\infty_{G_{\bar{x}}}(S^r_{\bar{z}}, W)$. Then f extends equivariantly to a smooth G-equivariant map $\tilde{f} : G(S^r_{\bar{z}}) \subset V \times \mathbb{R}^s \to W$. Since $\mathcal{F}$ generates $P_G(V, W)$, it follows from Schwarz' theorem that $\mathcal{F}$ generates $C^\infty_G(G(S^r_{\bar{z}}), W)$ as a $C^\infty(G(S^r_{\bar{z}}))^G$-module. Thus we may write $\tilde{f} = \sum_j f_j F_j$. Now restrict to $S^r_{\bar{z}}$. □

DEFINITION 6.13.2. Suppose that $f \in C^\infty_G(V \times \mathbb{R}^s, W)$. Given $z = (x, s) \in V \times \mathbb{R}^s$, we say that f is G_x-transverse to $0 \in W$ at z if $f|(V_x \times \mathbb{R}^s)$ is G_x-transverse to $0 \in W$ at z.

LEMMA 6.13.3 (cf Bierstone [**14**, Proposition 6.1]). *Suppose that $f \in C^\infty_G(V \times \mathbb{R}^s, W)$. Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a minimal set of homogeneous generators for $P_G(V, W)$. Suppose that K is a G-invariant subset of $V \times \mathbb{R}^s$ and $\Gamma^{\mathcal{F}}_f$ is transverse to $\Sigma^{\mathcal{F}}$ on K. Then f is G_x-transverse to $0 \in W$ for all $(x, s) \in K$.*

PROOF. Let $\bar{z} = (\bar{x}, \bar{s}) \in K$ and $S \subset V_{\bar{x}} \times \mathbb{R}^s$ be a slice for the action of G at $\bar{z}$. Set $H = G_{\bar{x}}$. The map $\Gamma_f : S \times_H G \subset V \times \mathbb{R}^s \to V \times \mathbb{R}^k$ is transverse to Σ on $(S \times_H G) \cap K$. Writing $\Gamma_f(x, s) = (x, \gamma_f(x, s))$, it follows by the G-invariance of γ_f that $\Gamma_f : S \times_H G \to V \times \mathbb{R}^k$ is transverse to Σ if and only if $\Gamma_f|S : S \subset V_{\bar{x}} \times \mathbb{R}^s \to V^H \times \mathbb{R}^k$ is transverse to $\Sigma^H = \Sigma \cap (V^H \times \mathbb{R}^k)$ on the H-invariant set $K \cap S$. Let $\overline{\mathcal{F}}$ be the set of inhomogeneous H-equivariant generators given by lemma 6.13.1. Let $\tilde{\Sigma} \subset S \times \mathbb{R}^k$ denote the associated zero variety defined by $\overline{\mathcal{F}}$. Then $\tilde{\Sigma} = \Sigma \cap (S \times \mathbb{R}^k)$ and so $\Gamma_f|S$ is transverse to $\tilde{\Sigma}$ at $\bar{z}$. It follows from lemma 6.7.5 that f is $G_{\bar{x}}$-transverse to $0 \in W$ at $\bar{z}$. □

Applying theorem 6.10.6, we deduce that the conclusion of lemma 6.13.3 does not depend on the choice $\mathcal{F}$ of generators. Consequently, we may now give an unrestricted definition of G-transversality on representations.

DEFINITION 6.13.4. Let $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ and $K \subset V \times \mathbb{R}^s$. We say f is G-transverse to $0 \in W$ on K if f is G_z-transverse to $0 \in W$ for all $z \in K$. Symbolically, we write this as "$f \pitchfork_G 0$ along K". (We may replace $V \times \mathbb{R}^s$ by a non-empty G-invariant open subset of $V \times \mathbb{R}^s$.)

As a corollary of lemma 6.13.3 we have openness of G-transversality.

COROLLARY 6.13.5. *Suppose that $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ and $f \pitchfork_G 0$ at $(0, 0) \in V \times \mathbb{R}^s$. There exists an open G-invariant neighbourhood U of $(0, 0) \in V \times \mathbb{R}^s$ such that for all $z = (x, s) \in U$, f is G_x-transverse to $0 \in W$ at z.*

COROLLARY 6.13.6 (Stratumwise transversality). *Let $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ and $K \subset V \times \mathbb{R}^s$. Suppose that $f \pitchfork_G 0$ on K. Then f is stratumwise transverse to $0 \in W$ on K.*

PROOF. Lemma 6.10.10. □

COROLLARY 6.13.7. *Let $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ and suppose f is G-transverse to $0 \in W$ at $(0, 0) \in V \times \mathbb{R}^s$. Then we can choose a G-invariant neighbourhood U of $(0, 0) \in V \times \mathbb{R}^s$ such that $f^{-1}(0) \cap U_\tau = \emptyset$ whenever $i_\tau(V, W) + s < 0$.*

PROOF. We have $i_\tau(V \times \mathbb{R}^s, W) = \tau(V, W) + s$. Apply corollary 6.13.6. □

REMARK 6.13.8. We do not need any consideration of G-transversality to show that we can always perturb f to obtain stratumwise transversality. The important property here is that if $f \pitchfork_G 0$ on a compact set $K \subset V \times \mathbb{R}^s$, then stratumwise transversality is an open condition on K: it persists under C^d-small perturbations of f.

6.14. Global definitions and results

6.14.1. G-transversality on a manifold. At the risk of some repetition, we start by recalling from section 6.5 the basic setup for defining G-transversality for maps between manifolds. Assume M, N are Riemannian G-manifolds and that P is a G-invariant submanifold of N. Let $f : M \to N$ be a smooth G-invariant map. Suppose that $x \in M$ and $f(x) \in P$. Since f is G-equivariant, $G_x \subset G_{f(x)}$. Hence $T_{f(x)}N$ has the structure of an orthogonal G_x-representation. Let $W_1 = T_{f(x)}P$ and $W_2 W_1^\perp$. As G_x-representations, $T_{f(x)}N = W_1 \oplus W_2$. Using Bochner's theorem (theorem 3.4.9), we choose a G_x-equivariant diffeomorphism ψ of a G_x-invariant open neighbourhood D of $f(x) \in N$ onto an open neighbourhood $B \times C$ of $(0,0) \in W_1 \oplus W_2$ so that $\psi(f(x)) = (0,0)$ and $\psi(D \cap P) = B \times \{0\}$. Let $\pi : W_1 \oplus W_2 \to W_2$ denote the projection on W_2.

Turning now to M, choose a slice S_x for the action of G at x so that $f(S_x) \subset D$. Let (V, G_x) be the G_x-representation induced on the normal space $T_x Gx^\perp$. Choose a G_x-equivariant diffeomorphism ϕ of S_x onto an open neighbourhood A of $0 \in V$. We may assume $\phi(x) = 0$. Define $\tilde{f} : A \subset V \to B \times C \subset W_1 \oplus W_2$ by

$$\tilde{f} = \psi f \phi^{-1}.$$

The map $\tilde{f}$ is smooth, G_x-equivariant and satisfies $\tilde{f}(0) = 0$. Conversely, given any smooth G_x-equivariant map $h : A \subset V \to B \times C \subset W_1 \oplus W_2$, h extends uniquely to a smooth G-equivariant map $h^\star : GS_x \to N$ ($h^\star(\gamma y) = \gamma\psi^{-1}h(\phi y)$, all $y \in S_x$, $\gamma \in G$). Obviously, $\tilde{f}^\star = f|G(S_x)$.

If we define $F = \pi \tilde{f} F$, then $F : A \subset V \to C \subset W_2$ is a smooth G_x-equivariant map and $F(0) = 0$. Observe that if $y \in S_x$ then $f(y) \in P$ if and only if

$$F(\phi(y)) = 0.$$

In particular, every zero z of F determines a unique G-orbit $G(\phi^{-1}(z)) \subset GS_z \subset M$ mapped by f to P. Conversely every G-orbit $\alpha \subset GS_x$ which is mapped to P by f determines a unique zero of F.

We are now in a position to give a local definition of G-transversality on manifolds. We say that f is G-transverse to P at x if either $f(x) \notin P$ or $f(x) \in P$ and, with the previous notation, $F : A \subset V \to C \subset W_2$ is G_x-transverse to 0 at $0 \in A$. This definition depends on several choices: The G_x-linearizing coordinate system at $f(x)$, the choice of a slice S_x, and the choice of the linearizing coordinate system $\phi : S_x \to A \subset V$. In addition, there is the question of invariance under G-translations: we need to show that f is G-transverse to P at x if and only if f is G-transverse to P at gx.

Proposition 6.14.1. *G-transversality is invariantly defined on manifolds, independent of choices. The set of points where a map $f : M \to N$ is G-transverse to $P \subset N$ is a G-invariant subset of M.*

Proof. Let $f \in C^\infty_G(M,N)$ and suppose that $x \in M$ and $f(x) \in P$. With the notation established above, suppose that $F = \pi\psi f \phi^{-1} : A \subset V \to C \subset W_2$ is

G_x-transverse to 0 at $0 \in A$. By propositions 6.11.3, 6.11.1, the G_x-transversality of F is independent of the choices of ψ and ϕ. If we take a different slice, say S'_x through x, then a choice of local G_x-equivariant section of $\sigma : U \subset G/G_x \to G$, allows us to define a G_x-equivariant diffeomorphism between S_x and S'_x and we again find by proposition 6.11.1 that the G_x-transversality of F is independent of the choice of slice S_x. Hence we have shown that the G-transversality of f to P at x is invariantly defined. Finally, it is easy to check that $f \pitchfork_G P$ at x if and only if $f \pitchfork_G P$ at gx, for all $g \in G$. $\square$

PROPOSITION 6.14.2. *Let M, N be G manifolds and P be a closed G-invariant submanifold of N. If $f \pitchfork_G P$ at x, then $f \pitchfork_G P$ at all points y in some open neighbourhood of x.*

PROOF. Immediate from corollary 6.13.5. $\square$

PROPOSITION 6.14.3. *Let M, N be G manifolds and P be a G-invariant submanifold of N. If $f \pitchfork_G P$ then f is stratumwise transverse to P.*

PROOF. The result follows from lemma 6.10.10. $\square$

THEOREM 6.14.4. *Let M, N be G manifolds, P be a G-invariant submanifold of N and set $\mathcal{G} = \{f \in C^\infty_G(M, N) \mid f \pitchfork_G P\}$.*

(1) *If P is closed, $\mathcal{G}$ is an open subset of $C^\infty_G(M, N)$ in the Whitney C^∞-topology. If M is compact, openness holds in the C^∞-topology.*
(2) *$\mathcal{G}$ is a residual subset of $C^\infty_G(M, N)$ (in either the C^∞- or Whitney C^∞-topology).*
(3) *If M is compact, P closed and $f_t \in \mathcal{G}$, $t \in [0, 1]$, is a smooth family, then there exists a continuous equivariant isotopy $H_t : M \to M$, $t \in [0, 1]$, such that $H_0 = I_M$ and $H_t(f_t^{-1}(P)) = f_0^{-1}(P)$.*

PROOF. (1) We show that the complement of $\mathcal{G}$ in $C^\infty_G(M, N)$ is closed in the Whitney C^∞-topology. The proof we give applies without change if M is compact and we use the C^∞-topology. Suppose then that $(f_n) \subset C^\infty_G(M, N) \setminus \mathcal{G}$ converges to $F \in C^\infty_G(M, N)$. It suffices to prove $F \notin \mathcal{G}$. By lemma 6.2.1(3), there exists a compact subset K of M and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $f_n = F$ on $M \setminus K$. Since $f_n \notin \mathcal{G}$, for each n, we can choose $x_n \in M$ such that f_n is not G-transverse to P at x_n. Choosing a subsequence if necessary, we can assume that $x_n \in K$, $n \geq n_0$ (if not, it already follows that $F \notin \mathcal{G}$). But now K is compact and so, choosing a further subsequence if necessary, we can assume $x_n \to x \in K$. We claim F is not G-transverse to P at x. Suppose the contrary, then by proposition 6.14.2 F will be G-transverse to P on a compact G-invariant neighbourhood U of x. Applying the openness theorem for transversality to a Whitney stratified set, we see that for sufficiently large n, f_n will be G-transverse to P on U. Contradiction, since $x_n \in U$ for large n. Hence $F \notin \mathcal{G}$.

For the density of $\mathcal{G}$ in $C^\infty_G(M, N)$, suppose first that P is closed. Choose a locally finite cover $\{K_i \mid i \in \mathbb{N}\}$ of M by compact G-invariant sets. Using the density theorem for maps transverse to a Whitney regular stratification, our characterization of G-transversality in term of transversality to a Whitney regular stratification of $\mathbb{U}$, and lemma 6.13.3, we may show that $\mathcal{W}_i = \{f \in C^\infty_G(M,N) \mid f \pitchfork_G P \text{ on } K_i\}$ is an open dense set of $C^\infty_G(M,N)$. (Note that this does not require us to perturb the *invariant* maps Γ_f so that they are transverse to the universal variety). Now $\mathcal{G} = \cap_{i\geq 0}\mathcal{W}_i$ proving (2) since $C^\infty_G(M,N)$ is a Baire space (in either the C^∞- or Whitney C^∞-topology – lemma 6.2.1). If P is not closed, we can still represent $\mathcal{G}$ as a countable intersection of open dense sets – cover P by a locally finite set (within P) of G-invariant compact submanifolds, possibly with boundary.

We indicate the proof of the isotopy theorem. Let $f \in \mathcal{G}$. Choose a finite cover of M by slice tubular neighbourhood pairs $GS_i \subset G\bar{T}_i$). We use the Thom-Mather theorem to construct isotopies locally, supported in GT_i, and then (equivariantly) patch together to obtain global isotopy of M. We omit the details □

Remark 6.14.5. We refer to Bierstone [**14**] for alternative proofs of parts of theorem 6.14.4 (see also the notes at the end of the chapter).

Exercise 6.14.6. Extend as far as possible the theory of G-transversality to the category of proper G-manifolds.

6.15. Solutions with specific isotropy type

We conclude this chapter with an investigation of the constraints on the isotropy type of solutions imposed by G-transversality at a point. We restrict attention to equivariant maps between G-representations and include some examples that relate to equivariant (reversible) systems.

Let $\mathcal{F}$ be a minimal set of homogeneous generators for $P_G(V,W)$. Suppose that $\tau \in \mathcal{O}^\star(V,G)$. Define the closed semialgebraic subset R_τ of $\mathbb{R}^k$ by

$$R_\tau = \partial\Sigma_\tau \cap \mathbb{R}^k = \overline{\Sigma}_\tau \cap \mathbb{R}^k.$$

Just as in the proof of theorem 6.10.6, lemma 6.7.4 implies that R_τ is naturally defined as a closed semialgebraic subset of $\mathbb{U}$, independent of the choice of generator set $\mathcal{F}$. Knowledge about $\mathcal{R}(V,W) = \{R_\tau \mid \tau \in \mathcal{O}^\star\}$ gives information about isotropy type of generic solutions to equivariant families in $C^\infty_G(V \times \mathbb{R}^s, W)$.

For $\tau \neq (G)$, define

$$r_\tau = \dim(R_\tau).$$

Lemma 6.15.1. *Let $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ and suppose that $f \pitchfork_G 0$ along the set $K \subset \mathbb{R}^s$. If $s + r_\tau < k = k(V,W)$, then there exists an open G-invariant neighbourhood U of K in $V \times \mathbb{R}^s$ such that $f^{-1}(0) \cap U$ contains no points of isotropy type τ.*

Proof. By theorem 6.10.1, R_τ is a union of $\mathcal{A}$-strata. Now apply theorem 6.10.6. □

LEMMA 6.15.2. *Assume $V^G = W^G = \{0\}$. Let $\tau \in \mathcal{O}^\star(V,G)$. Then*

$$k - e_\tau \leq r_\tau \leq \min\{k, k - n_\tau + i_\tau - 1\}.$$

PROOF. Let $H \in \tau \in \mathcal{O}(V,G)$ and consider $F : V_\tau^H \times \mathbb{R}^k \to W^H$. For $x \in V_\tau^H$, the kernel of the linear map $t \mapsto F(x,t)$ is a linear subspace $K(x)$ of $\mathbb{R}^k$ of dimension equal to $k - e_\tau$. Consider $K(x)$ as point in the Grassmann variety $\mathbf{Gr}_p(\mathbb{R}^k)$ of $p = k - e_\tau$-dimensional subspaces of $\mathbb{R}^k$. Replacing x by λx and letting $\lambda \to 0$ we see that each limit point of $K(\lambda x)$ is a $k - e_\tau$-dimensional subspace of $\mathbb{R}^k$ contained in R_τ. Hence $r_\tau \geq k - e_\tau$. For the upper bound, let $P : V \to V/G$ be the orbit map associated to a minimal set of homogeneous generators for $P(V)^G$. Then Σ_τ/G is a semialgebraic subset of $V/G \times \mathbb{R}^k \subset \mathbb{R}^\ell \times \mathbb{R}^k$ of dimension $k - n_\tau + i_\tau$. Hence $\dim(\partial(\Sigma_\tau/G)) \leq k - n_\tau + i_\tau - 1$ and so $\dim(\partial\Sigma_\tau \cap \mathbb{R}^k) \leq \min\{k, k - n_\tau + i_\tau - 1\}$. □

EXAMPLES 6.15.3. (1) Suppose that $V = W$ and that V is a sum of non-trivial absolutely irreducible representations. Let $F_1, \ldots, F_{k_1}$ be a vector space basis for $L_G(V,V)$. Then $F_1, \ldots, F_{k_1}$ extends to a minimal set $\{F_1, \ldots, F_k\}$ of homogeneous generators for $P_G(V,W)$. Let $\Delta \subset \mathbb{R}^k$ be the set of $t \in \mathbb{R}^k$ such that $F(t) = \sum_{j=1}^{k_1} t_j F_j \in L_G(V,V)$ is singular. Since Δ is the zero set of $\det(F(t))$, Δ is a codimension one algebraic subset of $\mathbb{R}^k$. If $t \in \mathbb{R}^k \setminus \Delta$, then $x = 0$ is an isolated zero of $F_t(x) = 0$. Hence the top dimensional stratum of $\mathcal{A}$ is equal to $\mathbb{R}^k \setminus \Delta$. If $\tau \in \mathcal{O}^\star(V,G)$, then $R_\tau \subset \Delta$ and the estimate of lemma 6.15.2 reads

$$k - d_\tau \leq r_\tau \leq k - n_\tau - 1.$$

When V is absolutely irreducible and $L_G(V,V)$ is one dimensional, $\Delta = \{t \mid t_1 = 0\}$ and each R_τ is contained in this hyperplane. Note that if $n_\tau > 0$, then $r_\tau < k - 1$. In particular, for 1-parameter families of normalized vector fields, there will generically be no branches of *equilibria* of isotropy type τ if $n_\tau > 0$. This can happen even if τ is a maximal isotropy type (see [**124**]).
(2) If $V = W$ and V is irreducible of complex type, then $L_G(V,V)$ is generated by $F_1 = I_V, F_2 = \imath I_V$. The corresponding set $\Delta \subset \mathbb{R}^k$ defined by the singular linear maps is the codimension two linear subspace of $\mathbb{R}^k$ defined by $t_1 = t_2 = 0$. Hence $k - d_\tau \leq r_\tau \leq k - 2$. Note that if $r_\tau \leq k - 2$ then whenever $f \in C_G^\infty(V \times \mathbb{R}, W)$ and $f \pitchfork_G 0$ at $(0,0) \in V \times \mathbb{R}$, then there will be an open neighbourhood of $(0,0) \in V \times \mathbb{R}$ contains no zeros of f of isotropy type τ. This applies, for example, if $V = W = \mathbb{C}$ and $G = \mathrm{SO}(2)$ acts non-trivially. Generically, in 1-parameter families, there will be no 'equilibrium' SO(2)-orbits.
(3) Suppose that $V = W$ and V is absolutely irreducible. Let $H \in \tau \in \mathcal{O}(V,G)$ and suppose that $\dim(V^H) = 1$ (V^H is an axis of symmetry). Then $d_\tau = 1$ and so, by (1) and lemma 6.15.2 we see that $r_\tau = k - 1$. In this case it is easy to check directly that R_τ equals the hyperplane $t_1 = 0$. Hence branches of solutions of isotropy type τ will be seen in all 1-parameter families $f \in \mathcal{V}_0(V,G)$. This argument gives a simple geometric proof of the equivariant branching lemma.

(4) In general, R_τ will usually not be an algebraic subset of $\mathbb{R}^k$. As an example, consider the representation $(\mathbb{R}^3, G)$, where $G \subset H_3$ is given by $G = \Delta_3 \rtimes \mathbb{Z}_3$. It is straightforward to verify that a minimal set of homogeneous generators for $P_G^{(3)}(\mathbb{R}^3, \mathbb{R}^3)$ is given by

$$\begin{aligned} F_1(x,y,z) &= (x,y,z), \\ F_2(x,y,z) &= (xy^2, yz^2, zx^2), \\ F_3(x,y,z) &= (xz^2, yx^2, zy^2). \end{aligned}$$

These polynomials do not generate the $P(\mathbb{R}^3)^G$-module $P_G(\mathbb{R}^3, \mathbb{R}^3)$: the fifth order equivariant (x^5, y^5, z^5) cannot be written as a linear combination over $P(\mathbb{R}^3)^G$ of F_1, F_2, F_3. However, since we know that bifurcation problems on $(\mathbb{R}^3, G)$ are determined by third order terms, we are able to determine those sets R_τ which are of codimension one in $\mathbb{R}^k$. The isotropy types $(G_{(1,0,0)})$ and $(G_{(1,1,1)})$ are both maximal and define one dimensional fixed point spaces. Therefore $R_{(G_{(1,0,0)})} = R_{(G_{(1,1,1)})} = \{t_1 = 0\}$. The isotropy type $(G_{(1,1,0)})$ is submaximal. We solve $F(t,x,y,z) = t_1 F_1(x,y,z) + t_2 F_2(x,y,z) + t_3 F_3(x,y,z) = 0$ subject to $z = 0$ and $xy \neq 0$. We find that

$$\begin{aligned} t_1 x + t_2 x y^2 &= 0, \\ t_1 y + t_3 y x^2 &= 0. \end{aligned}$$

(If we had taken account of higher order equivariants they would appear in these equations as terms of of order at least 5 in x, y and would not affect the following analysis.) Since we assume $xy \neq 0$, we have to solve $t_1 + t_2 y^2 = t_1 + t_3 x^2 = 0$. Eliminating t_1, we have $t_2 y^2 = t_3 x^2$ and so there are no (real) solutions if $t_2 t_3 < 0$. If $t_2 t_3 > 0$, we can always find solutions. These will be of the form $y = \pm\sqrt{-t_1/t_2}$, $x = \pm\sqrt{-t_1/t_3}$ and will meet the parameter plane $\mathbb{R}^3$ along $t_1 = 0$. Hence

$$R_{(G_{(1,1,0)})} = \{(t_1, t_2, t_3) \mid t_1 = 0, t_2 t_3 \geq 0\}$$

and so R_τ is a proper *semialgebraic* conical subset of $t_1 = 0$.

(5) It is easy to construct examples for which $r_\tau = k$ for all $\tau \in \mathcal{O}(V, G)$. For example, suppose $p, q > 1$ are coprime integers. Let $\mathrm{SO}(2)$ act on $V = \mathbb{C}$ as multiplication by $e^{\imath p\theta}$ and on $W = \mathbb{C}$ as multiplication by $e^{\imath q\theta}$. We have $P_{\mathrm{SO}(2)}(V, W) = \{0\}$, $k = 0$ and $r_\tau = \mathbb{R}^k$ for all $\tau \in \mathcal{O}(V, \mathrm{SO}(2))$.

(6) The finite reflection group $W = \Delta_4' \rtimes S_4$ is an index 2 subgroup of the hyperoctahedral group $G = H_4$. Generators for $P(\mathbb{R}^4)^W$ are well-known (for details and references see [**70**]) and given by

$$p_1(x) = \frac{1}{2}\|x\|^2,\ p_2(x) = \frac{1}{4}\sum_{i=1}^4 x_i^4,\ p_3(x) = \frac{1}{6}\sum_{i=1}^6 x_i^6,\ p_4(x) = x_1 x_2 x_3 x_4.$$

Corresponding generators for the equivariants are given by $\phi_i = \mathrm{grad}(p_i)$. Hence every smooth W-equivariant map $f : \mathbb{R}^4 \to \mathbb{R}^4$ may be written in the form

$$f = \sum_{j=1}^{4} f_i(p_1, \dots, p_4) F_i, \tag{6.5}$$

where the $f_i \in C^\infty(\mathbb{R}^4)$ are uniquely determined by f. We define two, non-isomorphic, representations of $G = \Delta_4 \rtimes S_4$ on $\mathbb{R}^4$. The first representation of G will be the standard representation $\rho : G = H_4 \to \mathrm{O}(4)$ of H_4. For the second representation, define the homomorphism $\sigma : G \to \mathrm{O}(1) = \mathbb{Z}_2$ by mapping W to $+1$ and $G \setminus W$ to -1. We define $\rho_\sigma : G \to \mathrm{O}(4)$ by $\rho_\sigma(g) = \sigma(g)\rho(g)$. The new representation ρ_σ of G is absolutely irreducible and is not isomorphic to the first representation. We write the first representation as $(\mathbb{R}^4, G)$, the second as $(\mathbb{R}^4_\sigma, G)$. Since $G \supset W$, every $P \in P_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$ may be written in the form (6.5). Although $F_4 \in P_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$, the remaining generators $F_1, F_2, F_3 \notin P_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$. In order that $f_i(p_1, \dots, p_4)F_i \in P_{H_4}(\mathbb{R}^4, \mathbb{R}^4_\sigma)$, $i \neq 4$, it is necessary and sufficient that $f_i(p_1, \dots, p_4)(gx) = -f_i(p_1, \dots, p_4)(x)$, for all g such that $\sigma(g) = -1$. Similarly, $f_4(p_1, \dots, p_4)F_4 \in P_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$ only if $f_4(p_1, \dots, p_4)(gx) = f_4(p_1, \dots, p_4)(x)$, for all g such that $\sigma(g) = -1$. If we define

$$\bar{F}_i = p_4 F_i,\ i = 1, 2, 3,\ \bar{F}_4 = F_4,$$

then it is not hard to show that $\{\bar{F}_1, \dots, \bar{F}_4\}$ generates the $P(\mathbb{R}^4)^G$-module $P_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$. In particular, every $f \in C^\infty_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$ may be written (uniquely) in the form

$$f = \sum_{j=1}^{4} f_i(p_1, p_2, p_3, p_4^2) \bar{F}_i, \tag{6.6}$$

where $f_i \in C^\infty(\mathbb{R}^4)$. Here we have used the fact that p_1, p_2, p_3, p_4^2 generate $P(\mathbb{R}^4)^{H_4}$. Elements of $X \in C^\infty_G(\mathbb{R}^4, \mathbb{R}^4_\sigma)$ are reversible equivariant vector fields:

$$X(gv) = \sigma(g) g X(v), \quad (g \in G, v \in \mathbb{R}^4).$$

Let $\boldsymbol{\vartheta}(x,t) = \sum_{j=1}^4 t_j \bar{F}_j(x)$. Then $(x_1, x_2, x_3, x_4, t) \in \Sigma = \boldsymbol{\vartheta}^{-1}(0)$ if any two of x_1, x_2, x_3, x_4 are zero. Hence

$$R_\tau = \mathbb{R}^4, \text{ if } \tau = (G_{1,0,0,0}), (G_{1,1,0,0}), (G_{1,2,0,0}).$$

It is not hard to compute $\mathcal{R}(\mathbb{R}^4, \mathbb{R}^4_\sigma)$. For example, we have

$$R_{(G_{(1,1,1,0)})} = R_{(G_{(1,1,1,1)})} = \{t_4 = 0\},\ R_{(G_{(1,1,2,2)})} = \{t_1, t_4 = 0\},$$
$$R_{(G_{(1,1,2,3)})} = \{t_1, t_2, t_4 = 0\},\ R_{(G_{(1,2,3,4)})} = \{(0,0,0,0)\}.$$

All of this extends easily to the infinite family $\Delta'_k \rtimes S_k \subset \Delta_k \rtimes S_k$, $k \geq 4$.

(7) Let $G = \Delta_3 \rtimes \mathbb{Z}_3$ and $H = \Delta'_3 \rtimes \mathbb{Z}_3$ (the group of orientation preserving symmetries of the tetrahedron). We define two non-isomorphic representations $\rho, \zeta : G \to \mathrm{O}(3)$. The representation ρ will be the standard representation of G as a subgroup of H_3. For the second representation, take the standard action of H

on $\mathbb{R}^3$ and extend this representation to a representation ζ of G by requiring that $(-1,1,1) \in \Delta_3$ act on $\mathbb{R}^3$ by $(-1,1,1)(x,y,z) = (x,-y,-z)$. In other words, if we define the homomorphism $\eta : G = \Delta_3 \rtimes \mathbb{Z}_3 \to \mathrm{O}(1) = \mathbb{Z}_2$ by $\eta(\delta,\sigma) = \det(\delta)$, then $\zeta : G \to \mathrm{O}(3)$ is defined by $\zeta(g) = \eta(g)\rho(g)$. In particular, G-equivariant maps $\mathbb{R}^3 \to \mathbb{R}^3$ may be regarded as reversible equivariant vector fields on $\mathbb{R}^3$. The space $P_G(\mathbb{R}^3,\mathbb{R}^3)$ is a linear subspace of $P_H(\mathbb{R}^3,\mathbb{R}^3)$. Let $P = (P_1,P_2,P_3) \in P_G(\mathbb{R}^3,\mathbb{R}^3)$. Then $P_1(-x,y,z) = P_1(x,y,z)$, $P_1(x,-y,z) = -P_1(x,y,z)$ and $P_1(x,y,-z) = -P_1(x,y,z)$. Hence $P_1(-x,-y,-z) = P_1(x,y,z)$ and so P_1 is a sum of terms of even degree. By the H-equivariance we have $P_1(-x,-y,z) = -P_1(x,y,z) = -P_1(x,-y,-z)$. Now $P_1(x,y,-0) = -P_1(x,y,0)$ and so $P_1(x,y,0) = 0$. Similarly, $P_1(x,0,z) = 0$. Hence $P_1(x,y,z) = yzQ_1(x,y,z)$, where Q_1 is of even degree. Using the relations we have on P_1, we may write $Q_1(x,y,z) = R_1(x^2,y^2,z^2)$. If $f \in C^\infty_G(\mathbb{R}^3,\mathbb{R}^3)$, we may find a smooth map $r \in C^\infty(\mathbb{R}^3)$ such that

$$f(x,y,z) = \begin{pmatrix} r(x^2,y^2,z^2)yz \\ r(y^2,z^2,x^2)zx \\ r(z^2,x^2,y^2)xy \end{pmatrix}.$$

It is now easy to work out generators for the module of equivariants to fourth order (which is all we will need). Up to terms of degree 4, the universal polynomial is given by

$$\boldsymbol{\vartheta}(\mathbf{x},\mathbf{t}) = t_1 \begin{pmatrix} yz \\ zx \\ xy \end{pmatrix} + t_2 \begin{pmatrix} y^3z \\ z^3x \\ x^3y \end{pmatrix} + t_3 \begin{pmatrix} yz^3 \\ zx^3 \\ xy^3 \end{pmatrix}, \quad (\mathbf{x},\mathbf{t}) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Let $\Sigma = \boldsymbol{\vartheta}^{-1}(0)$ and X denote the union of the x-, y- and z-axes. Obviously, $X \times \mathbb{R}^3 \subset \Sigma$. If $t_1 \neq 0$ then the germ of $\boldsymbol{\vartheta}^{-1}(0,\mathbf{t}) \subset V$ equals the germ of X at zero. If $xyz \neq 0$, it is easy to show that if $t_2 + t_3 \neq 0$, then the solutions of $\boldsymbol{\vartheta}(\mathbf{x},\mathbf{t}) = 0$ satisfy $x^2 = y^2 = z^2$ and are given explicitly by $x^2 = \frac{-t_1}{t_2+t_3}$. If we let τ be the isotopy type of non-zero points on the line $\mathbb{R}(x,x,x)$, then $R_\tau = \{\mathbf{t} \mid t_1 = 0\}$. Next we look for solutions lying in the coordinate planes. Solutions in the plane $x = 0$ with $yz \neq 0$ are given as solutions of

$$t_1 + t_2y^2 + t_3z^2 = 0.$$

If $t_2t_3 > 0$, we obtain, for fixed $t_1 \neq 0$ either the empty set or an ellipse of solutions meeting both the y and z axes in a pair of points. If $t_2t_3 < 0$, we obtain, for fixed $t_1 \neq 0$ a hyperbola of solutions which meets either the y-axis or the z-axis (but not both axes). Any point on these solutions curves, not lying on an axis, has isotropy type $\kappa = (\langle \mathrm{diag}(-1,1,1)\rangle)$ Hence $R_\kappa = \{\mathbf{t} \mid t_1 = 0\}$. With a little more work, the codimension one stratum of the natural stratification of $\mathbb{U}$ is seen to be the semialgebraic set $\{\mathbf{t} \mid t_1 = 0, t_2 + t_3, t_2t_3 \neq 0\}$.

(8) In the previous example, it was the case that for a generic G-equivariant reversible vector field on $\mathbb{R}^3$ the germ of the zero set at the origin was always the

same: the germ of the union of the coordinate axes at zero. There is a corresponding result for G-equivariant vector fields X on any representation (V, G). If $V^G = \{0\}$, then the generic situation is that X has an isolated zero at the origin (even hyperbolic). If $\dim(V^G) > 0$, then generically we expect there to be no zero at the origin. If G is finite then we expect that for generic X the zero set of X will be a discrete subset of V. In our final example, we show that for generic reversible G-equivariant vector fields, with G finite and no parameters, the germ of the zero set may vary along V^G. We let $G = \Delta_3 \rtimes \mathbb{Z}_3$ act on $\mathbb{R}^3$ in the standard way and $\eta : G = \Delta_3 \rtimes \mathbb{Z}_3 \to \mathrm{O}(1) = \mathbb{Z}_2$ be as defined in the previous example. We define two inequivalent representations of G on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. The first representation, on the domain, will be the product of the given representation of G on $\mathbb{R}^3$ with the trivial representation of G on $\mathbb{R}$. We have $\mathbb{R}^4 = \{0\} \times \mathbb{R}$, The second representation, on the range, will be the product of the given representation of G on $\mathbb{R}^3$ with the representation of G on $\mathbb{R}$ defined by η. If $f \in C^\infty_G(\mathbb{R}^4, \mathbb{R}^4)$, it is straightforward to show that there exist smooth functions $r, s : \mathbb{R}^4 \to \mathbb{R}$ such that

$$f(x, y, z, u) = \begin{pmatrix} r(x^2, y^2, z^2, u)x \\ r(y^2, z^2, x^2, u)y \\ r(z^2, x^2, y^2, u)z \\ s(x^2, y^2, z^2, u)xyz \end{pmatrix}. \tag{6.7}$$

Up to terms of degree 3, the universal polynomial is given by

$$\boldsymbol{\vartheta}(\mathbf{x}, \mathbf{t}) = t_1 \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} xy^2 \\ yz^2 \\ zx^2 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} xz^2 \\ yx^2 \\ zx^2 \\ 0 \end{pmatrix} + t_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ xyz \end{pmatrix},$$

where $\mathbf{x} = (x, y, z, u) \in \mathbb{R}^4$, $\mathbf{t} \in \mathbb{R}^4$.

We look for solutions with isotropy type $\nu = (G_{(1,1,0,2)})$ – these live on the coordinate planes. Without loss of generality we assume $z = 0$. Automatically the u-component of F vanishes and we have to solve

$$\begin{aligned} t_1 x + t_2 xy^2 &= 0, \\ t_1 y + t_3 yx^2 &= 0. \end{aligned}$$

Solutions (x, y), with $xy \neq 0$. can only exist if $t_2 t_3 \geq 0$ (see example 4).

If $f \in C^\infty_G(\mathbb{R}^4, \mathbb{R}^4)$, then $f(0, 0, 0, u) = 0$, all $u \in \mathbb{R}$. Write $f \in C^\infty_G(\mathbb{R}^4, \mathbb{R}^4)$ in the form given by (6.7). Set $R(u) = r(0, 0, 0, u)$. For generic $r \in C^\infty(\mathbb{R}^4)$, R will have isolated non-singular zeros. Let u_0 be a non-singular zero of R. The germ of the zero set of f at $(0, 0, 0, u_0)$ will contain points of isotropy type ν if and only if

$$\frac{\partial r}{\partial y}(0, 0, 0, u_0) \frac{\partial r}{\partial z}(0, 0, 0, u_0) \geq 0.$$

Consequently, if $f \pitchfork_G 0$ along $\{0\} \times \mathbb{R}$, the germ of the zero set of f at distinct zeros $(0,0,0,u_1)$, $(0,0,0,u_2)$ may be different. As an explicit example take $r(x^2, y^2, z^2, u) = u^2 - 1 + y^2 + (2u-1)z^2$ and $s \equiv 1$. The zero sets at $(0,0,0,\pm 1)$ have different topological type.

EXERCISE 6.15.4. Investigate 1-parameter bifurcation theory for the reversible vector fields defined in examples 6.15.3(6). In particular, compare the stabilities for branches along $\mathbb{R}(1,1,1,1)$ and $\mathbb{R}(-1,1,1,1)$.

6.16. Notes on chapter 6

The theory of G-transversality or equivariant general position was developed independently by Bierstone [**14**] and the author [**50**] in 1976-77. Bierstone's definition of equivariant general position [**14**] is given in terms of transversality to the canonical stratification of the universal variety and leads to straightforward natural proofs of the main theorems using Thom-Mather theory of stratifications. The author's definition of G-transversality is less transparent (at least in [**50**], which is difficult to read) and is based on ideas from equisingularity. The definitions are, however, equivalent [**51**]. Subsequently, Bierstone developed the theory of equivariant jet transversality and applied it to prove stability theorems on smooth equivariant maps between manifolds [**15**] similar to those proved by Mather for general smooth maps. On the other hand the author's interest was in the theory of equivariant dynamical systems and his initial applications included, for example, an equivariant version of the Kupka-Smale density theorem [**52**]. Later in 1988, it became clear that equivariant transversality had the potential for significant applications to equivariant bifurcation theory. With Roger Richardson, a complete description was given of branching patterns for a large class of finite reflection groups using ideas based on equivariant transversality [**70**] (see Stewart [**165**] for an overview). The way in which equivariant transversality was being used in bifurcation theory tended to reflect the equisingularity definition of G-transversality rather than the definition based on transversality to the universal variety. This point of view was developed in [**57, 60, 62**] and, together with results from equivariant jet transversality, enabled the proof of quite general genericity and stability theorems for equivariant bifurcation.

In this chapter we have taken some care to develop ideas from [**50**] which have potential application to equivariant bifurcation theory. In particular, we showed how it is possible to give a natural definition of G-transversality in terms of transversality to the intrinsic stratification $\mathcal{A}$ of $\mathbb{U}$. As we see in the next chapter, this characterization has powerful applications to equivariant bifurcation theory. We also developed some of the work in [**50**] so as to show that there is a natural symmetry group $\mathcal{G}$ of $\mathbb{U}$ which preserves $\mathcal{A}$. In some cases, knowledge of $\mathcal{G}$ determines $\mathcal{A}$ quite precisely.

CHAPTER 7

Applications of G-transversality to bifurcation theory I

In this chapter we apply equivariant transversality to equivariant bifurcation theory. Our focus will be on steady state bifurcation theory for 1-parameter families of equivariant vector fields defined on an absolutely irreducible representation (V, G), where G is a finite group. We also briefly consider the theory for equivariant reversible vector fields [**31**]. Most of what we do applies when G is a general compact Lie group and we sketch this extension at the end of the chapter. As part of our exposition, we include definitions and results on jet transversality for equivariant maps. This theory was developed by Bierstone in connection with his work generalizing Mather's theory of stable mappings to equivariant maps [**15**]. All the results we need will be local in character and we shall not develop the theory of jet transversality for G-manifolds.

In chapter 10, we allow for G to be a compact (non-finite) Lie group, and for (V, G) to be irreducible over the complexes. We then consider the natural questions of branching to relative equilibria, limit cycles (the equivariant Hopf bifurcation) and relative periodic orbits.

7.1. Weak stability and Determinacy

7.1.1. Generic 1-parameter steady-state bifurcation theory. Let G be a finite group. In this section we sketch how the generic 1-parameter steady state bifurcation theory of equivariant vector fields on a representation (V, G) can be reduced to the study of normalized families of equivariant vector fields on an absolutely irreducible representation (V, G). For simplicity, we assume (V, G) is finite dimensional, even though the most interesting case from the point of view of applications is when (V, G) is an infinite dimensional G-representation. If we are only interested in determining branches of equilibria – as opposed to the associated dynamics – it is possible to avoid this process of reduction, see remarks 7.1.10(3) and also [**84**]. In what follows we only consider bifurcation of the trivial solution $x = 0$ for a family $X \in C^\infty_G(V \times \mathbb{R}, V)$ (for bifurcation from relative equilibria, see chapter 10).

Suppose that (V, G) is a finite-dimensional G-representation and $V^G = \{0\}$ ((V, G) contains no proper trivial subrepresentations). Let $X \in C^\infty_G(V \times \mathbb{R}, V)$ be a 1-parameter family of vector fields on V. Since $V^G = \{0\}$, we have by G-equivariance and Taylor's theorem that

$$X(\mathbf{x}, \lambda) = A(\lambda)\mathbf{x} + O(\|\mathbf{x}\|^2), \quad (\mathbf{x}, \lambda) \in V \times \mathbb{R},$$

where $A(\lambda) \in L_G(V, V)$. We regard $(0, \lambda_0)$ (or just λ_0) as a bifurcation point for the family if $A(\lambda_0)$ is singular (has a zero eigenvalue[1]). In terms of the isotypic decomposition of (V, G) (theorem 2.7.10), we may write

$$L_G(V, V) = \bigoplus_{i=1}^{n} L_G(V_i^{p_i}, V_i^{p_i}),$$

where $\{(V_i, G) \mid i = 1, \ldots, n\}$ is a set of inequivalent proper $\mathbb{R}$-irreducible representations of G and $p_i \geq 1$, $i = 1, \ldots, n$. Hence $A(\lambda) = \bigoplus_{i=1}^n A_i(\lambda)$, where $A_i(\lambda) \in L_G(V_i^{p_i}, V_i^{p_i})$, $i = 1, \ldots, n$.

Relabelling, suppose (V_i, G) is absolutely irreducible, $1 \leq i \leq a$. Set $\dim(V_i) = n_i$. By proposition 2.7.18, $L_G(V_i^{p_i}, V_i^{p_i}) \cong M(p_i, p_i; \mathbb{R})$. In particular, every eigenvalue of $A_i(\lambda_0)$ has multiplicity divisible by n_i. Let Σ_i denote the closed subset of $M(p_i, p_i; \mathbb{R})$ consisting of singular matrices. The set $\Sigma_i^1 \subset \Sigma_i$ consisting of matrices of rank $p_i - 1$ is a codimension 1 submanifold of $M(p_i, p_i; \mathbb{R})$ and $\Sigma_i \setminus \Sigma_i^1$ is of codimension at least 2 (Σ_i^2 may be written as a union of submanifolds of increasing codimension [**1**]). Identify $L_G(V_i^{p_i}, V_i^{p_i})$ with $M(p_i, p_i; \mathbb{R})$. Using our codimension estimates, an application of transversality theory shows that we can make a C^∞-small perturbation of X so that

(1) $A_i \pitchfork \Sigma_i^1$ on $\mathbb{R}$ and $A_i(\mathbb{R}) \cap \Sigma_i^2 = \emptyset$, for all $1 \leq i \leq a$.
(2) If $A_{i_0}(\lambda) \in \Sigma_{i_0}$, then $A_i(\lambda) \notin \Sigma_i$, $i \neq i_0$, $1 \leq i \leq a$.

(We achieve (1) by elementary transversality theory applied to each A_i. Using the fact that points λ such that $A_i(\lambda) \in \Sigma_i^1$ are isolated for each i, we achieve (2) by a further perturbation of A_i.)

Suppose (V_i, G) is irreducible of complex type. In this case, $L_G(V_i^{p_i}, V_i^{p_i}) \cong M(p_i, p_i; \mathbb{C})$ and the set Σ_i of singular matrices is a submanifold of $M(p_i, p_i; \mathbb{C})$ of *complex* codimension 1. By transversality theory we can make a C^∞-small perturbation of X so that $A_i(\mathbb{R}) \cap \Sigma_i = \emptyset$. Similarly, if (V_i, G) is irreducible of quaternionic type, we can always assume $A_i(\mathbb{R}) \cap \Sigma_i = \emptyset$.

Let K be a compact interval[2] containing the origin of $\mathbb{R}$. Our transversality arguments show there is an open and dense subset $\mathcal{Z}(K)$ of $C_G^\infty(V \times \mathbb{R}, V)$ (C^∞-topology) such that if $X \in \mathcal{Z}(K)$ then

(1) Bifurcation points for X (on $\{0\} \times K$) are isolated in K.
(2) If $\lambda_0 \in K$ is a bifurcation point, there is a smooth curve $\eta(\lambda)$ of eigenvalues of multiplicity $k \geq 1$ for $DX_\lambda(0)$ such that (a) $\eta(\lambda_0) = 0$, (b) $\mathrm{kernel}(DX_{\lambda_0}(0))$ is isomorphic to an absolutely irreducible subrepresentation of (V, G) of degree k, and (c) $\eta'(\lambda_0) \neq 0$.

Suppose that $X \in \mathcal{Z}(K)$ and $\lambda_0 \in K$ is a bifurcation point. Let (W, G) denote the absolutely irreducible subrepresentation of (V, G) given by (2) above.

[1]We do not consider the Hopf bifurcation in this chapter and so we regard points where eigenvalues are pure imaginary as regular points.

[2]If we take the Whitney C^∞-topology, we can assume K is a closed interval

Given $1 \leq r < \infty$, we may apply the equivariant version of the centre manifold theorem to construct a C^r G-invariant submanifold $C = C_r \subset V \times \mathbb{R}$ such that

(1) $(0, \lambda_0) \in C$ and the tangent space to C at $(0, \lambda_0)$ is equal to the direct sum of (W, G) and the trivial representation $(\{0\} \times \mathbb{R}, G)$.
(2) $\tilde{X}(x, \lambda) = (X(x, \lambda), 1)$ is everywhere tangent to C.

(For the centre manifold theorem see [**88**].)

This process allows us to investigate bifurcation and changes in the dynamics at bifurcation points of $X \in \mathcal{Z}(K)$ by reducing to the study of equivariant families $X \in C^r_G(W \times \mathbb{R}, W)$, where $2 \leq r < \infty$ and (W, G) is absolutely irreducible. Just as in chapter 4, it is no loss of generality to restrict attention to families X which are in the normalized form

$$X(\mathbf{x}, \lambda) = \lambda \mathbf{x} + g(\mathbf{x}, \lambda),$$

where $g(\mathbf{x}, \lambda) = O(\|\mathbf{x}\|^2)$, uniformly for λ in some compact interval containing the origin.

While it is well-known we cannot assume the centre manifold C is smooth – that is C^∞ – we can require C to be C^r for any $r < \infty$. This suffices for the determination of the signed indexed branching patterns as G-equivariant bifurcation problems on an absolutely irreducible representation are generically *strongly determined* [**60**, §3.9]. That is, there exists $d < \infty$ and an open dense subset $\mathcal{S}(d) \subset P^{(d)}_G(V, V)$ such that if X is C^d and $J^d(X) = j^d X_0(0) \in \mathcal{S}(d)$, then $F(x, \lambda) = \lambda x + J^d(X)(x)$ is stable and X has a signed indexed branching pattern isomorphic to the signed indexed branching pattern of F. In particular, the signed indexed branching pattern of X is unchanged by the addition of higher order G-equivariant terms to X. The proof of this result requires techniques beyond the scope of these notes and we refer the reader to [**60**] for details (we discuss strong determinacy in chapter 10). In many cases it is not hard to compute d. For example, all of the examples we considered in chapter 4 had $d = 2$ or $d = 3$. Issues with dynamics are more subtle. For example, equivariant transversality is very much a smooth theory though it is possible to prove stability of G-transverse intersections under perturbation by sufficiently differentiable G-equivariant maps.

7.1.2. Steady-state bifurcation theory on an absolutely irreducible representation. Let (V, G) be an absolutely irreducible representation of the finite group G. Let $\mathcal{V}_0 = \mathcal{V}_0(V, G) \subset C^\infty(V \times \mathbb{R}, V)$ denote the space of normalized 1-parameter families of smooth G-equivariant vector fields on V.

Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a minimal set of homogeneous generators for the $P(V)^G$-module $P_G(V, V)$. As usual, we suppose that $F_1 = I_V$ and $1 < d_2 \leq \ldots \leq d_k$, where $\deg(F_i) = d_i$. We define

$$\mathcal{K}_G(V) = \{X \in \mathcal{V}_0 \mid X \pitchfork_G 0, \text{ at } 0 \in V\}.$$

Following chapter 6, let $\mathcal{A} = \{A_j \mid 0 \leq j \leq k\}$ denote the stratification of $\mathbb{U} \cong \mathbb{R}^k$ induced from the minimal Whitney stratification of the universal variety $\Sigma^{\mathcal{F}}$.

LEMMA 7.1.1. $A_j \subset \mathbb{R}^{k-1}$, $0 \leq j < k$.

PROOF. If $t_1 \neq 0$, then (x,t) is a non-singular zero of $x \mapsto \boldsymbol{\vartheta}(x,t)$ and so $(0,t) \notin \overline{\Sigma}_\tau \cap \mathbb{R}^k$, $\tau \neq (G)$. $\square$

REMARK 7.1.2. Lemma 7.1.1 implies that A_{k-1} is an open semialgebraic subset of $\mathbb{R}^{k-1}$. In all examples known to us, $A_k \cap \mathbb{R}^{k-1} = \emptyset$ and so A_{k-1} is ain open and dense subset of $\mathbb{R}^{k-1}$. If $A_k \cap \mathbb{R}^{k-1} \neq \emptyset$, there exist stable families in $\mathcal{V}_0(V,G)$ for which there are *no* nontrivial branches as λ increases through zero. To allow for this possibility, we will enlarge A_{k-1} by adding to A_{k-1} the (open) subset $A_k \cap \mathbb{R}^{k-1}$ of $\mathbb{R}^{k-1}$. We continue to let $\mathcal{A}$ denote the associated Whitney stratification of $\mathbb{U} \cong \mathbb{R}^k$. With this convention, we always have A_{k-1} is an open and dense semialgebraic subset of $\mathbb{R}^{k-1}$.

EXERCISE 7.1.3. Show that if (V,G) has an odd dimensional fixed point space, then A_{k-1} is a dense subset of $\mathbb{R}^{k-1}$ (Hint: Use Cicogna's extension of the equivariant branching lemma [**34**]).

Given $X \in \mathcal{V}_0$, let $\gamma(X) : \mathbb{R} \to \mathbb{R}^k$ denote the 'coefficient' map defined by $\gamma(X)(\lambda) = (\lambda, f_2(0,\lambda), \ldots, f_k(0,\lambda))$, where $X(x,\lambda) = \sum_j f_j(x,\lambda) F_j(x)$.

LEMMA 7.1.4. *Let $X \in \mathcal{V}_0$. Then $X \in \mathcal{K}_G(V)$ if and only if $\gamma(X)(0) \in A_{k-1}$.*

PROOF. If $X \in \mathcal{K}_G(V)$, then $\gamma(X) \pitchfork \mathcal{A}$ at $\lambda = 0$. Since $\gamma(X)(0) \in \cup_{i<k} A_i = \mathbb{R}^{k-1}$, $\gamma(X) \pitchfork \mathcal{A}$ at $\lambda = 0$ if and only if $\gamma(X) \pitchfork A_i$, $i < k$. That is, $\gamma(X)(0) \in A_{k-1}$ (since $\operatorname{codim}(A_i) = k - i$). $\square$

For $X \in \mathcal{V}_0$, let $\mathbf{Z}(X)$ denote the zero set of X.

THEOREM 7.1.5.

(1) *$\mathcal{K}_G(V)$ is an open and dense subset of $\mathcal{V}_0$.*
(2) *If $X \in \mathcal{K}_G(V)$, then X satisfies the branching condition B1.*
(3) *We may choose an open neighborhood $\mathcal{U}$ of X in $\mathcal{V}_0$ such that if $\{X_t \mid t \in [0,1]\}$ is any continuous path in $\mathcal{U}$ with $X_0 = X$, there is an open neighborhood W of $(0,0)$ in $V \times \mathbb{R}$ and an (equivariant) isotopy $\{K_t : W \to V \times \mathbb{R} \mid t \in [0,1]\}$ of (continuous) embeddings satisfying*
    (a) *K_0 is the inclusion of W in $V \times \mathbb{R}$.*
    (b) *$K_t(W \cap \mathbf{Z}(X)) = \mathbf{Z}(X_t) \cap K_t(W)$, all $t \in [0,1]$.*
(4) *$\mathcal{K}_G(V) \subset \mathcal{S}_w(V,G)$.*

PROOF. An application of theorem 6.14.4 gives statements (1,3) and (1,2,3) together imply (4). It remains to prove (2). Fix a minimal set $\mathcal{F}$ of homogeneous polynomial generators for $P_G(V,V)$ and let $\mathcal{A}_\mathcal{F}$ be the corresponding Whitney stratification of $\mathbb{R}^k$. Let $X \in \mathcal{K}_G(V)$ and suppose that $\gamma(X)(0) \in R_\tau = \mathbb{R}^k \cap \partial\Sigma_\tau$ (if no such $\tau \neq (G)$ exists, there are no nontrivial branches and there is nothing to prove). Since $\gamma(X) \pitchfork \mathcal{A}_\mathcal{F}$, $\gamma(X)(0)$ lies in a codimension one stratum C of $\mathcal{A}_\mathcal{F}$. Since $\mathcal{A}_\mathcal{F}$ is a union of strata from the canonical stratification $\mathcal{S}$ of Σ

(theorem 6.10.1), the frontier condition implies that (C, Σ_τ) is Whitney regular and so locally trivial along C (we are not claiming $\Sigma_\tau \in \mathcal{S}$). Hence we may choose open semialgebraic neighbourhoods W of $(0, \gamma(f)(0)) \in V \times \mathbb{R}^k$, W_1 of $(0,0) \in V \times \mathbb{R}$ and W_2 of $\gamma(X)(0)$ in $C \subset \mathbb{R}^{k-1}$ such that

(a) The intersection of W with Σ_τ consists of finitely many connected components $S_1, \dots, S_N$.
(b) If we set $K_j = (W_1 \times \{\gamma(X)(0)\}) \cap S_j$, then K_j is a (smooth) semialgebraic arc with boundary point $\{\gamma(X)(0)\}$, $j = 1, \dots, N$.
(c) There is a homeomorphism h of W onto $W_1 \times W_2$ which is equal to the identity on $\{0\} \times W_2$ and maps S_j onto $K_j \times W_2$, $j = 1, \dots, N$.

Since $\Gamma_X \pitchfork \Sigma$ at $(0,0)$ and $\Gamma_X(0,0) = (0, \gamma(X)(0))$, we may choose an open neighbourhood U of $(0,0) \in V \times \mathbb{R}$ such that $\Gamma_X \pitchfork \Sigma_\tau$ on U and $\Gamma_X(U) \subset W$. For $j = 1, \dots, N$, define $C_j = (\Gamma_X^{-1}|U)^{-1}(\overline{S_j})$. Each C_j has the structure of a 1-dimensional Whitney stratified set. If X is analytic, the curve selection lemma [**12, 20**] yields C^1 arcs $\gamma_j : [0, \delta] \to C_j$ such that $\gamma_j(0,0) = (0,0)$ and $\gamma_j'(0) \neq 0$. That is, each C_j is a solution branch. If we do not assume the analyticity of X, then we may use a result of Pawlucki [**140**] which implies that each semialgebraic set $W_2 \cup S_j$ has the structure of a C^1 submanifold of W (Pawlucki's result requires that $\dim(S_j) = \dim(W_2) + 1$). Standard transversality theory implies that C_j is a C^1 submanifold with boundary $(0,0)$ and the result follows as before. We apply the same argument to all nontrivial isotropy types τ for which $\gamma(X)(0) \in R_\tau$. □

7.1.3. Symmetry breaking isotropy types. If $\tau \in \mathcal{O}^\star(V, G)$, then $R_\tau = \overline{\Sigma}_\tau \cap \mathbb{R}^k$ and $r_\tau = \dim(R_\tau)$ (section 6.15). By lemma 6.15.1, we have

$$k - d_\tau \leq r_\tau \leq k - 1. \tag{7.1}$$

LEMMA 7.1.6. *For all $\tau \in \mathcal{O}^\star(V, G)$, $\mathcal{A}$ induces a semialgebraic Whitney stratification $\mathcal{A}_\tau$ of R_τ. (We assume that $A_k = \mathbb{R}^k \setminus \mathbb{R}^{k-1}$, see remark 7.1.2.)*

PROOF. Immediate from theorem 6.10.1 and the definition of $\mathcal{A}$. □

PROPOSITION 7.1.7. *(Notation as above.) Let $X \in \mathcal{K}_G(V)$ and $\tau \in \mathcal{O}^\star(V, G)$.*

(1) *The map $\gamma(X) : \mathbb{R} \to \mathbb{R}^k$ is transverse to $\mathcal{A}_\tau$.*
(2) *If $r_\tau < k - 1$, then $\Sigma(X)$ contains no branches of isotropy type τ.*
(3) *If $r_\tau = k - 1$ and $\gamma(X)(0) \in R_\tau$, then there is a branch of equilibria of isotropy type τ in $\Sigma(X)$.*

PROOF. Statements (1,2) follow from lemma 7.1.6. For (3), observe that if $\gamma(X)(0) \in R_\tau$, then $\gamma(X)(0)$ lies in a codimension 1 stratum of $\mathcal{A}_\tau$ and theorem 7.1.5 implies that there is a corresponding branch of equilibria of isotropy type τ in $\Sigma(X)$. □

COROLLARY 7.1.8. *(Notation as above.) Let $\tau \in \mathcal{O}^\star(V, G)$.*

(1) *τ is a symmetry breaking isotropy type if and only if $r_\tau = k - 1$.*

(2) *τ is generically symmetry breaking if and only if $R_\tau = \mathbb{R}^{k-1}$.*

7.1.4. Weak determinacy. Set $d = d_k$ (the maximum degree of polynomials in a minimal homogenous set of generators for $P_G(V,V)$). Let $P_G^{(d)}(V,V)_0 = \{P \in P_G^{(d)}(V,V) \mid DP(0) = 0\}$. Let $\Pi_0 : P_G^{(d)}(V,V)_0 \to \mathbb{U}$ be the restriction of the projection $\Pi : P_G(V,V) \to P_G(V,V)/\mathfrak{M}P_G(V,V) = \mathbb{U}$ to $P_G^{(d)}(V,V)_0$. Observe that Π_0 maps $P_G^{(d)}(V,V)_0$ onto $\mathbb{U}_1 \cong \mathbb{R}^{k-1}$ and $\mathbb{U}_1 = P_G^{(d)}(V,V)_0/\mathfrak{M}P_G(V,V)$ is an invariantly defined subspace of $\mathbb{U}$. Define $\mathcal{R}_w(d) \subset P_G^{(d)}(V,V)_0$ by

$$\mathcal{R}_w(d) = \{P \in P_G^{(d)}(V,V)_0 \mid \Pi_0(P) \in A_{k-1}\}.$$

Since A_{k-1} is an open and dense semialgebraic subset of $\mathbb{U}_1$, it follows that $\mathcal{R}(d)$ is an open and dense semialgebraic subset of $P_G^{(d)}(V,V)_0$. We recall that if $X \in C_G^\infty(V \times \mathbb{R}, V)$, then $J^d(X) \in P_G^{(d)}(V,V)$ is the degree d Taylor polynomial of X_0 at the origin. That is, $J^d(X)(x) = \sum_{j=2}^d D^j X(0)(x^j)/j!$.

LEMMA 7.1.9. *We have*

$$\mathcal{K}_G(V) = \{X \in \mathcal{V}_0 \mid J^d(X) \in \mathcal{R}_w(d)\}$$

and so G-equivariant bifurcation problems on (V,G) are weakly d_w-determined, where $d_w \le d_k$.

PROOF. Immediate from the definitions. □

REMARKS 7.1.10. (1) A consequence of lemma 7.1.9 is that as far as weak stability is concerned, we can always assume that $X_\lambda(x) = \lambda x + g(x)$, where $g = O(\|x\|^2)$ is *independent* of λ.
(2) If $X \in \mathcal{K}_G(V)$ then $X_a = X + a\|x\|^2 x \in \mathcal{K}_G(V)$ for all $a \in \mathbb{R}$. Indeed, we can add any radial term of order greater than or equal to 3 without changing the weak stability of X. Consequently, if $P_G^2(V,V) = \{0\}$, we can always choose $a \ll 0$ and use methods based on the invariant sphere theorem to classify branching patterns. Of course, varying a can be expected to change both stabilities and direction of branching.
(3) If (V,G) is not absolutely irreducible and we just assume that $V^G = \{0\}$, much of our analysis continues to apply. In particular, after a change of coordinates, we can make the generic assumption that at the bifurcation point $\lambda = 0$ we have $DX_0(0) = 0 \oplus A : V_1 \oplus V_2 \to V_1 \oplus V_2$, where V_1 is absolutely irreducible, $DX_0(0)|V_2$ is non-singular. If we let $\pi_1 : V \to V_1$ denote the projection on V_1, we may write $\pi_1 DX_0(0)|V_1 = \eta(\lambda) I_{V_1}$, where η is smooth near $\lambda = 0$. If we make the additional generic assumption that $\eta'(0) \neq 0$ then, after a smooth local change of coordinates, we may write

$$X(\mathbf{x}, \lambda) = (\lambda \mathbf{x}_1, A(\lambda)(\mathbf{x}_2)) + (g_1(\mathbf{x}, \lambda), g_2(\mathbf{x}, \lambda)),$$

where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in V$, $A(0) \in \mathrm{GL}(V_2)$, $g_1, g_2 = O(\|\mathbf{x}\|^2)$ and $g_2(\mathbf{x}_1, 0, \lambda) = 0$, for $\mathbf{x}_1, \lambda$ in a neighbourhood of $(0,0) \in V_1 \times \mathbb{R}$. Let $\mathcal{V}_0$ denote the set of smooth

G-equivariant families which are in this normalized form. Just as before, we define $\mathcal{K}_G(V) = \{X \in \mathcal{V}_0 \mid X \pitchfork_G 0, \text{ at } 0 \in V\}$. Using lemma 6.6.9, it is easy to prove that d-determinacy holds, where d is again the maximal degree of a minimal homogenous set of generators for $P_G(V_1, V_1)$.
(5) Suppose that (V, G) is absolutely irreducible. A consequence of lemma 7.1.9 is that as far as the weak stability and determinacy of elements of $\mathcal{V}_0$ are concerned, the structure of the $\mathbb{R}$-algebra G-invariants $P(V)^G$ is *irrelevant*. As we see later, similar comments hold when we come to consider general compact Lie groups, relative equilibria and the equivariant Hopf bifurcation. This is one of the main reasons why we prefer not to work at the orbit space level where the invariants, of course, play a major role. When we come to analyse stabilities along branches, invariants enter into stability computations. However, for generic families, we can always assume that the higher order terms are independent of the parameter λ and that the coefficient functions f_j for $X \in \mathcal{V}_0$ depend linearly on the elements of a generating set for the invariants. These observations allow one to make effective computations for generic families that would be quite intractable if we simply worked with truncations of the Taylor series. We gave one illustration of the effectiveness of this approach in our analysis of the Hopf bifurcation using blowing-up techniques.

7.1.5. Weak stability of equivariant reversible vector fields. Suppose we are given a pair of representations $\rho, \sigma : G \to \mathrm{O}(V)$ of the (finite) group G. Let ${}_\rho V$ denote V with the action on V determined by ρ. We similarly define ${}_\sigma V$. To start with we assume that ${}_\rho V^G = {}_\sigma V^G = \{0\}$.

We consider G-equivariant vector fields $X : {}_\rho V \to {}_\sigma V$. In the simplest case, we take the *same* action of G on source and target. This amounts to looking at G-equivariant vector fields defined on a G-representation. In the case of reversible equivariant vector fields, we fix an index two subgroup K of G and consider representations ρ, σ which restrict to the same representation on K but differ by a sign on $G \setminus K$. That is, $\rho(g) = -\sigma(g)$, $g \in G \setminus K$.

Let $\mathcal{F} = \{F_1, \dots, F_k\}$ denote a minimal set of homogeneous generators for the $P({}_\rho V)^G$-module $P_G({}_\rho V, {}_\sigma V)$. Set degree$(F_j) = d_j$ and index the F_j so that $1 \le d_1 \le \dots \le d_k$. As we are not assuming that ${}_\rho V$, ${}_\sigma V$ contain any isomorphic subrepresentations, it is possible that $d_1 \ge 2$. Let $\Sigma \subset {}_\rho V \times \mathbb{R}^k$ denote the universal variety.

Let $\mathcal{V} = C^\infty_G({}_\rho V \times \mathbb{R}, {}_\sigma V)$. If $X \in \mathcal{V}$ and ${}_\rho V$ and ${}_\sigma V$ are inequivalent representations, then X_λ has a singular zero at the origin ($\det(DX_\lambda(0)) = 0$) for all $\lambda \in \mathbb{R}$. Hence every point on the line $\{0\} \times \mathbb{R} \subset {}_\rho V \times \mathbb{R}$ is a 'bifurcation point' for the family X according to our earlier definition of bifurcation point. In order to handle this situation, we need to redefine what we mean by bifurcation point. Roughly speaking, a bifurcation point λ_0 will be a value of λ where the topological type of the germ of $\mathbf{Z}(X_\lambda) = X_\lambda^{-1}(0)$ at zero changes.

Let $\mathcal{A} = \{A_j \mid 0 \leq j \leq k\}$ denote the canonical stratification of $\mathbb{U} \cong \mathbb{R}^k$ induced from the minimal Whitney stratification $\mathcal{S}$ of the universal variety Σ ($\mathcal{A}$ may not be a union of $\mathcal{S}$ strata.) Given $X \in \mathcal{V}_0$, let $\gamma(X) : \mathbb{R} \to \mathbb{R}^k$ denote the coefficient map $\gamma(X)(\lambda) = (f_1(0,\lambda), f_2(0,\lambda), \ldots, f_k(0,\lambda))$, where $X(x,\lambda) = \sum_{j=1}^k f_j(x,\lambda)F_j(x)$.

DEFINITION 7.1.11. We say that λ is a *regular* point for the family $X \in \mathcal{V}$ if $\gamma(X)(\lambda) \notin \cup_{j=1}^{k-1} A_j$. If $\gamma(X)(\lambda) \in \cup_{j=1}^{k-1} A_j$, we refer to λ as a bifurcation point.

Let $\mathcal{V}_0$ be the subset of $\mathcal{V}$ consisting of families which have a bifurcation point at $\lambda = 0$. Define

$$\mathcal{K}_G({}_\rho V, {}_\sigma V) = \{X \in \mathcal{V}_0 \mid \gamma(X)(0) \in A_{k-1} \text{ and } \gamma(X) \pitchfork A_{k-1} \text{ at } 0\}.$$

LEMMA 7.1.12. (1) *If $X \in \mathcal{K}_G({}_\rho V, {}_\sigma V)$, then $\lambda = 0$ is an isolated bifurcation point of X.*
(2) $\mathcal{K}_G({}_\rho V, {}_\sigma V) = \{X \in \mathcal{V}_0 \mid X \pitchfork_G 0$ *at* $(0,0)\}$.
(3) $\mathcal{K}_G({}_\rho V, {}_\sigma V)$ *is an open and dense subset of* $\mathcal{V}_0$.

PROOF. If $\gamma(X)(0) \in A_{k-1}$ and $\gamma(X) \pitchfork A_{k-1}$ at 0, then there exists an open interval I containing $0 \in \mathbb{R}$ such that $(\gamma(X)|I)^{-1}(A_{k-1}) = \{0\}$. Since $\mathcal{A}$ is a Whitney stratification, we may suppose that I is chosen sufficiently small so that $(\gamma(X)|I)^{-1}(A_j) = \emptyset$, $j < k-1$, proving (1). Statements (2,3) follow in the usual way from theorem 6.10.6 and properties of equivariant transversality. $\square$

An application of the isotopy theorem for equivariant transversality yields the following stability result.

LEMMA 7.1.13. *If $X \in \mathcal{K}_G({}_\rho V, {}_\sigma V)$, we may choose an open neighborhood $\mathcal{U}$ of X in $\mathcal{K}_G({}_\rho V, {}_\sigma V)$ such that if $\{X_t \mid t \in [0,1]\}$ is any continuous path in $\mathcal{U}$ with $X_0 = X$, there is an open neighborhood W of $(0,0)$ in ${}_\rho V \times \mathbb{R}$ and a G-equivariant isotopy $\{K_t : W \to {}_\rho V \times \mathbb{R} \mid t \in [0,1]\}$ of (continuous) embeddings satisfying*
(a) *K_0 is the inclusion of W in ${}_\rho V \times \mathbb{R}$.*
(b) *$K_t(W \cap \mathbf{Z}(X)) = \mathbf{Z}(X_t) \cap K_t(W)$, all $t \in [0,1]$.*

As we did for equivariant vector fields, we may define weak stability and weak determinacy. Families in $\mathcal{K}_G({}_\rho V, {}_\sigma V)$ will be weakly stable and 1-parameter families of G-equivariant vector fields from ${}_\rho V$ to ${}_\sigma V$ will be weakly d-determined, where $d \leq d_k$.

We assumed that ${}_\sigma V^G = \{0\}$. If ${}_\sigma V^G$ is 1-dimensional, then bifurcation points for a family $X \in \mathcal{V}$ are defined exactly as before. In this case if λ is not a bifurcation point then $X_\lambda(0) \neq 0$. If $\dim({}_\sigma V^G) > 1$, then there are no bifurcation points in generic 1-parameter families. We can allow for $\dim({}_\rho V^G) > 0$ by considering multiparameter families on a representation ${}_\rho V^G$ with ${}_\rho V^G = \{0\}$. In this case it is natural to introduce the idea of a distinguished bifurcation parameter (cf [**83**]).

EXAMPLE 7.1.14. We consider 1-parameter families of reversible equivariant vector fields between the inequivalent representations ${}_\rho V$, ${}_\sigma V$ of H_4 on $\mathbb{R}^4$ defined in examples 6.15.3(6). We recall that a minimal set of homogeneous polynomial generators for $P_{H_4}({}_\rho V, {}_\sigma V)$ was given by

$$\begin{aligned} F_1(x_1,\dots,x_4) &= x_1x_2x_3x_4(x_1,\dots,x_4), \\ F_2(x_1,\dots,x_4) &= x_1x_2x_3x_4(x_1^3,\dots,x_4^3), \\ F_3(x_1,\dots,x_4) &= x_1x_2x_3x_4(x_1^5,\dots,x_4^5), \\ F_4(x_1,\dots,x_4) &= (x_2x_3x_4,\dots,x_1x_2x_3). \end{aligned}$$

It is straightforward to verify that if $\mathcal{A} = \{A_0,\dots,A_4\}$ is the canonical stratification of $\mathbb{R}^4$, then $A_0 \cup \dots \cup A_3 = \{t \in \mathbb{R}^4 \mid t_4 = 0\}$ and $A_3 = \{t \mid t_4 = 0, t_1 \neq 0\}$. If $X \in C^\infty_G({}_\rho V \times \mathbb{R}, {}_\sigma V)$, we may write

$$X(x,\lambda) = \sum_{j=1}^{4} f_j(x,\lambda) F_j(x).$$

Hence λ is a bifurcation point for X if $f_4(0,\lambda) = 0$. If we suppose that $X \in \mathcal{V}_0$, then $X \in \mathcal{K}_G({}_\rho V, {}_\sigma V)$ if and only if $f_1(0,0) \neq 0$ and $\frac{\partial f_1}{\partial \lambda}(0,0) \neq 0$. In this case 1-parameter families of H_4-equivariant vector fields from ${}_\rho V$ to ${}_\sigma V$ are weakly 4-determined. We refer to [**31**] for many more examples and references.

7.2. Jet transversality

In this section we describe some of the main definitions and results of Bierstone's jet transversality theory for equivariant maps [**15**]. We start by recalling the theory when there is no symmetry (for details and proofs see [**1, 92, 81**]).

Let M, N be smooth manifolds and $r \in \mathbb{N}$. Let $J^r(M,N)$ denote the bundle of r-jets from M to N (see chapter 6, section 6.2.1). If $f \in C^\infty(M,N)$, let $j^r f : M \to J^r(M,N)$ denote the r-jet extension map.

We recall from chapter 6 the local structure of $J^r(M,N)$. Suppose V, W are finite dimensional vector spaces. For $j \in \mathbb{N}$, let $L^j_s(V,W)$ denote the space of symmetric j-linear maps from V to W and note that $L^j_s(V,W) \approx P^j(V,W)$ by mapping the symmetric j-linear map A to the polynomial P defined by $P(x) = A(x,x,\dots,x)/j! = A(x^j)/j!$ For $r \in \mathbb{N}$, we define the jet space $J^r(V,W) = V \times \prod_{j=0}^{r} L^j_s(V,W)$. If we let $P^{(r)}(V,W)_0$ denote the vector space of of polynomial maps from V to W which are of degree r and vanish at the origin of V, then

$$J^r(V,W) \approx V \times W \times P^{(r)}(V,W)_0 \approx V \times P^{(r)}(V,W).$$

If $f \in C^\infty(V,W)$, then

$$j^r f(x) = (x, f(x), Df(x), \dots, D^r f(x)) \in J^r(V,W).$$

If we view $J^r(V,W)$ as $V \times P^{(r)}(V,W)$, then $j^r f(x) = (x, T_r f(x))$, where $T_r f(x)$ is the Taylor polynomial of degree r for f at x. We often identify $T_r f(x)$ with the r-jet of f at x.

THEOREM 7.2.1 (Thom jet transversality theorem). *Let $r \in \mathbb{N}$ and suppose that Q is a smooth submanifold of $J^r(M,N)$. Take the Whitney C^∞-topology on $C^\infty(M,N)$.*

(1) *$\{f \in C^\infty(M,N) \mid j^r f \pitchfork Q\}$ is a dense (residual) subset of $C^\infty(M,N)$. (This is also true if we take the C^∞-topology on $C^\infty(M,N)$.)*
(2) *If Q is closed, $\{f \in C^\infty(M,N) \mid j^r f \pitchfork Q\}$ is an open subset of $C^\infty(M,N)$. (This is also true if M is compact and we take the C^∞-topology on $C^\infty(M,N)$.)*

7.2.1. An equivariant version of Thom's jet transversality theorem. We now review some the basic ideas underlying Bierstone's generalization of the jet transversality theorem to equivariant maps. Our main interest will lie with the jet space $J^r(V,W)$, where V,W are G-representations. The theory in this case is a little simpler and it will not be necessary to verify that definitions are invariant under coordinate changes nor to assume that maps are defined on proper open subsets of the representation. For this reason we omit many details and proofs (which can be found in Bierstone's original paper [**15**]) and instead focus on those parts of the theory which suffice for our intended applications to equivariant bifurcation theory. In particular, we always assume that generators are homogeneous and defined on representations, rather than on G-invariant open sets. For our applications we shall mainly be interested in the equivariant jet transversality theorem for the 1-jet extension map and we generally only give full details of proofs for this case. Although we present an abbreviated version of the theory, details will be complete for our intended applications.

The equivariant version of Thom's transversality theorem is easily formulated. We require that M,N are smooth G-manifolds. The jet bundle $J^r(M,N)$ then has the natural structure of a G-manifold. Let Q be a G-invariant submanifold of $J^r(M,N)$ and consider the set $\{f \in C^\infty_G(M,N) \mid j^r f \pitchfork_G Q\}$. The whole problem rests with finding the right definition of $\pitchfork_G$. It turns out that a simple-minded extension of our earlier definition of $\pitchfork_G$ to jet spaces does not work.

We now describe the local set up for equivariant transversality. Basically, we will be describing a factorization of the r-jet extension map $j^r f : V \to J^r(V,W)$ through an intermediate space $V \times \mathbb{R}^K$. We start by assuming that $r = 1$.

Let V,W be finite-dimensional G-representations. For $r \geq 0$, $J^r(V,W)$ has the natural structure of a G-representation and $J^r(V,W)^G \approx V^G \times P^{(r)}_G(V,W)$. Let $\mathcal{P} = \{p_1,\dots,p_\ell\}$ be a minimal set of homogeneous polynomial generators for the $\mathbb{R}$-algebra $P(V)^G$ and $P = (p_1,\dots,p_\ell) : V \to \mathbb{R}^\ell$ denote the associated orbit map. Let $\mathcal{F} = \{F_1,\dots,F_k\}$ be a minimal set of homogeneous polynomial generators for the $P(V)^G$-module $P_G(V,W)$.

Let $\mathbf{P}_1(\mathbb{R}^\ell,\mathbb{R}^k)$ denote the set of affine linear maps from $\mathbb{R}^\ell$ to $\mathbb{R}^k$. Note that

$$\mathbf{P}_1(\mathbb{R}^\ell,\mathbb{R}^k) \approx \mathbb{R}^k \oplus L(\mathbb{R}^\ell,\mathbb{R}^k). \tag{7.2}$$

We represent elements $\mathbf{t}^1 \in L(\mathbb{R}^\ell, \mathbb{R}^k)$ as a k-tuple $(\mathbf{t}_1^1, \dots, \mathbf{t}_k^1)$, where each component $\mathbf{t}_i^1 = [t_{ij}^1]$ is a linear map of $\mathbb{R}^\ell$ onto the ith component of $\mathbb{R}^k$.

Every $f \in C_G^\infty(V, W)$ may be written

$$f(x) = \sum_{i=1}^k g_i(P(x))F_i(x),$$

where each coefficient $g_i \in C^\infty(V)^G$. (Using remark 6.6.14 we may require that the coefficients g_i depend continuously on f, C^∞-topology – however, we will not need this result).

We define $H_1 = H_1^{\mathcal{P},\mathcal{F}}(f) : \mathbb{R}^\ell \to \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k)$ by

$$H_1(s) = (g_1(s), \dots, g_k(s); Dg_1(s), \dots, Dg_k(s)), \quad (s \in \mathbb{R}^\ell).$$

Let $U_1 = U_1^{\mathcal{F},\mathcal{P}} : V \times \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k) \to J^1(V, W)$ be the map defined by

$$U_1(x, \mathbf{t}) = \left(x, \sum_{i=1}^k t_i^0 F_i(x), \sum_{i=1}^k [\mathbf{t}_i^1 DP(x)\, F_i(x) + t_i^0 DF_i(x)] \right),$$

where $\mathbf{t} = (\mathbf{t}^0, \mathbf{t}^1) = (t_1^0, \dots, t_k^0; \mathbf{t}_1^1, \dots, \mathbf{t}_k^1) \in \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k)$. Note that $\mathbf{t}_i^1 DP(x)$ is the composition of $DP(x) : V \to \mathbb{R}^\ell$ with the linear functional $\mathbf{t}_i^1 : \mathbb{R}^\ell \to \mathbb{R}$. Consequently, for fixed x, each term $\mathbf{t}_i^1 DP(x)\, F_i(x)$ defines a linear map from V to W.

Lemma 7.2.2. *With the previous notation, we have*

$$j^1 f = U_1 \circ (I, H_1 \circ P).$$

Proof. The results follows on differentiating $f(x) = \sum_{j=1}^k g_j(P(x))F_j(x)$ and applying the chain rule and the definitions of H_1 and U_1. $\square$

Now suppose $r > 1$. Define $\mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k) = \mathbb{R}^k \oplus \oplus_{j=1}^r L_s^j(\mathbb{R}^\ell, \mathbb{R}^k)$. Given $f = \sum g_j \circ P\, F_j \in C_G^\infty(V, W)$, we define $H_r = H_r^{\mathcal{P},\mathcal{F}}(f) : \mathbb{R}^\ell \to \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$ by $H_r(s) = (g_1(s), \dots, D^j g_i(s), \dots, D^r g_k(s))$, $s \in \mathbb{R}^\ell$. We may define a map $U_r : V \times \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k) \to J^r(V, W)$, linear in $\mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$-variables and independent of f, so that $j^r f = U_r \circ (I, H_r \circ P)$. The explicit definition of U_r depends on the rather complicated expression for the rth derivative of a composite of vector valued functions (Faá di Bruno's formula). More specifically, U_r is obtained by differentiating the expression $f(x) = \sum_{j=1}^k g_j(P(x))F_j(x)$ r-times and everywhere replacing derivatives of the g_j by the appropriate dummy variable from $\mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$. We refer to [**15**] or [**51**, §3] for details on the explicit form of U_r.

Exercise 7.2.3. Find the expression for U_2.

Let Q be a G-invariant closed semialgebraic[3] subset of $J^r(V,W)$. Set $\Lambda = U_r^{-1}(Q) \subset V \times \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$. Since U_r is a polynomial, Λ is a closed semialgebraic set. Give Λ the canonical Whitney regular stratification.

DEFINITION 7.2.4 ([**15**, Proposition 7.4]). (Notation as above.) Let $f \in C^\infty_G(V,W)$. The jet extension map $j^r f : V \to J^r(V,W)$ is in *equivariant general position to* Q on $A \subset V$ if $(I, H_r \circ P) : V \to V \times \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$ is transverse to Λ on A. In the sequel, we say that $j^r f$ is G-transverse to Q on A and write "$j^r f \pitchfork_G Q$ on A".

The next result is a special case of Bierstone's jet transversality theorem.

THEOREM 7.2.5 (Bierstone [**15**]). *Let Q be a G-invariant closed semialgebraic subset[4] of $J^r(V,W)$ and A be a closed G-invariant subset of V.*

(1) *The definition of $j^r f \pitchfork_G Q$ on A is well-defined and independent of choices of generating sets $p_1, \dots, p_\ell$, $F_1, \dots, F_k$ and representation of f in terms of polynomial invariants and equivariants.*
(2) *$\{f \in C^\infty_G(V,W) \mid j^r f \pitchfork_G Q$ on $A\}$ is an open and dense subset of $C^\infty_G(V,W)$ (Whitney C^∞-topology).*
(3) *(Isotopy theorem) There exists an open neighbourhood $\mathcal{U}$ of f in $C^\infty_G(V,W)$ such that if $g \in \mathcal{U}$, then (a) $j^r g \pitchfork_G Q$ on A, and (b) $(j^r g)^{-1}(Q)$ is continuously equivariantly isotopic to $(j^r f)^{-1}(Q)$.*

REMARKS 7.2.6. (1) A proof of theorem 7.2.5 is in [**15**]. It is also possible – and useful – to develop a proof along the lines of our exposition of equivariant transversality in Chapter 6. We develop some of the necessary local theory in subsequent sections.
(2) Openness of G-transversality holds: if $j^r f \pitchfork_G Q$ at $0 \in V$, then there is an open neighbourhood U of 0 such that $j^r f \pitchfork_{G_y} Q$, for all $y \in U$.

EXAMPLE 7.2.7. Let $H(V) \subset L(V,V)$ be the space of hyperbolic linear maps ($A \in H(V)$ if and only if all eigenvalues of A have real part non-zero). Define

$$Z_1 = Z_1(V) = \{(x, 0, A) \in J^1(V,V) = V \times V \times L(V,V) \mid A \notin H(V)\}.$$

Clearly Z_1 is a closed, nowhere dense, semialgebraic subset of $J^1(V,V)$. Let G be a finite group and (V,G) be a G-representation. Suppose $X \in C^\infty_G(V,V)$ and regard X as a smooth G-equivariant vector field on V. The following conditions are equivalent.

(1) All equilibria of X are hyperbolic.
(2) $j^1 X(V) \cap Z_1 = \emptyset$.

[3]Semianalytic will work just as well here. However, unless we can prove the existence of a canonical Whitney regular stratification, we cannot allow Q to be a general closed G-invariant submanifold.

[4]In [**15**] it is required that Q is a submanifold – for our context, this is an unnecessary restriction

(3) $j^1X \pitchfork_G Z_1$ on V.

It is obvious that (1) $\iff$ (2) $\implies$ (3). It remains to prove (3) $\implies$ (2). While it is possible to prove this directly from the definition of equivariant transversality, it is easier to infer the result using the isotopy theorem. Suppose that $j^1X \pitchfork_G Z_1$. We have $U^{-1}(Z_1) \subset \Sigma \times L(\mathbb{R}^\ell, \mathbb{R}^k)$, where $\Sigma \subset V \times \mathbb{R}^k$ is the universal variety. Suppose first that $(I, H \circ P)(x) \in (\Sigma \times L(\mathbb{R}^\ell, \mathbb{R}^k)) \setminus U^{-1}(Z_1)$. Then x is a hyperbolic equilibrium of X and so $j^1X(x) \notin Z_1$. On the other hand if $(I, H \circ P)(x) \in U^{-1}(Z_1)$ then since $(I, H \circ P) \pitchfork U^{-1}(Z_1)$, it is automatic that $(I, H \circ P) \pitchfork \Sigma \times L(\mathbb{R}^\ell, \mathbb{R}^k)$ at x. That is, $j^1X \pitchfork_G Z_1$ on V implies that $(I, H \circ P) \pitchfork \Sigma \times L(\mathbb{R}^\ell, \mathbb{R}^k)$ on V. Since the codimension of $\Sigma \times L(\mathbb{R}^\ell, \mathbb{R}^k)$ in $V \times \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k)$ is equal to $\dim(V)$, it follows that the equilibria of X are isolated. Suppose that $X(x_0) = 0$. If $DX(x_0) \notin H(V)$, it follows that $(x_0, 0, DX(x_0)) \in Z_1$. We can make a C^∞ small perturbation $\bar{X}$ of X supported on a neighbourhood D of x_0 disjoint from the remaining equilibria of X so that (a) the only zero of $\bar{X}$ on D is x_0, and (b) $D\bar{X}(x_0) \in H(V)$. Applying this argument at every non-hyperbolic zero of X, we can find arbitrarily C^∞-small perturbations $\bar{X}$ of X such that $\bar{X}$ has the same equilibria as X and all equilibria of $\bar{X}$ are hyperbolic. It follows from the isotopy theorem that every equilibrium of X must be hyperbolic (otherwise $(j^1X)^{-1}(Z_1) \neq \emptyset$ but $(j^1\bar{X})^{-1}(Z_1) = \emptyset$ for $\bar{X}$ arbitrarily close to X).

Remark 7.2.8. The argument of example 7.2.7 implies that the codimension of $U^{-1}(Z_1)$ in $V \times \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k)$ is at least $\dim(V) + 1$ (we emphasize G is *finite*). We give a more formal proof shortly.

7.2.2. Invariance lemmas. If we ignore questions about the invariance of the jet transversality definition – that is, the dependence of the definition on the choices of generators $\mathcal{P}$ and $\mathcal{F}$ – then the main issues involved in proving theorem 7.2.5 are as follows. In order to prove density, we would like to apply standard results on density of smooth mappings transverse to a Whitney stratified set. The problem here – just as in the case of G-transversality – is that the map $H_r(f)$ is G-invariant. This difficulty is easily overcome along either the lines described in Bierstone [**14, 15**] or by using the method described in chapter 6 and reformulating equivariant transversality in terms of transversality of a (general) smooth map to a stratification of the parameter space $\mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$, We describe the second approach later in the chapter. In order to prove openness, we first need to give a general definition of equivariant jet transversality for G-manifolds. This is standard and done in terms of slices and local charts (we need the submanifold $Q \subset J^r(M, N)$ to define an analytic subset of the local model so that we can apply Mather's results on minimal stratifications). Next we need to prove openness of the definition and show that if $j^rf \pitchfork_G Q$ at $0 \in V$, then $j^rf \pitchfork_{G_y} Q$ for all y in some open neighbourhood of 0. It is then easy to prove the openness statement of theorem 7.2.5. In this section we describe two of the basic invariance results. Using these results we can make our exposition relatively self-contained.

LEMMA 7.2.9. [**15**, Lemma 5.2] *Let (V, G), (W, G) be representations. Suppose that $\mathcal{P} = \{p_1, \dots, p_\ell\}$ and $\mathcal{Q} = \{q_1, \dots, q_{\ell'}\}$ are minimal sets of homogeneous generators for $P(V)^G$ and let P, Q denote the corresponding orbit maps. Then*

(1) *$\ell = \ell'$ and if we label the elements of $\mathcal{P}, \mathcal{Q}$ in ascending order of degree (so that $deg(p_i) \leq deg(p_{i+1})$), then $deg(p_i) = deg(q_i) = e_i$, $1 \leq i \leq \ell$.*
(2) *There is a polynomial automorphism $\phi : \mathbb{R}^\ell \to \mathbb{R}^\ell$ such that $P = \phi \circ Q$.*
(3) *For all $k \in \mathbb{N}$, ϕ induces a G-equivariant polynomial automorphism $\mathbf{\Phi}_r$ of $V \times \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$ such that $U_r^{\mathcal{P},\mathcal{F}} = U_r^{\mathcal{Q},\mathcal{F}} \circ \mathbf{\Phi}_r$, where $\mathcal{F}$ is a set of polynomial generators for $P_G(V, W)$.*

PROOF. (1) This is elementary and uses the homogeneity of the generators. (2) Denote coordinates on $\mathbb{R}^\ell$ by $(y_1, \dots, y_\ell)$. For $1 \leq j \leq \ell$, we may choose polynomials $\phi_j : \mathbb{R}^\ell \to \mathbb{R}$ such that $q_j = \phi_j(p_1, \dots, p_k)$. Further, we may require that each ϕ_j is weighted homogeneous of degree e_j, where the y_i-coordinate of ϕ_j is given weight e_i. Let $\phi = (\phi_1, \dots, \phi_\ell) : \mathbb{R}^\ell \to \mathbb{R}^\ell$. Obviously, $P = \phi \circ Q$. Group coordinates according to weight so that $(y_1, \dots, y_\ell) = (\mathbf{y}^1, \dots, \mathbf{y}^s)$, where $\mathbf{y}^j = (y_{i_j+1}, \dots, y_{k_j})$ and $e_{i_j+1} = \dots = e_{i_{k_j}} = \mathbf{e}_j$. Write $\phi = (\boldsymbol{\phi}^1, \dots, \boldsymbol{\phi}^s)$. Then we have $\boldsymbol{\phi}^j(\mathbf{y}^1, \dots, \mathbf{y}^s) = \boldsymbol{A}^j(\mathbf{y}^j) + \boldsymbol{P}^j(\mathbf{y}^1, \dots, \mathbf{y}^{j-1})$, where $\boldsymbol{P}^j$ is a weighted homogeneous of degree $\mathbf{e}_j$ and $\boldsymbol{A}^j \in \mathrm{GL}(k_j - i_j, \mathbb{R})$. It follows that ϕ is a polynomial automorphism of $\mathbb{R}^\ell$.
(3) We assume $r = 1$ and refer to [**15**] for the general case. We define the polynomial automorphism $\mathbf{\Phi}_1$ of $V \times \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k)$ by

$$\mathbf{\Phi}_1(x, \mathbf{t}^0, \mathbf{t}^1) = (x, \mathbf{t}^0, \mathbf{t}^1 D\phi(Q(x))).$$

We leave it to the reader to verify that $U_1^{\mathcal{P},\mathcal{F}} = U_1^{\mathcal{Q},\mathcal{F}} \circ \mathbf{\Phi}_1$. $\square$

EXERCISE 7.2.10. Prove (3) of lemma 7.2.9 in case $r = 2$.

LEMMA 7.2.11. [**15**, Lemma 5.4] *Let $\mathcal{F} = \{F_1, \dots, F_k\}$ and $\mathcal{G} = \{G_1, \dots, G_k\}$ be minimal sets of homogeneous generators for the $P(V)^G$-module $P_G(V, W)$. Let $\mathcal{P}$ be a minimal set of homogeneous generators for the $\mathbb{R}$-algebra $P(V)^G$. For $r \geq 0$, there exists a G-equivariant polynomial automorphism $\mathbf{A}_r$ of $V \times \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$ such that $U_r^{\mathcal{P},\mathcal{F}} = U_r^{\mathcal{P},\mathcal{G}} \circ \mathbf{A}_r$.*

PROOF. The case $r = 0$ is lemma 6.7.4. We prove the case $r = 1$ and refer the reader to [**15**] for the general case. As in the proof of lemma 6.7.4, we may choose $p_{ij} \in P(V)^G$ such that $F_i = \sum_{i=1}^k p_{ij} G_j$, $i = 1, \dots, k$, where $P = [p_{ij}]$ is invertible on V with P^{-1} having polynomial entries. Since $\mathcal{P}$ is a set of homogeneous generators for $P(V)^G$, we may choose $\lambda_{ij} \in P(\mathbb{R}^\ell)$ so that $p_{ij} = \lambda_{ji} \circ P$, $1 \leq i, j \leq k$. Set $\Lambda = [\lambda_{ij}]$. Define

$$\mathbf{A}_1(x, \mathbf{t}^0, \mathbf{t}^1) = (x, \Lambda(P(x))(\mathbf{t}^0), \Lambda(P(x))(\mathbf{t}^1) + d\Lambda(P(x))(\mathbf{t}^0)),$$

where the $k \times \ell$-matrix $d\Lambda(P(x))(\mathbf{t}^0)$ has ith row equal to $\sum_{j=1}^k t_j^0 D\lambda_{ij}(P(x))$. We leave it to the reader to verify that $\mathbf{A}_1$ has the claimed properties. $\square$

EXERCISE 7.2.12. Prove lemma 7.2.11 in case $r = 2$ (this is combination of the Faá di Bruno and Leibniz formulas).

The next result, which is immediate from lemmas 7.2.9, 7.2.11, shows that equivariant jet transversality is well-defined – at least if we restrict attention to minimal sets of homogeneous generators.

PROPOSITION 7.2.13. *Let $(V,G),(W,G)$ be G-representations. Let $r \in \mathbb{N}$ and suppose Q is a closed G-invariant semialgebraic subset of $J^r(V,W)$. Let $\mathcal{P},\mathcal{Q}$ be minimal sets of homogeneous generators for the $\mathbb{R}$-algebra $P(V)^G$ and $\mathcal{F}$, $\mathcal{G}$ be minimal sets of homogeneous generators for the $P(V)^G$-module $P_G(V,W)$. Let A be a G-invariant subset of V. The map $(I,H_r^{\mathcal{P},\mathcal{F}}) : V \to V \times \mathbf{P}_r(\mathbb{R}^k,\mathbb{R}^\ell)$ is transverse to the canonical stratification of $(U_r^{\mathcal{P},\mathcal{F}})^{-1}(Q)$ if and only if $(I,H_r^{\mathcal{Q},\mathcal{G}}) : V \to V \times \mathbf{P}_r(\mathbb{R}^k,\mathbb{R}^\ell)$ is transverse to the canonical stratification of $(U_r^{\mathcal{Q},\mathcal{G}})^{-1}(Q)$.*

REMARK 7.2.14. Using a similar technique to that used to prove the jet transversality theorem of Thom, one may show that if Q is a closed G-invariant semialgebraic subset of $J^r(V,W)$, and A is a closed subset of V^G, then $\{f \in C^\infty_G(V,W) \mid j^r f \pitchfork_G Q \text{ on } A\}$ is an open and dense subset of $C^\infty_G(V,W)$ (Whitney C^∞-topology). We refer to [**15**, section 9] for details. In the next section, we prove this using Thom's jet transversality theorem. This partial result will suffice for our applications to equivariant bifurcation theory.

7.3. Equivariant jet transversality for families

In this section we formulate equivariant jet transversality so that it is applicable to smooth equivariant families $f_t : V \to W$, $t \in \mathbb{R}^s$. In particular, we will be looking at submanifolds Q of $J^r(V \times \mathbb{R}^s, W)$ that are independent of the $\mathbb{R}^s$ coordinate and so may be regarded as submanifolds of $J^r(V,W)$ rather than $J^r(V \times \mathbb{R}^s, W)$.

As our main interest lies in the analysis of codimension one equivariant bifurcations, we shall assume that $V^G = \{0\}$ and, where necessary, that (V,G) is irreducible.

Fix minimal homogeneous sets of generators $\mathcal{P} = \{p_1,\dots,p_\ell\}$ for $P(V)^G$ and $\mathcal{F} = \{F_1,\dots,F_k\}$ for $P_G(V,W)$.

Let $r \geq 0$. For $s \geq 0$, there is a natural projection map $\pi_r : J^r(V \times \mathbb{R}^s, W) \to J^r(V,W)$ defined by restriction (if $A \in L^q_s(V \times \mathbb{R}^s, W)$, $r \geq q \geq 0$, then $\pi_r(A) \in L^q_s(V,W)$ is the map defined by $\pi_r(A)(v^q) = A((v,0)^q)$).

Let $j_1^r f : V \times \mathbb{R}^s \to J^r(V,W)$ be defined by

$$j_1^r f(x,\boldsymbol{\lambda}) = j^r f_{\boldsymbol{\lambda}}(x),\ (x,\boldsymbol{\lambda}) \in V \times \mathbb{R}^s.$$

For all $f \in C^\infty_G(V \times \mathbb{R}^s, W)$, we have

$$\pi_r \circ j^r f = j_1^r f.$$

We may factorize $j^r_1 f : V \times \mathbb{R}^s \to J^r(V,W)$ as $j^r_1 f = \tilde{U}_r \circ (\tilde{I}, \tilde{H}_r)$, where

$$\begin{aligned} \tilde{I} : V \times \mathbb{R}^s &\to V,\ (x, \boldsymbol{\lambda}) \mapsto x, \\ \tilde{H}_r : V \times \mathbb{R}^s &\to \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k), \\ \tilde{U}_r : V \times \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k) &\to J^r(V,W). \end{aligned}$$

Just as for U_r, the definition of $\tilde{U}_r$ depends on the chain rule. We give the explicit formulas only for $r = 1$.

$$\begin{aligned} \tilde{H}_r(x, \boldsymbol{\lambda}) &= ((g_1(Px, \boldsymbol{\lambda}), \dots, g_k(Px, \boldsymbol{\lambda})), (Dg_{1,\boldsymbol{\lambda}}(Px), \dots, Dg_{k,\boldsymbol{\lambda}}(Px))), \\ \tilde{U}_r(x, \boldsymbol{\lambda}, \mathbf{t}^0, \mathbf{t}^1) &= (x, \sum_{i=1}^k t^0_i F_i(x), \sum_{i=1}^k [\mathbf{t}^1_i DP(x)\, F_i(x) + t^0_i DF_i(x)]). \end{aligned}$$

Definition 7.3.1. Let Q be a closed G-invariant semialgebraic subset of $J^r(V,W)$ and A be a closed subset of $V \times \mathbb{R}^s$. Given $f \in C^\infty_G(V \times \mathbb{R}^s, W)$, we say $j^r_1 f$ is G-transverse to Q on A if $(I, \tilde{H}_r)$ is transverse to the canonical stratification of $\tilde{U}_r^{-1}(Q)$ along A. We write this $j^r_1 f \pitchfork_G Q$ (along A).

As an immediate consequence of our definitions and constructions we have

Lemma 7.3.2. *Let Q be a closed G-invariant semialgebraic subset of $J^r(V,W)$, A be a closed subset of $V \times \mathbb{R}^s$, and $f \in C^\infty_G(V \times \mathbb{R}^s, W)$. Then $j^r_1 f \pitchfork_G Q$ on A if and only if $j^r f \pitchfork_G \pi_r^{-1}(Q)$ on A.*

Given $f \in C^\infty_G(V \times \mathbb{R}^s, W)$, define $\gamma^{\mathcal{P},\mathcal{F}}_r(f) = \gamma_r(f) : \mathbb{R}^s \to \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$ by

$$\gamma_r(f)(\boldsymbol{\lambda}) = \tilde{H}_r(f)(0, \boldsymbol{\lambda}),\ (\boldsymbol{\lambda} \in \mathbb{R}^s).$$

Note that if $f = \sum g_i P\, F_i$, then $\gamma_r(f)(\boldsymbol{\lambda})$ is the r-jet (in $\mathbb{R}^\ell$-variables) at $\mathbf{u} = 0$ of the map $g_{\boldsymbol{\lambda}}(\mathbf{u}) = (g_1(\mathbf{u}, \boldsymbol{\lambda}), \dots, g_k(\mathbf{u}, \boldsymbol{\lambda}))$. In particular, there are no symmetry constraints on the values taken by $\gamma_r(f)$.

Let Q be a closed G-invariant semialgebraic subset of $J^r(V,W)$. By theorem 6.10.1, the canonical stratification of $\tilde{U}_r^{-1}(Q)$ (or $U_r^{-1}(Q)$) induces a Whitney regular stratification $\mathcal{A}^{\mathcal{P},\mathcal{F}}_r(Q) = \mathcal{A}_r(Q)$ of $\tilde{U}_r^{-1}(Q)^G \subset \mathbf{P}_r(\mathbb{R}^k, \mathbb{R}^\ell)$. We extend $\mathcal{A}_r(Q)$ to a Whitney stratification of $\mathbf{P}_r(\mathbb{R}^k, \mathbb{R}^\ell)$ by taking the top dimensional stratum to to be $\mathbf{P}_r(\mathbb{R}^k, \mathbb{R}^\ell) \setminus \tilde{U}_r^{-1}(Q)^G$.

Proposition 7.3.3. *Let Q be a closed G-invariant semialgebraic subset of $J^r(V,W)$, A be a closed subset of $\mathbb{R}^s \subset V \times \mathbb{R}^s$, and $f \in C^\infty_G(V \times \mathbb{R}^s, W)$. Then $j^r f \pitchfork_G \pi_r^{-1}(Q)$ on A if and only if $\gamma_r(f) \pitchfork \mathcal{A}_r(Q)$ on A.*

Proof. By definition of $\mathcal{A}_r(Q)$, $\gamma_r(f) \pitchfork \mathcal{A}_r(Q)$ on A if and only if $\tilde{H}_r(f) \pitchfork \tilde{U}_r^{-1}(Q)$ on A. Now use lemma 7.3.2. $\square$

Corollary 7.3.4. *Let Q be a closed G-invariant semialgebraic subset of $J^r(V,W)$, and A be a closed subset of $\mathbb{R}^s \subset V \times \mathbb{R}^s$. Then $\{f \in C^\infty_G(V \times \mathbb{R}^s, W) \mid j^r f \pitchfork_G \pi_r^{-1}(Q)$ on $A\}$ is an open dense subset of $C^\infty_G(V \times \mathbb{R}^s, W)$ (Whitney C^∞-topology).*

PROOF. We obtain density by applying the Thom jet transversality theorem to the r-jet extension map $j^r g_{\boldsymbol{\lambda}}$ along A. For openness, we start by noting that since $\mathcal{A}_r(Q)$ is a Whitney stratification of a closed subset of $\mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$, the set of $(g_1, \ldots, g_k) \in C^\infty(\mathbb{R}^\ell \times \mathbb{R}^s)^k$ for which $j^r g_{\boldsymbol{\lambda}} \pitchfork \mathcal{A}_r(Q)$ along A is open. But the map $C^\infty(\mathbb{R}^\ell \times \mathbb{R}^s)^k \to C^\infty_G(V \times \mathbb{R}^s, W)$, $(g_1, \ldots, g_k) \mapsto \sum g_i P\, F_i$ is open by the Open Mapping Theorem (see the proof of lemma 6.6.11). This 'local' result in the C^∞-topology implies openness in the Whitney C^∞-topology. □

Proposition 7.3.3 and corollary 7.3.4 suffice for our applications to equivariant bifurcation theory.

7.3.1. Intrinsic formulation of jet transversality. In the remainder of this section we briefly sketch how we can formulate the definitions for equivariant jet transversality in a more intrinsic way so as to parallel what we did in chapter 6 for equivariant transversality. We will not use any of these results in the sequel.

Let $\mathfrak{M} = \{p \in P(V)^G \mid p(0) = 0\}$. For $r \geq 0$, let $\mathfrak{M}^{r+1} \lhd P(V)^G$ be the ideal generated by all $r+1$-fold products of elements in $\mathfrak{M}$. We similarly define the ideal $\mathfrak{M}^{r+1}_\infty \lhd C^\infty(V)^G$.

LEMMA 7.3.5. *Let $r \geq 0$ and set $\mathbb{U}_r = P_G(V, W)/\mathfrak{M}^{r+1} P_G(V, W)$.*

(1) $C^\infty_G(V, W)/\mathfrak{M}^{r+1}_\infty C^\infty_G(V, W) \approx \mathbb{U}_r$ *(as vector spaces).*
(2) *A choice of minimal sets of homogeneous generators $\mathcal{F}$ for $P_G(V, W)$ and $\mathcal{P}$ for $P(V)^G$ determines a vector space surjection*
$$\chi^{\mathcal{F},\mathcal{P}}_r : \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k) \to \mathbb{U}_r.$$
$\chi^{\mathcal{F},\mathcal{P}}_r$ *is a linear isomorphism if $r = 0$.*

PROOF. Since $P_G(V, W) \subset C^\infty_G(V, W)$, there is a natural homomorphism of $P_G(V, W)$ in $C^\infty_G(V, W)/\mathfrak{M}^{r+1}_\infty C^\infty_G(V, W)$. If $P \in P^d_G(V, W)$ maps to zero in $C^\infty_G(V, W)/\mathfrak{M}^{r+1}_\infty C^\infty_G(V, W)$ then $P \in \mathfrak{M}^{r+1} P_G(V, W)$. Hence the natural map $P_G(V, W)/\mathfrak{M}^{r+1} P_G(V, W) \to C^\infty_G(V, W)/\mathfrak{M}^{r+1}_\infty C^\infty_G(V, W)$ is injective. Surjectivity follows from Schwarz's theorem, proving (1). If $\mathbf{t} = (\mathbf{c}, (\mathbf{A}^1, \ldots, \mathbf{A}^r)) \in \mathbb{R}^k \oplus \oplus_{i=1}^r L^i_s(\mathbb{R}^\ell, \mathbb{R}^k)$, we define
$$\hat{\chi}^{\mathcal{F},\mathcal{P}}_r(\mathbf{t}) = \sum_j (c_j + \sum_i \mathbf{A}^i_j \circ P) F_j \in P_G(V, W),$$
Quotienting by $\mathfrak{M}^{r+1} P_G(V, W)$, we see that $\hat{\chi}^{\mathcal{F},\mathcal{P}}_r$ determines the required surjection $\chi^{\mathcal{F},\mathcal{P}}_r : \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k) \to \mathbb{U}_r$. It follows from lemma 6.6.5 that if $r = 0$, then $\chi^{\mathcal{F},\mathcal{P}}_r$ is a linear isomorphism. □

REMARK 7.3.6. In general $\chi^{\mathcal{F},\mathcal{P}}_r$ will not be a linear isomorphism if $r > 0$. However, if (V, G) is the standard representation of a finite reflection group, then there are no relations in $\mathcal{P}$. If there are no relations in $\mathcal{F}$ (for example, if $(W, G) = (V, G)$ and (V, G) is a finite reflection group), then $\chi^{\mathcal{F},\mathcal{P}}_r$ is linear isomorphism for all $r \geq 0$.

Let $\Pi_r : C^\infty_G(V, W) \to \mathbb{U}_r$ be the projection map given by Lemma 7.3.5. Given $s \geq 0$, define $\Pi_r : C^\infty_G(V \times \mathbb{R}^s, W) \to C^\infty(\mathbb{R}^s, \mathbb{U}_r)$ by $\Pi_r(f)(t) = \Pi_r(f_t)$, $t \in \mathbb{R}^s$.

LEMMA 7.3.7. *The map* $\Pi_r : C^\infty_G(V \times \mathbb{R}^s, W) \to C^\infty(\mathbb{R}^s, \mathbb{U}_r)$ *is continuous with respect to the* C^∞*-topologies on* $C^\infty_G(V \times \mathbb{R}^s, W)$ *and* $C^\infty(\mathbb{R}^s, \mathbb{U}_r)$*.*

PROOF. After choosing generating sets $\mathcal{F}, \mathcal{P}$, the result may be proved along the lines of lemma 6.6.11. Alternatively, we can use results on the existence of continuous linear sections (see remark 6.6.14). □

Assume now that $V^G = \{0\}$. Let Q be a closed G-invariant semialgebraic subset of $J^r(V, W)$. Give $\tilde{U}_r^{-1}(Q) \subset V \times \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$ the canonical stratification and let $\mathcal{A}_r(Q)$ denote the induced stratification of $\tilde{U}_r^{-1}(Q)^G \subset \mathbf{P}_r(\mathbb{R}^\ell, \mathbb{R}^k)$. Set $\Lambda_r(Q) = \chi_r^{\mathcal{F},\mathcal{P}}(\tilde{U}_r^{-1}(Q)^G)$.

LEMMA 7.3.8. *Let* $\mathcal{A}^\star_r(Q) = \chi_r^{\mathcal{F},\mathcal{P}}(\mathcal{A}_r(Q))$*. Then* $\mathcal{A}^\star_r(Q)$ *is a well-defined Whitney stratification of* $\Lambda_r(Q)$ *which is independent of the choice of generating sets* $\mathcal{F}, \mathcal{P}$*. If* $A \subset \mathbb{R}^s$ *and* $f \in C^\infty_G(V \times \mathbb{R}^s, W)$ *then* $j^r f \pitchfork_G Q$ *along* A *if and only if* $\Pi_r(f) \pitchfork \mathcal{A}^\star_r(Q)$ *along* A*.*

PROOF. We indicate the proof in case $r = 1$. Let $\Theta \subset L(\mathbb{R}^\ell, \mathbb{R}^k)$ be the linear subspace consisting of all matrices $\boldsymbol{\theta} = [\theta_{ij}]$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{\ell} \theta_{ij} p_j F_i = 0 \ \ (\text{in } P_G(V, W)).$$

(Of course, Θ may just consist of the zero matrix in which case χ_1 is a linear isomorphism.) If $\boldsymbol{\theta} \in \Theta$, we define the G-equivariant polynomial automorphism $K(\boldsymbol{\theta})$ of $V \times \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k)$ by

$$K(\boldsymbol{\theta})(x, \mathbf{t}^0, \mathbf{t}^1) = (x, K_0(x, \mathbf{t}^0, \boldsymbol{\theta}), K_1(x, \mathbf{t}^1, \boldsymbol{\theta})),$$

where

$$\begin{aligned} K_0(x, \mathbf{t}^0, \boldsymbol{\theta})_i &= t^0_i + \sum_{j=1}^{\ell} \theta_{ij} p_j(x), \\ K_1(x, \mathbf{t}^1, \boldsymbol{\theta})_{ij} &= t^1_{ij} + \theta_{ij}. \end{aligned}$$

For all $\boldsymbol{\theta} \in \Theta$, $\tilde{U}_1 \circ K(\boldsymbol{\theta}) = \tilde{U}_1$. It follows that $K(\boldsymbol{\theta})(\tilde{U}_1^{-1}(Q)) = \tilde{U}_1^{-1}(Q)$. Hence $\mathcal{A}^\star_1(Q)$ is a well-defined Whitney stratification of $\Lambda_1(Q)$. The independence of $\mathcal{A}^\star_1(Q)$ from the choice of generating sets follows from proposition 7.2.13. □

7.4. Stability and determinacy

Throughout this section we assume that (V, G) is an absolutely irreducible G-representation of the finite group G. Let $\mathcal{V}_0 = \mathcal{V}_0(V, G) \subset C^\infty_G(V \times \mathbb{R}, V)$ denote the space of normalized 1-parameter families of vector fields on V. We recall some notation and definitions from example 7.2.7. Let $H(V)$ denote the open dense semialgebraic subset of $L(V,V)$ consisting of hyperbolic linear maps and define

$$Z_1 = \{(x,0,A) \in J^1(V,V) \mid A \notin H(V)\}.$$

Then $\Sigma_1 = U_1^{-1}(Z_1)$ is a closed G-invariant semialgebraic subset of $V \times \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k)$ and $\mathrm{codim}(\Sigma_1) \geq \dim(V) + 1$ (remark 7.2.8 and below).

We define

$$\begin{aligned} \mathcal{K}_G^1(V) &= \{X \in \mathcal{K}_G(V) \mid j_1^1 X \pitchfork_G Z_1 \text{ at } (0,0) \in V \times \mathbb{R}\} \\ &= \{X \in \mathcal{V}_0(V,G) \mid j^0 X = X \pitchfork_G 0,\ j_1^1 X \pitchfork_G Z_1 \text{ at } (0,0) \in V \times \mathbb{R}\}. \end{aligned}$$

LEMMA 7.4.1. (1) *$\mathcal{K}_G^1(V)$ is an open and dense subset of $\mathcal{V}_0(V,G)$.*
(2) *If $X \in \mathcal{K}_G^1(V)$, then X satisfies the branching conditions (B1, B2, B3).*

PROOF. (1) is immediate from the equivariant jet transversality theorem – corollary 7.3.4 suffices. (2) Since $X \in \mathcal{K}_G(V)$, X satisfies (B1), theorem 7.1.5(2). Since $j_1^1 X \pitchfork_G Z_1$ at $(0,0)$, we may use either example 7.2.7 or $\mathrm{codim}(\Sigma_1) \geq \dim(V)+1$ to deduce that that $(0,0) \in V \times \mathbb{R}$ is an isolated point of $(j_1^1 X)^{-1}(Z_1)$. Hence X satisfies (B2,B3). $\square$

THEOREM 7.4.2. *$\mathcal{K}_G^1(V) \subset \mathcal{S}(V,G)$ (stable families). In particular, $\mathcal{S}(V,G)$ is an open and dense subset of $\mathcal{V}_0(V,G)$.*

PROOF. Suppose that $X \in \mathcal{K}_G^1(V)$. Since $X \in \mathcal{K}_G(V)$, we can choose an open neighbourhood $\mathcal{U}$ of X in $\mathcal{K}_G(V)$ such that if $Y \in \mathcal{U}$, then (a) Y has a branching pattern $\Sigma(Y)$, and (b) there is a continuous path in $\mathcal{U}$ joining Y to X. In particular, X and Y have isomorphic branching patterns by the isotopy theorem. Since $\mathcal{K}_G^1(V)$ is open, we may suppose $\mathcal{U} \subset \mathcal{K}_G^1(V)$. Hence, by lemma 7.4.1, every $Y \in \mathcal{U}$ has a signed indexed branching pattern $\Sigma^\star(Y)$. Since we can connect Y to X by a continuous path in $\mathcal{U}$, lemma 7.4.1 implies that X and Y have isomorphic signed indexed branching patterns and so $\mathcal{K}_G^1(V) \subset \mathcal{S}(V,G)$. $\square$

Let d_V be the maximum degree of the polynomials in a minimal set of homogeneous generators for $P_G(V,V)$ and e_V be the maximum degree of the polynomials in a minimal set of homogeneous generators for $P(V)^G$. Set $D_V = d_V + e_V$ and note that d_V, e_V and D_V depend only on the isomorphism class of the representation G (and not on the choice of generating sets).

THEOREM 7.4.3. *Equivariant bifurcation problems on (V,G) are d-determined, where $d \leq D_V$.*

PROOF. We already know that G-equivariant bifurcation problems on (V,G) are weakly d-determined, where $d \leq d_V$ (lemma 7.1.9). Fix minimal sets $\mathcal{F}$ and $\mathcal{P}$ of homogeneous generators for $P_G(V,V)$ and $P(V)^G$ respectively. By proposition 7.3.3, $X \in \mathcal{K}_G^1(V)$ if and only if $X \in \mathcal{K}_G(V)$ and $\gamma_1(X) \pitchfork \mathcal{A}_1^{\mathcal{P},\mathcal{F}}$ at $0 \in \mathbb{R}$, where $\mathcal{A}_1^{\mathcal{P},\mathcal{F}}$ is the Whitney stratification of $\Sigma_1^G = \tilde{U}_1^{-1}(Z_1)^G \subset \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k)$

induced from the canonical stratification of $\Sigma_1 = \tilde{U}_1^{-1}(Z_1)$. We may write

$$X(x,\lambda) = \sum_{j=1}^{k}(c_j + \sum_{i=1}^{\ell} c_{ji}p_i(x))F_j(x).$$

where $c_j \in C^\infty(\mathbb{R})$, $c_{ji} \in C^\infty(V \times \mathbb{R})^G$. Since $X \in \mathcal{K}_G(V)$, $c_1(0) = 0$ and $(c_2,\dots,c_k)(0) \in A_{k-1}$, where A_{k-1} is the codimension 1 stratum of the stratification of $\mathbb{R}^k$ induced from the canonical stratification of Σ. Let $B \subset \mathbf{P}_1(\mathbb{R}^\ell,\mathbb{R}^k)$ denote the codimension 1 stratum of the stratification $\mathcal{A}_1^{\mathcal{P},\mathcal{F}}$. Since $\gamma_1(X)(0) \in \Sigma_1^G$, we have $\gamma_1(X) \pitchfork \mathcal{A}_1^{\mathcal{P},\mathcal{F}}$ at 0 if and only if $(c_2(0),\dots,c_k(0),(c_{ij}(0,0))) \in B$. That is, $X \in \mathcal{K}_G^1(V)$ if and only if $(c_2,\dots,c_k)(0) \in A_{k-1}$ and $(c_2(0),\dots,c_k(0),(c_{ij}(0,0))) \in B$. These are open and dense conditions on the D_V-jet of X_0 at 0. More formally, let $P_G^{(D_V)}(V,V)_0$ denote the subspace of $P_G^{(D_V)}(V,V)$ consisting of polynomials P such that $DP(0) = 0$. We define $\mathcal{R}(D_V) \subset P_G^{(D_V)}(V,V)_0$ to consist of polynomials P such that

$$P(x) = \sum_{j=2}^{k} c_jF_j(x) + \sum_{j=1}^{k}\sum_{i=1}^{\ell} c_{ji}p_i(x)F_j(x) + Q(x),$$

where $(c_2,\dots,c_k) \in A_{k-1}$, $(c_2,\dots,c_k,(c_{ij}(0))) \in B$, and $Q \in \mathfrak{m}^2P_G(V,W)$. Clearly $\mathcal{R}(D_V)$ defines an open and dense set of $P_G^{(D_V)}(V,V)_0$. If $X \in \mathcal{V}_0$ then X is stable if $j^{D_V}X_0(0) \in \mathcal{R}(D_V)$. Hence equivariant bifurcation problems on (V,G) are d-determined, where $d \leq D_V$. $\square$

7.4.1. A reformulation of the stability criterion. It is natural to ask whether $j_1^1X \pitchfork_G Z_1$ at $(0,0) \in V \times \mathbb{R}$ implies that $X \in \mathcal{K}_G(V)$. This is certainly the case when there is no assumption of G-equivariance. In [**60, 57**] we approached the proof of theorem 7.4.3 slightly differently. Rather than work with the twin conditions $X \pitchfork_G 0$ at $(0,0) \in V\times\mathbb{R}$ and $j_1^1X \pitchfork_G Z_1$ at $(0,0) \in V\times\mathbb{R}$, we defined a new Whitney stratification $\mathcal{Q}^\star$ of $\Sigma \times L(\mathbb{R}^\ell,\mathbb{R}^k) \subset V \times \mathbf{P}_1(\mathbb{R}^\ell,\mathbb{R}^k)$ and corresponding induced stratification $\mathcal{A}_1^\star$ of $\mathbf{P}_1(\mathbb{R}^\ell,\mathbb{R}^k)$ such that (a) Σ_1^G is a union of $\mathcal{Q}^\star$-strata and (b) the transversality of $\gamma_1(X)$ to $\mathcal{A}_1^\star$ implies that $X \in \mathcal{K}_G(V)$. As this approach sheds some light on the structure of Σ_1, we recall the main ideas (this section is not used elsewhere in the book).

If $\tau \in \mathcal{O}(V,G)$, Σ_τ is a k-dimensional submanifold of $V \times \mathbb{R}^k$ (lemma 6.9.4). A point $\xi \in \Sigma_\tau \subset \Sigma$ is *singular* if $\xi \in \overline{\Sigma}_\eta$ for some $\eta \neq \tau$. Otherwise ξ is *regular*. Let Σ_r denote the set of all regular points of Σ and $\Sigma_s = \Sigma \setminus \Sigma_r$ denote the set of singular points.

LEMMA 7.4.4. (1) $\Sigma_s \times L(\mathbb{R}^\ell,\mathbb{R}^k) \subset \Sigma_1$.

(2) $\Sigma_1 = \Sigma_s \times L(\mathbb{R}^\ell,\mathbb{R}^k) \cup \Upsilon_1$, *where* $\Upsilon_1 = \{(x,\mathbf{t}^0,\mathbf{t}^1) \in Z_1 \mid (x,\mathbf{t}^0) \in \Sigma_r\}$.

In particular, if $X = \sum_{i=1}^k f_iF_i$, *where* $f_i \in C^\infty(V)^G$, $1 \leq i \leq k$, *and* $(x,(f_1(x),\dots,f_k(x))) \in \Sigma_s$, *then* $j^1X(x) \in Z_1$.

Proof. Let $((x_0, \mathbf{t}_0^0), [a_{ij}]) \in \Sigma_s \times L(\mathbb{R}^\ell, \mathbb{R}^k)$. Define $J : V \times \mathbb{R}^k \to V$ by

$$J(x,t) = \sum_{i=1}^{k}(t_i + \sum_{j=1}^{\ell} a_{ij}(p_j(x) - p_j(x_0))F_i(x),$$

and define $F \in C_G^\infty(V,V)$ by $F(x) = J(x, \mathbf{t}_0^0)$. Since $F(x_0) = 0$ and $D_1J(x_0, \mathbf{t}_0^0) = DF(x_0)$, we see that if $DF(x_0)$ is nonsingular then $(x_0, \mathbf{t}_0^0)$ is a regular point of $J^{-1}(0)$. Now $\boldsymbol{\vartheta} = J \circ A$, where A is the G-equivariant diffeomorphism of $V \times \mathbb{R}^k$ defined by

$$A(x,t) = (x, (t_1 - \sum_{j=1}^{\ell} a_{1j}(p_j(x) - p_j(x_0)), \dots, \sum_{j=1}^{\ell} a_{kj}(p_j(x) - p_j(x_0))).$$

Since $A(x_0, \mathbf{t}_0^0) = (x_0, \mathbf{t}_0^0)$, it follows that if $(x_0, \mathbf{t}_0^0)$ is a regular point of $J^{-1}(0)$ then $(x_0, \mathbf{t}_0^0) \in \Sigma_r$. This contradicts our assumption that $(x_0, \mathbf{t}^0) \in \Sigma_s$ and so $DF(x_0)$ must be singular proving that $\Sigma_s \times L(\mathbb{R}^\ell, \mathbb{R}^k) \subset \Sigma_1$. Notice that this argument shows that if $X \in C_G^\infty(V,V)$ and $\Gamma_X(x_0) \in \Sigma_s$ then $DX(x_0)$ is singular. □

Lemma 7.4.5. *(Notation as in lemma 7.4.4.)*

(1) $\partial\Upsilon_1 \subset \Sigma_s \times L(\mathbb{R}^\ell, \mathbb{R}^k)$ *and* $dim(\partial\Upsilon_1) \leq k + k\ell - 2$.
(2) $dim(\Sigma_s \times L(\mathbb{R}^\ell, \mathbb{R}^k)) \leq k + k\ell - 1$.
(3) *The canonical stratification of* Σ_1 *restricts to the canonical stratification of* Υ_1.

Proof. (1) Since $\Upsilon_1 = \Sigma_1 \cap (\Sigma_r \times L(\mathbb{R}^\ell, \mathbb{R}^k))$ and Σ_1 is closed, $\partial\Upsilon_1 \subset \Sigma_s \times L(\mathbb{R}^\ell, \mathbb{R}^k)$. Since Υ_1 is semialgebraic of dimension $k+k\ell-1$, $\dim(\partial\Upsilon_1) \leq k+k\ell-2$. (2) follows since $\dim(\Sigma_s) \leq k-1$ and (3) is implied by (1) since $\Sigma_s \times L(\mathbb{R}^\ell, \mathbb{R}^k)$ is a closed subset of Σ_1. □

Corollary 7.4.6. *Let* $X \in \mathcal{K}_G^1(V)$. *Then there exists a neighbourhood* U *of* $(0,0) \in V \times \mathbb{R}$ *such that for all nonzero* $(x, \lambda) \in U$ *such that* $X_\lambda(x) = 0$, $DX_\lambda(x)$ *is nonsingular.*

Proof. If $X \in \mathcal{K}_G^1(V)$, then $j_1^1 X \pitchfork Z_1$ at $(0,0)$. Since $(0, \tilde{H}_1(0,0)) \in \Sigma_s \times L(\mathbb{R}^\ell, \mathbb{R}^k)$, lemma 7.4.5 (1,2) implies that we can choose a neighbourhood U of $(0,0) \in V \times \mathbb{R}$ such that (a) $H_1(x,\lambda) \notin \Upsilon_1$, all $(x,\lambda) \in U$ and (b) $H_1(x,\lambda) \in \Sigma_s \times L(\mathbb{R}^\ell, \mathbb{R}^k)$ if and only if $(x,\lambda) = (0,0)$. □

Proposition 7.4.7. *There exists a Whitney regular stratification* $\mathcal{K}$ *of* $\Sigma \times L(\mathbb{R}^\ell, \mathbb{R}^k) \subset V \times \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k)$ *such that*

$$\mathcal{K}_G^1(V) = \{X \in \mathcal{V}_0(V,G) \mid (I, \tilde{H}_1(X)) \pitchfork \mathcal{K} \text{ at } (0,0) \in V \times \mathbb{R}\}.$$

If we let $\mathcal{A}_1^\star$ *denote the induced stratification of* $\mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k)$ *then for all* $X \in \mathcal{V}_0(V,G)$, $X \in \mathcal{K}_G^1(V)$ *if and only if* $\gamma_1(X) \pitchfork \mathcal{A}_1^\star$ *at* $0 \in \mathbb{R}$.

PROOF. By lemma 7.4.4, Σ_1 is the disjoint union of $\Sigma_s \times L(\mathbb{R}^\ell, \mathbb{R}^k)$ and Υ_1. Give Σ_1 the canonical stratification $\mathcal{C}$. Applying lemma 7.4.5, we see that the canonical stratification of Υ_1 is a union of strata from $\mathcal{C}$. In what follows we will not change any of the strata comprising Υ_1. Set $\Sigma^0 = \Sigma \times L(\mathbb{R}^\ell, \mathbb{R}^k)$. Then $\Sigma_1 \subset \Sigma^0$ and $\dim(\Sigma^0) = m$, where $m = k + k\ell$. We define $K_m = \Sigma_r \times L(\mathbb{R}^\ell, \mathbb{R}^k) \setminus \partial\Upsilon_1$. This will be the top dimensional stratum of our new stratification of Σ^0. Observe that since $\dim(\Upsilon_1) = m-1$, $\dim(\partial\Upsilon_1) \leq m-2$ and so $\dim(K_m) = m$. Since K_m is an open subset of $\Sigma_r \times L(\mathbb{R}^\ell, \mathbb{R}^k)$, every point of K_m is regular. We now carry out the procedure for the canonical stratification of Σ^0 subject to K_m being the top-dimensional stratum. Observe that no strata in Υ_1 lie in the boundary of K_m. Let Σ_{rs} denote the set of regular points in the singular set Σ_s. If $\dim(\Sigma_s) < k-1$, we take K_{m-1} to be the union of the $(m-1)$-dimensional strata in Υ_1. Otherwise, we excise from $T_{m-1} = (\Sigma_{rs} \times L(\mathbb{R}^\ell, \mathbb{R}^k)) \setminus \partial\Upsilon_1$ the closure of the set of all points where Whitney regularity fails for the pair (T_{m-1}, Y_m). Adding in the $m-1$-dimensional strata from Υ_1, we thereby construct K_{m-1}. The process continues using the obvious induction – the method is the same as that used to construct the canonical stratification. $\square$

REMARK 7.4.8. If (Σ_{rs}, Σ_r) satisfies the Whitney regularity conditions (or if the points where (b)-regularity fails lie in $\partial\Upsilon_1$), then $K_{m-1} = T_{m-1}$. In this case that if $j_1^1 X \pitchfork_G Z_1$ then we automatically have $X \pitchfork_G 0$ at $(0,0)$ (this is always the case for non-equivariant maps). Unfortunately, we do not know the extent to which we can expect to see points in the primary stratification of Σ where (b)-regularity fails. We know of no examples where the primary and canonical stratifications differ. However, this cannot be regarded as serious evidence for the two stratifications to coincide. Most of the examples we have are determined by either the homogeneous quadratic or cubic terms. This is typically not the case for say the higher degree irreducible representations of SO(3) but then we have no reasonable way of computing the universal variety let alone verifying Whitney regularity. In a similar vein, solution branches are necessarily C^1 (branching condition B1) but we have no examples where solution branches fail to be smooth C^∞. Note that even if (local) connected strata pairs (U, V) in the canonical stratification of Σ are analytically locally trivial, it does not follow without further work that the associated C^1-manifold $U \cup V$ is smooth.

EXERCISE 7.4.9. Show that the canonical stratification of the universal variety associated to the representation $(\mathbb{R}^3, \mathbb{Z}_3 \rtimes S_3)$ is not locally analytically trivial.

7.4.2. Example: computations for $(\mathbb{R}, \mathbf{D}_4)$.

EXAMPLE 7.4.10. We carry out detailed computations for a simple example. Take $G = H_2$ (the dihedral group $\mathbf{D}_4$) acting in the standard way on $V = \mathbb{R}^2$. Denote coordinates on V by (x, y). We write a general point in V, without coordinates, in bold as $\mathbf{x}$. It is helpful to use some simple notation for the four isotropy types of the action of $\mathbf{D}_4$ on V. We denote the isotropy type of the origin by t, at non-zero points $(x, 0)$ by h, at (x, x), $x \neq 0$, by d, and at (x, y), $xy(x^2 - y^2) \neq 0$, by e. As basis for $P(V)^{H_2}$ we take $p_1(x, y) = (x^2 + y^2)$ and $p_2(x, y) = (x^4 + y^4)/4$. A basis $\{F_1, F_2\}$ for the $P(\mathbb{R}^2)^{H_2}$-module $P_{H_2}(V, V)$ is given by the gradients of p_1, p_2:

$$F_1(x, y) = (x, y), \quad F_2(x, y) = (x^3, y^3).$$

Since $k = \ell = 2$, we have $\mathbf{P}_1(\mathbb{R}^2, \mathbb{R}^2) = \mathbb{R}^2 \oplus L(\mathbb{R}^2, \mathbb{R}^2)$. Let $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$, $\mathbf{A} = [a_{ij}] \in L(\mathbb{R}^2, \mathbb{R}^2)$. Then $U_1(\mathbf{x}, \mathbf{t}, \mathbf{A}) = (x, \sum_{j=1}^2 t_j F_j(x), \sum_{i=1}^4 S_i(\mathbf{t}, \mathbf{A}))$, where

$$S_1(\mathbf{t}, \mathbf{A}) = t_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; S_2(\mathbf{t}, \mathbf{A}) = t_2 \begin{pmatrix} 3x^2 & 0 \\ 0 & 3y^2 \end{pmatrix},$$

$$\begin{aligned} S_3(\mathbf{t}, \mathbf{A}) &= \begin{pmatrix} a_{11}x^2 + a_{12}x^4 & a_{11}xy + a_{12}xy^3 \\ a_{11}xy + a_{12}x^3y & a_{11}y^2 + a_{12}y^4 \end{pmatrix}, \\ S_4(\mathbf{t}, \mathbf{A}) &= \begin{pmatrix} a_{11}x^4 + a_{12}x^6 & a_{11}x^3y + a_{12}x^3y^3 \\ a_{11}xy^3 + a_{12}x^3y^3 & a_{11}y^4 + a_{12}y^6 \end{pmatrix}. \end{aligned}$$

The universal variety $\Sigma \subset V \times \mathbb{R}^2$ is the common zero locus of

$$t_1 x + t_2 x^3 = 0, \quad t_1 y + t_2 y^3 = 0.$$

We write $\Sigma = \Sigma_t \cup \overline{\Sigma}_h \cup \overline{\Sigma}_d \cup \overline{\Sigma}_e$. Each of these subsets of Σ may be given explicitly in coordinate form. For example,

$$\begin{aligned} \Sigma_t &= \{(0, 0, t_1, t_2) \mid (t_1, t_2) \in \mathbb{R}^2\}, \\ \overline{\Sigma}_d &= \{(x, \pm x, t_1, t_2) \in V \times \mathbb{R}^2 \mid t_1 + t_2 x^2 = 0\}, \\ \overline{\Sigma}_e &= \{(x, y, 0, 0) \in V \times \mathbb{R}^2 \mid (x, y) \in V\}. \end{aligned}$$

We write $U_1 : V \times \mathbf{P}_1(\mathbb{R}^2, \mathbb{R}^2) \to V \times V \times L(V, V)$ in the form

$$U_1(\mathbf{x}, \mathbf{t}, \mathbf{A}) = (\mathbf{x}, T(\mathbf{x}, \mathbf{t}), L(\mathbf{x}, \mathbf{t}, \mathbf{A})).$$

Computing $L(\mathbf{x}, \mathbf{t}, \mathbf{A})$, we find that

$$L(\mathbf{x}, \mathbf{t}, \mathbf{A}) = \begin{pmatrix} t_1 + 3t_2x^2 + a_{11}x^2 & a_{11}xy \\ a_{11}xy & t_1 + 3t_2y^2 + a_{11}y^2 \end{pmatrix} + O(\|\mathbf{x}\|^4).$$

The $O(\|\mathbf{x}\|^4)$ terms are independent of $\mathbf{t}$ and depend linearly on a_{12}, a_{21}, a_{22}. Next we compute $L(\mathbf{x}, \mathbf{t}, \mathbf{A})$ for $(\mathbf{x}, \mathbf{t}) \in \Sigma$. As we shall only be interested in the case when $f \pitchfork_G 0$ at $(0, 0)$, we assume $t_2 \neq 0$. If $(\mathbf{x}, \mathbf{t}) \in \Sigma_t$, then $L(\mathbf{x}, \mathbf{t}, \mathbf{A}) = t_1 I_V$. Hence a necessary condition for $(\mathbf{t}, \mathbf{A}) \in \Sigma_1 = U_1^{-1}(Z_1)$ is $t_1 = 0$. Suppose $(\mathbf{x}, \mathbf{t}) \in \overline{\Sigma}_d$. We have $x = y$ and $t_1 + t_2 x^2 = 0$. Substituting in our expression for $L(\mathbf{x}, \mathbf{t}, \mathbf{A})$ we find that

$$L(\mathbf{x}, \mathbf{t}, \mathbf{A}) = \begin{pmatrix} (2t_2 + a_{11})x^2 & a_{11}x^2 \\ a_{11}x^2 & (2t_2 + a_{11})x^2 \end{pmatrix} + O(x^4).$$

Provided that $t_1, t_2 + a_{11}, t_2 \neq 0$, we may choose $r > 0$ such that if $0 < |x| < r$, $t_1 \neq 0$, then $L(\mathbf{x}, \mathbf{t}, \mathbf{A}) \in H(V)$. If $t_2(t_2 + a_{11}) = 0$, then although we can choose $a_{12}, a_{21}, a_{22}, r > 0$ so that $L(\mathbf{x}, \mathbf{t}, \mathbf{A}) \in H(V)$, $t_1 \neq 0$, $0 < |x| < r$, this condition will not be stable under perturbations of $\mathbf{t}, \mathbf{A}$. A similar analysis holds if $(\mathbf{x}, \mathbf{t}) \in \overline{\Sigma}_h \cup \overline{\Sigma}_e$ and we find that $L(\mathbf{x}, \mathbf{t}, \mathbf{A}) \in H(V)$ provided that $t_1, 2t_2 + a_{11} \neq 0$. If $2t_2 + a_{11} = 0$, we can choose a_{12}, a_{21}, a_{22} so that $L(\mathbf{x}, \mathbf{t}, \mathbf{A}) \in H(V)$, $t_1 \neq 0$. Again, this condition will not be stable under perturbations of $\mathbf{t}, \mathbf{A}$. As a result of our analysis the top dimensional stratum B of $\Sigma_1^G \subset \mathbb{R}^2 \times L(\mathbb{R}^2, \mathbb{R}^2)$ is given by

$$B = \{(\mathbf{t}, \mathbf{A}) \mid t_1 = 0, t_2 \neq 0, t_2 + a_{11}, 2t_2 + a_{11} \neq 0\}.$$

These are exactly the conditions given in proposition 4.5.4 (note that the a of proposition 4.5.4 is related to a_{11} by $a = a_{11}/2$). For this example it is easy to see that $j_1^1 X \pitchfork_G Z_1$ at $0 \in V$ implies that $X \in \mathcal{K}_G(V)$.

REMARK 7.4.11. The computations of example 7.4.10 are not recommended as a good way of actually working out the order of determinacy for G-equivariant bifurcation problems on an absolutely irreducible representation. The methods used in chapter 4 to establish 3-determinacy for subrepresentations of the hyperoctahedral group are far easier (and more powerful) as we only have to consider *one* invariant in addition to the cubic generators in $\mathcal{F}$. However, theorem 7.4.3 has two important consequences. Firstly, in order to determine stabilities of branches in generic equivariant bifurcations, it is only necessary to consider coefficients f_j of the equivariants F_j which are *affine linear* in the polynomial invariants. That is, $f_j(x) = c_j + \sum_{i=1}^{\ell} c_{ji} p_i(x)$, $c_j c_{ji} \in \mathbb{R}$. Secondly, in order to deduce that a branch of solutions of a generic equivariant bifurcation will be hyperbolic it suffices to check that if the branch has isotropy H, then zeros with isotropy H of a generic equivariant vector field will be hyperbolic. Indeed, this condition implies that the corresponding top-dimensional strata of Σ^G has codimension 1.

7.5. Higher order versions of G-transversality

Suppose M, N are smooth manifolds and that Q is a submanifold of N. Let $f \in C^\infty(M, N)$ and suppose that $f \pitchfork Q$. It is easy to show that for all $r \geq 0$, $j^r f \pitchfork J^r(M, Q)$, where $J^r(M, Q)$ is regarded as a submanifold of $J^r(M, N)$. In fact, if $f \pitchfork Q$ and $\dim(Q) < \dim(N)$, we always have $j^r f(M) \cap J^r(M, Q) = \emptyset$, $r \geq 1$. As observed by Bierstone [**14, 15**], this result fails in the equivariant category if we use the definition of G-transversality for maps given in chapter 6. That is, $f \pitchfork_G Q$ does not imply $j^r f \pitchfork_G J^r(M, Q)$. Although this failure does not seem important for the study of local equivariant bifurcation theory, it is important for the stability and classification theory of equivariant maps (and it *may* be significant in the study of reversible equivariant vector fields and higher codimension problems). We illustrate with two examples.

EXAMPLES 7.5.1. (1) (Bierstone [**14, 15**]) Let SO(2) act on $\mathbb{C}$ as multiplication by $e^{\imath\theta}$ and on $\mathbb{C}^2$ as multiplication by $(e^{\imath\theta}, e^{2\imath\theta})$. A minimal set of homogeneous generators for the $P(\mathbb{C})^{\mathrm{SO}(2)}$-module $P_{\mathrm{SO}(2)}(\mathbb{C}, \mathbb{C}^2)$ is given by $F_1(z) = (z, 0)$, $F_2(z) = (0, z^2)$. Every $f \in C^\infty_{\mathrm{SO}(2)}(\mathbb{C}, \mathbb{C}^2)$ may be written (uniquely) in the form $f(z) = (\alpha(|z|^2)z, \beta(|z|^2)z^2)$ where $\alpha, \beta : \mathbb{R} \to \mathbb{C}$ are smooth. We have $f \pitchfork_{\mathrm{SO}(2)} 0$ at $0 \in \mathbb{C}$ if and only if $\alpha(0)$ and $\beta(0)$ are not both zero. However, if $\alpha(0) = 0$, $Df(0)$ is singular at zero. If we add the requirement that $j^1 f \pitchfork J^1(\mathbb{C}, \{0\})$, then $\alpha(0) \neq 0$ (and so $j^1 f(0) \notin J^1(\mathbb{C}, \{0\})$). On the other hand if $\alpha(0) = 0$, then $j^1 f(0) \in J^1(\mathbb{C}, \{0\})$ and $j^1 f$ is not SO(2)-transverse to $J^1(\mathbb{C}, \{0\})$ at $z = 0$. Since $j^1 f(0) \notin J^1(\mathbb{C}, \{0\})$ if $\alpha(0) \neq 0$, it is trivial that $j^r f(0) \notin J^r(\mathbb{C}, \{0\})$ for all $r \geq 1$ if $\alpha(0) \neq 0$ and so this sequence stabilizes at $r = 1$. If instead we took the SO(2)-action on $\mathbb{C}^2$ given as multiplication by $(e^{n\imath\theta}, e^{(n+1)\imath\theta})$, we may write $f \in C^\infty_{\mathrm{SO}(2)}(\mathbb{C}, \mathbb{C}^2)$ uniquely as $f(z) = (\alpha(|z|^2)z^n, \beta(|z|^2)z^{n+1})$. We have $f \pitchfork_{\mathrm{SO}(2)} 0$ at $0 \in \mathbb{C}$ if and only if $\alpha(0)$ and $\beta(0)$ are not both zero. We find that $j^n f \pitchfork_{\mathrm{SO}(2)} J^n(\mathbb{C}, \{0\})$ provided $\alpha(0) \neq 0$. We then have $j^r f(0) \notin J^r(\mathbb{C}, \{0\})$, $r \geq n$, and so $j^r f \pitchfork_{\mathrm{SO}(2)} J^r(\mathbb{C}, \{0\})$, for all $r \geq n$.
(2) Let SO(2) act on $V = \mathbb{C}^2$ as multiplication by $(e^{2\imath\theta}, e^{3\imath\theta})$ and on $W = \mathbb{C}^2$ as multiplication by $(e^{5\imath\theta}, e^{7\imath\theta})$. A minimal set of homogeneous generators for the $P(V)^{\mathrm{SO}(2)}$-module $P_{\mathrm{SO}(2)}(V, W)$ is given by

$$F_1(z_1, z_2) = (z_1 z_2, 0), \quad F_2(z_1, z_2) = (0, z_1^2 z_2), \quad F_3(z_1, z_2) = (0, \bar{z}_1 z_2^3)$$
$$F_4(z_1, z_2) = (\bar{z}_1^2 z_2, 0) \quad F_5(z_1, z_2) = (z_1^4 \bar{z}_2, 0), \quad F_6(z_1, z_2) = (0, z_1^5 \bar{z}_2).$$

The universal variety $\Sigma \subset V \times \mathbb{C}^6$ may be written as $\Sigma = (V \times \{0\}) \cup (X \times \mathbb{C}^6)$, where $X = \{(z_1, z_2) \mid z_1 z_2 = 0\}$. The canonical stratification $\mathcal{A}$ of $\mathbb{C}^6$ is given by $\mathcal{A} = (\mathbb{C}^6 \setminus \{0\}) \cup \{0\}$. If $f \in C^\infty_{\mathrm{SO}(2)}(V, W)$, we may write $f(z_1, z_2) = \sum_{i=1}^6 f_i(z_1, z_2) F_i(z_1, z_2)$. We have $f \pitchfork_{\mathrm{SO}(2)} 0$ at $0 \in V$ if and only if at least one of the $f_1(0, 0), \ldots, f_6(0, 0)$ is non-zero. On the other hand, $j^2 f \pitchfork_{\mathrm{SO}(2)} J^2(\mathbb{C}^2, 0)$ at $0 \in V$ if and only if $f_1(0, 0) \neq 0$. If $f_1(0, 0) \neq 0$ then $j^r f \pitchfork_{\mathrm{SO}(2)} J^r(\mathbb{C}^2, 0)$ for all $r \geq 1$.

DEFINITION 7.5.2 ([**15**, section 8]). Let M, N be smooth G-manifolds, Q be a closed G-invariant submanifold of N and $A \subset M$. A map $f \in C^\infty_G(M, N)$ is in *rth-approximation to equivariant general position to Q along A* if for $q = 0, \ldots, r$, $j^q f \pitchfork_G J^q(N, Q)$ along A. If f is in rth-approximation to equivariant general position to Q along A for all $r \geq 0$, then f *is in equivariant general position to Q along A*.

We adopt some new notation. If $f \in C^\infty_G(M, N)$ is in rth approximation to equivariant general position to Q along $A \subset M$, we write "$f \pitchfork_{G,r} Q$ on A". If f is in equivariant general position to Q along A, we write "$f \pitchfork_{G,\infty} Q$ along A". We remark that $f \pitchfork_G Q$ along $A \iff f \pitchfork_{G,0} Q$ along A.

Bierstone proves [**15**, Theorem 8.4] that if (V, G), (W, G) are G-representations then there exists $r = r(V, W)$ such that $f \pitchfork_{G,r} 0$ at $0 \in V$ implies that $f \pitchfork_{G,\infty} 0$

at $0 \in V$. It follows that the sequence of approximation conditions always stabilizes on a compact subset of N. From this it follows that the openness and density theorem for G-transversality extends to $\pitchfork_{G,\infty}$ [**15**, Theorem 8.6].

7.6. Extensions to the case of non-finite G

Most of what we have done in this chapter for finite groups G extends without difficulty to general compact Lie groups G. The main differences are that new definitions of branch and indexed branching pattern are required and some statements on dimension need to be modified. In this section we describe the main definitions and results. We will be brief on details of proofs as much of what we do is quite similar (though simpler) to the more general theory we develop in chapter 10 for studying branches of relative equilibria, relative periodic orbits and limit cycles.

7.6.1. Equilibrium G-orbits. We briefly review definitions and elementary properties of equilibrium group orbits. Suppose that G is compact Lie group and M a smooth G-manifold. For each $\tau \in \mathcal{O}(M,G)$, choose $H \in \tau$ and set $\Delta_\tau = G/H$ – a representative G-orbit with isotropy type τ. We recall our earlier notations that $\dim(\Delta_\tau) = g_\tau$ and $\dim(N(H)/H) = n_\tau$.

DEFINITION 7.6.1. Let $X \in C^\infty_G(TM)$. A G-orbit $\alpha \subset M$ is an *equilibrium G-orbit* of X if $X|\alpha \equiv 0$.

REMARK 7.6.2. If $X \in C^\infty_G(TM)$ and there exists $x \in M$ such that $X(x) = 0$, then Gx is an equilibrium G-orbit.

LEMMA 7.6.3. *Let α be an equilibrium G-orbit of $X \in C^\infty_G(TM)$. For all $x \in \alpha$, $T_xX|T_x\alpha = 0$. In particular, if α has isotropy type τ then 0 is an eigenvalue of T_xX with multiplicity at least g_τ.*

PROOF. Immediate since $X|\alpha \equiv 0$. □

DEFINITION 7.6.4. An equilibrium G-orbit $\alpha \subset M_\tau$ of $X \in C^\infty_G(TM)$ is *transversally hyperbolic* or *generic* if for any (each) $x \in \alpha$ there are precisely $\dim(M) - g_\tau$ eigenvalues of $T_xX : T_xM \to T_xM$ which have real part nonzero. The *index* of α, $\mathrm{ind}(X,\alpha)$, is defined to be the number of eigenvalues of T_xX (counting multiplicities) which have real part less than or equal to zero.

REMARKS 7.6.5. (1) If $\alpha \subset M_\tau$ is an equilibrium G-orbit of $X \in C^\infty_G(TM)$, then 0 is an eigenvalue of T_xX of multiplicity at least g_τ. Hence there can never be more than $\dim(M) - g_\tau$ eigenvalues of T_xX with nonzero real part.
(2) If α is transversally hyperbolic, then α is *normally hyperbolic* [**93**]. In fact normal hyperbolicity is immediate as dynamics is trivial restricted to α and so the normal behaviour automatically dominates the tangential behaviour.
(3) Suppose that $\alpha \subset M_\tau$ is transversally hyperbolic for X. It follows from the stability of normal hyperbolicity [**93**], or directly, that there exists an open

neighbourhood U of X in $C^\infty_G(TM)$ (C^1-topology), and a G-invariant open neighbourhood V of α in M such that every $X' \in U$ has a unique flow-invariant G-orbit $\alpha' \subset V$ which is normally hyperbolic. Unless $n_\tau = 0$, α' will generally *not* be an equilibrium G-orbit.
(4) We generally use the term 'transversally hyperbolic' rather than 'generic' for equilibrium G-orbits. Later, when we investigate relative equilibria (flow-invariant G-orbits), we always use the term 'generic' for normally hyperbolic relative equilibria.

EXERCISE 7.6.6. (1) Assuming the necessary results on normal hyperbolicity [**93**], show that if α is a transversally hyperbolic equilibrium G-orbit for X, then there exist G-invariant stable and unstable manifolds $W^s(\alpha)$, $W^u(\alpha)$ for α. Verify that $W^s(\alpha)$, $W^u(\alpha)$ intersect transversally along α and that $\mathrm{ind}(X, \alpha) = \dim(W^s(\alpha))$. (These results may also be proved using slices without recourse to the theory of normal hyperbolicity [**52**]. We indicate how in the next chapter.)
(2) Let $\alpha \subset M_\tau$ be a transversally hyperbolic equilibrium G-orbit of X. Show that α is an attractor if and only if $\mathrm{ind}(X, \alpha) = \dim(M)$.

7.6.2. Branching and stability for compact Lie groups. For the remainder of the section we assume G is a compact Lie group and (V, G) is an absolutely irreducible G-representation.

DEFINITION 7.6.7. Let $X \in \mathcal{V}_0(V, G)$ and $\tau \in \mathcal{O}(V, G)$. A branch of equilibria of isotropy type τ for X at $(0, 0) \in V \times \mathbb{R}$ is a C^1 G-equivariant map

$$\phi = (\mathbf{x}, \lambda) : [0, \delta] \times \Delta_\tau \to V \times \mathbb{R},$$

satisfying

(1) $\phi(0, u) = (0, 0)$, all $u \in \Delta_\tau$.
(2) For all $s \in (0, \delta]$, $\alpha_s = \mathbf{x}(s, \Delta_\tau)$ is an equilibrium G-orbit of $X_{\lambda(s)}$ of isotropy type τ.
(3) For every $u \in \Delta_\tau$, the map $\phi_u : [0, \delta] \to V \times \mathbb{R}$, $s \mapsto \phi(s, u)$, is a C^1-embedding.

If α_s is transversally hyperbolic for $X_{\lambda(s)}$, $s \neq 0$, the branch ϕ is a *branch of transversally hyperbolic equilibrium G-orbits*.

EXAMPLE 7.6.8. Define maps $c^+, c^- : [0, \infty) \to V \times \mathbb{R}$ by $c^\pm(s) = (0, \pm s)$, $s \in [0, \infty)$. The maps $c^\pm$ define the trivial branches of equilibrium G-orbits for any $X \in \mathcal{V}_0(V, G)$. Both branches are transversally hyperbolic.

We regard two branches of equilibrium G-orbits as equivalent if they differ only by a local reparameterization (see section 10.1.3 for more a precise definition of what is meant by 'local reparameterization in this context). Let $[\phi]$ denote the equivalence class of the branch ϕ. It is easy to verify that if a branch ϕ has a parameterization as a branch of transversally hyperbolic G-orbits, then the same is true for all parameterizations of ϕ. It follows that we may talk about the equivalence class ϕ as being transversally hyperbolic.

DEFINITION 7.6.9. Given $X \in \mathcal{V}_0(V, G)$, we let $\Sigma(X)$ denote the set of all equivalence classes of nontrivial branches of equilibrium G-orbits. We call $\Sigma(X)$ the *branching pattern* of X.

REMARK 7.6.10. Unlike the earlier definition we gave, where we assumed G was finite, our new definition of branching pattern is framed in terms of G-orbits and so $\Sigma(X)$ has only the structure of a trivial G-set. However, we regard each equivalence class in $\Sigma(X)$ as being labelled with the isotropy type of the branch.

DEFINITION 7.6.11. Suppose $\phi = (\mathbf{x}, \lambda) : [0, \delta] \times \Delta_\tau \to V \times \mathbb{R}$ is a branch of transversally hyperbolic G-orbits for $X \in \mathcal{V}_0(V, G)$. The branch is supercritical (or forward) if $\lambda'(s) > 0$ for all $s \in [0, \delta]$. If $\lambda'(s) < 0$, the branch is subcritical (or backward).

LEMMA 7.6.12. *Let $X \in \mathcal{V}_0(V, G)$ and suppose $\phi = (\mathbf{x}, \lambda) : [0, \delta] \times \Delta_\tau \to V \times \mathbb{R}$ is a branch of transversally hyperbolic G-orbits for X. Then $\lambda'(s) \neq 0$, $s \in (0, \delta]$. In particular, ϕ is either supercritical or subcritical.*

PROOF. We refer the reader to the proof of lemma 10.1.13. □

Suppose that $\phi = (\mathbf{x}, \lambda) : [0, \delta] \times \Delta_\tau \to V \times \mathbb{R}$ is a branch of transversally hyperbolic G-orbits for $X \in \mathcal{V}_0(V, G)$. By continuity, $\mathrm{ind}(X_{\lambda(s)}, \mathbf{x}(s, \Delta_\tau))$ is constant on $(0, \delta]$. Hence we may define the index of ϕ, $\mathrm{ind}(\phi)$, to be the common value of the indices of the nontrivial equilibrium G-orbits along the branch. Just as for finite G, the index function is well defined for all equivalence classes $[\phi] \in \Sigma(X)$ corresponding to transversally hyperbolic branches. Using lemma 7.6.12, we similarly define $\mathrm{sgn}([\phi]) \in \{\pm 1\}$ whenever $[\phi] \in \Sigma(X)$ is the equivalence class of a transversally hyperbolic branch.

Following section 4.2, we may formulate branching conditions on normalized families.

Condition B1

There exists a finite set $\phi_1, \dots \phi_{r+2}$ of solution branches, with images $C_1, \dots C_{r+2}$ ($C_j = \mathrm{image}(\phi_j)$), such that

(1) $\Sigma(X) = \{[\phi_1], ..., [\phi_r]\}$ and $[\phi_{r+1}] = [c^+], [\phi_{r+2}] = [c^-]$.
(2) If $i \neq j$, then $C_i \cap C_j = \{(0, 0)\}$.
(3) There is a neighbourhood N of $(0, 0)$ in $V \times \mathbb{R}$ such that

$$\mathbf{Z}(X) \cap N = \cup_{j=1}^{r+2} C_j$$

Condition B2

Every $[\phi] \in \Sigma(X)$ is a branch of transversally hyperbolic equilibrium G-orbits.

PROPOSITION 7.6.13. *Let $\mathcal{K}_G(V) = \{X \in \mathcal{V}_0 \mid X \pitchfork_G 0, \text{ at } 0 \in V\}$.*

(1) *$\mathcal{K}_G(V)$ is an open and dense subset of $\mathcal{V}_0$.*
(2) *If $X \in \mathcal{K}_G(V)$, then X satisfies the branching condition B1.*

PROOF. The proof is similar to that of theorem 7.1.5. In order to verify that the X satisfies the branching condition B1, we use the argument of the proof of theorem 7.1.5 applied to Σ_τ^H, where $H \in \tau$. $\square$

REMARK 7.6.14. There is an isotopy result similar to that of theorem 7.1.5.

Along exactly the same lines as in section 7.1, we may show that G-equivariant bifurcation problems on (V, G) are weakly d-determined, where d is the maximal degree of polynomials in a minimal set of homogeneous generators for $P_G(V, V)$, and that $\mathcal{K}_G(V)$ is contained in the space of weakly stable families. Of more interest perhaps are the computations determining which isotropy types can be expected to occur in the branching patterns of generic 1-parameter families.

Following section 6.15, given $\tau \in \mathcal{O}^\star(V, G)$, we define the closed semialgebraic subset R_τ of $\mathbb{R}^k$ by $R_\tau = \partial\Sigma_\tau \cap \mathbb{R}^k$. An isotropy type will not occur as the isotropy of a branch of equilibrium G-orbits for $X \in \mathcal{K}_G(V)$ if $\mathrm{codim}(R_\tau) > 1$. The isotropy type will be symmetry breaking if $\mathrm{codim}(R_\tau) = 1$ and generically symmetry breaking if R_τ is the hyperplane $t_1 = 0$.

PROPOSITION 7.6.15. *Let* $\tau \in \mathcal{O}^\star(V, G)$. *Then* $\mathrm{codim}(R_\tau) \geq 1 + n_\tau$. *Consequently, if* $n_\tau > 0$, τ *is not a symmetry breaking isotropy type.*

PROOF. The result follows by lemma 6.15.2. $\square$

For the remainder of the section we discuss the issue of stabilities along branches of equilibrium G-orbits.

Let $L_S(V, V) = L(V, V) \setminus H(V)$. If $A \in L_S(V, V)$ then A has at least one eigenvalue on the imaginary axis. For each $\tau \in \mathcal{O}(V, G)$, let $L_\tau(V, V)$ denote the G-invariant subset of $L(V, V)$ consisting of maps that have at least $g_\tau + 1$ eigenvalues on the imaginary axis (counting multiplicities). If $g_\tau = 0$ then $L_\tau(V, V) = L_S(V, V)$; otherwise $L_\tau(V, V)$ is a proper subset of $L_S(V, V)$.

LEMMA 7.6.16. *For all* $\tau \in \mathcal{O}(V, G)$, $L_\tau(V, V)$ *is a closed* G*-invariant semialgebraic subset of* $L(V, V)$.

PROOF. Set $g_\tau = m$ and let $A \in L(V, V)$ have characteristic polynomial $p(\lambda)$. Suppose $\dim(V) = n$ so that $p(\lambda)$ is of degree n. Then $A \in L_\tau(V, V)$ if and only if there exists a factorization

$$p(\lambda) = \lambda^p \prod_{i=1}^{s} (\lambda^2 + \alpha_i)^{q_i} q(\lambda),$$

where $p + 2\sum_{i=1}^s q_i = m$, $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_s$, and q has no roots on the imaginary axis. From this one may easily show that the set of characteristic polynomials which have this root structure determines a semialgebraic subset $S_{p,q_1,\ldots,q_s}$ of $\mathbb{R}^n$ (the coefficient space for characteristic polynomials). Take the (finite) union of $S_{p,q_1,\ldots,q_s}$ over all possible $p, q_1, \ldots, q_s$ such that $p + 2\sum_{i=1}^s q_i = m$. The resulting set S is a closed semialgebraic subset of $\mathbb{R}^n$. But now $L_\tau(V, V) =$

$c^{-1}(S)$, where $c : L(V,V) \to \mathbb{R}^n$ is the polynomial map which sends A to the coefficients of the characteristic polynomial of A. $\square$

For $\tau \in \mathcal{O}(V,G)$, define

$$Z_1(\tau) = \{(x, 0, A) \in J^1(V,V) \mid x \in V_\tau,\ A \in L_\tau(V,V)\}.$$

We see from lemma 7.6.16 that $Z_1(\tau)$ is a G-invariant semialgebraic subset of $J^1(V,V)$. Set $Z_1 = \cup_{\tau\in\mathcal{O}(V,G)} Z_1(\tau) \subset J^1(V,V)$.

Lemma 7.6.17. *Z_1 is a closed G-invariant semialgebraic subset of $J^1(V,V)$. If G is finite, Z_1 coincides with Z_1 as defined in example 7.2.7 and section 7.4.*

Proof. It is obvious from the construction that Z_1 is a G-invariant semialgebraic subset of $J^1(V,V)$. If G is finite, $Z_1 = V \times \{0\} \times L_S(V,V)$, and the definition is the same as our earlier definition. It remains to prove that Z_1 is closed. For this, it suffices to observe that

$$\begin{aligned} \partial Z_1(\tau) &= \partial V_\tau \times \{0\} \times L_\tau(V,V), \\ &= \cup_{\mu>\tau} V_\mu \times \{0\} \times L_\tau(V,V), \\ &\subset \cup_{\mu>\tau} V_\mu \times \{0\} \times L_\mu(V,V), \end{aligned}$$

since if $\mu > \tau$, $g_\mu \le g_\tau$ and so $L_\mu(V,V) \subset L_\tau(V,V)$. $\square$

Just as we did for G finite, we define

$$\begin{aligned} \mathcal{K}^1_G(V) &= \{X \in \mathcal{K}_G(V) \mid j^1_1 X \pitchfork_G Z_1 \text{ at } (0,0) \in V \times \mathbb{R}\}, \\ &= \{X \in \mathcal{V}_0(V,G) \mid j^0 X \pitchfork_G 0,\ j^1_1 X \pitchfork_G Z_1 \text{ at } (0,0) \in V \times \mathbb{R}\}. \end{aligned}$$

The proofs of the following results are all similar to those of the corresponding results when G is finite.

Lemma 7.6.18. (1) *$\mathcal{K}^1_G(V)$ is an open and dense subset of $\mathcal{V}_0(V,G)$.*
(2) *If $X \in \mathcal{K}^1_G(V)$, then X satisfies the branching conditions (B1, B2).*

Theorem 7.6.19. *$\mathcal{K}^1_G(V) \subset \mathcal{S}(V,G)$ (stable families).*

Theorem 7.6.20. *Equivariant bifurcation problems on (V,G) are d-determined, where $d \le d_k$.*

7.7. Notes on chapter 7

The equivariant version of Thom's jet transversality theorem was proved by Bierstone and appears in [**15**]. The result was used in his work on the stability theory of smooth equivariant maps [**15**] which completed the programme started by Poenaru [**143**] and Ronga [**150**] to prove Mather's stability theorems for smooth equivariant maps. Applications to equivariant bifurcation theory were first given in [**57**] and developed further in [**60**]. While equivariant jet transversality is technically complicated to set up on account of the use of Faá di Bruno's formula and the factorization scheme, applications to equivariant bifurcation theory are relatively straightforward. This is because arguments are local, generators can be assumed homogeneous and, at least for the codimension one theory, it is only necessary to look at the 1-jet extension map. Methods generalize straightforwardly to the study of relative equilibria (see chapter 10 and [**60**]).

CHAPTER 8

Equivariant Dynamics

In this chapter we investigate of the theory of equivariant dynamical systems on smooth G-manifolds. We assume throughout that G is a compact Lie group and M is a smooth G-manifold.

Much of the chapter will be devoted to a careful study of the dynamics of equivariant diffeomorphisms and flows on and near dynamically invariant group orbits. We apply some of these ideas in chapter 10 where we study equivariant bifurcation to and from relative equilibria. After developing the local theory, we state and sketch the proofs of a number of global theorems on equivariant dynamical systems including the equivariant version of the Kupka-Smale theorem.

For simplicity, we always assume maps and vector fields are smooth, that is C^∞. Of course, many (though not all) results hold under the assumption of C^r, $2 \leq r < \infty$, sometimes with a loss of one order of differentiability. If M is compact we take the C^∞-topology on function spaces, if M is non-compact we take the Whitney C^∞-topology.

If M, N are smooth G-manifolds, let $C^\infty_G(M, N)$ denote the space of smooth equivariant maps from M to N. (Here as elsewhere, $C^\infty(M, N)$ will denote the space of all smooth maps from M to N and $C^\infty_G(M, N)$ will be a closed subspace of $C^\infty(M, N)$.) Let $\mathrm{Diff}_G(M)$ denote the space of smooth equivariant diffeomorphisms of M. We remark that $\mathrm{Diff}_G(M)$ is an open subspace of $C^\infty_G(M, M)$. Let $C^\infty_G(TM)$ denote the space of smooth equivariant vector fields on M. If $X \in C^\infty_G(TM)$, we let Φ^X_t (or just Φ^X) denote the associated flow of X. If M is compact, $\Phi^X_t \in \mathrm{Diff}_G(M)$, all $t \in \mathbb{R}$. Otherwise, $\Phi^X : \mathcal{D} \subset M \times \mathbb{R} \to M$ will be a smooth map defined on a (maximal) G-invariant open neighbourhood $\mathcal{D}$ of $M \times \{0\}$ in $M \times \mathbb{R}$. It is convenient for us to assume that when M is non-compact that Φ^X is defined on all of M. As least as far as the local theory is concerned this will be no loss of generality. Indeed, if $X \in C^\infty_G(TM)$ and K is any G-invariant compact subset of M we can always choose a strictly positive smooth G-invariant function $f \in C^\infty(M)^G$ such that $f \equiv 1$ on K and the flow of $fX \in C^\infty_G(TM)$ is defined for all t. Note that the phase portrait of Φ^{fX} is equal to that of X and that $fX = X$ on K.

8.1. Invariant G-orbits

We start by considering equivariant diffeomorphisms.

DEFINITION 8.1.1. Let $f \in \mathrm{Diff}_G(M)$ and $\alpha \subset M$ be a G-orbit. We say α is a *relative fixed set* for f if $f(\alpha) = \alpha$. If there exists a smallest $p > 1$ such that $f^p(\alpha) = \alpha$, then α is a *relative periodic orbit* of f of *relative prime period* p. We also refer to the f-orbit of α as a relative periodic orbit.

REMARKS 8.1.2. (1) If $f \in \mathrm{Diff}_G(M)$, $x \in M$ and $f(x) = x$, then $\alpha = Gx$ is a relative fixed set for f. In this case, we refer to α as a *G-orbit of fixed points* for f. Similarly if $f^q(x) = x$, for some $q > 0$, then $\alpha = Gx$ is a relative periodic orbit of f. If α has relative prime period p, then $p|q$ (α may be a relative fixed set for f – see the examples below).
(2) If α is a relative periodic orbit of f, relative prime period p, then so is $f^i(\alpha)$, $i \in \mathbb{Z}$.

EXAMPLES 8.1.3. (1) Let $M = \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ (the 2-torus) and let SO(2)-act on M by $e^{\imath t}(\theta, \psi) = (\theta + t, \psi)$. Given $\psi \in [0, 2\pi)$, set $\alpha_\psi = \mathrm{SO}(2)(0, \psi) = \mathbb{T} \times \{\psi\} \subset M$. For $u \in [0, 2\pi)$, define $f_u \in \mathrm{Diff}_{\mathrm{SO}(2)}(M)$ by $f_u(\theta, \psi) = (\theta + u, \psi + \frac{1}{2}\sin\psi)$. If $u = 0$, then $(0, 0)$ and $(0, \pi)$ are fixed points for f_0 and so α_0 and α_π are fixed sets for f_0. If $u \in (0, 2\pi)$, then α_0 and α_π are relative fixed sets for f_u. If $u/2\pi$ is irrational, then no point in α_0 and α_π is periodic for f_u. If we define $g_u \in \mathrm{Diff}_{\mathrm{SO}(2)}(M)$ by $g_u(\theta, \psi) = (\theta + u, \psi + \pi + \frac{1}{4}\sin 2\psi)$, then α_0 and $\alpha_{\pi/2}$ are distinct relative periodic orbits of g_u which are both of relative prime period 2. If $u/2\pi$ is irrational, then g_u will have no periodic points. If $u/2\pi$ is rational, points in $\alpha_0 \cup \alpha_{\pi/2}$ will be periodic but of prime period different from 2 (unless $u = 0$).
(2) Take the standard action of SO(2) on $\mathbb{T}$. Let $f \in \mathrm{Diff}_{\mathrm{SO}(2)}(\mathbb{T})$ be defined by $f(\theta) = \theta + \pi$. Then $f^2(\theta) = \theta$, for all $\theta \in \mathbb{T}$. In this case since $f(\theta) \in \mathrm{SO}(2)(\theta)$, we regard $\mathrm{SO}(2)(\theta)$ as a relative fixed set for f rather than a relative periodic orbit.

8.1.1. Vector fields and flows.

DEFINITION 8.1.4. Let $X \in C^\infty_G(TM)$ and $\alpha \subset M$ be a G-orbit. We say α is a *relative equilibrium* for X if α is Φ^X-invariant (equivalently, X is tangent to α). If $X|\alpha \equiv 0$, we refer to α as an equilibrium G-orbit (or G-orbit of equilibria).

EXAMPLE 8.1.5. Take the standard action of SO(2) on $\mathbb{C}$. Let $X \in \mathbb{C}^\infty_{\mathrm{SO}(2)}(\mathbb{C})$ be defined by $X(z) = a\imath z + (1 - |z|^2)z$, where $a \in \mathbb{R}$. The unit circle $|z| = 1$ is a relative equilibrium of X. If $a = 0$, $|z| = 1$ is an equilibrium G-orbit.

DEFINITION 8.1.6. A compact Φ^X- and G-invariant subset Σ of M is a *relative periodic orbit* of X if Σ is not a G-orbit and there exists $T > 0$ such that $\Sigma = G(\Phi^X_x([0, T])$. If $T > 0$ is minimal, we call T the *relative prime period* of Σ.

LEMMA 8.1.7. *Let Σ be a relative periodic orbit of X of relative prime period T.*

1. $\Sigma = G(\Phi^X_y([0, T])$ *for all* $y \in \Sigma$.
2. *All points of Σ have the same isotropy type.*

(3) Σ *is a smooth* G*-invariant submanifold of* M*.*
(4) Σ/G *is diffeomorphic to* S^1*.*
(5) Φ^X *(respectively,* X*) induces a flow* $\Phi^{X^\star}$ *(respectively, vector field* $X^\star$*) on* S^1 *and* S^1 *is a periodic orbit of* $\Phi^{X^\star}$ *of prime period* T*.*

PROOF. (1,2) follow from the G-equivariance of Φ^X. For (3), observe that $G\Phi^X_x(T) = Gx$ (otherwise $G(\Phi^X_x([0,T]))$ would be a proper subset of Σ or T would not be minimal). Hence $\Sigma = G(\Phi^X_x([0,T))$ and so the map $G/G_x \times [0,T) \to \Sigma$, $(g[G_x],t) \to g\Phi^X_x(t)$ is a G-equivariant embedding. Hence Σ has the structure of a smooth G-invariant submanifold of M. Since the action of G on Σ is monotypic, Σ/G has the structure of a smooth manifold which may be identified with $\mathbb{R}/\mathrm{T}\mathbb{Z} \approx S^1$ (see remark 3.1.15). For (5) it suffices to note that X drops down to a smooth vector field on the orbit space Σ/G. □

EXAMPLE 8.1.8. Let SO(2) act on the 2-torus $\mathbb{T}^2$ by $e^{\imath t}(\theta,\phi) = (\theta + t, \phi)$. Let X be the SO(2)-equivariant vector field on $\mathbb{T}^2$ defined by $X(\theta,\phi) = (a,b)$, $a,b \in \mathbb{R}$. Provided that $b \neq 0$, $\mathbb{T}^2$ is a relative periodic orbit of X and the induced flow on $\mathbb{T}^2/\mathrm{SO}(2) = S^1$ has period $2\pi/b$.

8.2. Stabilities and normal hyperbolicity

8.2.1. Diffeomorphisms. Let $f \in \mathrm{Diff}(M)$ and V be a compact invariant submanifold of M. We recall from [**93**] that V is *normally hyperbolic*[1] for f if there exists a Tf-invariant splitting $TV \oplus \mathbb{E}^u \oplus \mathbb{E}^s$ of $T_V M = TM|V$ into continuous subbundles such that, relative to some Riemannian metric on M, we have

$$\sup_{x\in V} \|Tf|\mathbb{E}^s_x\| \quad < \quad \inf_{x \in V} m(Tf|T_xV), \tag{8.1}$$

$$\sup_{x\in V} \|Tf|T_xV\| \quad < \quad \inf_{x \in V} m(Tf|\mathbb{E}^u_x). \tag{8.2}$$

$(m(A) = \inf\{\|AX\| \mid \|X\| = 1\} = \|A^{-1}\|^{-1}.)$

LEMMA 8.2.1. *Let* $f \in \mathit{Diff}_G(M)$ *and suppose that* V *is a compact* G*-invariant normally hyperbolic submanifold of* M*.*

(1) *The splitting* $TV \oplus \mathbb{E}^u \oplus \mathbb{E}^s$ *is a sum of continuous* G*-vector bundles.*
(2) *The normal hyperbolicity estimates (8.1,8.2) hold with respect to a smooth* G*-invariant Riemannian metric on* M*.*

PROOF. The estimates (8.1,8.2) continue to hold if we average the Riemannian metric over G, proving (1). It is shown in [**93**] that the hyperbolic estimates determine the bundles $\mathbb{E}^u$, $\mathbb{E}^s$ uniquely. Using the G-invariance of the metric, the equivariance of f and the G-invariance of V, the estimates hold relative to the splitting $TV \oplus g\mathbb{E}^u \oplus g\mathbb{E}^s$, for all $g \in G$. Hence $g\mathbb{E}^u = \mathbb{E}^u$, $g\mathbb{E}^s = \mathbb{E}^s$, $g \in G$. □

[1]Strictly "Immediately, absolutely 1-normally hyperbolic" in [**93**]

8.2.2. Flows. If Φ^X is a G-equivariant flow on M, a compact Φ^X- and G-invariant submanifold V of M is normally hyperbolic if, for some $s \neq 0$, V is normally hyperbolic for Φ^X_s. It is shown in [**93**, Theorem 2.4] that V is normally hyperbolic for Φ^X_t for all $t \neq 0$ and that the corresponding splittings of $T_V M$ are independent of t. Together with lemma 8.2.1, this shows that the splitting of $T_V M$ we get if V is Φ^X- and G-invariant is a splitting of continuous G-vector subbundles of $T_V M$.

8.2.3. Genericity.

DEFINITION 8.2.2. Let α be a relative fixed set for $f \in \mathrm{Diff}_G(M)$ (respectively, a relative equilibrium of $X \in C^\infty_G(TM)$). We say α is *generic* for f (respectively, X) if α is normally hyperbolic for f (respectively, X). We similarly define genericity for relative periodic orbits.

LEMMA 8.2.3. *Suppose that α is a relative fixed set for $f \in \mathrm{Diff}_G(M)$. We may choose a G-invariant Riemannian metric on M such that $\|Tf|T_y\alpha\| = 1$, for all $y \in \alpha$. In particular, if α is generic we may choose a G-invariant Riemannian metric on M such that the estimates (8.1,8.2) may be written equivalently as*

$$\sup_{x\in\alpha} \|Tf|\mathbb{E}^s_x\| \;<\; 1, \tag{8.3}$$

$$\inf_{x\in\alpha} \|Tf|\mathbb{E}^u\| \;>\; 1. \tag{8.4}$$

Moreover, the splitting $T_\alpha M = T\alpha \oplus \mathbb{E}^u \oplus \mathbb{E}^s$ will be a smooth orthogonal direct sum of smooth G-vector bundles.

Similar results hold for relative equilibria and relative periodic orbits.

PROOF. Fix $x \in \alpha$, set $G_x = H$ and identify α with G/H. We define an action of $G \times N(H)$ on α by

$$(g, n)(k[H]) = gk[H]n^{-1} = gkn^{-1}[H], \;\; (g, n) \in G \times N(H), k[H] \in \alpha.$$

Since f is G-equivariant, $f(x) = n^{-1}x$ for some $n \in N(H)$. Hence $f(k[H]) = (k, n)[H] = (e, n)k[H]$, all $k[H] \in \alpha$. If we average a Riemannian metric for α over $G \times N(H)$, then $Tf|T\alpha$ will be an isometry in the averaged metric and so $\|Tf|T_y\alpha\| = 1$, for all $y \in \alpha$.

The estimates (8.1,8.2) depend only on the norm of the Riemannian metric restricted to $T_\alpha M$. Averaging over $G \times N(H)$, we may assume that the metric is $G \times N(H)$-invariant on $T_\alpha M$. The metric may then be extended smoothly and G-equivariantly to all of M. Since $Tf|T\alpha$ is an isometry, estimate (8.1) becomes $\sup_{x\in\alpha} \|Tf|\mathbb{E}^s_x\| < 1$. Since $m(A) = \|A^{-1}\|^{-1} \leq \|A\|$, $\inf_{x\in\alpha} m(Tf|\mathbb{E}^u_x) \leq \inf_{x\in\alpha} \|Tf|\mathbb{E}^u_x\|$, proving (8.4). Since G acts transitively on α, $\mathbb{E}^u$, $\mathbb{E}^s$ are smooth G-vector bundles over α. Once we have the estimates (8.3,8.4) it is easy to choose the metric so that the bundles $T\alpha$, $\mathbb{E}^u$ and $\mathbb{E}^s$ are orthogonal.

The proof in case α is a relative equilibrium follows from the result for diffeomorphisms (or directly). For relative periodic orbits, it suffices to note that we can always reparameterize time so that the $T_x\Phi_t^X$ is an isometry tangent to flow lines (for diffeomorphisms, we use the fact that f cyclically permutes the group orbits constituting a relative periodic orbit). $\square$

REMARK 8.2.4. If Σ is a generic relative periodic orbit for the C^r-flow Φ_t, $1 \leq r < \infty$, then the splitting $T_\Sigma M = T\alpha \oplus \mathbb{E}^u \oplus \mathbb{E}^s$ will be only be a splitting of C^{r-1} G-vector bundles and cannot be assumed orthogonal. Since G acts transitively on G-orbits, the restriction of $T_\Sigma M$ to a G-orbit in Σ does always split as a sum of smooth G-vector bundles (which may be assumed orthogonal).

8.2.4. Stable and unstable manifolds. Suppose that Σ is a generic relative fixed set, relative equilibrium or relative periodic orbit. A consequence of the theory of normal hyperbolicity is that there exist transverse smooth local stable and unstable manifolds $W^s_{\mathrm{loc}}(\Sigma)$, $W^u_{\mathrm{loc}}(\Sigma)$ through Σ (these results can also be deduced from stable manifold theory for equilibria, fixed points and periodic orbits by looking at slices – we indicate how in exercise 8.3.27). We give the main results for diffeomorphisms and relative periodic orbits (the results for flows are completely analogous).

THEOREM 8.2.5 (Stable manifold theorem & Hartmans theorem [**52**]). *Let $f \in \mathrm{Diff}_G(M)$ and suppose that Σ is a generic relative periodic orbit for f. Then*

(a) *There exist smooth G-invariant locally f-invariant submanifolds $W^u_{loc}(\Sigma)$ and $W^s_{loc}(\Sigma)$ of M through Σ such that $T_\Sigma W^u_{loc}(\Sigma) = T\Sigma \oplus \mathbb{E}^u$ and $T_\Sigma W^s_{loc}(\Sigma) = T\Sigma \oplus \mathbb{E}^s$.*

(b) *There exists a G-invariant open neighbourhood U of Σ such that*

$$\begin{aligned} W^u_{loc}(\Sigma) &= \{z \in U \mid f^n(z) \in U, n \leq 0, \text{ and } d(f^n(z), \Sigma) \to 0, \text{ as } n \to -\infty\}, \\ W^s_{loc}(\Sigma) &= \{z \in U \mid f^n(z) \in U, n \geq 0, \text{ and } d(f^n(z), \Sigma) \to 0, \text{ as } n \to \infty\}. \end{aligned}$$

(c) *$W^u_{loc}(\Sigma)$ and $W^s_{loc}(\Sigma)$ have the structure of smooth G-equivariant fibrations over Σ (strong stable and unstable foliations). The fibre $W^{uu}_{loc}(\Sigma, p)$ at $p \in \Sigma$ is characterized by*

$$W^{uu}_{loc}(\Sigma, p) = \{z \in W^u_{loc}(\Sigma) \mid d(f^n(z), f^n(p)) \to 0, \text{ as } n \to -\infty\}.$$

Similarly for $W^{ss}_{loc}(\Sigma, p)$.

(d) *If we let $Nf = Tf|\mathbb{E}^u \oplus \mathbb{E}^s$, then f is conjugate to Nf near Σ by a G-equivariant homeomorphism.*

(e) *There exists an open neighbourhood $\mathcal{U}$ of $f \in \mathrm{Diff}_G(M)$ (C^1-topology), a G-invariant open neighbourhood V of Σ and continuous maps $\upsilon, \upsilon^u, \upsilon^s : \mathcal{U} \to \mathrm{Diff}_G(M)$ such that*
   (1) *$\upsilon(f) = \upsilon^u(f) = \upsilon^s(f) = I_M$,*
   (2) *For $g \in \mathcal{U}$, $\upsilon(g)(\Sigma) = \Sigma^g$ is contained in V and is a generic relative periodic orbit of g of the same relative period as Σ.*
   (3) *$\upsilon^u(g)(W^{uu}_{loc}(\Sigma, p)) = W^{uu}_{loc}(\Sigma^g, \upsilon(f)(p))$, $p \in \Sigma$, $g \in \mathcal{U}$. Similarly for W^u, W^{ss} and W^s.*

Granted the existence of local stable and unstable manifolds for a generic relative periodic orbit Σ we define global stable and unstable manifolds in the usual way by continuation. Thus,

$$W^u(\Sigma) = \cup_{n\geq 0} f^n W^u_{\mathrm{loc}}(\Sigma),\ W^s(\Sigma) = \cup_{n\leq 0} f^n W^u_{\mathrm{loc}}(\Sigma).$$

The unstable manifold $W^u(\Sigma)$ is an equivariantly injectively immersed G-invariant submanifold of M. More formally, we may construct an equivariant injective immersion $\xi^u(f) : \mathbb{E}^u \to M$ such that $\xi^u(f)(\mathbb{E}^u_0) = \Sigma$ and $\xi^u(f)(\mathbb{E}^u) = W^u(\Sigma)$. We may further require that $\xi^u(f)$ maps the fibers $\mathbb{E}^u_p$ onto the strong unstable sets $W^{uu}(\Sigma, p)$, $p \in \Sigma$. Similarly for $W^s(\Sigma)$ and for flows.

8.3. Diffeomorphism dynamics on G-orbits

Suppose that $\alpha \subset M$ is a relative fixed set for $f \in \mathrm{Diff}_G(M)$. Let $x \in \alpha$ and set $G_x = H$. Identifying α with G/H, we regard $f|\alpha$ as element of $\mathrm{Diff}_G(G/H)$. Let $C^0_G(G/H, G/H)$ denote the space of continuous G-equivariant maps of G/H.

LEMMA 8.3.1. *Let H be a compact subgroup of G and $f \in C^0_G(G/H, G/H)$.*

(1) $f \in \mathrm{Diff}_G(G/H)$.
(2) *There exists $n \in N(H)$ such that $f(g[H]) = gn[H] = g[H]n$, for all $g \in G$. In particular,* $\mathrm{Diff}_G(G/H) \approx N(H)/H$.
(3) *There exists a smallest $n(f) \geq 1$ such that $f^{n(f)}$ is G-equivariantly isotopic to the identity map of G/H.*
(4) *There exists $\beta_f \in C_G(H)_0$ such that $f^{n(f)}([H]) = \beta_f[H]$.*

PROOF. Since f is G-equivariant, $f(g[H]) = gf([H])$ for all $g \in G$. Hence f is smooth since the action of G on G/H is smooth (use a local section of G over G/H). Since $G_{f[H]} = H$, there exists $n \in N(H)$ such that $f([H]) = n[H]$ and so $f(g[H]) = gn[H]$, for all $g \in G$. Consequently, f is invertible ($f^{-1}(g[H]) = gn^{-1}[H]$) and so $f \in \mathrm{Diff}_G(G/H)$, proving (1). We have a natural antihomomorphism $\rho : N(H) \to \mathrm{Diff}_G(G/H)$ defined by $\rho(n) = f_n$, where $f_n(g[H]) = gn[H]$. Since $\rho(n) = \rho(e)$ if and only if $n[H] = [H]$, kernel$(\rho) = H$ and so $\mathrm{Diff}_G(G/H) \approx N(H)/H$, proving (2).

In order to prove (3,4), we start by recalling that $(G/H)^H = N(H)[H] \approx N(H)/H$. A necessary condition for $F \in \mathrm{Diff}_G(G/H)$ to be G-equivariantly isotopic to the identity is that $F([H]) \in (N(H)/H)_0[H]$ or, equivalently that $F([H]) \in C_G(H)_0[H]$ (by corollary 3.10.3). Let $n(f)$ be the smallest strictly positive integer such that there exists $c \in C_G(H)_0$ for which $f^{n(f)}([H]) = c[H]$. Choose $\xi \in \mathfrak{c}(\mathfrak{h})$ such that $c = \exp(\xi)$ and define $F_t \in \mathrm{Diff}_G(G/H)$ by $F_t([H]) = \exp(t\xi)[H]$, $t \in [0, 1]$. Then F_t defines the required equivariant isotopy between $f^{n(f)}$ and $I_{G/H}$. □

Given $f \in \mathrm{Diff}_G(G/H)$, $x \in G/H$, we define the closure of the f-orbit of x by

$$O_f(x) = \mathrm{closure}\{f^n(x) \mid n \in \mathbb{Z}\}.$$

By lemma 8.3.1(2), f determines a unique $\hat{n} \in N(H)/H$ such that

$$O_f([H]) = \langle \hat{n} \rangle [H],$$

where $\langle \hat{n} \rangle$ is the closed Abelian subgroup of $N(H)/H$ generated by $\hat{n}$. Now f induces a smooth map $\hat{f} : \langle \hat{n} \rangle \to \langle \hat{n} \rangle$ by $\hat{f}(g) = \hat{n}g$, $g \in \langle \hat{n} \rangle$. If we identify $\langle \hat{n} \rangle$ with $O_f([H]) = \langle \hat{n} \rangle [H] \subset G/H$, we may write $f(g[H]) = g\hat{f}(e)$, for all $g \in G$. Consequently, in order to describe the dynamics of equivariant diffeomorphisms of G/H, it suffices to classify all subgroups $\langle \hat{n} \rangle$, $\hat{n} \in N(H)/H$, together with the associated dynamics induced on $\langle \hat{n} \rangle$ by left translation by $\hat{n}$.

8.3.1. Cartan subgroups of a compact Lie group G.

Compact connected Lie groups. We recall the fundamental result about maximal tori in a compact *connected* Lie group (proofs and more details may be found in Bröcker and Dieck [**30**] or Adams [**2**]).

THEOREM 8.3.2. *Let G be a compact connected Lie group. Then there exists a toral subgroup $\mathbb{T}^r$ of G with the following properties.*

(1) *If A is an Abelian Lie subgroup of G containing $\mathbb{T}^r$, then $A = \mathbb{T}^r$ ($\mathbb{T}^r$ is a* maximal *Abelian subgroup).*
(2) *If $x \in G$, there exists $t \in G$ such that $x \in t\mathbb{T}^r t^{-1}$. In particular, every point of G lies in at least one maximal torus.*
(3) *If J is a maximal connected Abelian subgroup of G, then there exists $t \in G$ such that $J = t\mathbb{T}^r t^{-1}$.*
(4) *If J is a toral subgroup of G then J is contained in at least one maximal torus.*

We call the dimension of a maximal torus the rank *of G, denoted by* $\mathrm{rk}(G)$.

REMARK 8.3.3. If G is connected compact and Abelian, then $G \cong \mathbb{T}^g$, where $g = \dim(G)$ (proposition 1.5.16).

EXERCISE 8.3.4. (1) Show that a maximal compact Abelian subgroup of a compact connected Lie group need not be a torus. (Look at SO(3).)
(2) Let G be a compact connected Lie group. Show that the set of elements $g \in G$ such that $\langle g \rangle$ is a maximal torus is dense in G, Deduce that if H is a compact subgroup of G, $N(H)/H$ is connected, and $f \in \mathrm{Diff}_G(G/H)$, then there exist arbitrarily small perturbations f' of f in $\mathrm{Diff}_G(G/H)$ such that $O_{f'}([H])$ determines a maximal torus $\mathbf{T}$ in $N(H)/H$ and $f|O_{f'}([H])$ is left translation by a topological generator of $\mathbf{T}$.

Compact disconnected Lie groups. We shall need information about the structure of maximal compact Abelian subgroups of G when G is not connected. The theory we describe is due to G Segal. Details and proofs are in [**30**] (a brief presentation, assuming known the topological approach for connected G, is in [**58**]).

DEFINITION 8.3.5. A closed subgroup H of G is *monogenic* or *topologically cyclic* if there exists $h \in H$ such that $H = \langle h \rangle$.

REMARK 8.3.6. A monogenic subgroup is Abelian.

LEMMA 8.3.7. *A closed subgroup H of G is monogenic if and only if H is isomorphic to the produce of a torus group with a cyclic group.*

PROOF. If H is monogenic, then H is compact Abelian and $H/H_0 \cong \mathbb{Z}_p$ where p is the number of connected components of H. Hence $H \cong \mathbb{T}^s \times \mathbb{Z}_p$, where $s = \dim(H)$. For the converse, consider $H = \mathbb{T}^s \times \mathbb{Z}_p$. Let $\mathbb{Z}_p = \langle \rho \rangle$ and pick a point $u \in \mathbb{T}^s$ such that $\langle u \rangle = \mathbb{T}^s$ (Kronecker's theorem). Then $\langle (u, \rho) \rangle = \mathbb{T}^s \times \mathbb{Z}_p$. $\square$

Set $P = G/G_0$ and let $\Pi : G \to P$ denote the quotient map. Since $G_0 \lhd G$, P has the structure of a finite group. Let $\mathcal{Z} = \mathcal{Z}(G)$ denote the set of all cyclic subgroups of P. If $p \in P$, then $\langle p \rangle \in \mathcal{Z}$ and so every point of P lies in at least one cyclic subgroup.

DEFINITION 8.3.8. Let $X \in \mathcal{Z}$. A closed Abelian subgroup K of G is of *type* X if K is monogenic and $\Pi(K) = X$.

REMARK 8.3.9. If K is a type X subgroup of G then $K = \langle u \rangle$ for some $u \in G$, $\langle \Pi(u) \rangle = X$ and if $K \cong \mathbb{T}^s \times \mathbb{Z}_q$, then $|X||q$.

DEFINITION 8.3.10. Let $X \in \mathcal{Z}$. A *Cartan subgroup* of G of type X is a maximal closed Abelian subgroup of G of type X.

The next result generalizes the maximal torus theorem to non-connected compact Lie groups.

THEOREM 8.3.11. *Let $X \in \mathcal{Z}$.*

(1) *Any two Cartan subgroups of G of type X are conjugate subgroups of G.*
(2) *Suppose that $K = \langle u \rangle = \mathbb{T}^s \times \mathbb{Z}_m$ is a Cartan subgroup of G and that u lies in the connected component $G^\star$ of G. Then for every $x \in G^\star$, there exists $t \in G_0$ such that $x \in tKt^{-1}$.*

We define $\mathrm{rk}(G, X) = s$ to be the dimension of a Cartan subgroup of type X and $\mathrm{con}(G, X) = m$ to be the number of connected components of a Cartan subgroup of type X.

PROOF. See [**30**]. $\square$

COROLLARY 8.3.12. *Let $X \in \mathcal{Z}$ and suppose $|X| = p$. Let $K \subset G$ be a closed Abelian subgroup of type X. Then*

(1) *$K \cong \mathbb{T}^r \times \mathbb{Z}_s$, where $\mathrm{con}(G, X)|s$.*
(2) *There exists a Cartan subgroup K' of type X such that $K' \supset K$.*

PROOF. Since K is monogenic, it follows from lemma 8.3.7 that $K \cong \mathbb{T}^r \times \mathbb{Z}_s$, for some $s \geq 0$. Since $\mathbb{T}^r$ is connected and $\Pi(K) = X$, it follows that $\mathrm{con}(G, X)|s$, proving (1). In order to prove (2), choose any Cartan subgroup J of G which is of type X and suppose that $K = \langle u \rangle$. By theorem 8.3.11(2) that there exists $t \in G_0$ such that $u \in tJt^{-1}$. Hence K is contained in the Cartan subgroup tJt^{-1}. $\square$

EXAMPLES 8.3.13. (1) If $G = \times^n \mathrm{O}(2)$, then $G/G_0 \cong \mathbb{Z}_2^n$. For $0 \leq r \leq n$, let $H(r)$ be the subgroup of G generated by $(e,\ldots,e,\overbrace{\kappa,\ldots,\kappa}^{r})$, where κ is any element of $\mathrm{O}(2) \setminus \mathrm{SO}(2)$. Then $\Pi(H(r)) = X \in \mathcal{Z}(G)$, $|X| = 2$, and $\mathrm{rk}(G,X) = n - r$.
(2) Let $n \geq 2$. Then $\mathrm{O}(n)/\mathrm{SO}(n) \cong \mathbb{Z}_2$. We have

$$\mathrm{rk}(\mathrm{O}(n),\mathbb{Z}_2) = \begin{cases} \mathrm{rk}(\mathrm{SO}(n)) - 1 & \text{if } n \text{ is even} \\ \mathrm{rk}(\mathrm{SO}(n)) & \text{if } n \text{ is odd} \end{cases}$$

LEMMA 8.3.14. *Let $X \in \mathcal{Z}$ and suppose $|X| = q$. There exists a smallest $p \geq 1$ and $u \in G$, such that*

(1) $\langle u \rangle \cong \mathbb{Z}_{pq}$.
(2) $\Pi(\langle u \rangle) = X$.

PROOF. Pick $v \in G$ such that $\langle \Pi(v) \rangle = X$. By corollary 8.3.12, there exists a Cartan subgroup K of type X containing v. By lemma 8.3.7, $K = \mathbb{T}^s \times \mathbb{Z}_{pq}$, where $p \geq 1$, $s \geq 0$. Pick $(e,u) \in \mathbb{T}^s \times \mathbb{Z}_{pq}$ such that $\mathbb{Z}_{pq} = \langle u \rangle$. Then $\Pi(\langle u \rangle) = X$ and $|\langle u \rangle| = pq$. Minimality of p follows easily from corollary 8.3.12. $\square$

REMARK 8.3.15. Note that the group $\mathbb{T}^s \times \mathbb{Z}_{pq}$ constructed in the proof of lemma 8.3.14 meets G_0 in p connected components. Contrary to what is claimed in [**58**, Lemmas 4.1,4.2], we cannot generally find an element $u \in G$ which has order $|X|$ and satisfies $\langle \Pi(u) \rangle = X$ unless $G \cong G_0 \rtimes G/G_0$ (that is, G is a split extension of G_0).

EXERCISE 8.3.16 (see [**183**, Example 4.2]). Let H be a connected subgroup of $\mathrm{SO}(2n)$, $n \geq 1$ and assume that $-I \in H$. Embed H in $\mathrm{SO}(4n)$ as $\{(A, A^{-1}) \mid A \in H\}$. Given $B \in \mathrm{SO}(2n)$, define $\kappa = \begin{pmatrix} 0 & B \\ -B^{-1} & 0 \end{pmatrix} \in \mathrm{SO}(4n)$. Then $\kappa^4 = I$ and $\kappa^2 = -I \in H \subset \mathrm{SO}(4n)$. Assume that $B \in N_{\mathrm{SO}(2n)}(H)$. If we let G denote the subgroup of $\mathrm{SO}(4n)$ generated by H and κ, show that (a) $G_0 = H$, $G/G_0 \cong \mathbb{Z}_2$, (b) if $\Pi : G \to G/H$ denotes the quotient map then the order of $\Pi(\kappa)$ is always 2 and half the order of any $u \in G$ such that $\Pi(u) = \Pi(\kappa)$. (Explicit examples can be obtained for $n \geq 1$ by taking H to be a maximal torus of $\mathrm{SO}(2n)$ and $B \in N_{\mathrm{SO}(2n)}(H)$.)

DEFINITION 8.3.17. Let $X \in \mathcal{Z}$, $u \in G$ and $X = \langle \Pi(u) \rangle$. If u satisfies the conditions of lemma 8.3.14, we call u a *representative generator* of X.

REMARK 8.3.18. Every representative generator for X lies in at least one Cartan subgroup of type X.

EXERCISE 8.3.19. Suppose $X \in \mathcal{Z}(G)$ and K is a type X subgroup of G. Show that there exists $u \in K$ such that $|\langle \Pi(u) \rangle| = p|X|$, where p satisfies the minimality condition of lemma 8.3.14.

EXERCISE 8.3.20. Let $X \in \mathcal{Z}(G)$ and suppose that the connected component $G^\star$ of G maps by Π to a generator of X. Show that the subset of $G^\star$ consisting of u such that $\langle u \rangle$ is a Cartan subgroup of type X is dense in $G^\star$.

8.3.2. Dynamics on relative fixed sets & periodic orbits. Given $f \in \mathrm{Diff}_G(G/H)$, set $\Pi(O_f([H]) = X_f \in \mathcal{Z}(N(H)/H)$.

PROPOSITION 8.3.21. *Suppose that $f \in \mathrm{Diff}_G(G/H) \approx N(H)/H$. If C is the connected component of $N(H)/H$ containing f, then*

1. *For all $g \in C$, $\Pi(O_g([H])) = X_f$ (X_f depends only on the G-isotopy class of f in $\mathrm{Diff}_G(G/H)$).*
2. *There is a dense full-measure subset $C^\star$ of C such that if $g \in C^\star$, then $O_g([H])$ is a type X_f Cartan subgroup of $N(H)/H$. That is, $O_g([H]) \cong \mathbb{T}^r \times \mathbb{Z}_m$, where $r = \mathrm{rk}(N(H)/H, X_f)$, $m = \mathrm{con}(N(H)/H, X_f)$.*

PROOF. Part (1) is immediate since all points in C project to the (same) generator of X_f. For the density part of (2), use exercise 8.3.20. We omit the proof of the full measure statement which relies on Fubini's theorem (the measure will be Haar measure on $N(H)/H$). □

Before giving our next result, we briefly recall the definition of a smooth foliation. A smooth *foliation* $\mathcal{F}$ of a manifold M consists of a smoothly locally trivial partition of M into smooth submanifolds, all of the same dimension. As basic examples, the integral curves of a nowhere zero vector field on M define a 1-dimensional foliation ("flow-box" theorem) and the fibers of a smooth locally trivial fibre bundle $\pi : E \to X$ define a smooth foliation of E.

PROPOSITION 8.3.22. *Let $f \in \mathrm{Diff}_G(G/H)$. There is a unique smooth foliation $\mathcal{F} = \{\mathcal{F}_x \mid x \in G/H\}$ of G/H satisfying*

1. *$\mathcal{F}$ is G-invariant: $g\mathcal{F}_x = \mathcal{F}_{gx}$, for all $g \in G$, $x \in G/H$.*
2. *Leaves are f-invariant: $f(\mathcal{F}_x) = \mathcal{F}_x$, for all $x \in G/H$.*
3. *For each $x \in G/H$, $\mathcal{F}_x$ is naturally isomorphic to a compact Abelian subgroup of $N(G_x)/G_x$. $f|\mathcal{F}_x$ is transitive and acts by right translation by (u, η), where $\langle (u, \eta) \rangle \approx \mathcal{F}_x$.*
4. *For each $x \in G/H$, $\mathcal{F}_x \cong \mathbb{T}^r \times \mathbb{Z}_p$, where $r \leq \mathrm{rk}(N(H)/H, X_f)$ and $p = s\,\mathrm{con}(N(H)/H, X_f)$, $s \geq 1$.*

PROOF. For $x \in G/H$, we define $\mathcal{F}_x = O_f(x)$. The listed properties are immediate from our constructions and definitions. □

It remains to discuss the case of relative periodic orbits of an equivariant diffeomorphism. Suppose that $\alpha \subset M$ is a relative periodic orbit of relative prime period p. Identify α with G/H and define $\Sigma = \cup_{i=0}^{p-1} f^i(G/H)$. Set $f = f|\Sigma$. We may represent $f : \Sigma \to \Sigma$ as a skew product $\phi_c : \mathbb{Z}_p \times G/H \to \mathbb{Z}_p \times G/H$ over $\phi : \mathbb{Z}_p \to \mathbb{Z}_p$, where $\phi(r) = r + 1$, mod p, and $c : \mathbb{Z}_p \to N(H)$. That is,

$$\phi_c(r, g[H]) = (r + 1, g[H]c(r)).$$

Just as we did for relative fixed sets, we can to reduce to studying skew extensions $\phi_c : \mathbb{Z}_p \times N(H)/H \to \mathbb{Z}_p \times N(H)/H$, where ϕ is as previously defined and $c : \mathbb{Z}_p \to N(H)/H$. In other words, $\mathrm{Diff}_G(\Sigma)$ can be identified with $C^0(\mathbb{Z}_p, N(H)/H)$. Let $\eta_f = \prod_{i=0}^{p-1} c(r) \in N(H)/H$ and $X_f = \langle \Pi(\eta_f) \rangle \in \mathcal{Z}(N(H)/H)$.

Proposition 8.3.23. *(Notation as above)*

(1) *Suppose that $f, f' : \Sigma \to \Sigma$. If $\eta_f, \eta_{f'}$ lie in the same connected component C of $N(H)/H$ then $X_f = X_{f'}$.*
(2) *There is a dense full-measure subset $C^\star$ of C such that if $\eta_f \in C^\star$, then $O_f([H])$ meets each fiber $\{r\} \times N(H)/H$ in a type X_f Cartan subgroup of $N(H)/H$. In particular, every $f \in \mathrm{Diff}_G(\Sigma) \approx C^0(\mathbb{Z}_p, N(H)/H)$ can be approximated by $g \in \mathrm{Diff}_G(\Sigma)$ such that $O_g([H])$ meets the fibres of $\mathbb{Z}_p \to N(H)/H$ in a Cartan subgroup.*

Proof. The result follows from the corresponding results on relative fixed sets. Note that for the perturbation theory it suffices to perturb c at $r-1$. $\square$

8.3.3. Isotopy lemmas.

Definition 8.3.24 ([**54, 56**]). Let X be a G-space. A map $\chi : X \to G$ is *skew (G)-equivariant* if $\chi(gx) = g\chi(x)g^{-1}$ for all $x \in X$, $g \in G$.

Exercise 8.3.25. (1) Suppose that $f \in \mathrm{Diff}_G(M)$ and $\chi : M \to G$ is skew equivariant. Show that $\chi f \in \mathrm{Diff}_G(M)$ (we define $(\chi f)(x) = \chi(x)f(x)$, $x \in M$).
(2) Suppose that M is a compact G-manifold and all G-orbits have the same dimension, say $g > 0$. Suppose that $f : M \to M$ induces the identity map on the orbit space. Show that if f is smoothly isotopic to the identity map of M by an isotopy covering the identity map on M/G, then $f = \chi I_M$, where $\chi : M \to G$ is skew G-equivariant. (Hint: Observe that the assumptions imply that $f(x) \in N(G_x)_0 x$ for all $x \in M$. Use a covering by slices and the results of section 3.10.3. See also [**56**, Lemmas C,D].)
(3) Suppose that $(\mathbb{C}^2, \mathrm{SU}(2))$ is the standard representation of $\mathrm{SU}(2)$ on $\mathbb{C}^2$ (regard $\mathrm{SU}(2)$ as the group of complex matrices $\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, where $|a|^2 + |b|^2 = 1$). Show that if $A_\theta \in L_{\mathrm{SU}(2)}(\mathbb{C}^2, \mathbb{C}^2)$ is defined as complex multiplication by $e^{\imath\theta}$, then A_θ cannot be represented as $\chi I_{\mathbb{C}^2}$, where $\chi : \mathbb{C}^2 \to \mathrm{SU}(2)$ is a smooth skew equivariant map (see [**54**, Example 4]).

Suppose that H is a closed subgroup of G and $f \in \mathrm{Diff}_G(G/H)$. Let $\pi : E \to G/H$ be a smooth G-vector bundle over G/H. We denote the zero section of E by E_0.

Lemma 8.3.26. *(Notation as above.) Let $F \in \mathrm{Diff}_G(E)$ and suppose that $F|E_0 = f$ (that is, $f\pi|E_0 = \pi F|E_0$). We may choose a G-invariant open neighbourhood U of $E_0 \subset E$ such that $F|U$ is smoothly G-isotopic to $\bar{F} : U \to E$ where*

(1) $\bar{F}|E_0 = f$ *and* $\bar{F}$ *is fibre preserving (that is,* $f\pi|U = \pi\bar{F}|U$*).*
(2) $F = \chi\bar{F}$*, where* $\chi : U \to G$ *is a smooth skew* G*-equivariant map such that* $\chi(x) \in C(G_x)$*, for all* $x \in U$*.*

Proof. Let $x = [H] \in G/H$ and set $y = f(x)$. Since F is 1:1, $G_y = H$. The fiber E_y is a slice for the action of G on E at y. By lemma 3.10.6, we may choose an open neighbourhood V of $y \in G/H$ and an admissible local section $\sigma : V \subset G/H \to G$. Set $W = \pi^{-1}(V) \subset E$. We see from lemma 3.10.8 that the map $\rho^\sigma : E_y \times V \to W$ defined by $\rho^\sigma(z, v) = \sigma(v)z$ is an H-equivariant diffeomorphism such that for all $z \in E_y$ and $v = g[H] \in V$, $g \in N(G_z)$, we have $\sigma(v) \in C_G(G_z)_0$ and $\rho^\sigma(z, v) \in C_G(G_z)_0 z$ (note remark 3.10.9). Let $\nu : W \to G$ be the smooth map uniquely defined by the condition $\nu(\sigma(v)z) = \sigma(v)$, all $(z, v) \in E_y \times V$. Since ρ^σ is H-equivariant we have $\nu(hw) = h\nu(w)h^{-1}$, $w \in W$, $h \in H$. Choose an H-invariant open neighbourhood U_x of $0 \in E_x$ such that $F(U_x) \subset W$. Define $\bar{F}|E_x$ by $\bar{F}(u) = \nu(F(u))^{-1}F(u)$. Since ν is H-equivariant, and $\nu(F(u))^{-1} \in C(G_u)$, $\bar{F} : U_x \to E_y$ is well-defined and H-equivariant. Clearly $\bar{F}(U_x) \subset E_y$. Set $U = G(U_x)$. Then $\bar{F}$ extends G-equivariantly to a G-equivariant fibre preserving embedding $\bar{F} : U \to E$. Obviously, $\bar{F}|E_0 = f$. Shrinking V, U_x if necessary, we may suppose that there is a smooth map $\eta : V \to \mathfrak{g}$ such that $\sigma(v) = \exp(\eta(v))$, all $v \in V$. Define $F_t|E_x$ by $F_t(e) = \exp(\eta(-tv)F(e)$, $t \in [0, 1]$. Extending F_t G-equivariantly to U defines the required isotopy of G-equivariant embeddings between F and $\bar{F}$. Finally, it follows from the construction that we may write $F = \chi\bar{F}$, where $\chi : U \to G$ is smooth and $\chi|U_x$ is skew H-equivariant. Since $\bar{F}, F$ are G-equivariant, we deduce easily that χ is skew G-equivariant. □

Suppose that $f \in \mathrm{Diff}_G(M)$ and that $\alpha = G/H$ is a relative fixed set of f. As far as dynamics of f in a neighbourhood of α are concerned, lemma 8.3.26 implies that, modulo drifts along group orbits, it no loss of generality to assume f preserves a family of slices foliating a neighbourhood of α. Lifting to the associated normal bundle $\pi : E \to \alpha = G/H$, we may therefore assume that $f \in \mathrm{Diff}_G(E)$ is fibre preserving and covers $f|G/H$. The drift can be recovered (locally) by composing with the reciprocal of the skew equivariant map χ constructed in lemma 8.3.26.

Exercise 8.3.27. Show, using lemma 8.3.26, how theorem 8.2.5 can be deduced from the stable manifold theorem for hyperbolic fixed points and periodic points of a diffeomorphism.

Let $\mathrm{Diff}_G(E, f)$ denote the space of smooth G-equivariant fibre preserving maps $F : E \to E$ such that $F|E_0 = f$, where $f \in \mathrm{Diff}_G(G/H)$.

Lemma 8.3.28. *Let* $F \in \mathrm{Diff}_G(E, f)$*. There exists a smallest integer* $p \geq 1$*,* G*-equivariant isotopies* $f_t \in \mathrm{Diff}_G(G/H)$*,* $F_t \in \mathrm{Diff}_G(E, f_t)$*,* $t \in [0, 1]$*, and a smooth skew* G*-equivariant family* $\chi_t : G/H \to G$ *such that*

(1) $\chi_0 = e$*,* $F_0 = F$*,* $f_0 = f$*, and* $f_1^p = I_{G/H}$*.*
(2) $f_t = \chi_t f_0$*,* $F_t = \chi_t F_0$*,* $t \in [0, 1]$*.*

(3) $\chi_t(x) \in C(G_x)_0$, *for all* $x \in G/H$, $t \in [0,1]$.

The integer $p = \mathrm{con}(N(H)/H, X_f)$ *(notation of proposition 8.3.21).*

PROOF. Let f correspond to the element $\hat{n} \in N(H)/H$. Let $K = \mathbb{T}^s \times \mathbb{Z}_p$ be a Cartan subgroup of type X_f containing $\langle \hat{n} \rangle$. Choose $\hat{n}_1 \in K$ such that $\hat{n}_1, \hat{n}$ lie in the same connected component of K and $\hat{n}_1^p = e$. Let $\{k_t \in N(H) \mid t \in [0,1]\}$ be a smooth curve such that $\hat{k}_0 = \hat{n}$, $\hat{k}_1 = \hat{n}_1$. We define $f_t \in \mathrm{Diff}_G(G/H)$ by $f_t([H]) = k_t[H]$. Since $\hat{n}_1^p = \hat{k}_1^p = e$, $f_1^p = I_{G/H}$. Since $k_t = (k_t k_0^{-1})k_0$ and $k_t k_0^{-1} \in N(H)_0$, there are smooth curves $\eta_t \in \mathfrak{c}(\mathfrak{h})$ and $h_t \in H$ such that $k_t k_0^{-1} = \exp(\eta_t) h_t$, $t \in [0,1]$, and $\eta_0 = h_0 = e$. Hence $f_t([H]) = \exp(\eta_t) f([H])$. If we define $\chi_t([H]) = \exp(\eta_t)$, then χ_t extends skew G-equivariantly to a smooth map $\chi_t : G/H \to G$. Set $F_t = \chi_t F$, $t \in [0,1]$. $\square$

REMARK 8.3.29. Lemma 8.3.28 implies that, modulo drifts along group orbits, it is no loss of generality to assume that every relative fixed set of $f \in \mathrm{Diff}_G(M)$ is a G-orbit of periodic points.

8.3.4. Stabilities of relative fixed sets and periodic orbits. Let V be a real vector space and $A \in L(V, V)$. Let $\mathcal{E}(A) \subset \mathbb{C}$ denote the set of eigenvalues of A. We define the *reduced spectrum* of A by

$$\mathrm{spec}(A) = \{|\lambda| \mid \lambda \in \mathcal{E}(A)\}.$$

If $\mu \in \mathrm{spec}(A)$ then the multiplicity of μ is defined to be the sum of the multiplicities of all $\lambda \in \mathcal{E}(A)$ such that $|\lambda| = \mu$. For $m \in \mathbb{Z}$, we define $\mathrm{spec}(A)^m = \{\mu^m \mid \mu \in \mathrm{spec}(A)\}$. Of course, $\mathrm{spec}(A)^m = \mathrm{spec}(A^m)$.

EXAMPLE 8.3.30. If $A \in \mathrm{O}(V)$, then $\mathrm{spec}(A) = \{1\}$ (multiplicity is $\dim(V)$).

Let α be a relative fixed set of $f \in \mathrm{Diff}_G(M)$. Given $x \in \alpha$, suppose $nf(x) = x$, where $n \in N(G_x)$. Set $f_n = nf \in \mathrm{Diff}(M)$. Clearly, $f_n(x) = x$ and so $T_x f_n \in L(T_x M, T_x M)$. Unless G is Abelian, f_n will generally not be G-equivariant. Define

$$\mathbf{spec}(f, \alpha) = \mathrm{spec}(T_x(f_n)).$$

We call $\mathbf{spec}(f, \alpha)$ the *reduced spectrum of f along α*.

THEOREM 8.3.31. *(Notation as above.)* $\mathbf{spec}(f, \alpha)$ *is independent of the choice of* $x \in \alpha$, $n \in N(G_x)$. *Furthermore,*

(1) *The multiplicity of* $1 \in \mathbf{spec}(f, \alpha)$ *is greater than or equal to* $\dim(G/H)$.

(2) *If* $\tilde{f} \in \mathrm{Diff}_G(M)$ *induces the same map on* M/G *as* f, *then*

$$\mathbf{spec}(f, \alpha) = \mathbf{spec}(\tilde{f}, \alpha).$$

We break the proof of theorem 8.3.31 into several lemmas.

LEMMA 8.3.32. *Let* $f \in \mathrm{Diff}_G(G/H)$. *Then* $\mathbf{spec}(f, G/H) = \{1\}$ *and the multiplicity of* 1 *is equal to* $\dim(G/H)$.

Proof. As in the proof of lemma 8.2.3, take the $G \times N(H)$-action on G/H defined by $(g, n)k[H] = gkn^{-1}[H]$ and choose a $G \times N(H)$-invariant Riemannian metric ξ on G/H. If $f([H]) = n[H]$, then $f_n = n^{-1}f$ is an isometry with respect to ξ and so $T_{[H]}f_n \in \mathrm{O}(T_{[H]}G/H)$. □

Lemma 8.3.33. *Let (V, G) be a G-representation and suppose $A \in L_G(V, V)$ satisfies $A(Gx) = Gx$, for all $x \in V$. Then,*

(1) $\mathrm{spec}(A) = \{1\}$.
(2) *If $B \in L_G(V, V)$,* $\mathrm{spec}(AB) = \mathrm{spec}(B)$.

Proof. Statement (1) follows by noting that the condition on A implies that if $\oplus_i V_i$ is any direct sum decomposition of V as a sum of $\mathbb{R}$-irreducible representations then $A(V_i) = V_i$. Using the real isotopic decomposition of V (theorem 2.7.10), we may reduce (2) to the special case $V = W^p$, (W, G) is irreducible. Choose a decomposition $V = \oplus_i W$ with respect to which B is in upper triangular form (this will be over $\mathbb{R}$ or $\mathbb{C}$ according to whether (W, G) is irreducible of real, complex or quaternionic type respectively). The result follows since A is diagonal with respect to the decomposition $V = \oplus_i W$. □

Lemma 8.3.34. *Let K be a compact Abelian Lie group and (V, K) be a real representation of K. Then*

$$\mathrm{spec}(kA) = \mathrm{spec}(A),\ A \in L_K(V, V), k \in K.$$

Proof. Since K is Abelian, every $k \in K$ defines a K-equivariant linear isomorphism of V. Obviously $k(Kx) = Kx$, for all $x \in V$. Hence the result by lemma 8.3.33(2). □

Lemma 8.3.35. *Let H be a compact subgroup of the compact Abelian Lie group K. Suppose $\pi : E \to K/H$ is a smooth K-vector bundle over K/H and $A : E \to E$ is a smooth K-vector bundle map covering $a \in \mathrm{Diff}_K(K/H)$. For $x \in K/H$, choose $k \in K$ such that $ka(y) = y$. Then* $\mathrm{spec}(kA|E_x)$ *is independent of the choice of $x \in K/H$ and $k \in K$ and depends only on A.*

Proof. Since K is Abelian, $K_x = H$ for all $x \in K/H$. Hence each fiber E_x has the structure of an H-representation. If $ka(x) = x$ for some $x \in K/H$, then $ka(y) = y$ for all $y \in K/H$. Suppose $y = gx$. Since $kA = g(kA)g^{-1}$, $\mathrm{spec}(kA|E_x) = \mathrm{spec}(kA|E_y)$ and so it is enough to show that $\mathrm{spec}(kA|E_x)$ is independent of the choice of k. If $ka(x) = k'a(x)$, then $k'A = (k'k^{-1})kA$ and so $\mathrm{spec}(kA|E_x) = \mathrm{spec}(k'A|E_x)$ by lemma 8.3.34. □

Lemma 8.3.36. *Let H be a compact subgroup of G and $\pi : E \to G/H$ be a smooth G-vector bundle over G/H. Suppose that $A : E \to E$ is a G-vector bundle map covering $a \in \mathrm{Diff}_G(G/H)$. Given $x \in G/H$, choose $g \in G$ such that $ga(x) = x$. Then* $\mathrm{spec}(gA|E_x)$ *is independent of the choice of x and g and depends only on A.*

PROOF. As in lemma 8.3.35, the problem is to show that $\mathrm{spec}(gA|E_x)$ is independent of g. Suppose first that G is finite and that $ga(x) = \bar{g}a(x) = x$, where $g, \bar{g} \in G$. Then $g^n A^n = \bar{g}^n A^n$, $n \geq 0$. Choosing $n > 0$ so that $g^n = \bar{g}^n = e$, we see that $\mathrm{spec}((gA|E_x)^n) = \mathrm{spec}((\bar{g}A|E_x)^n) = \mathrm{spec}(A^n|E_x)$ and so $\mathrm{spec}(gA|E_x) = \mathrm{spec}(\bar{g}A|E_x)$. For the general case, suppose $x = [H] \in G/H$ and start by assuming that $N(H)$ is connected. Let g lie in a maximal torus $\mathbf{T}$ of $N(H)$ and suppose $\bar{g} \in \mathbf{T}$ satisfies $\bar{g}a(x) = x$. Then $g^{-1}\bar{g} \in H \cap \mathbf{T} = S$ which is compact Abelian. Since E_x is an S-representation, lemma 8.3.35 implies that $\mathrm{spec}(gA|E_x) = \mathrm{spec}(\bar{g}A|E_x)$. Any $\bar{g} \in N(H)$ satisfying $\bar{g}a(x) = x$ is a conjugate of some $g \in \mathbf{T}$ satisfying $ga(x) = x$. That is, there exists $p \in N(H)$ such that $\bar{g} = pgp^{-1}$. Since $pgp^{-1}A = p(gA)p^{-1}$, we have $\mathrm{spec}(gA|E_x) = \mathrm{spec}(\bar{g}A|E_x)$. Finally, if $N(H)$ is not connected, we can choose $m \geq 1$ such that $a^m(x) \in N(H)_0 x$. The previous argument then applies together with that given for finite groups. $\square$

REMARK 8.3.37. In future, we denote the common value of $\mathrm{spec}(gA|E_x)$ by $\mathrm{spec}(A)$ and refer to $\mathrm{spec}(A)$ as the reduced spectrum of the map A.

LEMMA 8.3.38. *Let α be a relatively fixed set of $f \in \mathrm{Diff}_G(M)$.*

(1) $\mathbf{spec}(f, \alpha)$ *is well-defined, independent of choices.*
(2) *The multiplicity of* $1 \in \mathbf{spec}(f, \alpha)$ *is greater than or equal to* $\dim(\alpha)$.
(3) $\mathbf{spec}(f^m, \alpha) = \{\mu^m \mid \mu \in \mathbf{spec}(f, \alpha)\} \stackrel{def}{=} \mathbf{spec}(f, \alpha)^m$, $m \in \mathbb{Z}$.

PROOF. (1) Take $E = T_\alpha M$, $A = Tf|T_\alpha M$, and apply lemma 8.3.36. (2) Observe that $T_\alpha M$ contains the Tf-invariant subbundle $T\alpha$ and apply lemma 8.3.32. (3) Let $x \in \alpha$ and choose $g \in G$ such that $gf(x) = x$. Since $(gTf)^m = g^m Tf^m$ and $g^m f^m(x) = x$, (3) is immediate from the definitions. $\square$

LEMMA 8.3.39. *Let $E \to G/H$ be a smooth G-vector bundle over G/H. Suppose $A : E \to E$ is a smooth G-vector bundle map covering $a \in \mathrm{Diff}_G(G/H)$ and $\chi : G/H \to G$ is smooth and skew G-equivariant. Then* $\mathrm{spec}(\chi A) = \mathrm{spec}(A)$.

PROOF. Immediate from the definition of $\mathrm{spec}(A)$. $\square$

LEMMA 8.3.40. *Let H be a compact subgroup of G and $\pi : E \to G/H$ be a smooth G-vector bundle over G/H. Suppose that $A, \bar{A} : E \to E$ are G-vector bundle maps covering $a, \bar{a} \in \mathrm{Diff}_G(G/H)$ and that $A, \bar{A}$ induce the same maps on E/G. Then* $\mathrm{spec}(A) = \mathrm{spec}(\bar{A})$.

PROOF. It suffices to find $m > 0$ such that $\mathrm{spec}(A^m) = \mathrm{spec}(\bar{A}^m)$. Using lemma 8.3.28, we may find $m > 0$ and a smooth skew G-equivariant map $\chi : G/H \to G$ such that $A^m = \chi\bar{A}^m$. By lemma 8.3.39, $\mathrm{spec}(\bar{A}^m) = \mathrm{spec}(\chi\bar{A}^m) = \mathrm{spec}(A^m)$. $\square$

Proof of theorem 8.3.31 We have already proved that **spec** is well-defined and that the multiplicity of $1 \in \mathbf{spec}(f, \alpha)$ is at least $\dim(\alpha)$ (lemma 8.3.38). For (2) of theorem 8.3.31 take $E = T_\alpha M$, $A = Tf|E$ in lemma 8.3.40. $\square$

As an immediate corollary of theorem 8.3.31 we have the following characterization of generic relative fixed sets and periodic orbits.

THEOREM 8.3.41. *Let $f \in \mathrm{Diff}_G(M)$.*

(1) *If α is a relative fixed set of f, then α is generic if and only if $1 \in$ $\mathbf{spec}(f, \alpha)$ has multiplicity equal to $\dim(\alpha)$.*
(2) *If α is a relative periodic orbit of f of relative prime period m, then α is generic if and only if $1 \in \mathbf{spec}(f^m, \alpha)$ has multiplicity equal to $\dim(\alpha)$.*

8.3.5. Perturbation theory. Let α be a relative fixed set of $f \in \mathrm{Diff}_G(M)$. Fix $x \in \alpha$, set $G_x = H$ and identify α with G/H so that x corresponds to $[H] \in G/H$. Let $N = T_\alpha M/T\alpha \to \alpha$ denote the normal bundle of α and $Nf : N \to N$ denote the map induced by $Tf|T_\alpha M$. The relative fixed set α is generic if and only if $1 \notin \mathrm{spec}(N_x f)$ (lemma 8.3.32).

Let $\rho : H \to \mathrm{GL}(N_x)$ denote the representation of H induced on the normal fibre N_x. Since $G_{f(x)} = H$, $N_x f : N_x \to N_{f(x)}$ defines an isomorphism of the H-representations (N_x, H) and $(N_{f(x)}, H)$. Pick $n \in N(H)$ such that $nf(x) = x$. Unless G is Abelian, the map $f_n = nf : N \to N$ will generally not be equivariant and, in particular, $N_x f_n : N_x \to N_x$ will not be H-equivariant. However, we can define a new representation $\bar{\rho}$ of H on N_x so that $N_x f_n : N_x \to N_x$ is H-equivariant with respect to the given representation ρ on the domain and the new representation $\bar{\rho}$ on the range. For this we define $\bar{\rho} : H \to \mathrm{GL}(N_x)$ by

$$\bar{\rho}(h) = nhn^{-1}, \; h \in H.$$

Since $Nf_n(\rho(h)v) = nhNf(v) = nhn^{-1}nNf(v) = \bar{\rho}(h)Nf_n(v), \; h \in H, v \in N_x$, $N_x f_n : N_x \to N_x$ is H-equivariant with respect to $\rho, \bar{\rho}$.

LEMMA 8.3.42. (1) *The representations $\rho, \bar{\rho}$ of H on N_x are isomorphic.*
(2) *The actions of $\rho, \bar{\rho}$ on N_x have the same H-orbits. In particular, a function on N_x is H-invariant with respect to ρ if and only if it is H-invariant with respect to $\bar{\rho}$.*

PROOF. (1) The map $Nf_n : N_x \to N_x$ intertwines ρ and $\bar{\rho}$. For (2) it suffices to note that since $n \in N(H)$, $nhn^{-1} \in H$, all $h \in H$ and so $\bar{\rho}(H)(v) = \rho(H)(v)$, for all $v \in N_x$. □

REMARKS 8.3.43. (1) Although the representations $\rho, \bar{\rho}$ on N_x are isomorphic and have the same H-orbits, the actions will generally differ unless $n \in C(H)$. In particular, the identity map $I : (N_x, \rho) \to (N_x, \bar{\rho})$ will generally not be H-equivariant. (See also section 10.7.)
(2) Let H be a closed subgroup of G, (V, H) be an H-representation and consider the G-vector bundle $p : G \times_H V \to G/H$. Suppose that $G_x = G_y = H$, $x, y \in G/H$. It is not generally true that the H-representations $(p^{-1}(x), H)$ and $(p^{-1}(y), H)$ are isomorphic. Examples are (implicit) in section 10.7.

PROPOSITION 8.3.44. *Suppose α is a relative fixed set of $f \in \mathrm{Diff}_G(M)$. Given any C^1-open neighbourhood $\mathcal{U}$ of f, and a G-invariant open neighbourhood U of α in M, we may choose $\bar{f} \in \mathcal{U}$ such that*

(1) *$f = \bar{f}$ on $M \setminus U$.*
(2) *α is a generic relative fixed set for $\bar{f}$.*

A similar result holds for relative periodic orbits.

PROOF. As there is nothing to prove if α is generic for f, we suppose α is not generic. Let $x \in \alpha$, set $G_x = H$ and choose $n \in N(H)$ so that $nf(x) = x$. Set $f_n = nf$. Let $S_x \subset T_x$ be differentiable slices at x such that $\overline{G(f(S_x))} \subset G(T_x)$. It is no of generality, by lemma 8.3.26 and theorem 8.3.31, to assume that $f(S_x) \subset T_{f(x)}$. Under this assumption, $f_n : S_x \to T_x$. Let $\lambda : S_x \to \mathbb{R}$ be a smooth H-invariant function on S_x which is equal to one near x and is zero near the boundary of S_x. By lemma 8.3.42(2) λ is H-invariant with respect to the action $\bar{\rho}$ on S_x defined by $\bar{\rho}(h) = nhn^{-1}$, $h \in H$. For $t \in [0, 1]$, define the map $f_n^t : S_x \to T_x$ by

$$f_n^t(y) = (t\lambda(y) + 1)f_n(y),\ y \in S_x.$$

We have $\mathrm{spec}(T_x f_n^t) = (t + 1)\mathrm{spec}(T_x f_n)$. In particular, we may choose $\varepsilon > 0$, so that $1 \notin \mathrm{spec}(T_x f_n^t)$ for $t \in (0, \varepsilon]$. Taking $\varepsilon > 0$ smaller if necessary, we may also require that f_n^t is a smooth H-equivariant embedding for all $t \in [0, \varepsilon]$ (we take the action of $\bar{\rho}$ on the range). Hence for $t \in [0, \varepsilon]$, $f^t = n^{-1} f_n^t : S_x \to T_{f(x)}$ is H-equivariant. Since $f^t = f$ near the boundary of S^x, f^t extends smoothly and G-equivalently to M. This proves the result since $1 \notin \mathrm{spec}(Nf^t)$, for all $t \in (0, \varepsilon]$. ☐

8.3.6. Genericity theorems for equivariant diffeomorphisms. For $p \in \mathbb{N}^+$, let $\mathcal{G}_1(M; p) \subset \mathrm{Diff}_G(M)$ denote the set of all diffeomorphisms f such that if α is a relative periodic orbit of f of relative prime period at most p, then α is generic. We set

$$\mathcal{G}_1(M) = \cap_{p \geq 1} \mathcal{G}_1(M; p).$$

If $f \in \mathcal{G}_1(M)$ then all relative periodic orbits of f are generic.

Let $f \in \mathcal{G}_1(M; p)$ and suppose that α is a relative periodic orbit of f of relative prime period $q \leq p$. We say α is $\star$-generic if the dimension of $\mathrm{O}_{f^q}(x)$, $x \in \alpha$, is maximal (that is, the orbits of f^q determine Cartan subgroups – see proposition 8.3.23). Let $\mathcal{G}_1^\star(M; p)$ denote the subset of $\mathcal{G}_1(M; p)$ consisting of diffeomorphisms such that all relative periodic orbits of relative prime period $q \leq p$ are $\star$-generic. We set

$$\mathcal{G}_1^\star(M) = \cap_{p \geq 1} \mathcal{G}_1^\star(M; p).$$

THEOREM 8.3.45 ([**52**]). *Suppose that M is a compact G-manifold.*

(1) *For all $p \in \mathbb{N}^+$, $\mathcal{G}_1(M; p)$ is an open and dense subset of $\mathrm{Diff}_G(M)$.*
(2) *$\mathcal{G}_1(M)$ is a residual subset of $\mathrm{Diff}_G(M)$.*
(3) *For all $p \in \mathbb{N}^+$, $\mathcal{G}_1^\star(M; p)$ is a dense subset of $\mathcal{G}_1(M; p)$.*

(4) $\mathcal{G}_1^\star(M)$ *is a dense subset of* $\mathrm{Diff}_G(M)$.

Similar results hold if M is not compact provided we take the Whitney C^∞-topology on $\mathrm{Diff}_G(M)$.

PROOF. (Sketch) (1) We start by proving that $\mathcal{G}_1(M;1)$ is open in the C^1-topology on $\mathrm{Diff}_G(M)$. Let $f \in \mathcal{G}_1(M;1)$. It suffices to construct a C^1-open neighbourhood of f contained in $\mathcal{G}_1(M;1)$. Since M is compact, f has finitely many relative fixed sets, say $\alpha_1, \ldots, \alpha_k$. By theorem 8.2.5, we can choose a C^1-open neighbourhood $\mathcal{U}$ of f and G-invariant open neighbourhoods U_i of α_i, $1 \le i \le k$, such that if $\bar{f} \in \mathcal{U}$, then each U_i contains a unique relative fixed set $\bar{\alpha}_i$ of $\bar{f}$ and $\bar{\alpha}_i$ is generic. Let $X = M \setminus \cup_i U_i$. Then X is a compact G-invariant subset of M. For $\bar{f} \in \mathcal{U}$, define $R_{\bar{f}} = \inf_{x\in X} d(\bar{f}(x), Gx)$ (d is any G-invariant metric on M compatible with the topology on M). The compactness of X implies that $R_f > 0$. We have $d(f(x), Gx) \le d(\bar{f}(x), f(x)) + d(\bar{f}(x), Gx)$ and so $R_{\bar{f}} \ge R_f - \inf_{x\in X} d(f(x), g(x)$. We may choose a C^0-open neighbourhood $\mathcal{U}^\star$ of f within $\mathcal{U}$ such that if $\bar{f} \in \mathcal{U}^\star$, then $R_{\bar{f}} \ge R_f/2 > 0$. Hence $\mathcal{U}^\star \subset \mathcal{G}_1(M;1)$ proving that $\mathcal{G}_1(M;1)$ is open in the C^1-topology. A similar argument shows that $\mathcal{G}_1(M;p)$ is open in the C^1-topology for all $p \ge 1$.

Next we prove $\mathcal{G}_1(M;p)$ is C^∞ dense in $\mathrm{Diff}_G(M)$. We start by assuming that G acts freely on M. Let $\tilde{f} \in \mathrm{Diff}(M/G)$ denote the map induced by f on the orbit space M/G. Since the action of G on M is free, $\widetilde{\mathrm{Diff}}(M/G) = \{\tilde{f} \mid f \in \mathrm{Diff}_G(M)\}$ is an open subset of $\mathrm{Diff}(M/G)$ and the map $f \mapsto \tilde{f}$ is continuous. Furthermore, if $f \in \mathrm{Diff}_G(M)$, we can find an open neighbourhood V of $\tilde{f} \in \mathrm{Diff}(M/G)$ and continuous local section $\xi : V \to \mathrm{Diff}_G(M)$ such that $f \in \xi(V)$ and

$$\xi(\tilde{g}) = g, \quad \text{for all } g \in \xi(V).$$

For $p \in \mathbb{N}^+$, let $\mathcal{G}_0(M/G;p) \subset \mathrm{Diff}(M/G)$ consist of all diffeomorphisms k such that the graph map $\mathrm{graph}(k^q) : M/G \to M/G \times M/G$, $x \mapsto (x, k^q(x))$, is transverse to the diagonal $\Delta(M/G) \subset M/G \times M/G$, $1 \le q \le p$. Using standard transversality arguments, $\mathcal{G}_0(M/G;p)$ is a C^1-open and C^∞-dense subset of $\mathrm{Diff}(M/G)$. Hence $\mathcal{G}_0(M;p) = \{f \in \mathrm{Diff}_G(M) \mid \tilde{f} \in \mathcal{G}_0(M/G;p)\}$ is a C^1-open and C^∞-dense subset of $\mathrm{Diff}_G(M)$ (density uses the existence of continuous local sections of $\mathrm{Diff}_G(M) \to \mathrm{Diff}(M/G)$). Now M is compact and so if $f \in \mathcal{G}_0(M;p)$ then f has finitely many relative periodic orbits of relative period at most p. Applying proposition 8.3.44 to each of these (stable) relative fixed sets, we deduce that $\mathcal{G}_1(M;p)$ is dense in $\mathcal{G}_0(M;p)$ and hence in $\mathrm{Diff}_G(M)$. If the action of G on M is monotypic then we may apply the same argument to the free $N(H)/H$-manifold M^H, where H is an isotropy group for the action of G on M. Finally, suppose the action of G is not monotypic. Let $f \in \mathrm{Diff}_G(M)$. It suffices to show that $\mathcal{U} \cap \mathcal{G}_1(M;p) \neq \emptyset$, where $\mathcal{U}$ is an open neighbourhood of f. Choose a filtration $M_1 \subset M_2 \subset \ldots \subset M_N = M$ of M defined by the isotropy strata. Each set M_i will be a compact G-invariant subset of M, $N_i = M_{i+1} \setminus M_i$ will be a union of orbit strata and $\partial N_i = M_i$, $1 \le i < N$ (we refer to section 3.7.1).

Note that M_1 consists of maximal isotropy strata and is a compact G-invariant submanifold of M (the dimensions of components may vary). Our argument proceeds by an upward induction on the filtration. We start with M_1. Since M_1 is a disjoint union of compact monotypic G-manifolds, there is an open and dense subset $\mathcal{G}_1^1(M;p)$ of $\mathrm{Diff}_G(M)$ such that if $k \in \mathcal{G}_1^1(M;p)$, then $k|M_1$ has finitely many relative periodic orbits of relative prime period at most p. It follows from proposition 8.3.44 that we may require that these orbits are generic as relative periodic orbits of $k : M \to M$. Choose $k_1 \in \mathcal{G}_1^1(M;p) \cap \mathcal{U}$. By theorem 8.2.5 we can choose a C^1-open neighbourhood $\mathcal{U}_1 \subset \mathcal{U}$ of k_1 and a G-invariant open neighbourhood U_1 of $M_1 \subset M$ such that if $\bar{k} \in \mathcal{U}_1$ then every relative periodic orbit of $\bar{k}$ of relative prime period at most p that meets U_1 is generic and a subset of M_1. This completes the first inductive step. At the nth step, we will have constructed $k_n \in \mathcal{U}$, a C^1-open neighbourhood $\mathcal{U}_n \subset \mathcal{U}$ of k_n, and a G-invariant open neighbourhood U_n of $M_n \subset M$ such that if $\bar{k} \in \mathcal{U}_n$, then every relative periodic orbit of $\bar{k}$ of relative prime period at most p that meets U_n is generic and a subset of M_n. Suppose $n < N$ (otherwise we are done). Then $M_{n+1} \setminus U_n$ is a finite union of compact disjoint monotypic G-manifolds. The previous argument therefore applies to complete the inductive step.

Statement (2) follows since $\mathrm{Diff}_G(M)$ is a Baire space. Finally, we prove (3,4). We start by making a choice for each $r \geq 0$ of a metric d_r on $\mathrm{Diff}_G(M)$ that defines the C^r topology. We require that $d_r \leq d_{r+1}$, $r \geq 0$. Let $f \in \mathcal{G}_1(M)$. The set of relative periodic orbits of f is countable. For $n \geq 1$, let $P(f,n) \subset M$ denote the set of relative periodic orbits of f of relatively prime period exactly n. It follows from theorem 8.2.5 that, for each $n \geq 1$, we may construct an open G-invariant neighbourhood U_n of $P(f,n)$ such that $U_n \cap \cup_{j=1}^{n-1} P(f,j) = \emptyset$. Let $\varepsilon > 0$ and $r \in \mathbb{N}^+$. Using propositions 8.3.23, 8.3.44, we may construct $(f_n) \subset \mathcal{G}_1(M)$ such that

(1) $f_0 = f$.
(2) For $n \geq 0$, the maps induced by f, f_n on M/G are equal.
(3) $f_n = f_{n-1}$ outside U_n, $n \geq 1$.
(4) $f_n \in \mathcal{G}_1^\star(M;n)$.
(5) $d_{r+n}(f_n, f_{n-1}) < \varepsilon/(2n^2)$, $n \geq 1$.

It follows from (5) that (f_n) converges to $\bar{f} \in \mathrm{Diff}_G(M)$. Since $f_n - f = (f_n - f_{n-1}) + \ldots + (f_1 - f)$, and $d_r \leq d_{n+r}$, it follows that $d_r(f_n, f) < \varepsilon$, $n \geq 1$, and so $d_r(\bar{f}, f) \leq \varepsilon$. Hence $\mathcal{G}_1^\star(M)$ is C^r-dense in $\mathcal{G}_1(M)$. Since this result holds for all $r \geq 1$, it follows that $\mathcal{G}_1^\star(M)$ is C^∞-dense in $\mathcal{G}_1(M)$. □

REMARK 8.3.46. The proof that $\mathcal{G}_1(M;p)$ is open and dense uses only elementary results (such as lemma 8.3.44) in combination with an upward induction on the isotropy filtration (this methods appears in [**52**]. The inductive method depends on proving openness and density results on each compact set M_i in the filtration $M_1 \subset \ldots \subset M_N = M$. A significantly less elementary proof can be based on equivariant transversality (including jet transversality). This proof amounts to an equivariant version of the approach used in Abraham and Robbin [**1**] in their proof of the Kupka-Smale theorem.

We now turn to the question of G-transversality of stable and unstable manifolds. Let $f \in \mathcal{G}_1(M)$. Suppose that Σ is a relatively periodic orbit of f – we assume Σ is f-invariant and so Σ is a finite union of G-orbits. Let $T\Sigma \oplus \mathbb{E}^u(\Sigma) \oplus \mathbb{E}^s(\Sigma)$ be the corresponding Tf-invariant splitting of $T_\Sigma M$. We may represent the stable manifold $W^u(\Sigma)$ as the image of a G-equivariant injective immersion $\xi^u_\Sigma : \mathbb{E}^u_\Sigma \to M$ mapping the zero section of $\mathbb{E}^u_\Sigma$ onto Σ. Similarly for $\xi^s_\Sigma : \mathbb{E}^s_\Sigma \to M$. Fix a G-invariant Riemannian metric on M. The bundles $\mathbb{E}^u(\Sigma), \mathbb{E}^s_\Sigma$ inherit the structure of Riemannian G-vector bundles from the induced Riemannian structure on $T\Sigma M$. For $T > 0$, we let $\mathbb{E}^u_\Sigma(T) = \{e \in \mathbb{E}^u_\Sigma \mid \|e\| \leq T\}$ denote the T-disk bundle of $\mathbb{E}^u_\Sigma$. We similarly define $\mathbb{E}^s_\Sigma(T)$. Set $W^u(\Sigma, T) = \xi^u_\Sigma(\mathbb{E}^u_\Sigma(T))$ and similarly define $W^s(\Sigma, T)$. Thus $W^u(\Sigma, T)$, $W^s(\Sigma, T)$ are compact G-invariant (embedded) submanifolds of M (with boundary).

Suppose that Σ_1, Σ_2 are relatively periodic orbits of f. We say that $W^u(\Sigma_1)$ is G-transverse to $W^s(\Sigma_2)$ if $\xi^u_{\Sigma_1} : \mathbb{E}^u(\Sigma_1) \to M$ is G-transverse to $W^s(\Sigma_2)$ (equivalently, if $\xi^s_{\Sigma_2} : \mathbb{E}^s(\Sigma_2) \to M$ is G-transverse to $W^u(\Sigma_1)$). Alternatively, and equivalently, we may require that $W^u(\Sigma_1, T)$ is G-transverse to $W^s(\Sigma_2, T)$ for all $T > 0$. If $W^u(\Sigma_1)$ is G-transverse to $W^s(\Sigma_2)$, we write $W^u(\Sigma_1) \pitchfork_G W^s(\Sigma_2)$.

Let $\mathcal{G}_2(M)$ be the subset of $\mathcal{G}_1(M)$ for which the stable and unstable manifolds of all relative periodic orbits meet G-transversally. Let $\mathcal{G}^\star_2(M) = \mathcal{G}_2(M) \cap \mathcal{G}^\star_1(M)$.

THEOREM 8.3.47 (Equivariant Kupka-Smale theorem [**52**]). *$\mathcal{G}_2(M)$ is a residual subset of $\mathrm{Diff}_G(M)$ and $\mathcal{G}^\star_2(M)$ is a C^∞-dense subset of $\mathrm{Diff}_G(M)$.*

PROOF. (Sketch) Let $p \in \mathbb{N}^+$ and $\mathcal{G}_2(M; p)$ denote the subset of $\mathcal{G}_1(M; p)$ consisting of diffeomorphisms for which $W^u(\Sigma_1, p)$ is G-transverse to $W^s(\Sigma_2, p)$ whenever Σ_1, Σ_2 are relative periodic orbits of relative prime period at most p. It follows easily from the openness of G-transversality that $\mathcal{G}_2(M; p)$ is an open subset of $\mathcal{G}_1(M; p)$. Some careful perturbation theory then shows that $\mathcal{G}_2(M; p)$ is dense in $\mathcal{G}_1(M; p)$ (see [**52**] for the case of flows – the method is based on the technique of Peixoto [**142**]). Finally $\mathcal{G}_2(M) = \cap_{p>0} \mathcal{G}_2(M; p)$. $\square$

8.4. Equivariant vector fields

Much of the theory of dynamics of equivariant flows on and near relative equilibria and periodic orbits is similar to, or follows from, the corresponding theory for diffeomorphisms. As as result, we often omit or severely abbreviate those proofs that are simple generalizations or reformulations of our results for diffeomorphisms and instead focus on new issues as and when they arise.

8.4.1. Relative equilibria.

We start with a result that gives dynamics on a relative equilibrium.

PROPOSITION 8.4.1. *Let H be a closed subgroup of G.*

(1) *Every G-equivariant vector field on G/H is smooth.*
(2) $C^\infty_G(T(G/H)) \approx \mathfrak{n}(\mathfrak{h})/\mathfrak{h} \approx \mathfrak{c}(\mathfrak{h})/\mathfrak{h} \cap \mathfrak{c}(\mathfrak{h})$ *and if X corresponds to $\eta \in \mathfrak{c}(\mathfrak{h})$, then for all $x \in G/H$, $t \in \mathbb{R}$ we have*

$$\begin{aligned} \Phi^X_t(x) &= x\exp(t\eta), \quad (x \in G/H, \text{ right } N(H)\text{-action on } G/H), \\ &= \exp(t\mathrm{Ad}(g)(\eta))x, \quad \text{where } x = g[H]. \end{aligned}$$

(3) *If $X \in C^\infty_G(T(G/H))$, there is a Φ^X-invariant foliation $\mathcal{F}^X = \{\mathcal{F}^X_x \mid x \in G/H\}$ of G/H by s-dimensional tori satisfying:*
   (a) $\mathcal{F}^X_x = \mathrm{closure}(\Phi^X_x(\mathbb{R}))$, $x \in G/H$.
   (b) $\mathcal{F}^X_{gx} = g\mathcal{F}^X_x$, *for all $g \in G$, $x \in X$.*
   (c) $1 \leq s \leq \mathrm{rk}(N(H)/H)$.
   (d) $\Phi^X|\mathcal{F}^X_x$ *is the identity, a periodic orbit or an irrational torus flow according to whether $s = 0, 1$ or $s > 1$ respectively.*
   (e) *The subset of $\mathfrak{c}(\mathfrak{h})$ defining flows for which $s = \mathrm{rk}(N(H)/H)$ is a full measure subset of $\mathfrak{c}(\mathfrak{h})$.*

PROOF. (1) If X is a G-equivariant vector field on G/H then $X(g[H]) = gX([H])$, for all $g \in G$. Since the left action of G on G/H is smooth, $X \in C^\infty_G(T(G/H))$. (2) A vector $X \in T_{[H]}G/H \approx \mathfrak{g}/\mathfrak{h}$ extends to a G-equivariant vector field on G/H if and only if $X \in (\mathfrak{g}/\mathfrak{h})^H \approx \mathfrak{n}(\mathfrak{h})/\mathfrak{h} \approx \mathfrak{c}(\mathfrak{h})/\mathfrak{c}(\mathfrak{h}) \cap \mathfrak{h}$, where the last isomorphism is the Lie algebra version of corollary 3.10.3. The expressions for Φ^X_t come from section 3.3 and remarks 3.10.1(2). (3) These statements are proved along similar lines to the corresponding results for relative fixed points. For example, $\Phi_x(\mathbb{R})$ inherits the structure of a connected Abelian group from $\mathbb{R}$ and so $\mathrm{closure}(\Phi^X_x(\mathbb{R}))$ is a compact connected Abelian group and therefore a torus (theorem 1.5.16). We leave the remaining details to the reader. ☐

8.4.2. Stability of relative equilibria. Suppose that α is a relative equilibrium of $X \in C^\infty_G(TM)$. We recall that α is generic if α is normally hyperbolic for the flow of X. We start by showing that normal hyperbolicity can be characterized in terms of spectral conditions on a linearization of X along α.

Let V be a real vector space, $A \in L(V, V)$ and $\mathcal{E}(A) \subset \mathbb{C}$ denote the set of eigenvalues of A. We define the *v-reduced spectrum* of A by

$$\mathrm{vspec}(A) = \{\mathrm{Re}(\lambda) \mid \lambda \in \mathcal{E}(A)\}.$$

If $\mu \in \mathrm{vspec}(A)$ then the multiplicity of μ is defined to be the sum of the multiplicities of all $\lambda \in \mathcal{E}(A)$ such that $\mathrm{Re}(\lambda) = \mu$. We remark that for all $t \in \mathbb{R}$,

$$\mathrm{spec}(\exp(At)) = \{\exp(t\mu) \mid \mu \in \mathrm{vspec}(A))\} \stackrel{\mathrm{def}}{=} \exp(t\,\mathrm{vspec}(A)). \tag{8.5}$$

LEMMA 8.4.2. *Let α be a relative equilibrium of $X \in C^\infty_G(TM)$. There exists $\tilde{X} \in C^\infty_G(TM)$ supported on a neighbourhood of α such that*
(1) *$\tilde{X}$ is everywhere tangent to G-orbits.*
(2) *$X - \tilde{X} \equiv 0$ on α.*

PROOF. Fix $x \in \alpha$ and set $G_x = H$. By proposition 8.4.1 we may choose $\eta \in \mathfrak{c}(\mathfrak{h})$ such that $\Phi_t^X(y) = y\exp(t\eta)$, $y \in \alpha$, $t \in \mathbb{R}$. Choose a slice S_x at x and a smooth H-invariant function $\lambda : S_x \to \mathbb{R}(\geq 0)$ which is equal to one near x and vanishes near the boundary of S_x. We define an H-equivariant vector field $\tilde{X}$ on S_x by $\tilde{X}(y) = \frac{d}{dt}(y\exp(t\lambda(y)\eta)|_{t=0}$. By construction, $X - \tilde{X} \equiv 0$ on α. Extend $\tilde{X}$ G-equivariantly to GS_x and then by zero to M. $\square$

DEFINITION 8.4.3. Let α be a relative equilibrium of $X \in C_G^\infty(TM)$. Let $\tilde{X}$ be an equivariant vector field satisfying the conditions of lemma 8.4.2. The *reduced v-spectrum of X along α* is defined by

$$\mathbf{vspec}(X, \alpha) = \mathrm{vspec}(T_x(X - \tilde{X})),$$

where $x \in \alpha$.

LEMMA 8.4.4. *(Notation as above)*

(1) $\mathbf{vspec}(X, \alpha)$ *is well-defined independent of the choice of* $\tilde{X}$ *and* $x \in \alpha$.
(2) $\mathbf{spec}(\Phi_t^X, \alpha) = \exp(t\,\mathbf{vspec}(X, \alpha))$.
(3) $0 \in \mathbf{vspec}(X, \alpha)$ *has multiplicity at least* $\dim(\alpha)$ *and* α *is generic if and only if the multiplicity of* 0 *equals* $\dim(\alpha)$.

PROOF. The result may either be proved directly or deduced from the corresponding results for diffeomorphisms using (8.5). $\square$

EXERCISE 8.4.5. (1) Suppose that $X \in C_G^\infty(TM)$ is everywhere tangent to G-orbits. Prove that if α is a G-orbit then $\mathbf{vspec}(X, \alpha) = \{0\}$.
(2) Following what we did for diffeomorphisms, it is useful to have a description of $\mathbf{vspec}(X, \alpha)$ that does not depend on the choice of a slice. Let α be a relative equilibrium of $X \in C_G^\infty(TM)$. Fix $x \in \alpha$, set $G_x = H$ and suppose that $X(x) = \frac{d}{dt}\exp(tc)(x)|_{t=0}$, where $c \in \mathfrak{c}(\mathfrak{h})$ (or $\mathfrak{n}(\mathfrak{h})$). Define $X^c \in C^\infty(TM)$ by $X^c(y) = \frac{d}{dt}\exp(tc)(y)|_{t=0}$, $y \in M$. Certainly X^c is tangent to α and vanishes at x. Generally, X^c will not be equivariant unless, for example, G is Abelian. Show that $\mathbf{vspec}(X, \alpha) = \mathrm{vspec}(T_x(X - X^c))$.
(3) Let (V, G) be a complex representation. Let $\omega \in \mathbb{R}$, $\omega \neq 0$, and define the linear G-equivariant vector X on V by $X(x) = \imath\omega x$. Show that if α is a relative equilibrium of X then $\mathbf{vspec}(X, \alpha) = \{0\}$ (note this result is trivial if $S^1 \subset G$).

Let α be a relative equilibrium of $X \in C_G^\infty(TM)$. Lemma 8.4.2 gives a decomposition $X = Z + \tilde{X}$, where $\tilde{X}$ is tangent to G-orbits and $Z|\alpha \equiv 0$. If one fixes a family $\{S_x \mid x \in \alpha\}$ of slices for α, Krupa [**105**] showed that it was possible to decompose $X|GS_x$ into a component X_T tangent to G-orbits and a 'normal' component X_N tangent to slices and necessarily vanishing along α. In essence Krupa's result allows one to factor out the drift along group orbits and analyse bifurcations from a relative equilibrium by determining the bifurcations of the normal vector field (see [**105, 82**] for some applications).

LEMMA 8.4.6 ([**105**, Theorems 2.1, 2.2]). *Let α be a relative equilibrium of $X \in C^\infty_G(TM)$. Fix a smooth family $\mathcal{S} = \{S_x \mid x \in \alpha\}$ of slices and set $U = \cup_{x\in\alpha} S_x$. There exist $X_T, X_N \in C^\infty_G(TU)$ such that*

(1) *$X = X_T + X_N$ on U.*
(2) *X_T is everywhere tangent to G-orbits and $X_T|\alpha = X|\alpha$.*
(3) *X_N is tangent to $\mathcal{S}$. That is, for all $y \in S_x$, $x \in \alpha$, $X_N(y) \in T_y S_x$. In particular, $X_N|\alpha \equiv 0$.*

Furthermore, there exists an open neighbourhood $W_X(U)$ of $U \times \{0\}$ in $U \times \mathbb{R}$ and a smooth map $\gamma : W_X(U) \to G$ such that

$$\Phi^X_t(y) = \gamma(y,t)\Phi^{X_N}_t(y,t),\ (y,t) \in W_X(U),$$

and $\gamma(y,t) = \exp(t\eta) \in C_G(G_y)_0$, $\eta \in \mathfrak{c}(\mathfrak{g}_y)$, $(y,t) \in W_X(U)$.

PROOF. Let $x \in \alpha$, set $G_x = H$ and identify α with G/H. Suppose that $\sigma : V \subset G/H \to G$ is an admissible section. For each $y \in S_x$, let $E_y = T_y(\sigma(V)y)$, $F_y = T_y S_y$. Since $E_{hy} = hE_y$ (using admissibility – see proof of lemma 8.3.26), and $F_{hy} = hF_y$, the families $\{E_y \mid y \in S_x\}$, $\{F_y \mid y \in S_x\}$ define smooth H-vector bundles over S_x. Extend by G-equivariance to smooth G-vector bundles $\pi_T : E \to U$, $\pi_N : F \to U$. We have $T_U M = E \oplus F$. Given $X \in C^\infty_G(TM)$, we have the unique decomposition $X|U = X_T + X_N$, where $X_T \in C^\infty_G(E)$, $X_N \in C^\infty_G(F)$. The final statement follows using proposition 8.4.1(2) and properties of admissible sections. $\square$

REMARKS 8.4.7. (1) The construction of the bundle E can also be done Lie algebraically using the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ – see the proof of Lemma 2.3 [**105**].
(2) Lemma 8.3.26 may be regarded as giving a 'tangent and normal' form for diffeomorphisms. Indeed, it is not hard to derive lemma 8.4.6 from lemma 8.3.26.
(3) There is an important variation of lemma 8.4.6 that gives a natural decomposition of the vector field in terms of a skew product. The 'tangent' vector field is then naturally defined on the group rather than a group orbit. We describe the basic idea and refer the reader to [**48**] for more details and applications (including to proper G-actions).

Identify the neighbourhood U of lemma 8.4.6 with the twisted product $G \times_H V$, where $V = T_x G x^\perp$, $G_x = H$. The orbit map $p : G \times V \to G \times_H V$ gives $G \times V$ the structure of a principal H-bundle over $G \times_H V$ (example 3.1.19). If $Z = (\alpha, \beta) \in C^\infty_{G\times H}(T(G \times V))$, then for all $(g, v) \in G \times V$ we may write

$$\begin{aligned} \alpha(g,v) &= g\mathbf{a}(v), \\ \beta(g,v) &= \beta(v), \end{aligned}$$

where $\mathbf{a} \in C^\infty_H(V, \mathfrak{g})$ and $\beta \in C^\infty_H(TV)$. The H-equivariance of $\mathbf{a}$ is given by

$$\mathbf{a}(hv) = h\mathbf{a}(v)h^{-1} = \mathrm{Ad}(h)\mathbf{a}(v),\ \ (h \in H, v \in V).$$

Consequently, every $G \times H$-equivariant vector field on $G \times V$ is a skew product over an H-equivariant vector field on V. Observe that if Φ_t^Z is the flow of Z, then we may write $\Phi_t^Z(g,v)) = (\phi_t^G(v,g), \phi_t^V(v))$ where ϕ_t^V is the flow of β and $\phi_t^G : G \times V \to G$. Since Z is $G \times H$-equivariant, Z induces a unique $\hat{Z} \in C_G^\infty(T(G \times_H V))$ and $\Phi_t^{\hat{Z}} = \phi_t^G \phi_t^V$. That is, $\Phi_t^{\hat{Z}}([g,v]) = \phi_t^G(g,v)\phi_t^V(v) = g\phi_t^G(e,v)\phi_t^V(v)$, for all $(g,v) \in G \times V$ (we identify $\phi_t^V(v)$ with $[e, \phi_t^V(v)] \in G \times_H V$).

Since the orbit map is submersive, every smooth G-equivariant vector field $X \in C_G^\infty(T(G \times_H V))$ lifts to a smooth $G \times H$-equivariant vector field $\tilde{X}$ on $G \times V$. However, the lift will be non-unique (unless H is finite). We can enforce uniqueness by fixing a $G \times H$-invariant Riemannian metric on $G \times V$. Given $(e,v) \in G \times V$, let $L = T_{(e,v)}H(e,v) \subset \mathfrak{g} \times V$ ($H(e,v) = (h^{-1}, hv) \mid h \in V\}$). Then $T_{(e,v)}p(L) = \{0\} \in T_{[e,v]}(G \times_H V)$ and $T_{(e,v)}p$ maps $L^\perp \subset \mathfrak{g} \times V$ isomorphically onto $T_{[e,v]}(G \times_H V)$. Hence given $X \in C_G^\infty(T(G \times_H V))$, we uniquely determine $\tilde{X} \in C_{G \times H}^\infty(T(G \times V))$ by requiring that $\tilde{X}(e,v) \in L^\perp$ is mapped to $X([e,v])$ by $T_{(e,v)}p$. Set $\tilde{X}(e,v) = (\mathbf{a}(v), \beta(v))$, $v \in V$. Let $\mathfrak{h}$ act on V by $kv = \frac{d}{dt}\exp(tk)(v)|_{t=0}$, $k \in \mathfrak{h}$. Our condition uniquely characterizing $\tilde{X}$ amounts to the condition $(\beta(v), kv)_V = (\mathbf{a}(v), k)_{\mathfrak{g}}$, for all $k \in \mathfrak{h}$. Finally, the components of $\tilde{X}$ determine a tangential and normal decomposition for the vector field X ($X_T = p_\star(\alpha, 0)$, $X_N = p_\star(0, \beta)$).

Exercise 8.4.8. (1) Let M be a Riemannian G-manifold and $X \in C_G^\infty(TM)$. Show that it is in general not possible to write $X = X_T + X_N$ where $X_T, X_N \in C_G^\infty(TM)$, X_T is tangent to G-orbits and X_N is orthogonal to G-orbits. (Hint: Look at the representation of SO(2) on $\mathbb{C}^2$ defined by $e^{\imath\theta}(z_1, z_2) = (e^{\imath\theta} z_1, e^{2\imath\theta} z_2)$.)//
(2) Investigate the skew product formulation given in remarks 8.4.7(3) in case $H \cong \mathbb{T}$ is the diagonal subgroup of $\mathbb{T}^2$ and (V, H) is the standard irreducible representation of $\mathbb{T}$ on $V = \mathbb{C}$.

As an immediate consequence of lemma 8.4.6, we have the following simple criterion for genericity of relative equilibria.

Lemma 8.4.9. *Let α be a relative equilibrium of $X \in C_G^\infty(TM)$. Fix a smooth slice family $\mathcal{S} = \{S_x \mid x \in \alpha\}$ and set $U = GS_x$. Let $X_T, X_N \in C_G^\infty(TU)$ be the associated tangent and normal forms of X given by lemma 8.4.6. Then α is generic if and only if $0 \in S_x$ is a hyperbolic zero of X_N for some (any) $x \in \alpha$.*

Proposition 8.4.10. *Suppose α is a relative equilibrium of $X \in C_G^\infty(TM)$. Given a C^∞-open neighbourhood $\mathcal{U}$ of X, and a G-invariant open neighbourhood U of α in M, we may choose $\bar{X} \in \mathcal{U}$ such that*

(1) *$X = \bar{X}$ on $M \setminus U$.*
(2) *α is a generic relative equilibrium for $\bar{X}$.*

Proof. The result may either be proved directly along the lines of proposition 8.4.10, using lemma 8.4.4 or, more simply, by using lemma 8.4.9. □

8.4.3. Classification of relative periodic orbits. Suppose that $\Sigma \subset M$ is a relative periodic orbit of $X \in C^\infty_G(TM)$ with relative prime period $T > 0$. We recall from lemma 8.1.7 that G acts monotypically on Σ and Σ/G is diffeomorphic to S^1. If Σ has isotropy type (H), then $p : \Sigma \to \Sigma/G$ is a locally trivial G-fibre bundle over S^1, fibre G/H. The vector field X induces a vector field $X^\star \in C^\infty(TS^1)$ such that S^1 is a periodic orbit of $X^\star$, prime period T.

We now consider the inverse problem. That is, given a non-zero vector field $X^\star$ on S^1, a group G, and a compact subgroup H of G, classify all relative periodic flows $\Phi^X_t : \Sigma \to \Sigma$ over $X^\star$, where Σ is a monotypic G-manifold with isotropy type (H). The first step is classify all monotypic G-spaces Σ with orbit space S^1.

We start by classifying free K-manifolds with orbit space S^1, for K a compact Lie group. This amounts to the well-understood problem of the classification of K-principal bundles over S^1. We review the basic definitions and results and refer the reader to [**26**] for the general theory.

Let $\mathrm{Prin}(K, S^1)$ denote the set of isomorphism classes of principal K-bundles over S^1. Then

$$\mathrm{Prin}(K, S^1) \approx \pi_0(K) \approx \pi_0(K/K_0).$$

($\pi_0(K/K_0) = |K/K_0|$ is the number of connected components of K/K_0.) For completeness, we give a direct and simple proof that $\mathrm{Prin}(K, S^1) \approx \pi_0(K/K_0)$.

LEMMA 8.4.11. *There is a natural bijection* $\chi : K/K_0 \to \mathrm{Prin}(K, S^1)$.

PROOF. Set $K/K_0 = P$ and let $\Pi : K \to P$ denote the quotient map. Identify S^1 with $[0, 1]/(0 = 1)$. Let $z \in P$ and pick $\zeta \in \Pi^{-1}(z)$. Let E^ζ be the free K-space defined by

$$E^\zeta = [0, 1] \times K/\sim, \tag{8.6}$$

where $(1, k) \sim (0, k\zeta)$, $k \in K$. If we let $\pi^\zeta : E^\zeta \to S^1$ denote the orbit map, then $p^\zeta : E^\zeta \to S^1$ is a K-principal bundle over S^1. If $\zeta' \in \Pi^{-1}(z)$ then there is a smooth path ζ_t connecting ζ to ζ' and standard elementary techniques show that E^ζ and $E^{\zeta'}$ are isomorphic as K-principal bundles over S^1. Hence we may define $\chi(z) \in \mathrm{Prin}(K, S^1)$ to be the isomorphism class of the bundle E^ζ. Conversely, suppose that $\pi : E \to S^1$ is a K-principal bundle. Then we may represent E as

$$E = [0, 1] \times K/\sim,$$

where the relation $\sim$ is a K-invariant equivalence relation identifying boundary points of $[0, 1] \times K$. That is, for each $k \in K$, there exists $\bar{k} \in K$ such that $(1, k) \sim (0, \bar{k})$ and $(1, gk) \sim (0, g\bar{k})$, all $g \in K$. If we let $\zeta = \bar{e}$, then E is isomorphic to $E^\zeta = \chi(\Pi(\zeta))$ as a K-principal bundle. Hence $\chi : K/K_0 \to \mathrm{Prin}(K, S^1)$ is a bijection. $\square$

Given $z \in P = K/K_0$, $\zeta \in \Pi^{-1}(z)$, let $\pi^\zeta : E^\zeta \to S^1$ be the principal K-bundle over S^1 defined by (8.6). We define a K-equivariant flow Θ^ζ_t on E^ζ by

$$\Theta^\zeta_t(\theta, k) = (\theta + t, k)/\sim, \ \ (t \in \mathbb{R}, k \in K).$$

We call Θ_t^ζ the *canonical flow on* E^ζ. (The canonical flow is a *suspension flow* over K/K_0 with roof function constant and equal to 1 – see section 9.4.)

LEMMA 8.4.12. *(Notation as above.) Let $z \in P$ and set $X = \langle z \rangle \in \mathcal{Z}(K)$. Suppose that Cartan subgroups C of type X have* $\mathrm{rk}(C, X) = s$*,* $\mathrm{ind}(C, X) = p$*.*

(1) *For all $x \in E^\zeta$, $\overline{\Theta^\zeta(x, \mathbb{R})} = \mathbf{T}_x$ is an r-dimensional torus, where $1 \le r \le s+1$. The restriction of Θ^ζ to $\mathbf{T}_x$ is conjugate to a linear torus flow (irrational if $r > 1$).*
(2) *If ζ is a representative generator of X, then every trajectory of Θ_t^ζ will be periodic of prime period p and $|X|$ divides p.*
(3) *For ζ lying in a full measure subset $K_z^\star$ of $\Pi^{-1}(z)$, and all $x \in E^\zeta$, $\overline{\Theta^\zeta(x, \mathbb{R})} = \mathbf{T}_x$ is an $(s+1)$-dimensional (maximal) torus.*

PROOF. The lemma follows from the results of section 8.3.2. $\square$

EXERCISE 8.4.13. (1) Let ζ, ζ' be representative generators of $X = \langle z \rangle \subset P$. Prove that the canonical flows are smoothly K-equivariantly topologically conjugate. Show also that there is a free $\mathrm{SO}(2)$-action on E^ζ such that (a) The $\mathrm{SO}(2)$-orbits are equal to the periodic orbits of Θ^ζ, (b) the actions of G and $\mathrm{SO}(2)$ commute, and (c) $G \times \mathrm{SO}(2)$ acts transitively on E^ζ.
(2) Extend (1) to show that that for general $\zeta \in \Pi^{-1}(z)$, there exists a free $\mathbb{T}^{s+1}$-action action on E^ζ with orbits equal to the closure of Θ^ζ-trajectories.

PROPOSITION 8.4.14. *Let H be a closed subgroup of G. Let* $\mathrm{FB}(G, H, S^1)$ *denote the set of isomorphism classes of G-fibre bundles over S^1 with fiber G/H. There exists a natural bijection*

$$\Delta : \mathrm{FB}(G, H, S^1) \to \mathrm{Prin}(N(H)/H, S^1).$$

Given $\Sigma \in \mathrm{FB}(G, H, S^1)$, let $z = \chi(\Delta(\Sigma)) \in (N(H)/H)/(N(H)/H)_0$, $X = \langle z \rangle$, and s be the rank of Cartan subgroups C of type X. Then the canonical flow Θ_t^ζ on E^ζ determines a unique flow Φ_t on Σ with a flow invariant foliation $\mathcal{F} = \{\mathcal{F}_x \mid x \in \Sigma\}$ of Σ satisfying

(1) *$\mathcal{F}_{gx} = g\mathcal{F}_x$, $g \in G$, $x \in \Sigma$.*
(2) *Each leaf $\mathcal{F}_x$ is diffeomorphic to an r-dimensional torus where $1 \le r \le s+1$ and the restriction of Φ_t to $\mathcal{F}_x$ is transitive and conjugate to a linear torus flow.*

PROOF. If $\Sigma \in \mathrm{FB}(G, H, S^1)$, we define $\Delta(\Sigma) = \Sigma^H$ and take the induced free $N(H)/H$-action on Σ^H. With this definition, $\Delta(\Sigma) \in \mathrm{Prin}(N(H)/H, S^1)$. Conversely, if $E \in \mathrm{Prin}(N(H)/H, S^1)$, we define $\Delta^{-1}(E) = \Sigma \in \mathrm{FB}(G, H, S^1)$ by $\Sigma = G \times_{N(H)} E$. The remaining statements of the proposition follow straightforwardly from lemma 8.4.12. $\square$

EXERCISE 8.4.15. Extend exercise 8.4.13 to the setup of proposition 8.4.14 – in particular, show that the leaves of the foliation $\mathcal{F}$ may be represented as group orbits of a smooth free toral action on Σ.

REMARK 8.4.16. Suppose that Σ is a relative periodic orbit of Φ^X and Σ has relative prime period $T > 0$. We may rescale time so that the induced vector field $X^\star$ on S^1 is the constant unit field. It then follows that there exists $\zeta \in \Pi^{-1}(\chi(\Delta(\Sigma))$ such that $\Phi_t^X : \Sigma \to \Sigma$ is conjugate, by a time preserving smooth G-equivariant diffeomorphism, to the canonical flow Θ_t^ζ on E^ζ. In particular, up to a time rescaling, lemma 8.4.12 and proposition 8.4.14 give a complete classification of relative periodic orbits.

8.4.4. Periodic orbits and symmetry. Suppose that γ is a periodic orbit of $X \in C_G^\infty(TM)$. Noting that isotropy groups are constant on trajectories of X, we let H denote the isotropy group of any point on γ. We define the symmetry group of γ by

$$G_\gamma = \{g \in G \mid g\gamma = \gamma\}.$$

LEMMA 8.4.17. (1) *$H \lhd G_\gamma$.*
(2) *G_γ/H is either isomorphic to $\mathbb{Z}_m$ or to S^1.*

PROOF. Let $g \in G_\gamma$. By G-equivariance and the 1-parameter group property of flows, there exists a unique smallest $t = t(g) \geq 0$ such that $gx = \Phi_t^X(x)$, for all $x \in \gamma$. For all $h \in H$, $ghg^{-1}x = \Phi_t^X(h\Phi_{-t}^X(x)) = x$ and so $H \lhd G_\gamma$. Since $gx = g'x$ if and only if $t(g) = t(g')$, we see easily that if $\inf_{g \in G_\gamma \setminus H} t(g) = T > 0$, then $G_\gamma/H \cong \mathbb{Z}_m$, where $m = \text{period}(\gamma)/T$. If $\inf_{g \in G_\gamma \setminus H} t(g) = 0$, then $G_\gamma/H \cong [0, \text{period}(\gamma)]/(0 = \text{period}(\gamma)) \cong S^1$. $\square$

REMARK 8.4.18. If $G_\gamma/H \cong S^1$, γ is called a *rotating wave*, and if $G_\gamma/H \cong \mathbb{Z}_m$, then γ is called a *discrete rotating wave*.

More generally, suppose that $\Phi_x^X(t)$ is a trajectory contained in a relative equilibrium or relative periodic orbit of $X \in C_G^\infty(TM)$. Let $\Lambda = \overline{\Phi_x^X(\mathbb{R})}$. Isotropy groups are constant on Λ and we set $G_x = H$, where x is any point of Λ. Let $G_\Lambda = \{g \in G \mid g\Lambda = \Lambda\}$. It is easy to show that $H \lhd G_\Lambda$. If G_Λ/H is connected then $G_\Lambda/H \cong \mathbb{T}^s$, $s \geq 0$. Otherwise, G_Λ/H will be isomorphic to $\mathbb{T}^s \times \mathbb{Z}_p$, where $s, p \geq 0$. All of this is a consequence of lemma 8.4.12.

8.4.5. Poincaré map for a relative periodic orbit. Just as for limit cycles of vector fields, we may construct a Poincaré first return map for relative periodic orbits of an equivariant vector field. In this way, we can reduce the analysis of dynamics in a neighbourhood of a relative periodic orbit to the study of an equivariant diffeomorphism in a neighbourhood of a relative fixed set. However, unlike the case of unconstrained vector fields, the imposition of symmetry on the vector field can lead to additional constraints on the Poincaré map that significantly affect the types of bifurcations that may occur. We return to this point in the chapter 10.

Let $X \in C_G^\infty(TM)$. Suppose that $\Sigma \subset M$ is a relative periodic orbit of X with relative prime period T. Let $\pi : \Sigma \to \Sigma/G = S^1 = [0, T]/(0 = T)$ denote the orbit map.

Let $q : N(\Sigma) \to \Sigma$ denote the normal bundle of Σ and fix an equivariant tubular neighbourhood $\rho : N(\Sigma) \to M$ of Σ in M (see proposition 3.4.8). Set $U = \rho(N(\Sigma))$ and note that U is a G-invariant open neighbourhood of Σ. Fix $\theta \in S^1$ and let $\alpha = \pi^{-1}(\theta) \subset \Sigma$. Set $N_\alpha = N(\Sigma)|\alpha$, $q_\alpha = q|N_\alpha$, and $D = q_\alpha(N_\alpha)$. The map $q_\alpha : N_\alpha \to D \subset M$ is a G-equivariant embedding onto the G-invariant submanifold $D \subset M$. Note that $D \pitchfork \Sigma$ and that $D \cap \Sigma = \alpha$. Since X is tangent to Σ and non-vanishing, we may suppose that q, D are chosen so that $X \pitchfork D$. In particular, $X|D$ is non-vanishing. Fix a G-invariant metric on N_α and let N_α^r denote the open r-disc bundle of N_α. For sufficiently small $r > 0$, it follows from the continuity of Φ^X that there exists $\varepsilon \in (0, T)$ such that for each $y \in q_\alpha(N_\alpha^r)$, there exists a unique $\tau(y) \in [T-\varepsilon, T+\varepsilon]$ such that $\Phi^X(y, \tau(y)) \in D$ and $\tau|\alpha \equiv T$. Set $D' = q_\alpha(N_\alpha^r)$. An application of the implicit function theorem (using $X \pitchfork D$) shows that $\tau : D' \to \mathbb{R}$ is smooth. We define the Poincaré map $P = P^X : D' \to D$ by

$$P(y) = \Phi^X(y, \tau(y)), \quad (y \in D').$$

(See figure 1.) We call (D, D', P, τ) a *Poincaré system* for Σ.

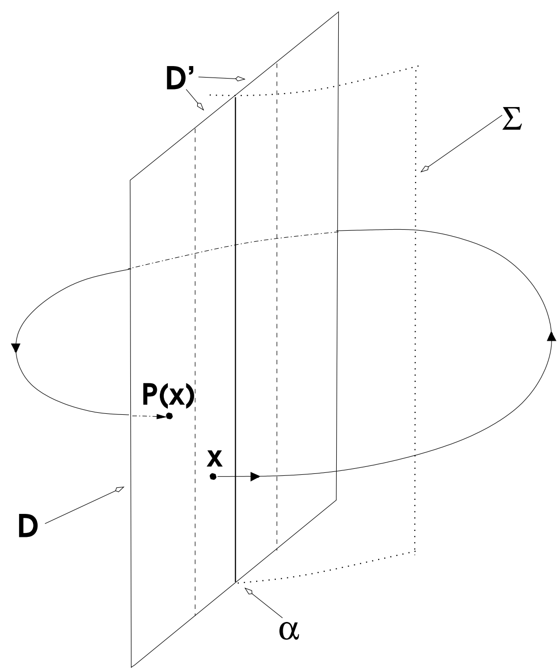


FIGURE 1. Poincaré system for Σ

As simple consequences of our construction we have

(a) $P : D' \to D$ is a smooth G-equivariant embedding.
(b) If we choose another Poincaré system $(\bar{D}, \bar{D}', \bar{P}, \bar{\tau})$ for Σ, then P and $\bar{P}$ are smoothly G-equivariantly conjugate near Σ. In particular, the germ

of the Poincaré map P along $\alpha = \Sigma \cap D$ is independent of choices up to smooth equivariant conjugacy.

(c) If we only assume X is C^r, $r \geq 1$, then P will be C^r.

(d) There exists an open neighbourhood $\mathcal{U}$ of $X \in C^\infty_G(TM)$, such that
   (1) $Y \pitchfork D$, for all $Y \in \mathcal{U}$.
   (2) $P^Y : D' \to D$ is well defined and smooth for all $Y \in \mathcal{U}$.
   (3) The map $C^\infty_G(TM) \to C^\infty(D', D)$, $Y \mapsto P^Y$, is continuous in the sense that for all compact $K \subset D'$ we can choose a compact subset $\bar{K}$ of GD such that for every $r \in \mathbb{N}^+$, continuity holds with respect to $\| \|^{\bar{K}}_r$ on $C^\infty_G(TM)$ and $\| \|^K_r$ on $C^\infty(D', D)$.

Proposition 8.4.19. *(Notation as above.) The following statements are equivalent.*

(1) *The relative periodic orbit Σ is generic.*

(2) *If (D, D', P, τ) is a* Poincaré system *for Σ, then the G-orbit $\alpha = \Sigma \cap D$ is a generic relative fixed set for P.*

Proof. The result may be proved in several ways. Most simply, the splitting $T_\Sigma M = T\Sigma \oplus \mathbb{E}^u \oplus \mathbb{E}^u$ restricts to give a TP-invariant splitting of $T_\alpha M$. Alternatively, we can equivariantly isotop the flow along G-orbits to a periodic flow Ψ_t, with Poincaré map $\bar{P}$, so that $\mathbf{spec}(P, \alpha) = \mathbf{spec}(\bar{P}, \alpha)$. The Floquet exponents of Ψ_t are equal to the exponentials of the eigenvalues of $\bar{P}^d$, where all points of α are of prime period d for $\bar{P}$. □

8.4.6. Perturbation lemmas for relative periodic orbits. Suppose that Σ is a relative periodic orbit of $X \in C^\infty_G(TM)$ which is not generic. In this section we show how we can find arbitrarily small perturbations Y of X for which Σ is a generic relative periodic orbit of Y. In order to construct the perturbation Y, we first perturb the Poincaré map P^X to $\bar{P}$ so that α is generic for $\bar{P}$ (relative to a Poincaré system for X and Σ). We then show that $\bar{P}$ is the Poincaré map of a vector field Y close to X. The proof that $\bar{P}$ can be realized as the Poincaré map of a vector field needs a preliminary lemma on equivariant isotopies.

Lemma 8.4.20. *Let X, Y be compact G-manifolds with boundary and $f : X \to Y$ be a smooth equivariant embedding such that $f(\partial X) \cap \partial Y = \emptyset$. Given $0 \leq a < b$, there exists an open neighbourhood $\mathcal{V}$ of f in $C^\infty_G(X, Y)$ such that if $g \in \mathcal{V}$ then*

(1) *g is an embedding and $g(\partial X) \cap \partial Y = \emptyset$.*

(2) *There exists a smooth equivariant isotopy $H : X \times [a, b] \to Y$ between f and g satisfying*
   (a) *$H_t(x) = f(x)$, whenever $f(x) = g(x)$.*
   (b) *$H_t = f$, for t close to a, $H_t = g$, for t close to b.*
   (c) *H depends continuously on g, C^r-topology, $1 \leq r \leq \infty$.*
   (d) *The t-derivative of H is C^∞ (in (x, t)-variables).*

Proof. Fix an equivariant Riemannian metric on Y. Let V be an open neighbourhood of ∂Y in Y such that $f(\partial X) \cap \bar{V} = \emptyset$. For $r > 0$, let $D_r(y)$ denote the open r-disk, centre y, in Y. Choose $r > 0$, so that $D_r(y) \subset Y \setminus \partial Y$, all $y \in Y \setminus V$. Choosing $r > 0$ smaller if necessary, we may suppose that $\exp_y$ maps the r-disk centre 0 in T_yY diffeomorphically onto $D_r(y)$ for all $y \in Y \setminus V$. Choose a C^1-open neighbourhood $\mathcal{V}$ of f in $C^\infty_G(X, Y)$ such that if $g \in \mathcal{V}$ then (a) g is an embedding, and (b) $d(f(x), g(x)) < r$. For each $x \in X$, there exists a unique $\xi(x) \in T_{f(x)}Y$, $\|\xi(x)\| < r$, such that $g(x) = \exp_{f(x)}(\xi(x))$. Since we may represent $\xi : X \to TY$ as the composition $(\exp_{f(x)})^{-1}g(x)$, it follows that ξ is smooth and depends continuously on g, C^r-topology, $1 \leq r \leq \infty$. Define

$$H_t(x) = \exp_{f(x)}(\kappa(t)\xi(x)),\ (x \in X, t \in [a, b]),$$

where $\kappa : [a, b] \to [0, 1]$ is C^∞ and equal to 0 near a and 1 near b. Clearly H satisfies all the required conditions. $\square$

Let Σ be a relative periodic orbit of $X \in C^\infty_G(TM)$ and (D, D', P, τ) be a Poincaré system for Σ. An open G-invariant neighbourhood U of Σ is *subordinate* to (D, D', P, τ) if $\bar{U} \subset \cup_{x \in D'} \Phi^X(x, [0, \rho(x)])$.

Lemma 8.4.21. *(Notation as above). Assume that U is an open G-invariant neighbourhood of Σ subordinate to (D, D', P, τ). Set $\alpha = \Sigma \cap D$. Fix a compact neighbourhood $K \subset U \cap D'$ of α and let $C^\infty_G(D', D; K)$ denote the subset of $C^\infty_G(D', D)$ consisting of smooth maps F such that $F = P$ on $D' \setminus K$.*

There exists an open neighbourhood $\mathcal{Q}$ of P in $C^\infty_G(D', D; K)$ and a map $\nu : \mathcal{Q} \to C^\infty_G(TM)$ such that

(1) *P' is an equivariant embedding for all $P' \in \mathcal{Q}$.*
(2) *ν is continuous with respect to the C^r-topologies on $\mathcal{Q}$ and $C^\infty_G(TM)$, $1 \leq r \leq \infty$.*
(3) *$\nu(P) = X$.*
(4) *$\nu(P') = X$ on $M \setminus U$, all $P' \in \mathcal{Q}$.*

Proof. Fix real numbers a, b such that $0 < a < b < \inf_{x \in D'} \rho(x)$. By lemma 8.4.20 we can choose a C^1-open neighbourhood $\mathcal{Q}$ of P in $C^\infty_G(D', D; K)$ such that if $P' \in \mathcal{Q}$, then (a) P' is an embedding, (b) there exists an isotopy $H : \bar{D}' \times [a, b] \to D$ between the inclusion $i : \bar{D}' \to D$ and $P^{-1}P'$. For $y \in D', t \in [0, \rho(P^{-1}(P'(y)))]$, define

$$\rho'(y) = \rho(P^{-1}(P'(y)),\ \ \tilde{\Phi}(y, t) = \Phi^X(H_t(y), t).$$

It follows from the definitions that

(a) If $y \in D' \setminus K$, $\tilde{\Phi}(y, t) = \Phi^X(y, t)$.
(b) If $P' = P$, then $\tilde{\Phi} = \Phi^X$.
(c) $\tilde{\Phi}(y, \rho'(y)) = \Phi^X(P^{-1}(P'(y)), \rho((P^{-1}(P'(y)))) = P'(y)$, for all $P' \in \mathcal{Q}$.

By the C^1-openness of embeddings that we may suppose $\mathcal{Q}$ is chosen sufficiently small so that if $P' \in \mathcal{Q}$, then $\tilde{\Phi}$ is an immersion and restricts to an embedding of $\cup_{y\in D'}\{y\}\times[0,\rho'(y))$ in $U\subset M$. We define a smooth G-equivariant vector field X' on the image of $\tilde{\Phi}$ by $X'(\tilde{\Phi}(y,t))=\frac{d}{dt}\tilde{\Phi}_y(t)$. Since $\tilde{\Phi}=\Phi^X$ outside $\cup_{y\in K}\tilde{\Phi}(y,[0,\rho'(y)])$, X' extends smoothly and G-equivariantly to M if we set $X'=X$ on $M\setminus U$. We define $\nu(P')=X'$, for $P'\in C^\infty_G(D',D;K)$. The remaining statements of the lemma are now immediate or follow directly from lemma 8.4.20. □

REMARK 8.4.22. If X is a C^r-vector field, $r<\infty$, then the vector field X' constructed in lemma 8.4.21 will generally only be C^{r-1}. Note that this problem does not arise if we work within the space of C^r-vector fields but require that X is (at least) C^{r+1}.

8.4.7. Genericity theorems. For $\tau\geq 0$, let $\mathcal{G}_1(TM;\tau)\subset C^\infty_G(TM)$ denote the set of all vector fields X such that if Σ is a relative periodic orbit of X of relative prime period at most τ, then Σ is generic (we regard $\mathcal{G}_1(TM;0)$ as consisting of vector fields all of whose relative equilibria are generic). We set

$$\mathcal{G}_1(TM)=\cap_{\tau\geq 0}\mathcal{G}_1(TM;\tau).$$

If $X\in\mathcal{G}_1(TM)$ then all relative periodic orbits of X are generic.

Let $X\in\mathcal{G}_1(TM;\tau)$ and suppose that Σ is a relative periodic orbit of X of relative prime period $\gamma\leq\tau$. We say Σ is $\star$-generic if the dimension of $\overline{\Phi_x(\mathbb{R})}$, $x\in\Sigma$, is maximal (see lemma 8.4.12). Let $\mathcal{G}_1^\star(TM;\tau)$ denote the subset of $\mathcal{G}_1(TM;\tau)$ consisting of vector fields such that all relative periodic orbits of relative prime period $\gamma\leq\tau$ are $\star$-generic. We set

$$\mathcal{G}_1^\star(TM)=\cap_{\tau\geq 0}\mathcal{G}_1^\star(TM;\tau).$$

THEOREM 8.4.23 ([**52**]). *Suppose that M is a compact G-manifold.*

(1) *For all $\tau\geq 0$, $\mathcal{G}_1(TM;\tau)$ is an open and dense subset of $C^\infty_G(TM)$.*
(2) *$\mathcal{G}_1(TM)$ is a residual subset of $C^\infty_G(TM)$.*
(3) *For all $\tau\geq 0$, $\mathcal{G}_1^\star(TM;\tau)$ is a dense subset of $\mathcal{G}_1(TM;\tau)$.*
(4) *$\mathcal{G}_1^\star(TM)$ is a dense subset of $C^\infty_G(TM)$.*

Similar results hold if M is not compact provided we take the Whitney C^∞-topology on $C^\infty_G(TM)$.

PROOF. The proof follows the same strategy as that of the corresponding result for diffeomorphisms. For details we refer the reader to [**52**, section 7]. □

Suppose that Σ is a generic relatively periodic orbit of $X\in\mathcal{G}_1(TM)$. Let $T\Sigma\oplus\mathbb{E}^u(\Sigma)\oplus\mathbb{E}^s(\Sigma)$ be the corresponding Φ^X_t-invariant splitting of $T_\Sigma M$. We may represent the stable manifold $W^u(\Sigma)$ as the image of a G-equivariant injective immersion $\xi^u_\Sigma:\mathbb{E}^u_\Sigma\to M$ mapping the zero section of $\mathbb{E}^u_\Sigma$ onto Σ. Similarly for $\xi^s_\Sigma:\mathbb{E}^s_\Sigma\to M$. Fix a G-invariant Riemannian metric on M. The bundles $\mathbb{E}^u(\Sigma),\mathbb{E}^s(\Sigma)$ inherit the structure of Riemannian G-vector bundles from the induced Riemannian structure on $T_\Sigma M$. For $T>0$, let $\mathbb{E}^u_\Sigma(T)=\{e\in\mathbb{E}^u_\Sigma\mid\|e\|\leq T\}$ denote the T-disk bundle of $\mathbb{E}^u_\Sigma$. We similarly define $\mathbb{E}^s_\Sigma(T)$. Set $W^u(\Sigma,T)=\xi^u_\Sigma(\mathbb{E}^u_\Sigma(T))$ and similarly define $W^s(\Sigma, T)$. Thus $W^u(\Sigma, T)$, $W^s(\Sigma, T)$ are compact G-invariant (embedded) submanifolds of M (with boundary).

Suppose that Σ_1, Σ_2 are relative periodic orbits of X. We say that $W^u(\Sigma_1)$ is G-transverse to $W^s(\Sigma_2)$ if $\xi^u_{\Sigma_1} : \mathbb{E}^u(\Sigma_1) \to M$ is G-transverse to $W^s(\Sigma_2)$ (equivalently, if $\xi^s_{\Sigma_2} : \mathbb{E}^s(\Sigma_2) \to M$ is G-transverse to $W^u(\Sigma_1)$). Alternatively, and equivalently, we may require that $W^u(\Sigma_1, T)$ is G-transverse to $W^s(\Sigma_2, T)$ for all $T > 0$. If $W^u(\Sigma_1)$ is G-transverse to $W^s(\Sigma_2)$. We write $W^u(\Sigma_1) \pitchfork_G W^s(\Sigma_2)$.

Let $\mathcal{G}_2(TM)$ be the subset of $\mathcal{G}_1(TM)$ for which the stable and unstable manifolds of all relative periodic orbits meet G-transversally. We similarly define $\mathcal{G}_2^\star(TM)$ $(= \mathcal{G}_2(TM) \cap \mathcal{G}_1^\star(TM))$.

THEOREM 8.4.24 (Equivariant Kupka-Smale theorem [**52**]). *$\mathcal{G}_2(TM)$ is a residual subset of $C^\infty_G(TM)$ and $\mathcal{G}_2^\star(TM)$ is a C^∞-dense subset of $C^\infty_G G(TM)$.*

PROOF. The proof is similar to that of the corresponding result for diffeomorphisms. For details we refer the reader to [**52**, sections 8,9]. ☐

8.5. Notes on chapter 8

Proposition 8.4.1, giving dynamics on relative equilibria, was originally proved in the author's thesis (Warwick, 1970, [**49**]) and first appeared in [**52**] together with results for relative fixed sets (for diffeomorphisms) and relative periodic orbits. The results for relative periodic orbits were only partial and correct and complete results first appeared in Krupa [**105**]. Krupa's paper also contains the statement and proof of the tangential and normal form for equivariant vector fields, a result that has proved useful in equivariant bifurcation theory. Results along similar lines for equivariant diffeomorphisms and relative fixed sets appear in [**56**, Lemmas C,D] and [**54**] (these results were not motivated by bifurcation theory). General results on dynamics near and on relative periodic orbits and fixed sets also appear in [**58**]. There have been some errors in the literature involving the minimal period of periodic orbits for flows in a fixed isotopy class of equivariant flows on a relative periodic orbit (see [**183**, Example 6.2]). In section 8.3.2 we give careful statements of results for relative fixed sets. Corresponding results for relative periodic orbits of flows follow using a Poincaré system (see also chapter 10). More recent works, for example [**48, 8, 134**], have emphasized proper actions by Lie groups and have used the skew product formulation for dynamics in a neighbourhood of a relative equilibrium (remark 8.4.7(3)). We say more about recent work in the notes to chapter 10.

The Kupka-Smale genericity theorems 8.3.47, 8.4.24 originally appeared in [**52**] – indeed proving these results was the (author's) incentive for developing G-transversality. Earlier versions of these theorems, without the G-transversality, appeared in [**49**].

CHAPTER 9

Dynamical systems on G-manifolds

In this chapter we describe a number of basic classes of equivariant dynamical systems on G-manifolds and indicate some of the main theorems and properties about the associated dynamics. We tend to give only brief details of proofs of the main results (proofs can be long and sometimes quite technical) and instead refer the reader to the original sources.

After a quick review of skew products – the easiest way to manufacture equivariant dynamics – we start by considering G-invariant Morse functions. We prove that every compact G-manifold admits G-Morse functions that determine a very regular decomposition of the manifold into G-handlebundles. Next we consider the equivariant version of subshifts of finite type and indicate the proof of an equivariant version of the C^0-density theorem of Shub and Smale – this uses our results on G-handlebundle decompositions. Finally, we look at symmetric solenoidal attractors and Anosov diffeomorphisms.

9.1. Skew products

We have already encountered skew products – for example as a model for dynamics near a relative equilibrium (remarks 8.4.7(3)).

9.1.1. Skew extensions of diffeomorphisms. Let M be a compact manifold and G be a compact Lie group. Let G act freely on $M \times G$ by $g(x, \gamma) = (x, g\gamma)$, $g \in G$, $(x, \gamma) \in M \times G$. Suppose $\phi \in \mathrm{Diff}(M)$ and $f \in C^\infty(M, G)$. We define $\phi_f \in \mathrm{Diff}_G(M \times G)$ by

$$\phi^f(x, g) = (\phi(x), gf(x)), \quad (x \in X, g \in G).$$

We call ϕ^f a *G-extension* or *skew-extension* of ϕ by G. We refer to f as a *cocycle*.

Exercise 9.1.1. Show that every $\psi \in \mathrm{Diff}_G(M \times G)$ may be represented as a skew extension. That is, there exist unique $\phi \in \mathrm{Diff}(M)$ and $f \in C^\infty(M, G)$ such that $\psi = \phi_f$.

For $n \in \mathbb{Z}$, $x \in M$, define

$$f^n(x) = \begin{cases} \Pi_{j=0}^{n-1} f(\phi^j(x)), & n > 0, \\ e, & n = 0, \\ (\Pi_n^{j=-1} f(\phi^j(x)))^{-1}, & n < 0. \end{cases} \tag{9.1}$$

Order of multiplication is from left to right ($f^2(x) = f(x)f(\phi(x))$). We have

$$(\phi^f)^n(x,g) = (\phi^n(x), gf^n(x)), \ \ (n \in \mathbb{Z}, (x,g) \in M \times G).$$

It is easy to make translations from dynamical properties of ϕ to properties of the G-equivariant diffeomorphism ϕ_f. For example if $x \in M$ is a point of prime period p for ϕ, then $\alpha = G(x,e)$ is a relative periodic orbit of ϕ^f of relative prime period p. If x is hyperbolic then α is generic (that is, normally hyperbolic). Dynamical properties of $\phi_f|\alpha$ will depend on the value of $f^n(x)$. Stable and unstable manifolds of hyperbolic periodic points of ϕ lift to G-invariant stable and unstable manifolds of generic relative periodic orbits. Transversality of invariant manifolds also lifts. Equivariant transversality is not an issue here. Since the G-action is free, equivariant transversality is equivalent to stratumwise transversality which is regular transversality.

EXERCISE 9.1.2. Assuming the Kupka-Smale theorem for diffeomorphisms, verify that $\mathcal{G}_2(M \times G)$ is a residual subset of $\mathrm{Diff}_G(M \times G)$. Show that if $\phi \in \mathrm{Diff}(M)$ is Kupka-Smale, then (1) $\phi_f \in \mathcal{G}_2(M \times G)$ for all cocycles f, and (2) there is a C^∞-dense subset of cocycles for which $\phi_f \in \mathcal{G}_2^\star(M \times G)$.

9.1.2. Principal G-extensions. Suppose that E is a compact G-manifold and G acts freely on E. Let $\Phi \in \mathrm{Diff}_G(E)$. If we let $M = E/G$ denote the orbit space, then M is a smooth compact manifold and Φ uniquely induces $\phi \in \mathrm{Diff}(M)$. We refer to $\Phi : E \to E$ as a principal G-extension of $\phi : M \to M$. The orbit map $p : E \to M$ gives E the structure of a principal G-bundle over M (see section 3.1.1). Let $C^\infty_G(E,G)$ denote the space of 'cocycles' $f : E \to G$ satisfying the equivariance condition

$$f(gv) = gf(v)g^{-1}, \ \ (g \in G, v \in E).$$

If $\Phi \in \mathrm{Diff}_G(E)$, $f \in C^\infty_G(E,G)$, then the map $f\Phi$ defined by

$$(f\Phi(v) = f(v)\Phi(v), \ \ (v \in E)$$

is a G-equivariant diffeomorphism of E.

EXERCISE 9.1.3. Show that every (skew) G-extension can be viewed as a principal G-extension. Verify that $C^\infty(M,G) \approx C^\infty_G(M \times G, G)$.

9.1.3. Skew products for flows and vector fields. Let M be a smooth manifold and G be a compact Lie group. Let $X \in C^\infty(TM)$ and $\xi \in C^\infty(M, \mathfrak{g})$. We define $X_\xi \in C^\infty_G(T(M \times G))$ by

$$X_\xi(x,g) = (X(x), g\xi(x)).$$

We call X_ξ a *skew product* vector field. More formally, X_ξ is a skew extension of X by G and ξ is a cocycle.

Let Φ_t be the flow of X_ξ. Then we may write $\Phi_t(x,g) = (\phi^X_t(x), g\gamma_t(x))$, where ϕ^X_t is the flow of X and $\gamma_t : M \to G$ satisfies the (flow) cocycle condition

$$\gamma_{t+s}(x) = \gamma_t(x)\gamma_s(\phi^X_t(x)), \ \ (x \in X, \ t, s \in \mathbb{R}).$$

Just as for diffeomorphisms, it is easy to translate dynamical properties of X to the skew product field X^ξ. In particular every equilibrium (respectively, limit cycle) of X corresponds to a relative equilibrium (respectively, relative periodic orbit) of X^ξ.

EXERCISE 9.1.4. (1) Show that every smooth G-equivariant flow on $M \times G$ is a skew product flow.
(2) Suppose that H is a closed subgroup of G and M is a smooth H-manifold. Let $\phi \in \mathrm{Diff}_H(M)$ and $f \in C^\infty(M, G)$. Find conditions on f that are sufficient for the skew-extension ϕ_f to induce a G-equivariant diffeomorphism of the twisted product $G \times_H M$. Similarly for flows.
(3) Develop the formal theory of principal skew product flows (that is, equivariant flows on a principal G-bundle).

REMARK 9.1.5. Skew products go back at least to Von Neumann and figure prominently in ergodic theory – both in classification theory and also in the construction of examples. In smooth ergodic theory, it is natural to take M to be a hyperbolic basic set (for example, $\phi : M \to M$ might be Anosov). Compact Lie group extensions of Anosov diffeomorphisms appear in the influential mid-1970's articles of Brin [**27, 28**] who gave conditions for ergodicity of G-extensions. Later, Brin, Katok and Feldman used circle extensions as part of their construction of Bernoulli diffeomorphisms on compact Riemannian manifolds [**29**]. Burns and Wilkinson [**32**] have shown that for ergodic compact group extensions of Anosov diffeomorphisms, ergodicity can persist under measure preserving smooth perturbations of ϕ^f, breaking the smooth skew product structure. The perturbed skew-extensions may be richly pathological [**152, 161**]. Results and references on smooth compact group extensions over basic hyperbolic sets may be found in [**66**].

9.2. Gradient dynamics

Throughout this section we suppose that M is a compact Riemannian G-manifold. The Riemannian metric on M will remain fixed. Given $f \in C^\infty(M)^G$, define

(a) $\Sigma_f = \{x \in M \mid df(x) = 0\}$ (the set of *critical points* of f).
(b) $C_f = f(\Sigma_f)$ (the set of *critical values* of f).
(c) $R_f = \mathbb{R} \setminus C_f$ (the set of *regular values* of f).

Since f is G-invariant, Σ_f is a closed G-invariant subset of M.

Let $\mathrm{grad}(f) \in C^\infty_G(TM)$ denote the gradient vector field of f. If we let $(\ ,\ )$ denote the inner product on fibres of TM determined by the Riemannian metric on M, then (by definition)

$$(\mathrm{grad}(f)(x), v) = df(x)(v), \quad (v \in T_xM, x \in M).$$

The zero set of $\mathrm{grad}(f)$ is equal to the singular set Σ_f of f. In particular, if $\alpha \subset \Sigma_f$ is a G-orbit then α is an equilibrium G-orbit of $\mathrm{grad}(f)$.

LEMMA 9.2.1. *Let $\alpha \subset \Sigma_f$ be a G-orbit. The following conditions are equivalent.*

(1) *The equilibrium orbit α is generic (for* $\mathrm{grad}(f)$*).*
(2) *If we choose a slice S_x at $x \in \alpha$, then x is a non-degenerate critical point of $f|S_x$.*
(3) *α is non-degenerate for f in the sense of equivariant Morse theory* [**176**].

If any of these conditions hold, we refer to α as a non-degenerate critical orbit of f.

PROOF. Exercise (we shall only need the equivalence of (1) and (2)). ☐

DEFINITION 9.2.2. Suppose that α is a non-degenerate critical orbit of f. The *index* of α, $\mathrm{ind}(\alpha, f)$, is defined to be the dimension of the stable manifold of $W^s(\alpha)$ (for $\mathrm{grad}(f)$).

DEFINITION 9.2.3. If all critical G-orbits of $f \in C^\infty(M)^G$ are non-degenerate, we say f is a *G-Morse function.*

It was shown by Wasserman [**176**] that every G-manifold admits a G-Morse function. Let $\mathcal{M}(M, G) \subset C^\infty(M)^G$ denote the set of G-Morse functions on M and $\mathcal{M}_T(M, G) \subset \mathcal{M}(M, G)$ denote the set of G-Morse functions for which the stable and unstable manifolds of equilibrium orbits of $\mathrm{grad}(f)$ meet G-transversally.

PROPOSITION 9.2.4. *$\mathcal{M}_T(M, G)$ is an open and dense subset of $\mathcal{M}(M, G)$.*

PROOF. The result uses similar techniques to those used to prove the Kupka-Smale theorem for equivariant flows – see [**52**]. Since there is no recurrence in the dynamics, the stable and unstable manifolds of critical orbits are embedded G-manifolds and there are no difficult issues concerning stability of G-transverse intersections. ☐

DEFINITION 9.2.5. A G-Morse function is *excellent* if stable and unstable manifolds of the equilibrium orbits of $\mathrm{grad}(f)$ meet transversally. We denote the set of excellent G-Morse functions by $\mathcal{M}_E(M, G)$.

PROPOSITION 9.2.6. *Every compact Riemannian G-manifold admits an excellent G-Morse function.*

PROOF. We give a sketch of the proof – complete details may be found in [**53**]. Let $M_1 \subset M_2 \subset \ldots \subset M_N = M$ be the filtration of M defined by the isotropy strata. We recall that each set M_i is a compact G-invariant subset of M, $N_i = M_{i+1} \setminus M_i$ will be a union of orbit strata and $\partial N_i = M_i$, $1 \leq i < N$ (see section 3.7.1). Note that M_1 consists of maximal isotropy strata and is a G-invariant submanifold of M (the dimensions of components may vary). Our construction goes by an upward induction on the filtration. At the rth step, suppose we have constructed an open G-invariant neighbourhood U_r of M_r and

$f_r \in C^\infty(M)^G$ such that (a) ∂U_r is smooth, (b) $\mathrm{grad}(f_r) \pitchfork \partial U_r$ and inward pointing, (c) if $\alpha \subset N_i \subset M_r$ is an equilibrium orbit, then α is generic and $W^s(\alpha) \pitchfork N_i$, (d) there are no equilibrium orbits in $U_r \setminus M_r$ (these conditions imply that the trajectory of $\mathrm{grad}(f)$ through any point of $\bar{U}_r \setminus M_r$ is forward asymptotic to M_r). To start the induction, choose a G-invariant tubular neighbourhood U_1 of $M_1 = N_0$ and define $\tilde{f}_1 \in C^\infty(U_1)^G$ to be minus the square of the distance from M_1. Extend $\tilde{f}_1$ smoothly and G-equivariantly to $\tilde{f}_1 \in C^\infty(M)^G$. Perturb $\tilde{f}_1$ to $f_1 \in C^\infty(M)^G$ so that $f_1|M_1$ is a G-Morse function (equivalently, f_1 induces a Morse function on M_1/G). We may do this so that conditions (a–d) hold. In particular, every equilibrium orbit of f meeting $\bar{U}_1$ will be a subset of M_1. This completes the first step of the induction. Suppose next that $1 < r < N$ and we have constructed U_r, f_r satisfying (a–d). Let $K \subset U_r$ be a G-invariant compact neighbourhood of M_r. Choose a G-invariant tubular neighbourhood V_r of $N_r \setminus K$ and consider the G-invariant neighbourhood $U_{r+1} = U_r \cup V_r$ of M_{r+1}. Smoothing corners, we may assume that ∂U_{r+1} is a smooth G-invariant submanifold of M. We define a new G-Morse function $\tilde{f}_{r+1} \pitchfork \partial U_{r+1}$ on U_{r+1} by patching together f_r on U_r with the negative of the square of the radius function for the tubular neighbourhood V_r and then extending smoothly and equivariantly to M. Finally, perturb $\tilde{f}_{r+1}$ to $f_{r+1} \in C^\infty(M)^G$ so that (a–d) hold. □

EXAMPLE 9.2.7. In general, a G-Morse function cannot be approximated by an excellent G-Morse function. As an example, take the $\mathbb{Z}_2$ action on S^2 which has the equator as fixed point set. Choose a smooth $\mathbb{Z}_2$-invariant Morse function on S^2 which has exactly two non-degenerate singular points p, q on the equator, both of index one. The equator will then be non-transversal saddle link connecting p, q. The connection cannot be removed by $\mathbb{Z}_2$-invariantly perturbing f.

9.2.1. Handlebundle decompositions of a G-manifold. We start by reviewing the definition of a G-handlebundle (we refer to Wasserman [**176**, section 4] for details we omit).

Let H be a closed subgroup of G and E, F be smooth Riemannian G-vector bundles over $\alpha = G/H$. Define

$$\mathbf{H}(E,F) = \{(v,w) \in E \oplus F \mid \|v\|, \|w\| \leq 1\}.$$

We call $\mathbf{H}(E,F)$ a *(G-) handlebundle of type* (E,F) *and index dim*(F). Note that $\mathbf{H}(E,F)$ is a G-manifold with boundary that has corners. We define

$$\begin{aligned}
\mathbf{C}(E,F) &= \{(v,w) \in \mathbf{H}(E,F) \mid w = 0\},\\
\mathbf{T}(E,F) &= \{(v,w) \in \mathbf{H}(E,F) \mid v = 0\},\\
\mathbf{B}(E,F) &= \{(v,w) \in \mathbf{H}(E,F) \mid \|v\| = 1\},\\
\mathbf{H}^\circ(E,F) &= \{(v,w) \in \mathbf{H}(E,F) \mid \|w\| < 1\},
\end{aligned}$$

and call $\mathbf{C}(E,F)$, $\mathbf{T}(E,F)$ the *core* and *transverse* bundles of $\mathbf{H}(E,F)$ respectively. The set $\mathbf{B}(E,F)$ is the *distinguished boundary* of $\mathbf{H}(E,F)$. Given $x \in \alpha$,

$w \in F_x$, we refer to $\{(v, w) \in (E \oplus F)_x \mid \|v\| \leq 1\}$ as a *core disk*. We similarly define transverse disks.

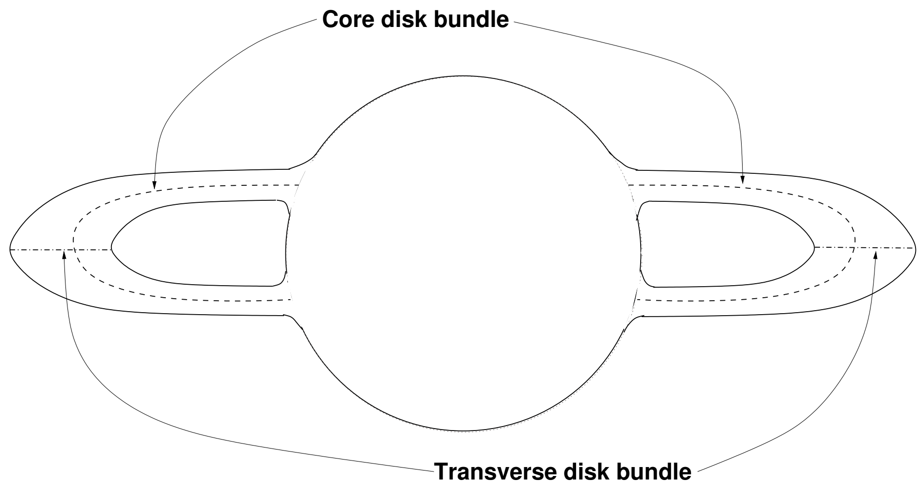


FIGURE 1. Attaching a handlebundle

Suppose that M is a compact G-manifold, boundary ∂M, and $\dim(M) = \dim(\mathbf{H}(E, F))$. Let $f : \mathbf{B}(E, F) \to \partial M$ be a G-equivariant embedding. We define $M \cup_f \mathbf{H}(E, F)$ to be the topological space defined by identifying points in ∂M and $\mathbf{H}(E, F)$ by f. We may give $M \cup_f \mathbf{H}^\circ(E, F)$ the structure of a smooth manifold. We can extend this smooth structure to $M \cup_f \mathbf{H}(E, F)$ by the process of "straightening the angle" along the corners (see Milnor [**127**]). We say that $M \cup_f \mathbf{H}(E, F)$ is the result of attaching a handlebundle of type (E, F) to M. In figure 1, we show the effect of attaching a ($\mathbb{Z}_2$-) handlebundle to a $\mathbb{Z}_2$-invariant 2-disk (note we have smoothed the corners).

Denote the isotropy strata of M by $M_1, \ldots, M_N$, where we have labelled so that $M_N = M_\Pi$ (principal isotropy stratum) and if $\partial M_i \cap M_j \neq \emptyset$ then $i > j$. In particular, M_1 is a compact G-invariant submanifold of M and $M^j = \cup_{i \leq j} M_i$ is a compact G-invariant subset of M for all $j \geq 1$.

THEOREM 9.2.8. *Let M be an m-dimensional compact Riemannian G-manifold. There exists $f \in \mathcal{M}_E(M, G)$ such that*

(1) $f \geq 0$.
(2) $f^{-1}([0, j])$ *is a closed neighbourhood of M^j with smooth boundary,* $1 \leq j \leq N$.
(3) $f^{-1}([j, j+1] \cap C_f) \subset M_{j+1}$, $j \geq 0$.
(4) *If $\alpha \subset M_j$ is a critical orbit of f, then*

$$f(\alpha) = j - \frac{k+1}{m+2},$$

where $k = \mathrm{ind}(\alpha, f)$.

In particular, if we set $W^j = f^{-1}([0, j])$, $j \geq 1$, *then* W^{j+1} *is obtained from* W^j *by successively attaching groups of handlebundles of common index* k, $0 \leq k \leq m$, *each of which is associated to a critical orbit of* $\mathrm{grad}(f)$ *lying in* M_{j+1}.

PROOF. The proof uses the technique of the proof of proposition 9.2.6 together with Smale [**163**, Theorem C]. (We refer to [**53**] for more details – the only issue beyond the method of proposition 9.2.6 is modifying the Morse function so that it takes prescribed values at critical points.) □

EXAMPLE 9.2.9. Let $\mathrm{SO}(2)\times\mathbb{Z}_2$ act on $\mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ as the sum of the standard representation of $\mathrm{SO}(2)$ on $\mathbb{R}^2$ and the non-trivial representation of $\mathbb{Z}_2$ on $\mathbb{R}$. The action extends smoothly to $S^3 = \mathbb{R}^3 \cup \{\infty\}$. We have the orbit decomposition $S^3 = S_1 \cup S_2 \cup S_3 \cup S_4$, where S_1 consists of the two fixed points of the action, S_2 is the set of points with isotropy $\mathbb{Z}_2$, S_3 is the set of points with isotropy $\mathrm{SO}(2)$ and S_4 is the set of points with trivial isotropy. We remark that $S_1 \cup S_2$ is diffeomorphic to S^2 (with the obvious $\mathrm{SO}(2)$-action) and $S_1 \cup S_3$ is diffeomorphic to S^1. We may construct an excellent $\mathrm{SO}(2) \times \mathbb{Z}_2$-Morse function f such that the two critical points of $f|S_1$ are of index 3 and $f(S_1) = \{1/5\}$, there is one critical orbit β for $f|S_2$ of index 2 and $f(\beta) = 7/5$, there is one critical orbit γ for $f|S_3$ of index 2 and $f(\gamma) = 12/5$, and one critical orbit δ for $f|S_4$ of index 1 and $f(\delta) = 18/5$. In figure 2 we have taken a section of $\mathbb{R}^3$ by the (x, z)-plane and shown the level surfaces $f^{-1}(j)$, $j = 1, 2, 3$, as well as representative critical points $a \in S_1$, $b \in \beta$, $c \in \gamma$ and $d \in \delta$.

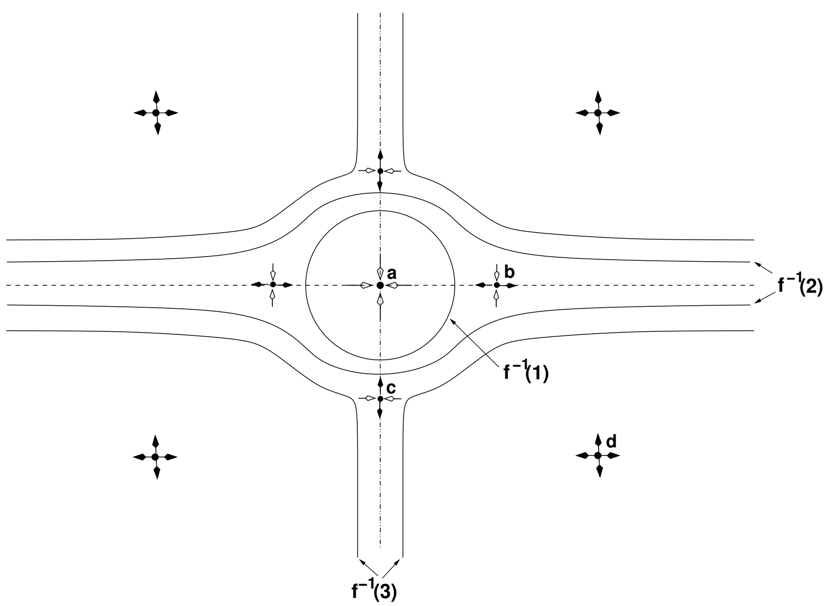


FIGURE 2. $S^1 \times \mathbb{Z}_2$ handlebundle decomposition for S^3

9.2.2. Triangulations and handlebundle decompositions. We conclude with a discussion of an alternative proof of the final statement of theorem 9.2.8.

If M is a compact G-manifold, then we may assume that M has the structure of a real analytic G-manifold (see remark 3.1.10). Give M the semianalytic[1] Whitney regular stratification by isotropy type. It follows by results of Schwarz [**154**] and Mather [**120**] that M/G has the structure of a semianalytic subset of some $\mathbb{R}^n$. If we give M/G the canonical semianalytic Whitney stratification and let $p : M \to M/G$ denote the orbit map, then p maps connected strata of M onto strata of M/G [**13**]. It follows from a result of Lojasiewicz [**116**] that we may triangulate M/G as a semianalytic set (see also Hardt [**90**] and [**121**]). Denote the triangulation of M/G by T and let T_k denote the k-skeleton of T, $0 \leq k \leq \dim(M/G)$. If we let $M^1 \subset \ldots \subset M^N$ denote a filtration of M by isotropy type (that is, the filtration is determined by imposing an order on isotropy types), then the triangulation T induces a triangulation T^j of each M^j/G. If we choose a G-invariant Riemannian metric on M and let ρ denote the induced metric on M/G, then given any $\varepsilon > 0$, we may choose the triangulation T so that each simplex of T is of ρ-diameter at most ε.

Let $\mathcal{T}$ denote the structure on M determined by $p^{-1}(T)$. If $F \in T$ is a p-face, with interior F°, then $\pi^{-1}(F^\circ)$ will consist of points of the same isotropy type. We may build a G-handlebundle decomposition of M using the structure $\mathcal{T}$ (for the non-equivariant case see Mazur [**123**]). Roughly speaking, the sets $p^{-1}(F)$, F a face, will correspond to core bundles of the G-handlebundle decomposition. We build the G-handlebundle decomposition by an upwards induction over the isotropy filtration. If $F \in T$ is a p-face, let $v_F \in F^\circ$ denote the barycentre of F ($\rho(x, \partial F) \leq \rho(v_F, \partial F)$ for all $x \in F$). The set T^1 determines a triangulation of the manifold M_1/G. We define an excellent G-Morse function f^1 on a G-invariant neighbourhood U_1 of M_1 which has critical orbits of index $m-p$ at $\pi^{-1}(v_F) \subset M^1$, where $v_F \in F \in T_1$ and F is a p-dimensional face. We can arrange all of this so that U_1 is a union of G-handlebundles, ∂U_1 is smooth, $\mathrm{grad}(f^1) \pitchfork \partial U_1$ and inward pointing, and $\pi^{-1}(v_F) \cap U_1 = \emptyset$ for all $v_F \notin M_1$. We induct up the filtration in the obvious way to construct the required G-handlebundle decomposition of M. The fact that the triangulation determines a triangulation T^j of each M^j/G allows us to carry through this process without explicitly constructing a G-Morse function. In figure 3 we show a simple example of this process for the $\mathbb{Z}_4$-action on S^2 defined by rotations about the x-axis. On the lefthand side of the figure we show a triangulation of the orbit space $S^2/\mathbb{Z}_4$ with four vertices, six edges and four faces. Two of the vertices correspond to the fixed points of the action on S^2. The associated $\mathbb{Z}_4$-handlebundle decomposition of S^2 will have a total of 10 handlebundles of index 2 (0-dimensional core), 24 handlebundles of index 1 (1-dimensional core) and 16 handlebundles of index 0 (2-dimensional core). In terms of a $\mathbb{Z}_4$-Morse function on S^2, the vertices will be sinks, centres of index 1

[1]The analytic version of semialgebraic – see [**119**]

handles will be saddles and centres of index 2 handles will be sources. Once one has the handlebundle decomposition of S^2, it is trivial to draw the level curves of the Morse function.

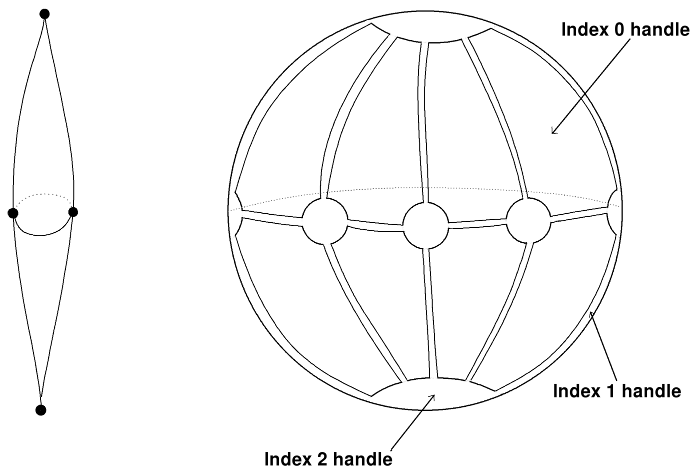


FIGURE 3. $\mathbb{Z}_4$-handlebundle decomposition for S^2

9.3. G-Subshifts of finite type

In this section, we assume some familiarity with basic definitions and results on subshifts of finite type [**139, 102**] (or topological Markov chains [**100**, chapter 1, section 9]). Our aim will be to describe an equivariant version of this theory. In particular, we give an equivariant version of Williams' result [**179**] on realizing subshifts of finite type as basic hyperbolic sets of a diffeomorphism. We also show that every C^1-equivariant diffeomorphism of a compact G-manifold can be equivariantly isotoped to an equivariantly Ω-stable diffeomorphism (the equivariant version of Shub and Sullivan's theorem [**160**]). There will be quite a few preliminaries, mainly concerned with the case when G is not finite.

Recall that for $n \geq 1$, $\mathbf{n} = \{1, 2, \ldots, n\}$. For $n \geq 2$, let $\Sigma_n = \mathbf{n}^{\mathbb{Z}}$ denote the space of all bi-infinite sequences $\mathbf{x} = (x_i)$, with $x_i \in \mathbf{n}$, $i \in \mathbb{Z}$. We give Σ_n the product topology (discrete topology on $\mathbf{n}$) and note that with this topology Σ_n is a compact metrizable space. If $\theta \in (0, 1)$, we define a metric d_θ on Σ_n by

$$d_\theta(\mathbf{x}, \mathbf{y}) = \theta^N,$$

where $N = N(\mathbf{x}, \mathbf{y}) = \max\{N \mid x_j = y_j, |j| < N\}$. The topology defined by d_θ coincides with the product topology on Σ_n. Let $\sigma : \Sigma_n \to \Sigma_n$ be the shift map defined by $\sigma(\mathbf{x})_i = x_{i+1}$, $i \in \mathbb{Z}, \mathbf{x} \in \Sigma_n$. The shift map defines a homeomorphism of Σ_n ('the full shift on n symbols').

G-subshifts of finite type, G is finite. Let G be a finite group and $\psi : G \to S_n$ be a representation of G into the symmetric group S_n of $\mathbf{n}$. We give Σ_n the structure of a G-space with G-action defined by

$$g\mathbf{x}_i = \psi(g)(x_i),\ i \in \mathbb{Z}, g \in G.$$

With this G-action, $\sigma : \Sigma_n \to \Sigma_n$ is an equivariant homeomorphism.

An $n \times n$ matrix A is called a 0-1 matrix if each entry $A(i,j) = a_{ij} \in \{0,1\}$. Let $M(n)$ denote the set of 0-1 matrices. We have an action of G on $M(n)$ defined by

$$g(A)(i,j) = A(\psi(g)(i), \psi(g)(j)),\ (g \in G, A \in M(n), i, j \in \mathbf{n}).$$

Set $M(n)^G = M(n;\psi)$. Given $A \in M(n;\psi)$ define

$$\Sigma_A = \{\mathbf{x} = (x_i) \in \mathbf{n}^{\mathbb{Z}} \mid A(x_i, x_{i+1}) = 1, i \in \mathbb{Z}\}.$$

Clearly Σ_A is a compact G- and σ-invariant subspace of Σ_n.

A matrix $A \in M(n)$ is *irreducible* (or transitive) if for all $i, j \in \mathbf{n}$, there exists $p = p(i,j)$ such that $A^p(i,j) > 0$.

EXERCISE 9.3.1. Show that $\sigma : \Sigma_A \to \Sigma_A$ is transitive (there exists a dense orbit) if and only if A is irreducible.

Henceforth we always assume A is irreducible. We refer to $\sigma : \Sigma_A \to \Sigma_A$ as a *subshift of finite type.*

We define the *period* of A to be the highest common factor of $\{n | A^n(i,i) > 0,\ i, j \in \mathbf{n}\}$. When A has period 1, A is called *aperiodic.* If A is aperiodic then $\sigma : \Sigma_A \to \Sigma_A$ is topologically mixing[2].

EXERCISE 9.3.2. Verify that σ is topologically mixing if and only if A is aperiodic (see [**139, 102**]).

DEFINITION 9.3.3. Let X be a compact metric G-space and $f : X \to X$ be a G-equivariant homeomorphism. We say that the pair (X, f) is a *G-subshift of finite type* if there exists a representation ψ in some S_n, an irreducible matrix $A \in M(n)^G$ and an equivariant homeomorphism $h : \Sigma_A \to X$ such that

$$h \circ \sigma = f \circ h.$$

REMARK 9.3.4. It is easy to see that the pair (Σ_A, σ) need not be uniquely determined by $f : X \to X$ (see also Williams [**179**]).

EXAMPLE 9.3.5. Let $\mathbb{Z}_2$ act on $\mathbb{R}^2$ as multiplication by $\pm I$. Referring to figure 4, we consider the oval $\mathbb{Z}_2$-symmetric region $\Omega \subset \mathbb{R}^2$ defined as the union of the rectangle Q and the half disks A and B. The rectangles P_1, P_2, P_3 and R_1, R_2, R_3 are contained in Q. We assume $-P_1 = P_3$, $-P_2 = P_2$, $-R_1 = R_3$ and $-R_2 = R_2$. We define a smooth $\mathbb{Z}_2$-equivariant equivariant embedding $f : \Omega \to \text{interior}(\Omega)$ such that $f|P_i : P_i \to R_i$ is an affine linear bijection, $i =$

[2] $\sigma : \Sigma_A \to \Sigma_A$ is topologically mixing if for every pair of non-empty open subsets U, V of Σ_A there exists $N = N(U,V) \in \mathbb{N}$ such that $\sigma^n(U) \cap V \neq \emptyset$, $n \geq N$.

$1, 2, 3$. We require that f has a hyperbolic attracting fixed point $u \in f(A)$ (and corresponding point $-u \in f(B)$) and that these are the only fixed points of f in $A \cup B$. Set $\Lambda = \cap_{i=-\infty}^{\infty} f^i(Q)$. Then $(\Lambda, f|\Lambda)$ is $\mathbb{Z}_2$-equivariantly topologically conjugate to the full $\mathbb{Z}_2$-shift $\sigma : \mathbf{3}^{\mathbb{Z}} \to \mathbf{3}^{\mathbb{Z}}$ where we take the representation $\psi : \mathbb{Z}_2 \to S_3$ defined by $\psi(-I)(1) = 3$, $\psi(-I)(2) = 2$ and $\psi(-I)(3) = 1$. We

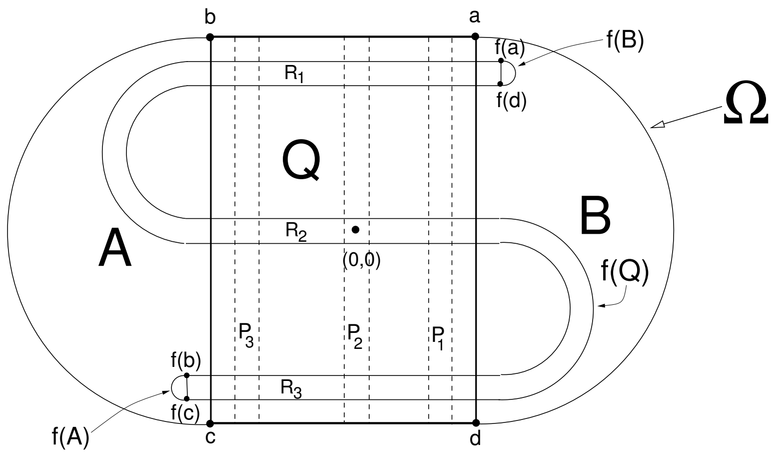


FIGURE 4. $\mathbb{Z}_2$-subshift of finite type

may extend f to an $\mathbb{Z}_2$-equivariant diffeomorphism of S^2. To this end, regard S^2 as $\mathbb{R}^2 \cup \{\infty\}$. Since f is $\mathbb{Z}_2$-equivariantly isotopic to the identity map of Ω, it follows from the equivariant isotopy theorem that f extends to a smooth $\mathbb{Z}_2$-equivariant diffeomorphism of S^2. We may require this extension to have a hyperbolic repellor at ∞ and $W^u(\infty) \supset S^2 \setminus \Omega$. The resulting diffeomorphism of S^2 is Axiom A and even $\mathbb{Z}_2$-structurally stable. This example is based on the construction described by Smale [**163**].

EXERCISE 9.3.6. (1) Show that for all $p > 0$, $\sigma^p : \Sigma_n \to \Sigma_n$ is topologically conjugate to the full shift on n^p symbols. That is, there exists a homeomorphism $H_p : \Sigma_n \to \Sigma_{n^p}$ such that $H_p \circ \sigma^p = \sigma \circ H_p$. Find and prove a similar result for subshifts of finite type.
(2) Show that if $A^p(i, j) > 0$ for all $i, j \in \mathbf{n}$, then $A^q(i, j) > 0$ for all $q \geq p$, $i, j \in \mathbf{n}$.
(3) Suppose that (Σ_n, σ) is a full G-shift on n-symbols and that G acts freely on Σ_n. Letting f denote the map induced by σ on Σ_n/G, show that $(\Sigma_n/G, f)$ is topologically conjugate to a full shift on n symbols. (This result fails if the action of G is not free – see [**68**, chapter 4].)

G-subshifts of finite type when G is compact. Let G be a compact Lie group and J, H be closed subgroups of G with $J \lhd H$ and H/J finite. Let $\tilde{\psi}$ be a representation of H/J in S_n. We have an induced representation $\psi : H \to S_n$. Let $A \in M(\tilde{\psi}; n)$ and $\sigma : \Sigma_A \to \Sigma_A$ denote the corresponding subshift of finite

type. The representation $\psi : H \to S_n$ induces an H-action on Σ_A. It follows that $\sigma : \Sigma_A \to \Sigma_A$ has the structure of an H-subshift of finite type (we allow H to be infinite).

Let $\chi : \Sigma_A \to G$ be a continuous skew H-equivariant map:

$$\chi(hx) = h\chi(x)h^{-1}, \ \ (x \in \Sigma_A, h \in H).$$

We form the twisted product $G \times_H \Sigma_A$ and define the map $\sigma_\chi : G \times_H \Sigma_A \to G \times_H \Sigma_A$ by

$$\sigma_\chi([g,x]) = [g\chi(x), \sigma(x)], \ \ ([g,x] \in G \times_H \Sigma_A).$$

Since χ is skew H-equivariant, σ_χ is well-defined and a G-equivariant homeomorphism of $G \times_H \Sigma_A$. We may extend χ to a skew G-equivariant map $\boldsymbol{\chi} : G \times_H \Sigma_A \to G$ if we define $\boldsymbol{\chi}([g,x]) = g\chi(x)g^{-1}$ and then

$$\sigma_\chi([g,x]) = \boldsymbol{\chi}([g,x])\boldsymbol{\sigma}([g,x]),$$

where $\boldsymbol{\sigma}([g,x]) = [g, \sigma(x)]$. With this notation, $\sigma_\chi = \boldsymbol{\chi}\boldsymbol{\sigma}$.

Definition 9.3.7. Let X be a compact metric G-space and suppose that $f : X \to X$ is a G-equivariant homeomorphism. We say that (X, f) is a G-*subshift of finite type* if there exist closed subgroups J, H of G, with $J \lhd H$ and H/J finite, a representation $\tilde{\psi} : H/J \to S_n$, $A \in M(n; \tilde{\psi})$, a skew H-equivariant map $\chi : \Sigma_A \to G$ and a G-equivariant homeomorphism $h : G \times_H \Sigma_A \to X$ such that

$$\begin{array}{ccc} G \times_H \Sigma_A & \xrightarrow{\boldsymbol{\chi}\boldsymbol{\sigma}} & G \times_H \Sigma_A \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{f} & X \end{array}$$

commutes

Example 9.3.8. Consider the representation $(\mathbb{C}^2, \mathrm{SO}(2))$ which is defined by $e^{\imath\theta}(z_1, z_2) = (e^{\imath\theta}z_1, e^{2\imath\theta}z_2)$. All nonzero points with zero z_1-coordinate have isotropy $\mathbb{Z}_2$. Let $u = (0,1) \in \mathbb{C}^2$ and set $W = (\mathbb{R}\imath u)^\perp$. As a $\mathbb{Z}_2$-representation, we may write $W = \mathbb{R}u \oplus V$, where V is naturally isomorphic to the z_1-axis. The group $\mathbb{Z}_2$ acts trivially on $\mathbb{R}u$ and as multiplication by $\pm I$ on V. Define $I_u = \{(1+s)u \mid s \in [-\frac{1}{2}, \frac{1}{2}]\} \subset \mathbb{R}u$ and let $S = I_u \times V \subset \mathbb{C}^2$. Then S is a slice at u for the action of $\mathrm{SO}(2)$. We define a smooth $\mathbb{Z}_2$-equivariant embedding $\phi : S \to S$ by

$$\phi((1+s)u, z_1) = ((1 - \frac{s}{2})u, f(z_1)), s \in [-\frac{1}{2}, \frac{1}{2}], \, z_1 \in V \approx \mathbb{C},$$

where $f : \mathbb{C} \to \mathbb{C}$ is the $\mathbb{Z}_2$-equivariant diffeomorphism constructed in example 9.3.5. Let $\chi : S \to \mathrm{SO}(2)$ be a smooth map which equals the identity element on a neighbourhood of the boundary of S (since $\mathrm{SO}(2)$ is Abelian, χ is automatically skew-equivariant). Set $U = \mathrm{SO}(2) \times_{\mathbb{Z}_2} S$ and define $F : U \to U$ by $F([e^{\imath\theta}, x]) = [e^{\imath\theta}\chi(x), \phi(x)]$. Using the equivariant isotopy extension theorem, we may extend F to $\hat{F} \in \mathrm{Diff}_{\mathrm{SO}(2)}(\mathbb{C}^2)$. Restricting $\hat{F}$ to U, we see that U contains two hyperbolic attracting relative equilibria and an SO(2)-subshift of finite type which is topologically conjugate to the 'full shift' $\boldsymbol{f}$ on $\mathrm{SO}(2) \times_{\mathbb{Z}_2} \mathbf{3}^{\mathbb{Z}}$. All SO(2)-orbits in $\mathrm{SO}(2) \times_{\mathbb{Z}_2} \mathbf{3}^{\mathbb{Z}}$ have dimension one. However, not all points have the same isotropy. There is precisely one singular SO(2)-orbit through $\bar{2}$ (the point corresponding to $u \in \mathbb{C}^2$ or the origin of V).

Classes of maps covering the identity map on orbit space. If Λ is a closed G-invariant subset of the G-manifold M, let $\mathbf{F}(\Lambda)$ denote the space of equivariant maps $f : \Lambda \to \Lambda$ satisfying

(a) $f(x) \in Gx$, for all $x \in \Lambda$ (f covers the identity map on Λ/G).
(b) f is the restriction of a smooth diffeomorphism of M.

Noting (b), we may define the C^r-topology on $\mathbf{F}(\Lambda)$, $0 \leq r \leq \infty$.

Suppose that H is a closed subgroup of G and (V, H) is an orthogonal representation. We assume H-orbits are finite (equivalently, H_0 acts trivially on V). Let $\pi : G \times_H V \to G/H$ denote the natural projection on G/H ($\pi([g, x]) = g[H]$) and recall that $\pi : G \times_H V \to G/H$ is a smooth G-vector bundle over G/H. Let D denote a closed disk centre 0 in V. Set $\mathbf{\Sigma} = G \times_H D$. Regard D as embedded in $\mathbf{\Sigma}$ as $\{[e, x] \mid x \in D\} = (\pi|\mathbf{\Sigma})^{-1}([H])$. Let $\mathbf{F}(\mathbf{\Sigma})$ denote the space of smooth G-equivariant diffeomorphisms of $\mathbf{\Sigma}$ covering the identity map on $\mathbf{\Sigma}/G$.

LEMMA 9.3.9. *Let U be an open neighbourhood of the identity element of G. There exists a C^0-open neighbourhood $\mathcal{N} = \mathcal{N}(U)$ of $I_{\mathbf{\Sigma}}$ in $\mathbf{F}(\mathbf{\Sigma})$ such that if $f \in \mathcal{N}$, then $f = \boldsymbol{\chi} I_{\mathbf{\Sigma}}$, where $\boldsymbol{\chi} : \mathbf{\Sigma} \to U \subset G$ is smooth and skew G-equivariant.*

PROOF. Using an admissible local section of $G \to G/H$, we can find a C^0-open neighbourhood $\mathcal{N}$ of $I_{\mathbf{\Sigma}}$ such that if $f \in \mathcal{N}$ then there exists a smooth skew G-equivariant map $\eta : \mathbf{\Sigma} \to U \subset G$ such that $\eta(x)f(x) \in \pi^{-1}(x)$, for all $x \in \mathbf{\Sigma}$ (see the proof of lemma 8.3.26). Let $D_P \subset D \subset \mathbf{\Sigma}$ denote the open and dense subset of D consisting of points of principal isotropy group J. We claim that for U sufficiently small $\eta(x)f(x) = x$, $x \in D_P$. Observe that since H-orbits are finite, $f_S = \{x \in D_P \mid \eta(x)f(x) \neq x\}$ is an open subset of D_P. Since D_P/H is connected, we either have f_S is empty or all of D_P. Since H-orbits are finite, if U is chosen sufficiently small then for all $f \in \mathcal{N}$, there exists $x \in D_P$ such that $\eta(x)f(x) = x$. Hence $\eta f = I_{\mathbf{\Sigma}}$, for all $f \in \mathcal{N}$. Take $\boldsymbol{\chi} = \eta^{-1}$. □

REMARK 9.3.10. We caution the reader that lemma 9.3.9 generally *fails* if the dimension of G-orbits in $\mathbf{\Sigma}$ varies – that is, if H_0 does not act trivially on V (for an example, see exercise 8.3.25(2)).

LEMMA 9.3.11. *If $f \in \mathbf{F}(\mathbf{\Sigma})$ is equivariantly isotopic to $I_{\mathbf{\Sigma}}$ through maps in $\mathbf{F}(\mathbf{\Sigma})$, then there exists a smooth skew equivariant map $\boldsymbol{\chi} : \mathbf{\Sigma} \to G$ such that $f = \boldsymbol{\chi} I_{\mathbf{\Sigma}}$.*

PROOF. By lemma 9.3.9, we may write $f = f_1 \circ \ldots \circ f_p$, where each $f_i = \boldsymbol{\chi}_i I_{\mathbf{\Sigma}}$ and $\boldsymbol{\chi}_i$ is skew equivariant. □

Define

$$J = \cap_{x\in V} H_x.$$

LEMMA 9.3.12. *(Notation as above.)*

(1) *J is a normal subgroup of H and $J \supset H_0$.*
(2) *H/J is finite and the H-action on V induces a linear action of H/J on V.*
(3) *The principal isotropy group of the action of H on V is constant, equal to J.*

PROOF. (1) Since $J = \cap_{x\in V} H_x$, $J \lhd H$; H_0 acts trivially on V and so $J \supset H_0$. (2) Immediate from (1). (3) The principal isotropy group for the action of H/J on V is constant (see exercise 3.7.10(1)) and so must be the identity by definition of J. $\square$

Given $n \in G$, let $[n, H] = \{nhn^{-1}h^{-1} \mid h \in H\}$. Define

$$K(J, H) = \{n \in N_G(J) \mid [n, H] \subset J\}.$$

REMARK 9.3.13. If H is finite and $J = \{e\}$, then $K(J, H) = C_G(H)$.

LEMMA 9.3.14. *(Notation as above.)*

(1) *$K(J, H)$ is a closed subgroup of $N_G(H)$.*
(2) *$J \lhd K(J, H)$.*

PROOF. Obviously, $K(J, H)$ is a closed subset of G containing e_G. If $[n, H] \subset J$, then $n \in N_G(H)$ since $Jh \subset H$, for all $h \in H$. Let $n, m \in K(J, H)$, $h \in H$. Then

$$\begin{aligned}
nmh(nm)^{-1}h^{-1} &= n(mhm^{-1}h^{-1})hn^{-1}h^{-1}, \\
&= nj_1hn^{-1}h^{-1}, \text{ for some } j_1 \in J, \\
&= j_2nhn^{-1}h^{-1}, \text{ where } j_2 \in J, \\
&= j_2j_3, \text{ for some } j_3 \in J, \\
&\in J.
\end{aligned}$$

Hence $K(J, H)$ is a subgroup of G. Since $J \lhd H$, $J \subset K$ and so, since $K \subset N_G(J)$, we have $J \lhd K(J, H)$. $\square$

Given $n \in K(J, H)$, define $\beta_n : G \times_H V \to G \times_H V$ by

$$\beta_n([g, v]) = [gn, v], \ [g, v] \in G \times_H V.$$

LEMMA 9.3.15. *(Notation as above.) For all $n \in K(J, H)$, $\beta_n : G \times_H V \to G \times_H V$ is a well-defined smooth G-vector bundle isomorphism of $G \times_H V$ such that $\beta_n(x) \in Gx$ for all $x \in G \times_H V$.*

PROOF. We must show that for all $x \in V$, $g \in G$, we have $\beta_n([g,x]) = \beta_n([gh^{-1},hx])$. We have

$$\begin{aligned}
\beta_n([gh^{-1},hx]) &= [gh^{-1}n,hx],\\
&= [gjnh^{-1},hx], \text{ since } h^{-1}nhn^{-1} \in J,\\
&= [gjn,x],\\
&= [gnj',x], \text{ since } J \lhd K(J,H),\\
&= [gn,j'x] = [gn,x],\\
&= \beta_n([g,x]).
\end{aligned}$$

Hence β_n is well-defined. $\square$

LEMMA 9.3.16. *Let $L : G\times_H V \to G\times_H V$ be a smooth G-vector bundle isomorphism covering $\ell \in \mathrm{Diff}_G(G/H)$ and such that $L(x) \in Gx$ for all $x \in G\times_H V$. There exists $n \in K(J,H)$ such that $L = \beta_n$. If we let p denote the number of connected components of a Cartan subgroup of $N_G(J)/J$ containing $n[J]$, then there exists $\hat{n} \in K(J,H)$, with $\hat{n}^p \in J$, such that L is smoothly equivariant isotopic to the G-vector bundle isomorphism $\hat{L} = \beta_{\hat{n}}$.*

PROOF. Regard V as embedded in $G\times_H V$ as $\{[e,v] \mid v \in V\}$. Let V_P denote the open and dense subset of V consisting of points of principal isotropy J (for the action of H on V). If $z \in V_P$, then $Lz = nz$ for some $n \in N_G(J)$. We claim that $Lx = nx$ for all $x \in V_P$ and hence, by continuity, for all $x \in V$. Let $V(z) = \{x \in V_P \mid Lx = nx\}$. Clearly $V(z)$ is open in V_P since H-orbits are finite and L covers $\ell \in \mathrm{Diff}_G(G/H)$. Since $V(z)$ is obviously nonempty and $L|V$, $x \mapsto nx$ are both linear, $V(z) = V$ and so $Lx = nx$ for all $x \in V$. In order that $Lx = nx$ extend to a G-equivariant map of $G\times_H V$, $L|V$ must be H-equivariant and so $[n,H] \subset J$. Hence $n \in K(J,H)$. Finally, suppose the Cartan subgroup C of $K(J,H)/J$ containing $n[J]$ is isomorphic to $\mathbb{T}^s \times \mathbb{Z}_p$. Choose $\hat{n} \in K(J,H)$ so that $\hat{n}[J] \in C$ is of order p and lies in the same connected component of $K(J,H)$ as n. Necessarily, $\hat{n}^p \in J$. Let n_t be a smooth curve in $K(J,H)$ joining n to $\hat{n}$. Using the previous lemma, we may define the required equivariant isotopy L_t of G-vector bundle isomorphisms of $G\times_H V$ by $L_t = \beta_{n_t}$, $t \in [0,1]$. $\square$

REMARKS 9.3.17. (1) In lemma 9.3.16, we cannot just quotient by J and assume that the principal isotropy group is trivial. This is because although $J \lhd K(J,H)$, it is not necessarily the case that $J \lhd G$. Of course, if we know the result when $J = \{e\}$, it is not hard to deduce the general result proved above.
(2) In lemma 9.3.16, we could have chosen $\hat{n}$ so that $\hat{n}^P = e$ – however, this would generally require P to be a multiple of the integer p defined above (take a Cartan subgroup $\mathbb{T}^r \times \mathbb{Z}_q$ of $K(J,H)$ containing n).

LEMMA 9.3.18. *Let $f \in \mathbf{F}(\mathbf{\Sigma})$. There exists a smooth skew G-equivariant map $\boldsymbol{\chi} : \mathbf{\Sigma} \to G$, $n, \hat{n} \in K(J,H)$ and a smallest strictly positive integer p such that*

(1) $f = \boldsymbol{\chi}\beta_n$.

(2) $\hat{n}^p \in J$.
(3) *f is smoothly G-equivariantly isotopic to $\beta_{\hat{n}}$ through elements of* $\mathbf{F}(\mathbf{\Sigma})$.

PROOF. Define $L : G \times_H V \to G \times_H V$ by

$$L(x) = \lim_{t \to 0} \frac{f(tx)}{t},$$

where tx is defined as multiplication by t in the fibre over $\pi(x)$ of the vector bundle $\pi : G \times_H V \to G/H$. It is straightforward to verify that L is well-defined and a smooth G-vector bundle isomorphism of $G \times_H V$ (L can be equivalently defined in terms of Tf restricted to vertical fibres along the zero section of the tangent bundle of $G \times_H V$). Since $f(x) \in Gx$ for all $x \in G \times_H V$, the same is true for L and so, by lemma 9.3.16, there exists $n \in K(J, H)$ such that $L = \beta_n$. We define a smooth G-equivariant isotopy f_t, $t \in [0, 1]$, between $L|\mathbf{\Sigma}$ and f by

$$f_t(x) = \begin{cases} t^{-1} f(tx) & \text{if } t \neq 0 \text{ and } x \in \mathbf{\Sigma} \\ L(x) & \text{if } t = 0 \text{ and } x \in \mathbf{\Sigma}. \end{cases}$$

Set $k = \beta_n^{-1} f \in \mathbf{F}(\mathbf{\Sigma})$. Since k is isotopic to the identity, we deduce from lemma 9.3.11 that there exists a smooth skew G-equivariant map $\boldsymbol{\chi} : \mathbf{\Sigma} \to G$ such that $k(x) = \boldsymbol{\chi}(x)x$, $x \in \mathbf{\Sigma}$. Hence $f = \boldsymbol{\chi}\beta_n$. The remaining statements of the lemma are immediate from lemma 9.3.16. ☐

DEFINITION 9.3.19. Suppose that $f : \mathbf{\Sigma} \to \mathbf{\Sigma}$ is a smooth G-equivariant diffeomorphism (or embedding) and that Λ is a compact G- and f-invariant subset of $\mathbf{\Sigma}$. We say that $f|\Lambda$ has *representable shift dynamics* if $\Lambda = G \times_H \Sigma_A$, for some 0-1 matrix A, and there exists $\eta \in \mathbf{F}(\mathbf{\Sigma})$ such that $f|\Lambda = \eta\boldsymbol{\sigma}$.

REMARKS 9.3.20. (1) If $f|\Lambda$ has representable shift dynamics, then the dynamics on the orbit space Λ/G is the same as dynamics on the orbit space of the G-subshift of finite type $\boldsymbol{\sigma} : G \times_H \Sigma_A \to G \times_H \Sigma_A$.
(2) If $f|\Lambda$ has representable shift dynamics, then we are realizing the H-subshift of finite type $\sigma : \Sigma_A \to \Sigma_A$ by a smooth mapping $g : D \to D$. That is, $g = \eta^{-1} f|D$ and $g|\Sigma_A = \sigma$ (recall $\mathbf{\Sigma} = G \times_H D$). Later we show that every H-subshift of finite type, with H finite, can be realized by a smooth H-equivariant map on a twisted product. For the present we assume this implicitly and investigate G-equivariant maps with representable shift dynamics.

THEOREM 9.3.21. *If $f|\Lambda$ has representable shift dynamics, f is G-equivariantly conjugate to a G-subshift of finite type.*

PROOF. Suppose that $f|\Lambda = \eta\boldsymbol{\sigma}$, where $\Lambda = G \times_H \Sigma_A$ and $\eta \in F(\Lambda)$. It follows from lemmas 9.3.18 and 9.3.11 that there exist $n \in K(J, H)$ and a smooth skew equivariant map $\boldsymbol{\chi} : \Lambda \to G$ such that $n^p \in J$ and $f = \boldsymbol{\chi}\beta_n\boldsymbol{\sigma}$. Let $\psi : H \to S_n$ denote the representation of H in S_n associated to the G-subshift $\boldsymbol{\sigma} : G \times_H \Sigma_A \to G \times_H \Sigma_A$. Let $q \geq 1$ be the smallest positive integer such that $n^q \in H$. Clearly $q|p$. If we define $\tilde{H} = \langle H, n \rangle$, then $H \lhd \tilde{H}$ and $\tilde{H}/H \cong \mathbb{Z}_q$.

Let S_{nq} be the symmetric group on the nq symbols $\{x_{11}, \ldots, x_{1q}, x_{21}, \ldots, x_{mq}\}$. Define a representation $\tilde{\psi} : \tilde{H} \to S_{nq}$ by

$$\begin{aligned} \tilde{\psi}(h)(x_{ij}) &= x_{uj}, \text{ where } \psi(h)(x_i) = x_u, \\ \tilde{\psi}(n)(x_{ij}) &= x_{ij+1}, \text{ if } j+1 \leq q, \\ \tilde{\psi}(n)(x_{iq}) &= x_{u1}, \text{ where } \psi(n^q)(x_i) = x_u. \end{aligned}$$

(In order to verify that these relations define $\tilde{\psi}$ as a group homomorphism we need $[n, H] \subset J$ and our assumption that J acts trivially on S_n.) Define the non-zero entries of $\tilde{A} \in M(mp; \tilde{\psi})$ by

$$\begin{aligned} \tilde{A}((i,j),(i',j')) &= 1 \text{ if } A(i,i') = 1 \text{ and } j' = j+1 \leq q, \\ &= 1 \text{ if } j = q,\ j' = 1 \text{ and } A(\psi(n^q)(i), i') = 1. \end{aligned}$$

Set $\tilde{\Lambda} = G \times_{\tilde{H}} \Sigma_{\tilde{A}}$ and let $\tilde{\sigma}$ denote the corresponding shift map on $\tilde{\Lambda}$. Define

$$k : \tilde{H} \times_H \Sigma_A \to \Sigma_{\tilde{A}}$$

by $k([\tilde{h}, \mathbf{x}]) = \tilde{\mathbf{x}}$, where $\tilde{\mathbf{x}}_i = \tilde{\psi}(\tilde{h})(\mathbf{x}_i)$, for all $\mathbf{x} \in \Sigma_A$, $\tilde{h} \in \tilde{H}$, $i \in \mathbb{Z}$. The map k defines an $\tilde{H}$-equivariant homeomorphism and k extends uniquely to a G-equivariant homeomorphism $k : \Lambda \to \tilde{\Lambda}$ which commutes with the shift maps defined on $\Lambda, \tilde{\Lambda}$. Now define $\tilde{f} = kfk^{-1}$. Then $\tilde{f} = \tilde{\boldsymbol{\chi}}\tilde{\sigma}$, where $\tilde{\boldsymbol{\chi}} = \boldsymbol{\chi}k^{-1}$ is skew equivariant. □

9.3.1. Stability and the realization of G-subshifts of finite type as basic sets of equivariant diffeomorphisms. Let M be a compact G-manifold. We recall (see [**163**]) that the Ω-set $\Omega(f)$ of $f \in \mathrm{Diff}(M)$ is the set of points $x \in M$ such that for every neighbourhood U of x, there exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. It is easily seen that $\Omega(f)$ is a compact f-invariant subset of M. If f is G-equivariant, then $\Omega(f)$ is a G-invariant subset of M.

Definition 9.3.22. Let f be a smooth G-equivariant diffeomorphism of the compact G-manifold M. We say that f is equivariantly Ω-stable if there exists a neighbourhood $\mathcal{N}$ of f in $\mathrm{Diff}_G(M)$ such that if $F \in \mathcal{N}$, there exists an equivariant homeomorphism $h : \Omega(f) \to \Omega(F)$ and a continuous skew equivariant map $\chi : \Omega(f) \to G$ such that

$$\chi(x)F(h(x)) = h(f(x)),\ (x \in \Omega(f)).$$

Remarks 9.3.23. (1) If G is finite, we can dispense with the skew equivariant map χ of the definition. If G is not finite, then even a hyperbolic attracting relative fixed point will generally require a nontrivial χ in the definition of Ω-stability so as to allow for drifts along G-orbits.
(2) The equality $\chi(x)F(h(x)) = h(f(x))$ determines a conjugacy at the orbit space level.

Definition 9.3.24. Let $f \in \mathrm{Diff}_G(M)$ and suppose that Λ is a compact G- and f-invariant subset of M. We say that Λ is *transversally hyperbolic* (for f) if

(1) All G-orbits in Λ have the same dimension.
(2) Λ is normally hyperbolic for f with centre bundle given by the G-action. That is, there exists a continuous G- and Tf-invariant splitting $T_\Lambda M = \mathbb{L} \oplus \mathbb{E}^u \oplus \mathbb{E}^s$ of G-vector bundles such that
   (a) $\mathbb{L}_x = T_x Gx$, for all $x \in \Lambda$.
   (b) Given a G-invariant Riemannian metric on TM, there exist $C > 0$, $\lambda \in (0,1)$ such that

$$\begin{aligned} \|T^m f(v)\| &\leq C\lambda^m \|v\|,\ v \in \mathbb{E}^s, m > 0, \\ \|T^{-m} f(v)\| &\leq C\lambda^m \|v\|,\ v \in \mathbb{E}^u, m > 0. \end{aligned}$$

Remark 9.3.25. More generally we say that a compact G- and f-invariant subset Λ is transversally hyperbolic if it can be written as a finite union of mutually disjoint pieces each of which satisfy the conditions of the definition. Note that if G is finite (or G-orbits are finite) then transversally hyperbolic is equivalent to hyperbolicity.

Theorem 9.3.26 (Isotopy, density and stability theorem). *Let M be a compact G-manifold and $f \in \mathrm{Diff}_G(M)$. Then f is smoothly equivariantly isotopic to an equivariant diffeomorphism $\tilde{f}$ satisfying*
(1) *$\Omega(\tilde{f})$ is transversally hyperbolic.*
(2) *$\Omega(\tilde{f})$ consists of a finite number of relative periodic points and G-subshifts of finite type.*
(3) *$\tilde{f}$ is equivariantly Ω-stable.*

Furthermore, the subset of $\mathrm{Diff}_G(M)$ consisting of diffeomorphisms satisfying (1,2,3) is C^0-dense in $\mathrm{Diff}_G(M)$.

Theorem 9.3.27 (Realization theorem). *Let (X, ϕ) be a G-subshift of finite type. There exists a compact connected G-manifold M and $f \in \mathrm{Diff}_G(M)$ such that f satisfies (1,2,3) of theorem 9.3.26 and one of the indecomposable pieces of $\Omega(f)$ is equivariantly topologically conjugate to (X, ϕ).*

Proof of theorem 9.3.26 In the case where there is no G-action, (1–3) of theorem were proved by Smale [**164**] using isotopies relative to a handlebody decomposition of M. The density statement is due to Shub [**159**] and the proof is given in Shub and Sullivan [**160**] where it is shown that every diffeomorphism on a compact manifold can be smoothly isotoped to a C^0-close structurally stable diffeomorphism. The method in this case depends on constructing a 'fitted' handlebody decomposition of M. The fitted condition is needed for structural stability; it is not needed for Ω-stability. The arguments in the equivariant case are completely analogous. We construct a G-handlebody decomposition of M using an excellent G-Morse function. The diameter of the handles can be made arbitrarily small using the triangulation technique described in section 9.2.2. The isotopy argument follows Smale [**164**]. In order to show that the invariant sets constructed are G-subshifts of finite type, we use theorem 9.3.21. We refer to [**56**] for more complete details and note that the G-handlebundle decompositions we constructed using excellent G-Morse functions allow us to perform the required equivariant isotopies without running into any obstructions caused by the presence of a G-action. Indeed, we could carry through the arguments at the orbit space level. $\square$

Proof of theorem 9.3.27 The proof is a straightforward equivariant generalization of the argument used by Williams [**179**]. We refer to [**56**] for detailed arguments. $\square$

REMARK 9.3.28. Oliveira proved the C^0-density and isotopy theorem for flows [**136**] and we would expect that his results (and techniques) extend to equivariant flows.

9.4. Suspensions

Let X be a G-space. In this section we review the suspension construction for equivariant dynamical systems. Specifically, we show that every equivariant homeomorphism $f : X \to X$ can be represented as the time one map of an equivariant flow Φ_t^f on a G-space X^f determined by f. If X is a G-manifold (possibly with boundary) and f is smooth, then X^f will be a G-manifold and Φ_t^f will be smooth.

Let G act on $X \times \mathbb{R}$ by $g(x,t) = (gx,t)$, $g \in G$, $(x,t) \in X \times \mathbb{R}$. We define the trivial G-equivariant flow T_t on $X \times \mathbb{R}$ by integrating the unit vector field along $\mathbb{R}$. That is,

$$T_t(x,s) = (x, s+t), \ \ (x,s) \in X \times \mathbb{R}, \ t \in \mathbb{R}.$$

We let X^f be the quotient space of $X \times \mathbb{R}$ defined by the equivalence relation

$$(f(x), s) \sim (x, s+1), \ (x,s) \in X \times \mathbb{R}.$$

Since f is G-equivariant, X^f has the structure of a G-space. If X is a smooth G-manifold, then T_t is smooth and X^f inherits the structure of a smooth G-manifold from that on on X. If we let $\pi : X \times \mathbb{R} \to X^f$ denote the quotient map, then T_t induces a G-equivariant flow Φ_t^f on X^f by $\Phi_t^f(\pi(x,s)) = \pi T_t(x,s)$, $(x,s) \in X \times \mathbb{R}$. We call X^f the *suspension* of X and Φ_t^f the *suspension flow*.

The space X^f can be represented explicitly as the cylinder $X \times [0,1]$ with the ends identified according to $(f(x), 0) \sim (x, 1)$. If f is the identity map of X, then $X^f = X \times \mathbb{T}$ (where we regard $\mathbb{T} = \mathbb{R}/\mathbb{Z}$). If f is not homotopic (strictly, isotopic) to the identity map, then X^f will not be homeomorphic to $X \times \mathbb{T}$. For example, if $X = [-1,1]$ and $f(x) = -x$, then X^f is the Mobius band (see figure 5).

REMARK 9.4.1. All the constructions we have given above work carry through if f is only a (smooth) embedding into X. In that case, the flow Φ_t^f will not generally be defined for all negative values of t. However the associated vector field will be well-defined and can be extended equivariantly to all of X^f (for

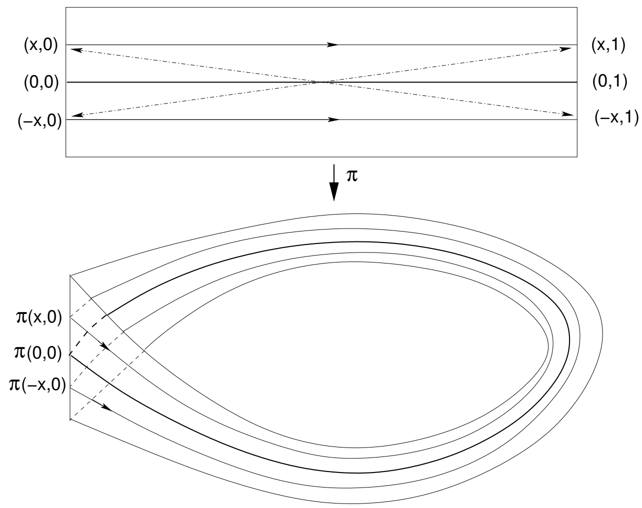


FIGURE 5. Suspension flow on the Mobius band

example, using the weak version of Whitney's extension theorem applicable to manifolds with boundary).

Dynamical properties of f lift to properties of the suspension flow. For example, every fixed point x of f corresponds to a periodic orbit of period 1 for Φ_t^f through $(x, 0)$. Points of period p for f correspond to periodic orbits for Φ_t^f of period p. More generally, a relative fixed point of f lifts to a relative equilibrium of Φ_t^f and a relative periodic orbit of f will lift to a relative periodic orbit of Φ_t^f with the same relative period. Similarly, hyperbolicity, transversality and G-transversality all lift (see exercises 9.4.3(8) for mixing properties).

We define an embedding of $i : X \to X^f$ by $i(x) = \pi(x, 0)$, $x \in X$. The embedded image $i(X) = X_0$ of X defines a *section* for the suspension flow. That is, for every $x \in X_0$, there exists a smallest $T(x) > 0$ such that $\phi^f_{T(x)}(x) \in X_0$. In this way we define a map $F : X_0 \to X_0$ by $F(x) = \Phi^f_{T(x)}(x)$. For the suspension flow constructed above, $T \equiv 1$ and $F = f$. However, the construction applies whenever a flow has a section. The construction is useful if the flow has local sections – for example in a neighbourhood of periodic orbit. The resulting map is just the *Poincaré* map for the periodic orbit.

Since the return times to a section need not be constant, it is useful to generalize our definition of suspension. Suppose that $r : X \to \mathbb{R}$ is strictly positive

(r is called a *roof* function). For $n \in \mathbb{Z}$, $n \neq 0$, $x \in X$, we define

$$\begin{aligned} r^n(x) &= \sum_{j=0}^{n-1} r(f^j(x)),\ n > 0, \\ &= -\sum_{j=-n}^{-1} r(f^j(x)),\ n < 0. \end{aligned}$$

We generate an equivalence relation $\sim$ on $X \times \mathbb{R}$ by

$$(f(x), s) \sim (x, s + r(x)),\ (x, s) \in X \times \mathbb{R}.$$

It follows from the definition of r^n that $(x,t) \sim (y,s)$ if and only either $x = y$, $s = t$ or there exists $n \in \mathbb{Z}$, $n \neq 0$, such that $y = f^n(x)$ and $t = s + r^n(x)$. We let X_r^f denote the quotient space of $X \times \mathbb{R}$ defined by $\sim$. The trivial flow on $X \times \mathbb{R}$ induces a flow $\Phi_t^{r,f}$ on X_r^f. Just as above, we let X_0 denote the image of X in X_r^f by the embedding $i(x) = (x, 0)$. The set X_0 defines a section for the flow $\Phi_t^{r,f}$ and clearly the first return times to the section X_0 are given by the roof function r. That is, for all $x \in X_0$, $\Phi_{r(x)}^{r,f}(x) = f(x) \in X_0$.

Example 9.4.2. Let Σ be a relative periodic orbit of $X \in C_G^\infty(TM)$. Let (D, D', P, τ) be a Poincaré system for Σ. We recall (section 8.4.5) that $P : D' \to D$ and $P(x) = \Phi^X(x, \tau(x)) \in D$. The flow Φ_t^X is smoothly equivariantly conjugate to the suspension flow $\Phi_t^{\tau,P}$ near Σ.

Exercise 9.4.3. (1) Let $f : X \to X$ be a homeomorphism. Suppose that $x_0 \in X$ has dense orbit in X. Show that the Φ_t^f-orbit of $(x_0, 0) \in X^f$ is dense in X^f (transitivity lifts).
(2) Regard $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $f : \mathbb{T} \to \mathbb{T}$ be defined by $f(\theta) = \theta + \alpha$, $\alpha \in [0,1)$. Show that X^f is the 2-torus $\mathbb{T}^2$. Describe the suspension flow in case (a) $\alpha = p/q$ (rational), (b) α irrational.
(3) Suppose that $f : X \to X$ is *minimal* (the only closed invariant subsets of X for f are X and the empty set). Is the suspension flow minimal? What about the time-T map $\Phi_T^f : X^f \to X^f$?
(4) Describe X^f and the suspension flow in case $f : \mathbb{T} \to \mathbb{T}$ is defined by $f(\theta) = -\theta$ (if we regard $\mathbb{T} = S^1 \subset \mathbb{C}$, f is complex conjugation: $f(z) = \bar{z}$.
(5) Show that if $f : X \to X$ is continuous, but not a homeomorphism, then the suspension construction defines Φ_t^f as a semiflow. Investigate in the case $f : \mathbb{T} \to \mathbb{T}$ is the tripling map defined by $f(\theta) = 3\theta$. The tripling map is $\mathbb{Z}_2$-equivariant with respect to the action defined by $\theta \mapsto \theta + 1/2$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$). What does this imply about Φ_t^f?
(6) Let $f : X \to X$ be a homeomorphism and $r : X \to \mathbb{R}$ be a roof function. Suppose that $x \in X$ has prime period p for f. Show that $\Phi_t^{r,f}$ has a periodic orbit γ through $(x, 0)$. What is the period of γ?
(7) Suppose that f is the map of (2) with α-irrational. Can you find a roof

function $r : S^1 \to \mathbb{R}$ such that the time-1 map of the suspension flow $\Phi_t^{r,f}$ has dense trajectories? is minimal?
(8) $f : X \to X$ is *topologically mixing* if for every pair of nonempty open subsets U, V of X, there exists $N = N(U,V) \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$, for all $n \geq N$. There is a similar definition of topological mixing for flows with the condition $f^n(U) \cap V \neq \emptyset$, for all $n \geq N$ replaced by $\Phi_t(U) \cap V \neq \emptyset$, for all $t \geq T(U,V)$. Show that the suspension flow $\Phi_t^{r,f}$ of a topologically mixing homeomorphism is not topologically mixing if r is constant.

9.5. The inverse limit: turning maps into a homeomorphisms

We continue to assume X is a G-space or G-manifold, as appropriate (for our applications, G will always be finite). Suppose that $f : X \to X$ is a continuous equivariant map which is onto but not necessarily a homeomorphism of X. In this section we review a process that enables us to generate an equivariant homeomorphism $\hat{f}$ from which one can reconstruct f. (The construction is a special case of the *inverse limit* [**45**, Appendix II].)

If f is not 1:1 then the inverse image $f^{-1}(x)$ of a point $x \in X$ will generally contain more than one point and so x will generally not have a 'unique' history. More formally, if $x_0 \in X$, it may be possible to find many sequences $(z_n)_{n\geq 1}$, such that $f(z_1) = x_0$, $f(z_2) = z_1, \ldots$. Typically, the number of points in $f^{-n}(x_0)$ may grow exponentially fast, even for quite simple maps f.

EXAMPLE 9.5.1. If we identify $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$, the tripling map $f : \mathbb{T} \to \mathbb{T}$ is defined by $f(\theta) = 3\theta$, mod $\mathbb{Z}$. A straightforward computation shows that for $n \geq 1$, $\psi \in [0,1)$,

$$f^{-n}(\psi) = \frac{\psi}{3^n} + \{\frac{p}{3^n} \mid 0 \leq p < 3^n\}.$$

Hence for all $\psi \in \mathbb{T}$, $f^{-n}(\psi)$ contains exactly 3^n points. Clearly, $f^{-n}(\psi)$ is a $\mathbb{Z}_2$-invariant subset of $\mathbb{T}$, where the $\mathbb{Z}_2$-action on $\mathbb{T}$ is defined by $\theta \mapsto \theta + 1/2$,

We shall define the space Σ of all possible histories of points in X and show that f naturally induces a homeomorphism $\hat{f}$ of Σ. The space Σ will inherit the structure of a G-space and the homeomorphism $\hat{f}$ will be G-equivariant.

It is convenient to assume in what follows that (X,d) is a compact metric G-space. Since X is compact, d is bounded on X and, replacing d by $d/\sup_{x,y} d(x,y)$, it is no loss of generality to assume that $d \leq 1$. Of course, we may and shall assume that the metric d is G-invariant.

Let $X_\infty = \Pi_{i=1}^\infty X$ denote the infinite product of X. We regard X_∞ as the space of all sequences $\mathbf{x} = (x_n)_{n\geq 0}$ of points of X,

$$X_\infty = \{(x_0, x_1, \ldots) \mid x_i \in X, \text{ all } i \geq 0\}.$$

We take the product topology on X_∞. It follows by Tychonoff's theorem that X_∞ is compact. The topology is metrizable and given by the metric d_∞ defined by

$$d_\infty(\mathbf{x},\mathbf{y}) = \sum_{j=0}^{\infty} 2^{-j-1} d(x_j, y_j), \quad (\mathbf{x} = (x_j), \mathbf{y} = (y_j)).$$

Observe that $d_\infty \leq 1$ on X_∞. The action of G on X extends to X_∞ by

$$g(x_0, x_1, \ldots) = (gx_0, gx_1), \quad (g \in G, (x_0, x_1, \ldots) \in X_\infty).$$

It is straightforward to verify that this action of G gives X_∞ the structure of a G-space and that d_∞ is a G-invariant metric on X_∞. For $n \geq 0$, define the continuous G-equivariant projection map $p_n : X_\infty \to X$ by

$$p_n(\mathbf{x}) = x_n, \ \mathbf{x} = (x_j).$$

The map f induces a continuous equivariant map $\hat{f} : X_\infty \to X_\infty$ by $\hat{f}(\mathbf{x})_j = f(x_j)$, $j \geq 0$ (the sequence (x_n) is mapped to the sequence $(f(x_n))$).

We define $\mathcal{S}(X, f) = \mathcal{S} \subset X_\infty$ to be the space of all sequences (x_n) such that

$$f(x_{n+1}) = x_n, \ n \geq 0.$$

Each sequence $(x_n) \in \mathcal{S}$ represents a possible history of the initial point x_0.

Lemma 9.5.2. *Let $f : X \to X$ be continuous, equivariant and onto. Then*

(a) *$\mathcal{S}(X, f)$ is a compact nonempty G-invariant subspace of X_∞.*
(b) *The map $\hat{f}$ restricts to an equivariant homeomorphism $\hat{f} : \mathcal{S} \to \mathcal{S}$.*
(c) *The projections $p_n : \mathcal{S} \to X$ are all onto.*
(d) *$fp_n = p_n\hat{f}$, $n \geq 1$.*

Proof. We prove (b) and leave the remaining statements to the reader. First observe that $\hat{f}$ is injective since $\hat{f}(x_0, x_1, \ldots) = (f(x_0), x_0, x_1, \ldots)$ and so if $\hat{f}(\mathbf{x}) = \hat{f}(\mathbf{y})$, we have $\mathbf{x} = \mathbf{y}$. On the other hand, if $\mathbf{x} = (x_0, x_1, \ldots) \in \mathcal{S}$, then $\hat{f}(x_1, x_2, \ldots) = \mathbf{x}$ since $f(x_1) = x_0$. Since $\hat{f}$ is a continuous bijection of compact metric spaces it follows that $\hat{f}$ is a homeomorphism. $\square$

Remarks 9.5.3. (1) $\mathcal{S}(X, f)$ is called the *inverse limit* (space) of $f : X \to X$ and is commonly denoted by $\varprojlim X$ or $\varprojlim f : X \to X$. We often refer to the map $\hat{f}$ as the *shift map* on $\varprojlim X$.
(2) Statement (a) of lemma 9.5.2 holds without the assumption that f is onto. However, $\mathcal{S}$ may consist of a single point.
(3) For (c) we need the compactness of X (or extra conditions). If X is not compact, and f not onto then $\mathcal{S}$ may be empty. Notice that (c) shows that $\mathcal{S}$ is at least as big as X.
(4) We can use (b,c,d) of the lemma to reconstruct f knowing $\hat{:}\mathcal{S} \to \mathcal{S}$. Specifically, given $x \in X$, we may, by (c), choose $\mathbf{x} \in \mathcal{S}$ such that $p_1(\mathbf{x}) = x$. We then define $f(x) = p_0(\mathbf{x})$. Observe that $f(x)$ is well-defined since $\hat{f}$ is a homeomorphism.

The inverse of $\hat{f} : \mathcal{S} \to \mathcal{S}$ is given by $(x_0, x_1, \ldots) \mapsto (x_1, x_2, \ldots)$.

Dynamical properties of $f : X \to X$ often lift nicely to the inverse limit. We recall some standard definitions.

DEFINITION 9.5.4. Let $f : X \to X$ be a continuous map.

(1) f is *transitive* if there exists $x_0 \in X$ such that the forward f-orbit $\{f^n(x) \mid n \in \mathbb{N}\}$ is dense in X.
(2) f is *topologically transitive* if for all non-empty open subset U, V of X, there exists $p = p(U, V) \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.
(3) f is *topologically mixing* if for all non-empty open subset U, V of X, there exists $N = N(U, V) \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$, for all $n \geq N$.

REMARKS 9.5.5. (1) If f is onto and X is perfect[3], then f is transitive if and only if f is topologically transitive. Obviously, topological mixing implies topologically transitive.
(2) Transitivity, topological transitivity and topological mixing are all invariants of topological conjugacy.

LEMMA 9.5.6. *Let X be a compact perfect metric space. A continuous map $f : X \to X$ is transitive if and only if f is topologically transitive. A necessary condition for (topological) transitivity is that f is surjective.*

PROOF. (Sketch.) This is a standard result in topological dynamics – though there are variations on the statement depending on the exact definition of transitive and the properties of the map f. If f is topologically transitive, then f must be surjective since X is assumed compact (either the open set $X \setminus f(X)$ consists of a single point, violating the assumption that X is perfect, or we can choose disjoint nonempty open subsets U, V of $X \setminus f(X)$). If f is transitive, but not surjective, then the open set $X \setminus f(X)$ would consist of a single point, contradicting the assumption that X is perfect. The proof that topological transitivity implies transitivity uses the Baire category theorem (and does not use X perfect). For the converse, we use the fact that there are no isolated points to show that if the forward-orbit of x under f is dense then given any $y \in X$, there is a subsequence of $(f^n(x))$ converging to y. $\square$

EXERCISE 9.5.7. Show that the condition that X be perfect is necessary in lemma 9.5.6. Specifically, find an example of a continuous map of a compact metric space which is transitive but *not* topologically transitive (according to our definitions of transitivity and topological transitivity).

Given $f : X \to X$ and an integer $p \geq 1$, we let $\mathrm{Per}_p(f)$ denote the subset of X consisting of points of period p for f. We let $\mathrm{Per}(f) = \cup_{p \geq 1} \mathrm{Per}_p(f)$ denote the set of all periodic points of f.

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[3] X is perfect if every point of X is an accumulation point of X. Equivalently, X has no isolated points.

PROPOSITION 9.5.8. *Let $f : X \to X$ be a continuous surjective map of the perfect compact metric space X.*

(a) *The inverse limit $\mathcal{S} = \mathcal{S}(X, f)$ is perfect.*
(b) *If $\mathrm{Per}(f)$ is dense in X then $\mathrm{Per}(\hat{f})$ is dense in $\mathcal{S}$.*
(c) *If f topologically transitive then $\hat{f}$ is topologically transitive.*
(d) *If f topologically mixing then $\hat{f}$ topologically mixing.*

PROOF. (Sketch) We leave the proofs of (a,c) and (d) to the reader and sketch the proof of (b). Suppose $q \in \mathrm{Per}_Q(f)$. If we set $q_i = f^i(q)$, $0 \le i \le Q-1$, then the periodic sequence $\mathbf{w}(q) = \overline{q_{Q-1}q_{Q-2}\cdots q_0} \in \mathcal{S}$ is a point of period Q for $\hat{f}$. Let $\mathbf{x} \in \mathcal{S}$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that $2^{-N} < \varepsilon$. Since X is compact, $f, \dots, f^N$ are uniformly continuous and so we can choose $\delta > 0$ such that if $d(x, y) < \delta$, then $d(f^n(x), f^n(y)) < \varepsilon$, $0 \le n \le N$. Since periodic points are dense in X, we may choose a periodic point $q \in X$ such that $d(q, x_N) < \delta$. Suppose q has prime period Q and write $N + 1 = rQ + s$, $s \in [0, Q-1]$. Define $\mathbf{w} = \sigma^{Q-s}(\mathbf{w}(q))$. and note that $w_N = q$. Hence, $d(w_N, x_N) < \delta$ and so $d_\infty(\mathbf{w}, \mathbf{x}) < \varepsilon$, proving (b). $\square$

REMARKS 9.5.9. (1) The assumptions of proposition 9.5.8 are unnecessarily strong. For example, the density of $\mathrm{Per}(\hat{f})$ follows without the assumption that X is perfect.
(2) Suppose that $f : X \to X$ is continuous and for every non-empty open subset U of X there exists $p = p(U) \in \mathbb{N}$ such that $f^p(U) = X$. It is immediate that f is topologically mixing and therefore $\hat{f} : \mathcal{S} \to \mathcal{S}$ will be topologically mixing and therefore topologically transitive (and transitive by lemma 9.5.6).

EXAMPLE 9.5.10. Let $f : \mathbb{T} \to \mathbb{T}$ be the tripling map defined by $d(\theta) = 3\theta$, mod $\mathbb{Z}$. Since f triples arc length, for any non-empty open subset U of S^1, there exists $p = p(U)$ such that $f^p(U) = \mathbb{T}$. Hence f is topologically transitive and therefore transitive since $\mathbb{T}$ is perfect. By example 9.5.1, $\mathrm{Per}(f)$ is dense in $\mathbb{T}$. By proposition 9.5.8, $\mathrm{Per}(\hat{f})$ is dense in $\mathcal{S}(\mathbb{T}, f)$ and $\hat{f}$ is topologically mixing.

EXERCISE 9.5.11. (1) Show that the inverse limit of the tripling map defined in the previous example has the structure of a compact topological group (multiplication defined coordinate wise). This group – known as the triadic solenoid – is far from Lie and does not embed in any orthogonal group $\mathrm{O}(n)$. (See also example 1.4.7 for the general case of p-adic solenoids. The triadic solenoid is the Pontryagin dual of the group $\mathbb{Z}[\frac{1}{3}]$ of triadic rationals.)
(2) Suppose that (X, G) is a compact perfect metric G-space and $f : X \to X$ is a G-equivariant continuous surjection.

(a) Suppose that H is the isotropy subgroup of a point in X and that $f|X^H : X^H \to X^H$ is onto. Show that $\mathcal{S}^H \neq \emptyset$.
(b) Show that if $\mathcal{S}$ contains a point with isotropy subgroup H, then there exists a point of X with isotropy group H. (You might prefer to assume

that G is finite, though the result is true provided that G is a compact Lie group).

The inverse limit construction gives a satisfactory way of manufacturing an equivariant homeomorphism from a continuous equivariant surjective map. Of course, the construction is abstract and gives no hint as to how one might realize the inverse limit as an invariant – even hyperbolic – set of a smooth equivariant map. This question was considered by Williams [**178, 180**] who gave sufficient conditions for the realization of the inverse limit as a hyperbolic attractor of a smooth diffeomorphism. Later, parts of Williams' theory were extended to equivariant maps by Field, Melbourne and Nicol [**64**]. In the next section we describe some of these extensions as well as some more recent work done by Jacobs [**97**].

9.6. Solenoidal attractors

In this section we prove a number of results about hyperbolic attractors for equivariant diffeomorphisms. Most of the results we present are based on the paper [**64**] which in turn was inspired by earlier work of Williams on 1-dimensional expanding maps and attractors [**178**].

We always assume the group G is finite. We recall some results from section 3.7. If M is a connected G-manifold then the principal isotropy group is a normal subgroup of G which acts trivially on M. Quotienting out by the principal isotropy group we may and shall assume that the principal isotropy group is trivial. Let M_Π denote the set of points with trivial isotropy. If there are no reflections for the action of G on M, then M_Π is a connected open and dense G-invariant subset of M. So as to simplify our exposition we usually assume that there are no reflections for the action of G on M and so M_Π is connected (this restriction was not made in [**64**]).

If Λ is a nonempty closed subset of the G-manifold M, we define the *symmetry group* G_Λ of Λ by

$$G_\Lambda = \{g \in G \mid g\Lambda = \Lambda\}.$$

Definition 9.6.1. Let M be a G-manifold, $f \in \mathrm{Diff}_G(M)$ and $\Lambda \subset M$ be a closed f-invariant subset of M. We say that Λ is an *attractor* if

(1) For every open neighbourhood U of Λ, we can choose an open neighbourhood $V \subset U$ of Λ such that $f^n(U) \subset V$, $n \geq 0$, and $\cap_{n\geq 0} f^n(V) = \Lambda$.
(2) $f|\Lambda$ is transitive.

We may similarly define attractors for flows (for flows, transitivity implies the attractor is connected).

Remark 9.6.2. As an immediate consequence of equivariance, we see that if Λ is an attractor for $f \in \mathrm{Diff}_G(M)$, then so is $g\Lambda$, for all $g \in G$.

The next lemma is a slight variant of a result originally proved by Chossat and Golubitsky in the context of (non-invertible) equivariant maps [**33**].

LEMMA 9.6.3. *Let Λ be an attractor for $f \in \mathrm{Diff}_G(M)$. Then if $g \in G$, $g\Lambda \cap \Lambda \neq \emptyset$ if and only if $g \in G_\Lambda$. A similar result holds for flows.*

PROOF. Let $g\Lambda \cap \Lambda \neq \emptyset$ for some $g \in G$. We are required to prove that $g\Lambda = \Lambda$. Suppose the contrary. We may assume that Λ is not finite – otherwise Λ is a periodic orbit and the result is trivial. Since f is invertible and transitive, Λ has no isolated points. Hence Λ is perfect and so (proof of lemma 9.5.6) $f|\Lambda$ is topologically transitive. Without loss of generality, suppose there exists $x \in \Lambda \setminus g\Lambda$. Since $g\Lambda$ is closed, we may choose an open neighbourhood W of x in M such that $\overline{W} \cap g\Lambda = \emptyset$. By remark 9.6.2, $g\Lambda$ is an attractor and so we may pick an open neighbourhood V of $g\Lambda$ such that $\cap_{n\geq 0} f^n(V) = g\Lambda$ and $f^n(V) \cap W = \emptyset$, for all $n \geq 0$. Since $f|\Lambda$ is topologically transitive, and $V \cap \Lambda \neq \emptyset$, there exists $n \geq 0$ such that $f^n(V) \cap W \neq \emptyset$. Contradiction. Hence $g\Lambda = \Lambda$. □

REMARKS 9.6.4. (1) It follows from lemma 9.6.3 that the G-orbit of an attractor Λ consists of $|G/G_\Lambda|$ distinct attractors each with symmetry group conjugate to G_Λ.
(2) It is not required in lemma 9.6.3 that Λ be compact.

We may now state our main results on the existence of attractors with specified symmetry group.

THEOREM 9.6.5. *(Assumptions as above.) Let M be a G-manifold, $\dim(M) \geq 4$, and H be a subgroup of G. There exists $f \in \mathrm{Diff}_G(M)$ such that f has a connected compact hyperbolic attractor $\Lambda \subset M_\Pi$ with symmetry group $G_\Lambda = H$.*

THEOREM 9.6.6. *(Assumptions as above.) Let M be a G-manifold, $\dim(M) \geq 5$, and H be a subgroup of G. There exists smooth G-equivariant flow Φ_t on M with connected compact hyperbolic attractor $\Lambda \subset M_\Pi$. with symmetry group $G_\Lambda = H$.*

REMARKS 9.6.7. (1) Both theorems give the existence of attractors consisting entirely of points of trivial isotropy. Later we shall give examples of hyperbolic attractors which contain points of nontrivial isotropy.
(2) If we allow the action of G on M to have reflections, then the theorems continue to hold if we assume there exists a connected component M_0 of M_Π which is H-invariant. Weaker theorems hold if we drop this assumption [**64**].

Our construction of hyperbolic symmetric attractors is based on an equivariant version of the branched 1-manifolds studied by Williams in his work on solenoidal attractors [**178**]. We call these smooth branched one-dimensional G-manifolds *smooth graphs*. Following Williams, we consider a class of expanding smooth G-equivariant maps $f : \Gamma \to \Gamma$ of a smooth graph Γ. For these maps we are able to realize the inverse limit $\mathcal{S}(\Gamma, f) = \varprojlim f : \Gamma \to \Gamma$ as a hyperbolic attractor of an equivariant diffeomorphism of an appropriately chosen G-manifold or representation.

9.6.1. Finite graphs. We start by recalling a few elementary definitions and facts about finite graphs (we refer to [**22**] for more details).

A *finite graph* Γ consists of a finite set of vertices and a finite set of edges that join pairs of vertices. If an edge lies in Γ, then the ends of the edge define vertices which must also lie in Γ. A subset $J \subset \Gamma$ is a *subgraph* if J is a graph and the vertices and edges of J are vertices and edges of Γ. A *path* in Γ is a finite sequence of (oriented) edges such that the initial vertex of each edge is the terminal vertex of the preceding edge. A graph is *connected* if there is a path between any two vertices. If each pair of vertices is joined by an edge, the graph is *completely connected*.

Each edge $E \in \Gamma$ may be given the structure of a metric space isometric to the unit interval. The length of a path in Γ is defined in the obvious way and we define the distance between two points in Γ as the length of the shortest path joining the points. This construction defines a metric ρ on each connected component of Γ. If we let D be the maximum diameter of the components of Γ and define the distance between distinct components to be $D+1$, then this metric gives Γ the structure of a compact metric space.

It is sometimes useful to regard edges as oriented. That is, each edge will be an edge from one vertex (the initial vertex) to another vertex (the terminal vertex). Given a finite set $\mathcal{V}$ of vertices, *the completely connected oriented graph* on $\mathcal{V}$ is the graph which has vertices $\mathcal{V}$ and exactly two edges, of opposite orientations, joining each distinct pair of vertices in $\mathcal{V}$. (If $\mathcal{V}$ consists of a single vertex the completely connected oriented graph is defined to be the graph consisting of one vertex and one edge.) More generally, we say a graph is a completely connected oriented graph if it contains the completely connected oriented graph on its vertex set.

Let v be a vertex and $B(v, \frac{1}{4}) \subset \Gamma$ be the ρ-ball radius one quarter, centre v. We define the *degree* of v to the the number of connected components of $B(v, \frac{1}{4}) \setminus \{v\}$ (note that a loop at v contributes 2 to the degree of v). A connected graph is *Eulerian* if it has at least one edge and each vertex has even degree. It is well-known that Eulerian graphs are characterized by the property that there exists a *Eulerian circuit*. That is, there is a (closed) path tracing through each edge exactly once and with the same initial and terminal point. More generally, there exists a *Eulerian path* tracing through each edge of the graph exactly once if and only if the graph is connected and there are either two vertices or any vertices of odd degree lie at the end points of the path. In particular, if Γ is Eulerian, then the graph defined by removing an edge E from Γ will have a Eulerian path provided that removing the edge does not disconnect Γ (that can only happen if Γ has two vertices and one edge).

EXAMPLE 9.6.8. The graph shown in figure 6(a) is the completely connected oriented graph on 3 vertices. Like all completely connected oriented graphs, it is Eulerian.

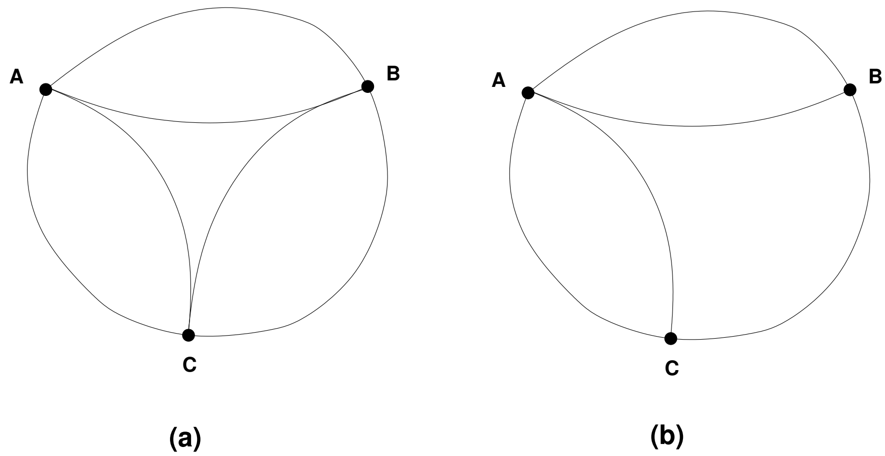


FIGURE 6. Eulerian circuits and paths

The graph shown in figure 6(b) has no Eulerian circuits. It has a Eulerian path joining the vertices B and C (but no Eulerian path joining any other pair of vertices).

EXERCISE 9.6.9. Let $E \in \Gamma$ be an edge and $\Gamma \setminus E$ be the graph defined as the closure of $\Gamma \setminus E$. Show that if Γ is a completely connected oriented graph, then in order to disconnect Γ it is necessary to remove at least eight edges. In particular, if Γ has fewer than four vertices, Γ cannot be disconnected by removing edges.

9.6.2. Branched 1-manifolds. Our definition of smooth graph is an adaptation of Williams [**178**] definition of branched 1-manifold. For our purposes, it is sufficient to consider compact Hausdorff branched 1-manifolds without boundary. We start by defining coordinate neighbourhoods. Fix a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{aligned} \phi(x) &= 0,\ x \leq 0, \\ &> 0,\ x > 0. \end{aligned}$$

(For example, take $\phi(x) = \exp(-1/x),\ x > 0$.) Given integers $p, q \geq 1$, we define the local branched 1-manifold $Y_{p,q} \subset \mathbb{R}^2$ by

$$Y_{p,q} = \{(x,y) \in \mathbb{R}^2 \mid y = i\phi(x),\ \text{or}\ y = j\phi(-x), i = 0,\dots,p-1, j = 0,\dots,q-1\}.$$

Note that $Y_{1,1}$ is the x-axis – and so is non-singular. If $p + q > 2$, then $Y_{p,q}$ has a *branch point* at the origin of $\mathbb{R}^2$ – see figure 7 for $Y_{2,2}$.

A branched 1-manifold Σ will consist of a (compact) Hausdorff topological space together with a differential atlas of coordinate neighbourhoods each of which is diffeomorphic to some $Y_{p,q}$. This definition gives Σ a smooth structure. We may define smooth endomorphisms of Σ in terms of the smooth structure.

We say that a point $z \in \Sigma$ is a point of type (p,q) if there is a chart $\psi : U \to Y_{p,q}$ where $\Psi(z) = 0$ and $p \geq q$. If $p + q > 2$, we say z is a branch point. We let

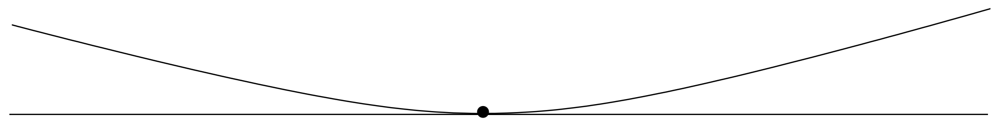


FIGURE 7. Local models for smooth graphs: $Y_{2,2}$

$B = B(\Sigma)$ denote the set of branch points. Since Σ is assumed compact, B is finite. A point of type (p, q) with $p + q = 2$ (equivalently $p = q = 1$) is a regular point. The set of regular points of Σ has the structure of a 1-manifold which is non-compact (unless $B(\Sigma) = \emptyset$).

Since each point in Σ has a neighbourhood which smoothly embeds in $\mathbb{R}^2$, it is easy to construct a smooth embedding of Σ in $\mathbb{R}^n$, $n \geq 3$ (embed neighbourhoods of branch points first, then connect the edges). If Σ is embedded in $\mathbb{R}^n$, we may define the tangent bundle $T\Sigma \subset T\mathbb{R}^n$ in the obvious way.

REMARKS 9.6.10. (1) For the situations we are interested in, it will always be possible to define a smooth immersion $f : [0, 1] \to \Sigma \subset \mathbb{R}^n$ such that $f([0, 1]) = \Sigma$ (typically f will be a smooth Eulerian circuit). We could use this as the basis for an alternative (but equivalent) approach to defining a smooth structure on Σ.
(2) For our applications, we will always be able to assume that branched 1-manifolds locally embed in $\mathbb{R}^2$. However, if we want to allow for smooth non-free group actions on branched 1-manifolds, the local models will typically only embed in representations of degree greater than 2. A simple example may be constructed by taking the representation $(\mathbb{R}^3, \mathbb{Z}_3)$ with one-dimensional fixed point space L and considering a $\mathbb{Z}_3$-invariant branched 1-manifold Σ which is tangent at the origin to L. There will be no local smooth embedding of a neighbourhood of $0 \in \Sigma$ in $\mathbb{R}^2$.

For our purposes it is easiest to think of a branched 1-manifold as embedded in $\mathbb{R}^n$ and a smooth map of a branched manifold as being the restriction of a smooth map defined on a neighbourhood of the embedded branched manifold. This is the approach we develop and use here.

9.6.3. Neighbourhoods of branched 1-manifolds. Let the branched 1-manifold Σ be smoothly embedded in $\mathbb{R}^n$. For our purposes, we need to be able to construct neighbourhoods of Σ which have smooth boundary and a smooth foliation transverse to Σ.

PROPOSITION 9.6.11. *Let Σ be a (compact) connected branched 1-manifold which is smoothly embedded in $\mathbb{R}^n$. Let W be an open neighbourhood of Σ. Then there exists an open connected neighbourhood U of Σ such that*

(1) $\overline{U} \subset W$.
(2) *∂U is smooth.*
(3) *There exists a smooth foliation $\mathcal{F} = \{\mathcal{F}_y \mid y \in \Sigma\}$ of U such that*

(a) *Each leaf $\mathcal{F}_y$ is an embedded $(n-1)$-dimensional disk.*
(b) $T_x\mathcal{F}_y \perp T_x\Sigma$, *for all* $x \in \Sigma \cap \mathcal{F}_y$, $y \in \Sigma$.
(c) *If a leaf* $\mathcal{L} \in \mathcal{F}$ *meets* Σ *at* $\{y_1, \ldots, y_k\}$, *then* $\mathcal{F}_{y_i} = \mathcal{L}$, $1 \le i \le k$.
(d) *If* $W^\star \subset W$ *is an open neighbourhood of* $\overline{U}$, *we may choose an open neighbourhood* $U^\star \subset W^\star$ *of* $\overline{U}$ *and a foliation* $\mathcal{F}^\star$ *of* $U^\star$ *by* $(n-1)$-*disks such that every leaf* $\mathcal{L}^\star \in \mathcal{F}^\star$ *meets* U *is a finite union of* $\mathcal{F}$ *leaves.*

In order to prove proposition 9.6.11, we start by looking at the local problem of constructing foliated neighbourhoods in $\mathbb{R}^2$ of $Y_{p,q}$.

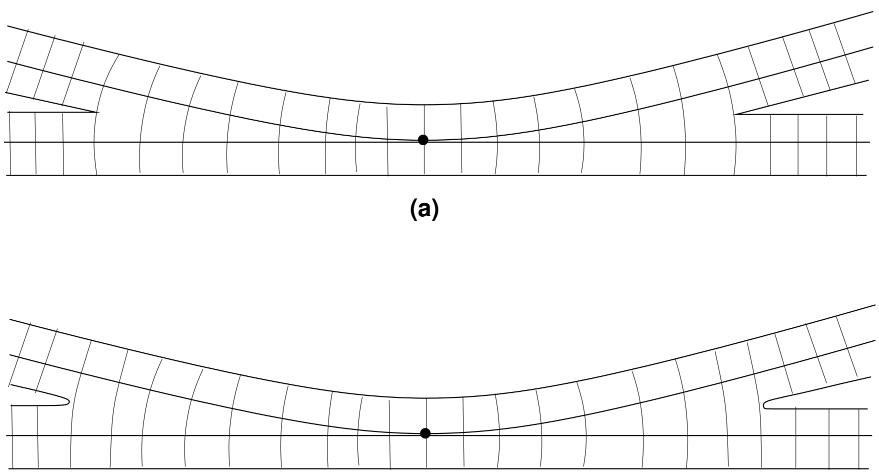
Let d denote the Euclidean distance on $\mathbb{R}^n$. Take the standard representation of $Y_{1,1}$ in $\mathbb{R}^2$ as the x-axis. Let $r > 0$. If we define

$$U_r(Y_{1,1}) = \{X \in \mathbb{R}^2 \mid d(X, Y_{1,1}) < r\},$$

then U_r is an open neighbourhood of $Y_{1,1}$ in $\mathbb{R}^2$ and we have a smooth foliation $\mathcal{F}(Y_{1,1}, r)$ of U_r defined by the family of intervals $I_x = \{(x, y) \mid |y| < r\}$, $x \in Y_{1,1}$. Obviously ∂U_r is smooth. If $p + q > 1$, we define

$$U_r = U_r(Y_{p,q}) = \{X \in \mathbb{R}^2 \mid d(X, Y_{p,q}) < r\}.$$

In this case, ∂U_r will have exactly $p + q - 2$ corners – see figure 8(a) for the case $p = q = 2$. All other points of ∂U_r are smooth.



(a)

(b)

FIGURE 8. The foliation of U_r and rounding corners

We take a smooth foliation of U_r by embedded open intervals I_y, $y \in Y_{p,q}$, such that I_y is everywhere perpendicular to $Y_{p,q}$. Note that different y lead to the same leaf if I_y meets $Y_{p,q}$ in more than one point. See figure 8(a). (We can make an explicit construction of the leaves by defining a smooth nonzero vector field on U_r which is everywhere orthogonal to $Y_{p,q}$.) We smooth the corners so that there is precisely one leaf meeting each smoothed corner tangentially (see figure 8(b)). In this way we obtain a foliated neighbourhood $U^\star \subset W$ of $Y_{p,q}$ with smooth boundary. We may require that U_r and $U^\star$ coincide outside of a (preassigned) neighbourhood of the corner points. Shrinking $U^\star$ to U, we easily satisfy the conditions of the proposition.

We may extend this construction to allow for $Y_{p,q} \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^n$. In this case, the leaves of the neighbourhood $\overline{U}$ will be $n-1$-disks except for the at most $p+q-2$-leaves which are tangential to rounded corner points. These singular leaves will correspond to closed disks touching at one point. Of course, for the open neighbourhood, we remove the points where the disks touch and recover a foliation by open $(n-1)$-disks. See figure 9 for the case $n = 3$ and $Y_{2,2}$.

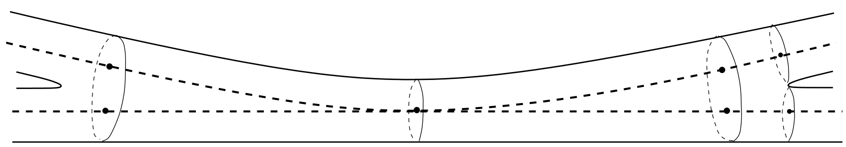


FIGURE 9. The foliation of a neighbourhood of $Y_{2,2}$ in dimension 3

It is straightforward to patch together the local foliations described above and so complete the proof of proposition 9.6.11.

9.6.4. Smooth graphs. Suppose that Γ is a finite connected graph with vertex set $\mathcal{V}$. Provided that the degree of each vertex is at least two, it is easy to give Γ the structure of a compact connected branched 1-manifold with branch set $B \subset \mathcal{V}$. Of course, there are many ways we can do this depending on how we arrange the edges at branch points. Conversely, every compact branched 1-manifold without boundary determines a unique connected graph with vertex set equal to the set of points of type not equal to $(1,1)$. Every vertex will have degree at least two.

We define a *smooth graph* to be a finite graph with designated branched 1-manifold structure. Implicit in our definition is the assumption that the degree of each vertex is at least two. Of course, we can remove this restriction if we allow for branched 1-manifolds with boundary (as was done in [**64**]).

If Γ is a smooth graph then every branch point of Γ is a vertex. Conversely, every vertex of Γ, except for vertices of type $(1,1)$, will be a branch point.

9.6.5. Group actions on graphs.

DEFINITION 9.6.12 (cf [**7, 64**]). Let H be a finite group. A graph Γ is an *H-graph* if

(1) H maps vertices to vertices, edges to edges.
(2) H acts freely on the set of edges.

If Γ is a smooth graph, then Γ is a smooth *H-graph* if Γ is an H-graph and H acts smoothly on Γ.

EXAMPLE 9.6.13. An important example of an H-graph is given by the *complete H-graph* $\Gamma(H)$ introduced in [**7**]. The graph $\Gamma(H)$ is defined to be the completely connected oriented graph with vertex set H. The left action of H on the vertex set H induces (and uniquely determines) an orientation preserving orientation preserving action of H on $\Gamma(H)$. If $E_{\gamma,\tau}$ denotes the edge joining γ to τ, $\gamma \neq \tau \in H$, then $hE_{\gamma,\tau}$ will be the edge $E_{h\gamma,h\tau}$. If we define $J = \{E_{e,h} \mid h \in H, h \neq e\}$, then J will be a (minimal) subgraph of $\Gamma(H)$ with H-orbit equal to $\Gamma(H)$. In the sequel, we refer to J as a *fundamental subgraph* of $\Gamma(H)$.

REMARK 9.6.14. If Γ is a smooth H-graph, then every point in the H-orbit of a vertex has the same degree and type.

DEFINITION 9.6.15. A smooth graph is *balanced* if the graph is connected and every vertex is of type (p,p), for some $p \geq 1$.

LEMMA 9.6.16. *The H-graph $\Gamma(H)$ may be given the structure of a smooth balanced H-graph.*

PROOF. It suffices to define coordinate neighbourhoods at each vertex of $\Gamma(H)$. Choose a fundamental subgraph J of $\Gamma(H)$ all of whose edges have the same initial vertex e. Denote the set of edges in J by $\{E_i = E_{e,h_i} \mid i = 1,\ldots,|H|\}$. Every edge $E \in \Gamma(H)$ may be written $E = hE_i$, for unique $i \in \{1,\ldots,|H|\}$ and $h \in H$. We require that the edge $E = hE_i$ correspond to the branch $(i-1)\phi$ in the coordinate neighbourhood at h and to the branch $(i-1)\psi$ in the coordinate neighbourhood at hh_i. It is easy to see that this construction gives $\Gamma(H)$ the structure of a smooth H-graph and that every vertex is of type $(|H|-1,|H|-1)$. □

REMARKS 9.6.17. (1) The smooth structure on $\Gamma(H)$ is not unique – it depends both on the choice of subgraph J and the ordering of the elements in J.
(2) In the sequel, we always regard $\Gamma(H)$ as having one of the smooth graph structures given by (the proof of) lemma 9.6.16.
(3) If $\Gamma^\star$ is an H-graph containing $\Gamma(H)$ with the same vertex set as $\Gamma(H)$, then we may give $\Gamma^\star$ the structure of a smooth balanced H-graph. The proof is the same as that of lemma 9.6.16.
(4) In some situations, it is possible to define smooth balanced structures on H-graphs when H acts transitively on the vertex set. However, this needs some care as if H permutes edges at a vertex, it may not be possible to prove that H is a smooth action (that is, the restriction of a smooth H-action defined on the space in which the H-graph is embedded).

EXAMPLE 9.6.18. In figure 10(a), we show those edges in a smooth balanced structure on $\Gamma(\mathbf{D}_2)$ which have the identity element as an end point. (The non-identity elements $a, b, c \in \Gamma(\mathbf{D}_2)$ satisfy $a^2 = b^2 = c^2 = e$, $ab = c$, $bc = a$, $ca = b$.)

In figure 10(b), we show a smooth structure on the oriented graph $\Gamma(\mathbb{Z}_3)$. Note

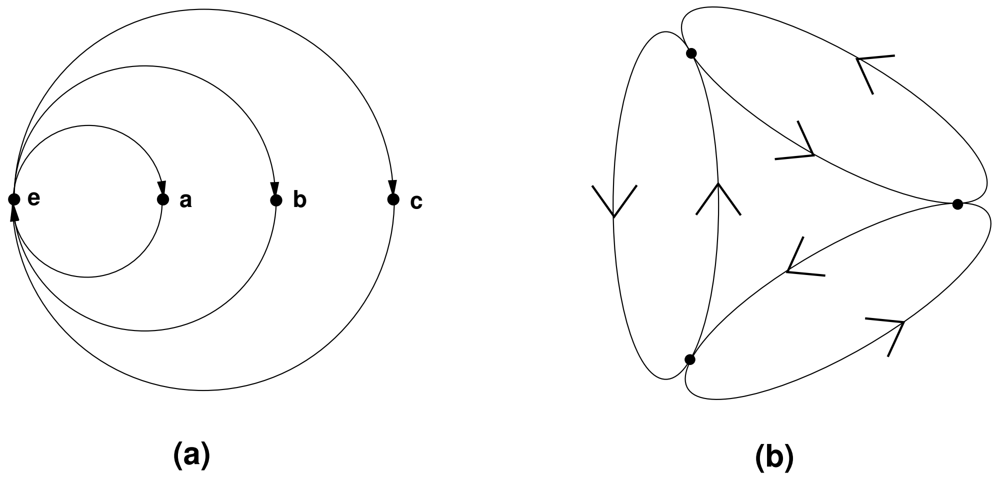


FIGURE 10. Smooth structures on graphs

that it is not possible to isometrically embed the graph in $(\mathbb{R}^2, \mathbb{Z}_3)$ though we can find an isometric embedding of $\Gamma(\mathbb{Z}_3)$ as a $\mathbb{Z}_3$-invariant subset of $(\mathbb{R}^3, \mathbb{Z}_3)$.

9.6.6. Twisted products and embeddings. Let H be a subgroup of the finite group G and Γ be an H-graph. If we let G act trivially on Γ and by left translation on G, then $G \times \Gamma$ has the structure of a G-graph. It follows that the twisted product $G \times_H \Gamma$ inherits the structure of a G-graph from that on $G \times \Gamma$. We omit the proof of the following lemma which summarizes the main properties of this construction.

LEMMA 9.6.19. *(Notation as above.)*

(1) *The mapping $i_\Gamma : \Gamma \to G \times_H \Gamma$, $x \mapsto [e, x]$, H-equivariantly embeds Γ as a subgraph of $G \times_H \Gamma$.*
(2) *If Γ is connected, then $G \times_H \Gamma$ has $|G|/|H|$ connected components.*
(3) *If J is a fundamental subgraph for Γ, then J is a fundamental subgraph for the G-graph $G \times_H \Gamma$.*
(4) *If Γ is a smooth H-graph, then $G \times_H \Gamma$ has the natural structure of a smooth G-graph such that i_Γ is a smooth embedding.*

Next we consider the problem of embedding a smooth H-graph in a G-manifold or representation. Following [**64**], we give a general embedding result. However, as our main interest will be in embedding the completely connected oriented graph $\Gamma(H)$, we could just as well have given only a direct construction of $\Gamma(H)$ as a smoothly equivariantly embedded graph.

PROPOSITION 9.6.20. *Let H be a subgroup of the finite group G. Let Γ be a smooth H-graph and suppose that H acts freely on Γ. Suppose that M is a* *connected G-manifold of dimension at least three and that the principal isotropy group for the action of G on M is trivial. If there exists a connected H-invariant component M_0 of M_Π, then we may construct a smooth H-equivariant embedding $\xi : \Gamma \to M_0 \subset M_\Pi$. Moreover, we may require that ξ extends to a smooth G-equivariant embedding of $G \times_H \Gamma$ in M_Π.*

PROOF. We start by choosing an H-equivariant embedding ξ of the vertex set $\mathcal{V}$ of Γ into M_0. For this it suffices to choose $|\mathcal{V}|/|H|$ points in M_0 which map to distinct points of the orbit space M_Π/H. Extend ξ G-equivariantly to the vertex set $G \times_H \mathcal{V}$ of $G \times_H \Gamma$, Restricting ξ to $\mathcal{V} \subset \Gamma$, extend ξ H-equivariantly to an H-invariant open neighbourhood U of $\mathcal{V}$ in Γ. We assume that U is chosen sufficiently small so that (a) each vertex in contained in one connected component of U, (b) distinct components of U are mapped to non-intersecting subsets of M_0 and (c) $\xi|U$ extends G-equivariantly to a smooth G-equivariant embedding of $GU = G \times_H U$ in M_Π. In particular, $\xi|U$ will be an H-equivariant smooth embedding. Extend $\xi|U$ smoothly to the edges in a fundamental graph for Γ so that $\xi(J \setminus U) \cap GU = \emptyset$ and $\xi(J) \subset M_0$. Extend by G-equivariance to all $G \times_H \Gamma$. We may do this so that the resulting map $\xi : G \times_H \Gamma \to M_\Pi$ is a smooth G-equivariant immersion and $\xi(G \times_H \setminus GU) \cap GU = \emptyset$. Finally, since $\dim(M) \geq 3$, we may perturb ξ to a smooth G-equivariant embedding. For this we consider the composition $q \circ \xi : G \times_H \Gamma \to M_\Pi/G$. Perturb $q \circ \xi$ outside of a closed neighbourhood $V \subset q(U)$ so that $q \circ \xi$ is injective and then lift back to obtain a G-equivariant embedding $\xi' : G \times_H \Gamma \to M_\Pi$ which will be equal to ξ on $q^{-1}(V)$. $\square$

REMARKS 9.6.21. (1) In general, we cannot embed smooth H-graphs in two dimensional manifolds. For example, if $m \geq 2$, there is no embedding of the completely connected oriented graph $\Gamma(\mathbf{D}_m)$ in the standard representation of $\mathbf{D}_m$ on $\mathbb{R}^2$. On the other hand, we can construct smooth graphs on $\mathbb{Z}_m$, $m \geq 2$, which embed in the standard representation of $\mathbb{Z}_m$ on $\mathbb{R}^2$ (for a not completely trivial example, see figure 10(b) in case $m = 3$.)
(2) If we are only interested in realizing the completely connected oriented graph $\Gamma(H)$ as a smoothly embedded graph in M_Π, we can give a very simple proof of proposition 9.6.6. Consider the smoothly embedded graph $K \subset M_0/H$ which has one vertex of type $(|H|-1, |H|-1)$ and $|H|-1$ edges. Lift K to M_0 as a connected H-graph. We leave it to the reader to fill in the routine details. Of course, we may do all this so that $\Gamma(H)$ equivariantly extends to the smoothly embedded G-graph $G \times_H \Sigma$. In this case we can even require that the embedding of $\Gamma(H)$ in M is isometric with respect to the graph metric on $\Gamma(H)$ and an H-invariant Riemannian structure on M (for this it is essential that $\dim(M) > 2$).

PROPOSITION 9.6.22. *Let H be a subgroup of the finite group G. Suppose that M is a connected Riemannian G-manifold of dimension at least three and that the principal isotropy group for the action of G on M is trivial. Suppose*

$\xi : \Gamma \to M_\Pi$ is a smooth H-equivariant embedding of the connected H-graph Γ which extends G-equivariantly to an embedding of $G \times_H \Gamma$ in M_Π There exists an open H-invariant connected neighbourhood U of $\Sigma = \xi(\Gamma)$ such that

(1) *$gU \cap U \neq \emptyset$ if and only if $g \in H$.*
(2) *∂U is smooth.*
(3) *There exists a smooth H-invariant foliation $\mathcal{F} = \{\mathcal{F}_y \mid y \in \Sigma\}$ of U such that*
   (a) *Each leaf $\mathcal{F}_y$ is an embedded $(n-1)$-dimensional disk.*
   (b) *$T_x\mathcal{F}_y \perp T_x\Sigma$, at all points $x \in \Sigma \cap \mathcal{F}_y$, all $y \in \Sigma$.*
   (c) *If a leaf $\mathcal{L} \in \mathcal{F}$ meets Σ at $\{y_1, \ldots, y_k\}$, then $\mathcal{F}_{y_i} = \mathcal{L}$, $1 \leq i \leq k$.*

PROOF. The proof is an equivariant version of that of proposition 9.6.11. □

9.6.7. Smooth Eulerian paths.

PROPOSITION 9.6.23. *Let Γ be a smooth connected graph containing at least one edge.*

(1) *There exists a smooth Eulerian circuit for Γ if and only if Γ is balanced.*
(2) *If $E \in \Gamma$ is an edge with distinct end points v_1, v_2, then there exists a smooth Eulerian path joining v_0 to v_1 in $\Gamma \setminus E$ if and only if Γ is balanced.*

PROOF. Since (1) is obviously equivalent to (2), it suffices to prove (1). First of all note that if there is a smooth Eulerian circuit for Γ, then Γ must be balanced: any smooth Eulerian circuit approaching a vertex from one side must exit from the other side in order to be smooth. Our proof of sufficiency goes by induction on the number of edges. If there is one edge, then the associated vertex is of type $(1,1)$, the edge defines a smooth loop and so the Eulerian circuit is given by the edge. Suppose the result is proved for all smooth balanced graphs with fewer than n edges. Let Γ be a graph with n-edges. If Γ contains a vertex of type $(1,1)$, we may remove the vertex to obtain a smooth balanced graph $\Gamma^\star$ with $(n-1)$ edges. Applying the inductive hypothesis we see that $\Gamma^\star$, and hence Γ, has a smooth Eulerian circuit. If Γ has just one vertex, then the vertex is of type (n,n) and it is easy to construct a smooth Eulerian circuit for Γ. If Γ contains an edge E such that E has common initial and terminal vertex v and the vertex v is of type $(1,1)$ for the graph $E \cup \{v\}$, then we may remove the edge E, construct a smooth Eulerian circuit for $\Gamma \setminus E$ and then extend to a smooth Eulerian circuit of Γ. Finally, if none of these special cases hold, we can find edges $E \neq F$ in Γ which share a common vertex v which is of type $(1,1)$ for $E \cup F$. We perturb the smooth curve $E \cup F$ off the vertex v, remove the vertex v from $E \cup F$, and thereby obtain a new smooth graph Γ' with $(n-1)$ edges. If Γ' is connected, we apply the inductive hypothesis to obtain a smooth Eulerian circuit of Γ' which determines a smooth Eulerian circuit of Γ in the obvious way. If Γ' is not connected, then the components of Γ' are both balanced and have fewer than n edges. Applying the inductive hypothesis, we obtain smooth Eulerian circuits for each component. These may be combined to define the required smooth Eulerian circuit of Γ. □

REMARK 9.6.24. Suppose that $\Gamma \subset \mathbb{R}^n$ is a smooth balanced connected graph containing at least one edge. By proposition 9.6.23, there exists a smooth immersion $\xi : S^1 \to \Gamma$ which is 1:1 outside the finite subset $\xi^{-1}(B(\Gamma))$.

9.6.8. Condition (W). Suppose that $\Gamma \subset \mathbb{R}^n$ is a smooth balanced graph with vertex set $\mathcal{V}$. Fix a Riemannian metric on $T\Gamma \subset T\mathbb{R}^n$ (for example, the Riemannian metric induced by the Euclidean metric on $\mathbb{R}^n$). We say that a smooth map $f : \Gamma \to \Gamma$ satisfies condition (W) if

(W1) f is an expanding immersion[4] and $f(\mathcal{V}) \subset \mathcal{V}$;
(W2) there exists $p \in \mathbb{N}$ such that $f^p(E) = \Gamma$ for every edge $E \in \Gamma$;
(W3) every point of Γ has a neighbourhood N such that $f(N)$ is an arc.

REMARKS 9.6.25. (1) Conditions (W1–3) are modelled on Williams' Axioms 1–3 [**178**, §3]. While we impose different conditions on vertices, it may be shown that the objects we construct are all realizable (up to conjugacy) within the framework developed in [**178**].
(2) Since $f(\mathcal{V}) \subset \mathcal{V}$, it follows that the f-image of an edge is always a finite union of edges. If we insist that all vertices are of type (p, p), with $p > 1$, then an expanding immersion of Γ automatically maps vertices to vertices.
(3) It follows easily from (W1,W2) that if N is any non-empty open subset of Γ, then there exists $p = p(N)$ such that $f^p(N) = \Gamma$.

PROPOSITION 9.6.26. *Let $\Gamma \subset \mathbb{R}^n$ be a smooth balanced graph with vertex set $\mathcal{V}$ and $f : \Gamma \to \Gamma$ be a smooth map satisfying (W1) and (W2).*

(1) *Periodic points of f are dense in Γ.*
(2) *The map f is topologically mixing (and therefore transitive).*

PROOF. Let $I \subset \Gamma \setminus \mathcal{V}$ be a closed arc. It follows by remarks 9.6.25(3) that there exists $p \in \mathbb{N}$ such that $f^p(I) \supset I$. Hence I contains a point of period p for f. It follows that $\mathrm{Per}(f)$ is dense in Γ. The second statement is immediate by remarks 9.6.25(3). □

EXAMPLE 9.6.27. Let $\mathbb{Z}_2$ act on $\mathbb{R}^2$ as $\pm I$. In figure 11, we show a smooth $\mathbb{Z}_2$-graph Γ which is $\mathbb{Z}_2$-equivariantly embedded in $\mathbb{R}^2$. The graph Γ has two vertices $\pm\alpha$ of type $(2, 2)$ and four oriented edges which we have labelled a_1, a_2, b_1, b_2. Note that the $\mathbb{Z}_2$ maps a_1 to a_2 and b_1 to b_2 preserving the orientations. We define a smooth $\mathbb{Z}_2$-equivariant map $f : \Gamma \to \Gamma$ satisfying (W1–3) by the rules

$$a_1 \mapsto a_1 b_2 a_2^{-1}, \qquad a_2 \mapsto a_2 b_1 a_1^{-1},$$
$$b_1 \mapsto b_1 a_2 b_2^{-1}, \qquad b_2 \mapsto b_2 a_1 b_1^{-1}.$$

The edge rules are to be read 'left to right'. Thus $a_1 \mapsto a_1 b_2 a_2^{-1}$ is to be interpreted as the map of the edge a_1 with image having initial point α and traversing the edges a_1, b_2 and a_2 (with reverse orientation). The terminal point of $f(a_1)$ is therefore $-\alpha$. We require that the map f is expanding (in this case

[4]There exists $\lambda > 1$ such that $\|Tf(v)\| \geq \lambda \|v\|$ for all $v \in T\Gamma$.

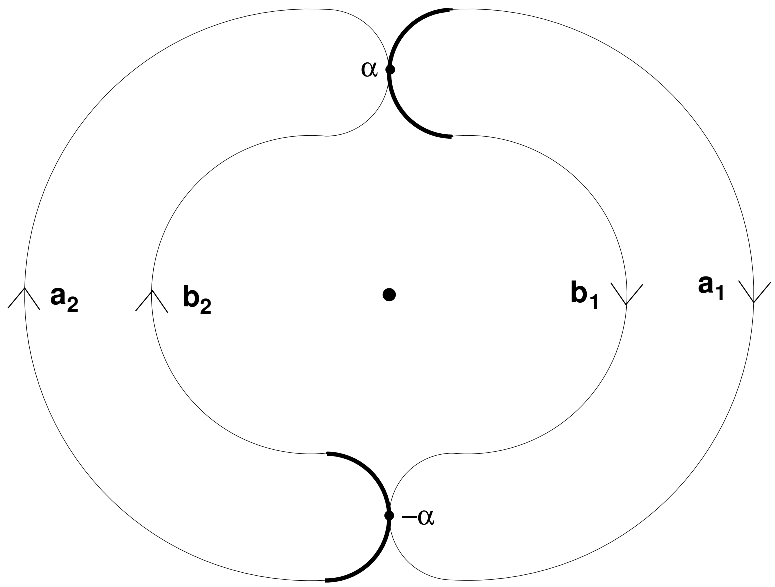


FIGURE 11. Smooth map of a $\mathbb{Z}_2$-graph

by a factor of approximately three). Clearly $f : \Gamma \to \Gamma$ satisfies (W1,2). To see that (W3) holds observe that neighbourhoods of $\pm\alpha$ map onto the highlighted arcs shown in figure 11.

LEMMA 9.6.28. *Let H be a finite group. There exists a smooth H-equivariant map $f : \Gamma(H) \to \Gamma(H)$ satisfying condition (W).*

In general, if we assume the conditions of proposition 9.6.22 hold (for a smooth map f of a smooth H-graph Γ satisfying condition (W)), we may require that f extends H-equivariantly to the foliated neighbourhood U of Γ and f collapses each leaf $\mathcal{F}_y$ of $\mathcal{F}$ to $f(y) \in \Gamma$.

PROOF. We prove in case $|H| \geq 3$ (the case $|H| = 2$, is trivial – but see also example 9.6.27). Fix a fundamental subgraph J for $\Gamma(H)$ consisting of edges beginning or ending at $e \in H$ so that the edges in J do not all lie on the same side of e. We choose edges $E_{1,\eta}, E_{\tau,1} \notin J$ that together define a smooth arc A_e through e. Since $|H| \geq 3$, we can require that $\tau \neq \eta^{-1}$. Setting $A_h = hA_e$, $h \in H$, we define a distinguished arc through each vertex h of $\Gamma(H)$. Since $\tau \neq \eta^{-1}$, no pair of distinguished arcs has an edge in common. Suppose that $E \in J$ joins e to σ. There is a unique choice of edges $I \subset A_e$, $F \in A_\sigma$ such that I, E lie on the same side of e and F, E lie on the same side of σ. By proposition 9.6.23, there exists a smooth Eulerian circuit γ_I which starts at e, traverses I and ends at e approaching from the side opposite to I. Similarly there is a smooth Eulerian circuit γ_F which starts at σ and ends by traversing the edge F. It follows that we may construct a smooth Eulerian path $\rho_E : E \to \Gamma(H)$ by combining γ_I, the path E and γ_F – see figure 12.

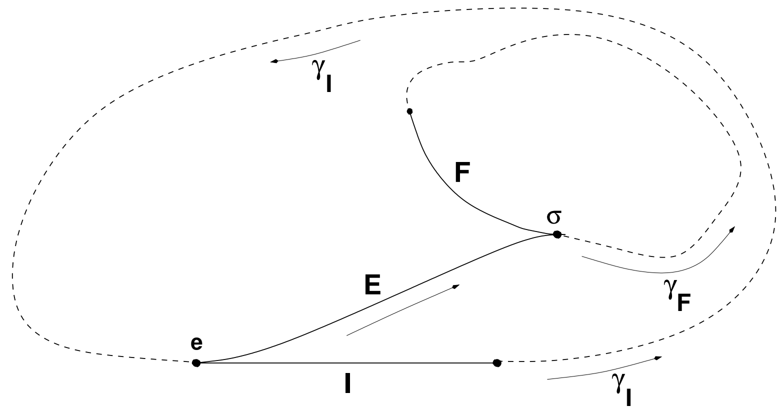


FIGURE 12. The smooth path ρ_E

In this way we may construct an expanding immersion $\rho_E : E \to \Gamma(H)$ for all $E \in J$. We may also require that the derivative of ρ_E near the end points of E is independent of $E \in J$. Equivariantly extending $\rho : J \to \Gamma(H)$ to $\Gamma(H)$, we obtain the required smooth H-equivariant map $f : \Gamma(H) \to \Gamma(H)$ satisfying condition (W).

The map f maps branch points to branch points. Hence, if we regard $\Gamma(H)$ as H-equivariantly embedded in M_Π, we can H-equivariantly deform f outside of a neighbourhood of the set of branch points so that f maps each leaf $\mathcal{F}_y$ to $f(y)$ – note that this is only an issue for those leaves that meet more than one edge of $\Gamma(H)$. The proof in the general case of maps of a smooth H-graph satisfying condition (W) exactly the same. $\square$

REMARK 9.6.29. This construction of f in the lemma is different from that used in the proof of lemma 3.18 [**64**]. Our construction has the advantage that the initial edge traversed by $f|E$ lies on the same side of the initial vertex as E. Similarly for the final vertex. Note that this property also holds for example 9.6.27.

DEFINITION 9.6.30. Let $f : \Gamma \to \Gamma$ be a smooth map of the smooth connected graph Γ and suppose f satisfies condition (W). We define the *solenoid* $\mathcal{S} = \mathcal{S}(\Gamma, f)$ to be the inverse limit of $f : \Gamma \to \Gamma$, together with the associated shift map $\hat{f} : \mathcal{S} \to \mathcal{S}$.

REMARKS 9.6.31. (1) If $f : \Gamma \to \Gamma$ is a smooth H-equivariant map of the balanced H-graph Γ, then the *solenoid* $\mathcal{S}$ inherits the structure of an H-space with respect to which the shift map is H-equivariant (see lemma 9.5.2).
(2) If $f : \Gamma \to \Gamma$ is a smooth map of the smooth connected graph Γ and f satisfies condition (W), then $\hat{f} : \mathcal{S} \to \mathcal{S}$ is (a) transitive, (b) topologically mixing and (c) $\mathrm{Per}(\hat{f})$ is dense in $\mathcal{S}$. These results follow from propositions 9.6.26 and 9.5.8.

9.6.9. Symmetric hyperbolic attractors – simply connected case. In this section we give our first results on the construction of connected hyperbolic attractors with specified symmetry group $H \subset G$. The constructions are equivariant versions of those originally made by Williams [**178**]. There are two steps. In step 1 we construct an H-equivariant embedding $f : U \to U$ where U is a neighbourhood of a smooth balanced H-graph $\Gamma \subset M$, M is a G-manifold, $\dim(M) \geq 4$, and f is a perturbation of a smooth H-equivariant map ϕ of the graph satisfying condition (W). All this is done in a way that preserves a foliation of U transverse to the graph. We define the attractor to be $\Lambda = \cap_{n \geq 0} f^n(U)$. Using standard methods, it is straightforward to verify that Λ has hyperbolic structure and $f|\Lambda$ is equivariantly conjugate to the shift map on the solenoid $\mathcal{S}(\Gamma, \phi)$. Only minor technicalities are introduced by the presence of the group action.

In the second step, we extend $f : U \to U$ to an G-equivariant diffeomorphism of M. We do this using the equivariant isotopy extension theorem. Here we must be careful as the orbit stratification of M can impose obstructions when we attempt to equivariantly isotop $f : U \to U$ to the identity map of U. More specifically, the principal isotropy stratum M_Π is always open and dense in M. If all the orbit strata M_τ, $\tau \neq \Pi$ are of codimension at least two, then M_Π is connected. However, even if M is simply connected (for example, a representation), M_Π may not be simply connected. As a simple example, if we let $\mathbb{Z}_2$ act on $\mathbb{R}^3$ by $(x, y, z) \mapsto (-x, -y, z)$, then the principal orbit stratum is $\mathbb{R}^3 \backslash$z-axis which is not simply connected. If $f : U \to U$ links with codimension 2 strata of M, it will generally not be possible to equivariantly isotop f to the identity map of U. However, if we assume that all orbit strata M_τ, $\tau \neq \Pi$ are of codimension at least three, then $f : U \to U$ cannot form links with $\cup_{\tau \neq \Pi} M_\tau$. This already suffices for applying the isotopy extension theorem if M is a G-representation. For general M, we need to work within a simply connected nonempty open H-invariant subset M_0 of M_Π.

THEOREM 9.6.32. *Let H be a subgroup of the finite group G and ϕ be a smooth H-equivariant map of the smooth balanced H-graph Γ which satisfies condition (W). Suppose that M is a G-manifold of dimension at least four and that the principal isotropy group for the action of G on M is trivial. Assume that there is a simply connected H-invariant connected component M_0 of M_Π. Then there exists $F \in \mathrm{Diff}_G(M)$ such that*

(1) *F has a connected H-invariant hyperbolic attractor $\Lambda \subset M_0$;*
(2) *the dynamics of $F|\Lambda$ are H-equivariantly conjugate to those of the shift map on the solenoid $\mathcal{S}(\Gamma, \phi)$;*
(3) *if $H \neq G$, $G \times_H \Lambda$ will be a disconnected hyperbolic attractor of F with $|G|/|H|$ components.*

PROOF. Fix a G-equivariant smooth embedding of $G \times_H \Gamma$ in GM_0 which maps Γ into M_0. Choose a smooth H-invariant foliation $\mathcal{F}$ of an H-invariant open connected neighbourhood U of $\Gamma \subset M_0$ with smooth boundary ∂U. We may require that if $g \in G$ then $g\bar{U} \cap \bar{U} \neq \emptyset$ if and only if $g \in H$. We may assume that $\mathcal{F}$ and ϕ are chosen so that ϕ extends to a smooth H-equivariant map $\phi : \bar{U} \to U$ sending $\mathcal{F}_y$ to $\phi(y)$ for all $y \in \Gamma$. In particular, if $z, z' \in \mathcal{F}_y \cap \Gamma$, then $\phi(z) = \phi(z')$. Let $J \subset \Gamma$ be a fundamental subgraph consisting of edges sharing a common vertex. Since $\dim(M_0) \geq 3$, we may perturb $\phi|J$ to $\tilde{\phi} : J \to U$ so that (a) $\tilde{\phi}$ is a smooth embedding and (b) if $z \in \mathcal{F}_y \cap J$, then $\tilde{\phi}(z) = \tilde{\phi}(y)$. After a additional perturbation satisfying (a,b), we may further require that the composite of the orbit map $M_\pi \to M_\Pi/G$ with $\tilde{\phi}$ is an embedding. Hence $\tilde{\phi}$ G-equivariantly extends to a smooth G-equivariant embedding $\tilde{\phi} : G\Gamma \to GU$ such that if $z \in \mathcal{F}_y$, then $\tilde{\phi}(z) = \tilde{\phi}(y)$. We now extend $\tilde{\phi}|\Gamma$ to a smooth H-equivariant embedding $\Phi : \bar{U} \to U$ which preserves the foliation $\mathcal{F}$ ($\Phi(\mathcal{F}_y) \subset \mathcal{F}_{\Phi(y)}$, all $y \in \Gamma$). This is easily done by mapping $\mathcal{F}_y$ isometrically into $\mathcal{F}_{\tilde{\phi}(y)}$ and then linearly contracting $\mathcal{F}_{\tilde{\phi}(y)}$ within $\mathcal{F}_{\tilde{\phi}(y)}$ to $\tilde{\phi}(y)$. We may assume the contraction μ, $0 < \mu \ll 1$ is uniform, independent of $y \in \Gamma$. Since $g\bar{U} \cap \bar{U} = \emptyset$ if $g \in G \setminus H$, we may G-equivariantly extend Φ to GU. Since $\dim(M_0) \geq 4$, we create no knots or links in the image of $\Phi(\Gamma)$ (or $\Phi(G\Gamma)$). Further, as M_0 is simply connected, $\Phi(\Gamma)$ (or $\Phi(G\Gamma)$) does not have any links with $M \setminus M_\Pi$. Hence there exists a smooth G-equivariant isotopy $h_t : \overline{GU} \to GM_0$ between Φ and $I_{\overline{GU}}$. Applying the isotopy extension theorem 3.6.1, we see that there exists $F \in \mathrm{Diff}_G(M)$ such that $F|GU = \Phi$.

Define $\Lambda = \cap_{n \geq 0} F^n(U)$. Clearly Λ is a compact H- and F-invariant subset of U. Using standard methods from the theory of hyperbolic sets, it is not hard to show that Λ has hyperbolic structure (for the method, see [**100**, chapter 17, section 1] and note that the foliation $\mathcal{F}$ already determines the contracting direction). We claim that $F|\Lambda$ is H-equivariantly topologically conjugate to the shift on the solenoid $\mathcal{S}(\Gamma, \phi)$.

Let $\mathbf{z} = (z_0, z_1, \ldots) \in \mathcal{S}(\Gamma, \phi)$. Define

$$\begin{aligned} h(\mathbf{z}) &= \cap_{n \geq 0} \overline{F^n(\mathcal{F}_{z_n})}, \\ &= \cap_{n \geq 0} F^n(\mathcal{F}_{z_n}). \end{aligned}$$

For $n \geq 0$, $\overline{F^n(\mathcal{F}_{z_n})}$ is a nonempty compact subset of $\mathcal{F}_{z_0}$. Since $\overline{F^n(\mathcal{F}_{z_n})} \supset \overline{F^{n+1}(\mathcal{F}_{z_{n+1}})}$, $n \geq 0$, $h(\mathbf{z}) \neq \emptyset$. Since F contracts leaves by the factor $\mu < 1$, the diameter of $\overline{F^n(\mathcal{F}_{z_n})} \to 0$ as $n \to \infty$ and so $h(\mathbf{z})$ consists of a single point. This construction defines a map $h : \mathcal{S}(\Gamma, \phi) \to \Lambda$. It is straightforward or trivial to verify that h is continuous, H-equivariant and that $h\hat{\phi} = Fh$. Since Λ, $\mathcal{S}(\Gamma, \phi)$ are compact metric spaces, it follows that in order to prove h is a homeomorphism it is enough to show that h is bijective. Suppose $\mathbf{z} \neq \mathbf{z}' \in \mathcal{S}$. There exists $j \geq 0$ such that $z_j \neq z'_j$. Hence $\mathcal{F}_{z_{j+1}} \cap \mathcal{F}_{z'_{j+1}} = \emptyset$ and so, since F is an embedding, we have $F^{j+1}(\mathcal{F}_{z_{j+1}}) \cap F^{j+1}(\mathcal{F}_{z'_{j+1}}) = \emptyset$. Hence $h(\mathbf{z}) \neq h(\mathbf{z}')$, proving that h is injective. Finally, let $Z \in \Lambda$. For $n \geq 0$, there exists a unique (connected) leaf $\mathcal{F}_n \in \mathcal{F}$ such that $F^{-n}(Z) \in \mathcal{F}_n$. Let $\mathcal{F}_{n+1} \cap \Gamma = \{y_1, \ldots, y_k\}$. By our construction of F, $F(y_1) = \ldots = F(y_k)$. Define $z_n = F(y_k) \in \Gamma$. In this way we define $H(Z) = (z_0, z_1, \ldots) \in \mathcal{S}$. Since $h(H(Z)) = Z$ for all $Z \in \Lambda$, h is surjective. □

9.6.10. Symmetric hyperbolic attractors – general case. We extend theorem 9.6.32 so as to allow for the case when M_Π contains no nonempty open simply connected H-invariant subsets. The issue will now be one of constructing a graph and graph map that will allow us to apply the equivariant isotopy extension theorem. A prototype of a suitable graph and graph map is given in example 9.6.27. Referring to figure 11, observe that $f(a_1)$ can be $\mathbb{Z}_2$-equivariantly isotoped back to a_1 (within $\mathbb{R}^2$), keeping the end points $\pm\alpha$ fixed. Similarly for the f-images of the remaining edges. This would not have been true if, for example, we had taken the $\mathbb{Z}_2$-graph consisting of the unit circle S^1, with vertex at $(1, 0)$. The map $f(z) = 3\theta$ is then $\mathbb{Z}_2$-equivariant and we may construct a 3-adic $\mathbb{Z}_2$-invariant solenoid in the solid torus $\mathbf{T} = S^1 \times D \subset \mathbb{C} \times \mathbb{R}$. An explicit $\mathbb{Z}_2$-equivariant embedding defining the solenoid may be given by

$$F(\theta, z) = (3\theta, z/8 + e^{\imath\theta}/4),\ (\theta, z) \in S^1 \times D,$$

where $D = \{z \mid |z| \leq 1\}$, we embed $S^1 \times D$ in $\mathbb{R}^3$ by $(\theta, x + \imath y) \mapsto ((1 + x)\cos(\theta), (1 + x)\sin(\theta), y)$, and the action of $\mathbb{Z}_2$ on $\mathbb{R}^3$ is given by $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3)$. Since F winds the core S^1 of $\mathbf{T}$ three times round the x_3-axis, it is not possible to $\mathbb{Z}_2$-equivariantly homotop $F : \mathbf{T} \to \mathbb{R}^3$ to the identity map without the image of $\mathbf{T}$ crossing the x_3-axis. But then the homotopy cannot be an equivariant isotopy as the isotropy of points on the x_3-axis is different from the isotropy of points on $\mathbf{T}$.

We start by constructing a 'doubled' version $\Gamma^\star(H)$ of the complete H-graph $\Gamma(H)$. As usual, we give $\Gamma(H)$ the structure of a smooth balanced H-graph and recall that $\Gamma(H)$ contains no edges with the same initial and final point. We define a new smooth balanced H-graph $\Gamma^\star(H)$ which has the same vertex set as $\Gamma(H)$ by assigning to each edge $E \in \Gamma(H)$, a new edge E' such that EE' is a smooth loop. Note that $\Gamma(H)$ is naturally a subgraph of $\Gamma^\star(H)$. We refer to $\Gamma^\star(H)$ as the *augmented* graph on G. If J is a fundamental subgraph of $\Gamma(H)$, all edges with common initial vertex e, then (with the obvious notation), $J^\star = J \cup J'$ is a fundamental subgraph of $\Gamma^\star(H)$.

Suppose that $P = E_1E_2 \ldots E_k$ is a path in $\Gamma^\star(H)$. We define $P' = E'_k \ldots E'_2E'_1$, where we adopt the convention that $E'' = E$. We define an equivalence relation $\sim$ on paths in $\Gamma^\star(H)$ by $P_1EE'P_2 \sim P_1P_2$.

LEMMA 9.6.33. *(Notation as above.) There exists a smooth H-equivariant map $\phi : \Gamma^\star(H) \to \Gamma^\star(H)$ such that*

(1) *ϕ satisfies condition W.*
(2) *ϕ fixes the vertex set of $\Gamma^\star(H)$.*
(3) *If $E \in \Gamma^\star(H)$ is an edge then $\phi(E) \sim E$.*

PROOF. We assume $|H| \geq 3$ (for $H = \mathbb{Z}_2$, see example 9.6.27 and the discussion above). We write $\Gamma^\star(H) = \Gamma(H) \cup \Gamma'(H)$. Following the proof of lemma 9.6.28, we choose a fundamental graph J for $\Gamma(H)$ and designated arc $A_e \subset \Gamma(H) \setminus J$ at the vertex $e \in H$. For each $E = E_{e,\sigma} \in J$, we let P denote the smooth path given by $P = IQ_1EQ_2F$. Here I is the edge of A_e, which is on the same side of e as E, F is the edge of $A_\sigma = \sigma A_e$ which is on the same side of σ as E and Q_1 and Q_2 are the paths γ_I, γ_F (see figure 12 and the proof of lemma 9.6.28). Observe that $II'EF'F$ is a smooth path and $II'EF'F \sim E$. Hence the smooth path $P^\star = IQ_1Q_1'I'EF'Q_2'Q_2F$ satisfies $P^\star \sim E$. This construction defines a smooth map $\phi : J \to \Gamma^\star(H)$. We extend ϕ to J' by defining $\phi(E') = \bar{I}\bar{I}'E'\bar{F}'\bar{F}$, where we have written $A_eI \cup \bar{I}$, $A_\sigma = F \cup \bar{F}$. We again have $\phi(E') \sim E'$. Extend ϕ H-equivariantly to $\Gamma^\star(H)$. It is straightforward to verify that ϕ is a smooth H-equivariant map satisfying condition (W) ($\phi^2(E) = \Gamma^\star(G)$ for every edge $E \in \Gamma^\star(H)$). The map ϕ fixes the vertex set of $\Gamma^\star(H)$ and for all edges $E \in \Gamma^\star(H)$, $\phi(E) \sim E$. $\square$

THEOREM 9.6.34. *Let H be a subgroup of the finite group G. Let M be a G-manifold of dimension at least four such that*

(1) *the principal isotropy group for the action of G on M is trivial;*
(2) *There is an H-invariant connected component M_0 of M_Π.*

There exists $f \in \mathrm{Diff}_G(M)$ such that f has a connected hyperbolic attractor $\Lambda \subset M_0$ satisfying

(a) *$G_\Lambda = H$;*
(b) *$F|\Lambda$ is H-equivariantly topological conjugate to the shift map on the solenoid $\mathcal{S}(\Gamma^\star(H), \phi)$, where $\phi : \Gamma^\star(H) \to \Gamma^\star(H)$ is the map given by lemma 9.6.33.*

PROOF. Choose a smooth G-equivariant embedding of $G \times_H \Gamma^\star(H)$ in M_Π such that (a) $\Gamma^\star(H)$ H-equivariantly embeds in M_0, and (b) each loop $E \cup E'$, $E \in \Gamma(H)$ an edge, is contractible in M_0. Choose an open H-invariant neighbourhood $W \subset M_0$ of $\Gamma^\star(H)$ such that (a) $g\bar{W} \cap \bar{W} = \emptyset$, $g \in G \backslash H$, and (b) Each loop $E \cup E'$, $E \in \Gamma(H)$ is contractible to a constant within W. Let $\phi : \Gamma^\star(H) \to \Gamma^\star(H)$ be the smooth graph map given by lemma 9.6.33. Just as in the proof of theorem 9.6.32, we may choose a compact H-invariant foliated neighbourhood $U \subset W$ of $\Gamma^\star(H)$ with smooth boundary $\partial U \subset W$. Granted these preliminaries, we carry through the same construction given in the proof of theorem 9.6.32 to obtain a smooth G-equivariant embedding of $\tilde{\phi} : GU \to GU$. It follows from lemma 9.6.33(2) that the $\tilde{\phi}$ image of each edge $E \in \Gamma^\star(H)$ can be H-equivariantly isotoped by an isotopy supported in W to I_E. Since we are assuming $\dim(M) \geq 4$, the H-equivariant embedding of $\tilde{\phi} : U \to U$ is H-equivariantly isotopic to the identity map on U by an isotopy supported in W. Now extend G-equivariantly to GU and apply the equivariant isotopy extension theorem. $\square$

REMARK 9.6.35. Condition (2) of theorem 9.6.34 is always satisfied if there are no reflections for the action of G on M. That is, if $\text{codim}(M_\tau) > 1$ for all isotropy types $\tau \neq \Pi$.

9.6.11. Examples in dimension 3.

$\mathbb{Z}_p$-actions, $p \geq 2$. Take the standard irreducible action of $\mathbb{Z}_p$ on $\mathbb{R}^2$, $p \geq 2$. This action extends to $\mathbb{R}^3$ if we take the trivial action of $\mathbb{Z}_p$ on the z-axis. Using the obvious generalization of the smooth $\mathbb{Z}_2$-graph described in example 9.6.27 (see figure 10(b) for the case $p = 3$), we may construct connected $\mathbb{Z}_p$-invariant hyperbolic attractors for $\mathbb{Z}_p$-equivariant diffeomorphisms of $\mathbb{R}^3$. This approach extends to $\mathbb{Z}_p$-manifolds of dimension 3 with $\dim(M^{\mathbb{Z}_p}) = 1$ (see also [**64**, lemma 4.5]).

$\mathbb{Z}_2$-action with fixed point set the origin. Take the $\mathbb{Z}_2$-action on $\mathbb{R}^3$ generated by $-I_{\mathbb{R}^3}$. In this case we can embed a smooth $\mathbb{Z}_2$-graph $\Gamma \subset \mathbb{R}^2$ in $\mathbb{R}^3$ so that Γ has the single vertex $\{(0,0,0)\}$ which is also the fixed point of the $\mathbb{Z}_2$-action (see figure 13). We define a smooth graph map ϕ satisfying condition (W) by the

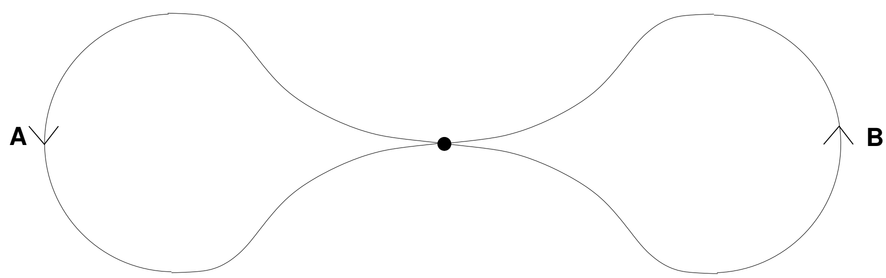


FIGURE 13. A graph with a non-free action of $\mathbb{Z}_2$

edge rules $\phi(A) = ABA^{-1}$, $\phi(B) = BAB^{-1}$. Taking care not create links, the method of proof of theorems 9.6.32, 9.6.34 yields a smooth $\mathbb{Z}_2$-equivariant diffeomorphism of $\mathbb{R}^3$ with connected $\mathbb{Z}_2$-invariant attractor Λ which is topologically conjugate to the solenoid $\mathcal{S}(\Gamma, \phi)$. In this case the action of $\mathbb{Z}_2$ on Λ is not free: there is a unique fixed point in Λ (the origin) for the $\mathbb{Z}_2$-action. Note that this example defines the same solenoid (up to topological conjugacy) as that defined by the first figure described in [**178**, Examples, pg 476] (Williams makes no use or mention of the underlying $\mathbb{Z}_2$-symmetry).

$\mathbf{D}_p$-actions, $p \geq 3$. Take the standard representation of $\mathbf{D}_p$ on $\mathbb{R}^2$. Extend to a representation on $\mathbb{R}^3$ so that $\mathbb{Z}_p \subset \mathbf{D}_p$ fixes the z-axis and all reflections in $\mathbf{D}_p$ act on the z-axis as multiplication by -1. This action has no reflection planes. With some care, one can prove that there exist $f \in \text{Diff}_{\mathbf{D}_p}(\mathbb{R}^3)$ which have connected hyperbolic attractors with $\mathbf{D}_p$-symmetry. Even more, it has been shown by Jacobs [**97**] that one may construct $f \in \text{Diff}_{\mathbf{D}_p}(\mathbb{R}^3)$ which have connected $\mathbf{D}_p$-symmetric hyperbolic attractors $\Lambda \subset \mathbb{R}^3$ such that the action of $\mathbf{D}_p$ on Λ

has $\mathbb{Z}_3$-orbits of fixed points. Similar results have been proved by Jacobs for the tetrahedral, cubical and icosahedral groups (the groups of orientation preserving symmetries). It is reasonable to expect that every every three-dimensional G-manifold satisfying the principal isotropy conditions of theorem 9.6.34 admits a hyperbolic H-symmetric connected solenoidal attractors, where H a finite subgroup of G fixing a connected component of M_Π. Verification of this conjecture would probably depend on a detailed study of finite group actions on 3-manifolds.

9.6.12. Non-free finite group actions on attractors. In general, if Λ is a G-symmetric attractor but G does not act freely on Λ, it is reasonable to expect a breakdown of hyperbolicity at the singular points of the G-action. The simplest, and best-known, example of this phenomenom occurs for the Lorenz flow. The Lorenz equations have a $\mathbb{Z}_2$-symmetry and this symmetry forces the existence of a singular point (equilibrium) on the Lorenz attractor. We briefly sketch an example of a $\mathbb{Z}_2$-equivariant diffeomorphism of $\mathbb{R}^3$ which has a $\mathbb{Z}_2$-invariant non-uniformly hyperbolic attractor (a slight variant of this example has been previously discussed in Coelho et al [**35**], see also [**69**, section 7]).

We take the same $\mathbb{Z}_2$-graph Γ as is shown in figure 13. This time, however, we define a smooth $\mathbb{Z}_2$-invariant graph map ψ according to the rules $\psi(A) = ABA$, $\psi(B) = BAB$. Although ψ satisfies conditions (W1,2), it does not satisfy (W3). We may construct a smooth ψ-invariant singular foliation $\mathcal{F}$ of a neighbourhood U of Γ in $\mathbb{R}^2$ – see figure 14. Observe that there is singular leaf containing the origin of $\mathbb{R}^2$. For our purposes it suffices that the foliation is smooth (in particular, smoothly locally trivial) at all points of the y-axis except the origin. The local triviality of the foliation along the x-axis will not concern us. Note that

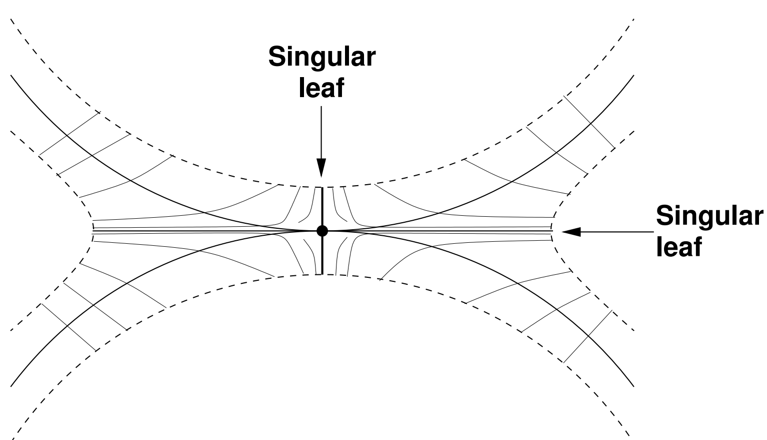


FIGURE 14. Foliation near the singular point

we can extend ψ to a diffeomorphism on some neighbourhood of the origin in U

so that ψ contracts leaves of $\mathcal{F}$. However, the contraction will *not* be uniform near the origin of $\mathbb{R}^2$. Indeed, while $\frac{\partial \psi}{\partial x}(0,0) = 1$ and $\frac{\partial^n \psi}{\partial x^n}(0,0) = 0$, for all $n \geq 2$, ψ will weakly expand Γ near the origin and weakly contract the horizontal component of the singular leaf. We embed the graph in $\mathbb{R}^3$ and take the $\mathbb{Z}_2$-action on $\mathbb{R}^3$ generated by multiplication by $-I_{\mathbb{R}^3}$. Extend the neighbourhood and foliation into $\mathbb{R}^3$ and extend and $\mathbb{Z}_2$-equivariantly perturb ψ to a $\mathbb{Z}_2$-equivariant $\mathcal{F}$-preserving embedding $\tilde{\psi}$ of U in U. In figure 15, we show part of the $\tilde{\psi}$-image of U inside U. The map $\tilde{\psi}$ extends to $F \in \mathrm{Diff}_{\mathbb{Z}_2}(\mathbb{R}^3)$ and we define $\Lambda = \cap_{n \geq 0} F^n(U)$. The set Λ is a $\mathbb{Z}_2$-invariant connected non-uniformly hyperbolic attractor for F. Since ψ satisfies conditions (W1,2), it follows that $F|\Lambda$ is transitive and $\mathrm{Per}(F)$

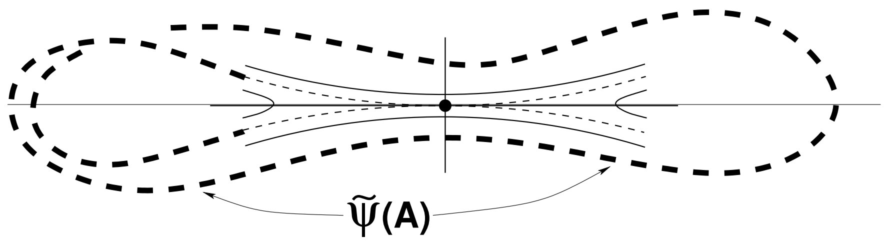


FIGURE 15. The $\tilde{\psi}$-images of a neighbourhood of 0 and the edge A.

is dense in Λ.

REMARK 9.6.36. From the topological point of view, the attractor Λ will be transitive, topologically mixing and have periodic points dense (this is shown by proving that $\Lambda \cong \mathcal{S}(\Gamma, \psi)$ and using propositions 9.6.26 and 9.5.8). When it comes to the existence of ergodic measures, matters are likely to be quite subtle and to depend on the differentiability of F. See the survey of Luzzatto [**117**, §4].

9.6.13. Symmetric hyperbolic attractors for flows. In this section we complete our investigation of symmetric attractors by proving a result on the existence of symmetric attractors for flows.

THEOREM 9.6.37. *Let H be a subgroup of the finite group G and suppose that M is a G-manifold,* $\dim(M) \geq 5$. *Assume that the principal isotropy group for the action of G on M is trivial and that there exists a connected component M_0 of M_Π that is H-invariant. Then there exists a smooth G-equivariant flow Φ_t on M which has an H-invariant connected hyperbolic attractor $\Lambda \subset M_0$. The flow $\Phi_t|\Lambda$ will be H-equivariantly conjugate to the (constant) suspension flow of a shift map on an H-solenoid.*

Suspension of a smooth graph. As usual we let $\Gamma(H)$ denote the complete H-graph and fix a smooth structure on $\Gamma(H)$. Let $\mathcal{G}(H)$ denote the product of $\Gamma(H)$ with S^1. Taking the trivial action of H on S^1, we give $\mathcal{G}(H)$ the structure of an H-space. We may extend the smooth structure on $\Gamma(H)$ to $\mathcal{G}(H)$ in the obvious way. If I is an open arc in S^1, we view $\Gamma(H) \times I \subset \mathcal{G}(H)$ as a 'ribboned' graph with ribbons touching along a common arc at each vertex $v \in \Gamma(H)$. We refer to $\{v\} \times S^1$ as a *vertex loop* (see figure 16). We let $\mathcal{G}^\star(H)$ denote the product of the augmented graph $\Gamma^\star(H) = \Gamma(H) \cup \Gamma'(H)$ with S^1 and note that $\mathcal{G}(H)$ is naturally an H-invariant subset of $\mathcal{G}^\star(H)$.

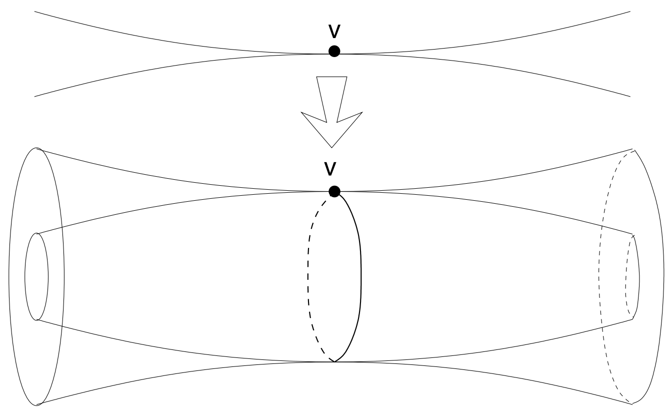


FIGURE 16. Graph suspension near a vertex of type $(2, 2)$

LEMMA 9.6.38. *(Notation and assumptions of theorem 9.6.37.)*

(1) *There exists a smooth H-equivariant embedding ξ of $\mathcal{G}(H)$ in M_0 which extends to a smooth G-equivariant embedding of $G \times_H \mathcal{G}(H)$ in $GM_0 \subset M_\Pi$.*
(2) *We can choose the H-equivariant embedding ξ of (1) so that*
   (a) *ξ extends to a smooth G-equivariant embedding $\xi^\star$ of $G \times_H \mathcal{G}^\star(H)$ in M_Π;*
   (b) *If $E \in \Gamma(H)$ is an edge, then $\xi^\star(S^1 \times (E \cup E'))$ is contractible in M_0.*
(3) *We may choose the embeddings $\xi, \xi^\star$ so that*
   (c) *$\xi(\Gamma(H))$ and $\xi(\Gamma^\star(H)$ are contained in a preassigned connected codimension one H-invariant submanifold Z of M_0.*
   (d) *$\xi(\mathcal{G}(H))$ and $\xi^\star(\mathcal{G}^\star(H))$ are transversal to Z and $\xi(\mathcal{G}(H)) \cap Z = \xi(\Gamma(H))$, $\xi^\star(\mathcal{G}^\star(H)) \cap Z = \xi^\star(\mathcal{G}^\star(H))$.*

PROOF. Let $v \in \Gamma(H)$ be a vertex and $V \subset \mathcal{G}(H)$ be an open connected neighbourhood of the vertex loop $\{v\} \times S^1$ (see figure 16). Since $n \geq 3$, we may construct a smooth embedding $\xi : V \to M_0$ such that $g\overline{\xi(V)} \cap \overline{V} = \emptyset$ for all $g \in G$, $g \neq e$. Hence ξ extends uniquely to a smooth G-equivariant embedding $\xi : \cup_{g \in G} gV \subset G \times_H \mathcal{G}(H) \to M_\Pi$ and that $\xi(HV) \subset M_0$. Let J be a fundamental subgraph for $\Gamma(H)$ and suppose $E \in J$ is an edge. The embedding ξ is already defined on a neighbourhood of the ends of the cylinder $E \times S^1$. Since $n \geq 4$, we may extend ξ so that ξ defines a smooth embedding of $E \times S^1$ in M_0 (if $n = 3$, it may not be possible to match the orientations of the S^1-fibres). Repeating this construction for all edges in J, we obtain a smooth H-equivariant immersion $\xi : \mathcal{G}(H) \to M_0$ which restricts to an embedding on a neighbourhood of the set of vertex loops as well as on the individual cylinders $E \times S^1$. The map ξ extends G-equivariantly to a smooth immersion $\xi : G \times_H \mathcal{G}(H) \to M_\Pi$ which restricts to a smooth embedding on a closed neighbourhood W of the set of vertex loops and embeds the cylinders $E \times S^1$ for all edges $E \in G \times_H \Gamma(H)$. Since G acts freely on M_Π and $n \geq 5$, we may G-equivariantly perturb ξ outside W so that $\xi : G \times_H \mathcal{G}(H) \to M_\Pi$ is an embedding. The proof of the second part of the lemma is similar (see the proof of theorem 9.6.34). Finally, (c) uses our earlier results for diffeomorphisms and (d) follows by the same arguments given above ($\Gamma^\star(H)/H$ is a smooth graph, one vertex, which can be embedded in a 2-disk in M_0/H and $\mathcal{G}^\star(H)/H$ can always be embedded in a 3-disk in M_0/H). $\square$

Remark 9.6.39. For the final statement of lemma 9.6.38, it is easy to construct connected codimension one H-invariant submanifolds of M_0. For example, choose any connected codimension one H-invariant submanifold Z' of M_0/H which contains the projection of a fundamental graph for $\Gamma(H)$. Lift Z' to $Z \subset M_0$.

9.6.14. A tubular neighbourhood of the embedded suspension. We continue with the assumptions of theorem 9.6.37. By lemma 9.6.38 we may choose an H-equivariant embedding of $\mathcal{G}^\star(H)$ in M_0 which extends to a G-equivariant embedding $\xi^\star$ of $G \times_H \mathcal{G}^\star(H)$ in M_Π. Identify $G \times_H \mathcal{G}^\star(H)$ with its $\xi^\star$-image in M_Π. Choose a G-invariant tubular neighbourhood $q : Q \to M_\Pi$ of $G \times_H \mathcal{G}(H) \subset G \times_H \mathcal{G}^\star(H)$ and set $N = q(Q)$. The natural projection of $G \times_H \mathcal{G}(H)$ onto S^1 induces a smooth G-equivariant fibration $\pi : N \to S^1$. Let $N_\theta \subset N$ denote the fibre over $\theta \in S^1$ (parameterizing by angle). By lemma 9.6.38 that we may choose $\xi^\star$ so that $\xi^\star(G \times_H (\Gamma^\star(H) \times \{\theta\})) \subset N_\theta$, for all $\theta \in S^1$. Let $\overline{N} \approx \overline{N}_0 \times S^1$ denote the H-invariant component of N containing $\mathcal{G}^\star(H)$.

Proof of 9.6.37 Since $\dim(M) \geq 5$, $\dim(\overline{N}_\theta) \geq 4$. Just as in the proof of theorem 9.6.34, lemma 9.6.33 implies that there exists a smooth H-equivariant graph map $\phi : \Gamma^\star(H) \times \{0\} \to \Gamma^\star(H) \times \{0\}$ which extends to $\tilde{\phi} : \overline{N}_0 \to \overline{N}_0$ such that (a) $\tilde{\phi}$ is H-equivariantly smoothly isotopic to the identity map of $\overline{N}_0$, and (b) $\tilde{\phi}$ has a unique connected H-invariant hyperbolic solenoidal attractor $\Lambda_0 \subset \overline{N}_0$. We may spread the isotopy round $\overline{N}$ and so construct a smooth H-equivariant flow Φ_t on $\overline{N}$ with solenoidal hyperbolic attractor conjugate to the suspension of $\tilde{}:\Lambda_0 \to \Lambda_0$. Finally, Φ_t extends smoothly and G-equivariantly to N and then to all of M. $\square$

Remarks 9.6.40. (1) We conjecture that provided the action of G on M has no reflections, theorem 9.6.37 holds for G-manifolds of dimension greater than or equal to four.
(2) It is well-known (see [**3, 64**]) that if $\phi : \Gamma \to \Gamma$ is a smooth graph map satisfying conditions (W1,2,3), then there exists a unique Lebesgue-equivalent

ϕ-invariant ergodic probability measure on Γ. In turn, we obtain an ergodic invariant probability measure on the associated solenoid. This measure will be mixing. The measure extends to the suspension given by theorem 9.6.37 and will be ergodic relative to the suspension flow. Although the flow cannot be mixing if we suspend by a constant roof function, it follows from results of Field et al. [**67**], that the suspended flow will be (rapid) mixing for a C^1-open and C^∞-dense space of roof functions.

9.6.15. Extensions to skew and twisted products. Suppose that H is a finite subgroup of the compact Lie group G. Let M be an H-manifold, $f \in \mathrm{Diff}_H(M)$ and $\chi : M \to G$ be a skew H-equivariant map ($\chi(hx) = h\chi(x)h^{-1}$, $h \in H$, $x \in M$. The map f extends to $f_\chi \in \mathrm{Diff}_G(G \times_H M)$ by

$$f_\chi([g,x]) = [g, \chi(x)f(x)],\ (g,x) \in G \times M.$$

If $\Lambda \subset M$ is an H-invariant hyperbolic attractor of f, then $G \times_H \Lambda$ will be a G-invariant *partially hyperbolic* attractor of $f_\chi([g,x])$. The centre foliation of $G \times_H \Lambda$ is given by the G-action and all leaves have dimension equal to that of G. If H acts freely on Λ – the case for most of the hyperbolic symmetric solenoidal attractors constructed above – then G will act freely on $G \times_H \Lambda$. If H does not act freely on Λ, then $G \times_H \Lambda$ will contain singular G-orbits.

It is shown in [**68**] that $f_\chi : G \times_H \Lambda \to G \times_H \Lambda$ will generically be stably ergodic relative to measures on $G \times_H \Lambda$ induced from Haar measure on G and an equilibrium state on Λ (we refer to [**68**] for precise details). It is also the case that $f_\chi | G \times_H \Lambda$ will generically be stably rapid mixing and satisfy various other statistical properties related to decay of correlations (see [**66, 44, 67, 68**] and note that Dolgopyat proves genericity of rapid mixing and the methods of [**66, 67**] yield stability of rapid mixing).

We may carry out similar twisted product constructions for flows. In this case, it can be shown that generically we have stable rapid decay of correlations of equivariant observations. We refer to [**65**] for more details and references.

Remark 9.6.41. We conclude this section on solenoidal attractors by remarking that it is possible to construct symmetric attractors using the expanding maps of higher dimensional (branched) manifolds (see [**180**]). For example, a non-singular matrix $A \in \mathrm{GL}(m, \mathbb{R})$ with integer entries and all eigenvalues of modulus greater than one, determines an expanding map $\hat{A}$ of $\mathbb{T}^m$. It is easy to construct examples where $\hat{A}$ is equivariant with respect to a finite group action on $\mathbb{T}^m$ (typically the group will be a subgroup of S_m). Similar methods to those described above will enable realization of the inverse limit as a hyperbolic attractor of a smooth equivariant diffeomorphism.

9.7. Equivariant Anosov diffeomorphisms

Let M be a compact connected Riemannian G-manifold, where G is a compact Lie group. Suppose that the isotropy groups for the action of G on M are all finite.

We have an associated smooth foliation $\mathcal{L}$ of M by G-orbits. Let $\pi : L \to M$ be the smooth G-vector subbundle of $\tau_M : TM \to M$ defined by $L_x = TGx$, $x \in M$. The bundle $\pi : L \to M$ is Tf-invariant for all $f \in \mathrm{Diff}_G(M)$.

DEFINITION 9.7.1. A diffeomorphism $f \in \mathrm{Diff}_G(M)$ is *G-Anosov* if there exists a continuous Tf-invariant splitting $\mathbb{E}^u \oplus \mathbb{E}^s \oplus L$ of TM together with constants C, λ, $0 < \lambda < 1$, such that

(1) $\|T^{-n}f(v^u)\| \leq C\lambda^n\|v^u\|$, for all $v^u \in \mathbb{E}^u$, $n \geq 0$.
(2) $\|T^n f(v^s)\| \leq C\lambda^n\|v^s\|$, for all $v^s \in \mathbb{E}^s$, $n \geq 0$.

REMARK 9.7.2. A diffeomorphism $f \in \mathrm{Diff}_G(M)$ is G-Anosov if f is partially hyperbolic on all of M with centre foliation given by the G-action. It is straightforward to verify that the definition is independent of the choice of Riemannian metric (the constant C depends on the metric) and that $\mathbb{E}^u, \mathbb{E}^s$ are G-invariant subbundles of TM.

DEFINITION 9.7.3. Let $f \in \mathrm{Diff}_G(M)$. We say f is *G-structurally stable* if there exists an open neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}_G(M)$ such that for all $\bar{f} \in \mathcal{U}$, there exist an equivariant homeomorphism $h : M \to M$ and a continuous skew G-equivariant map $\chi : M \to G$ such that

$$\chi(x)h(f(x)) = \bar{f}(h(x)), \quad (x \in M).$$

THEOREM 9.7.4. (1) *The space of G-Anosov diffeomorphisms is open in $\mathrm{Diff}_G(M)$ (C^1-topology).*
(2) *Every G-Anosov diffeomorphism is G-structurally stable.*

PROOF. The proof is standard and amounts to an equivariant version of the original proofs of Mather [**118**] and Moser [**131**]. We refer the reader to [**55**] for details on the equivariant case. □

We may give an analogous definition of a G-Anosov flow and may then prove a corresponding version of theorem 9.7.4 (of course, with a weaker definition of G-structurally stability, see [**55**, section 4]).

EXAMPLES 9.7.5. (1) If G is finite, then every G-Anosov diffeomorphism of a compact G-manifold M is Anosov. Conversely, it is well-known (see [**163**, Part IV]) that the centralizer $Z(f)$ of an Anosov diffeomorphism f is a discrete subgroup of $\mathrm{Diff}(M)$ and so if f is G-Anosov then G-orbits must be finite. Explicit examples of $\mathbb{Z}_2$-Anosov diffeomorphisms on tori are easily constructed by observing that if $A \in \mathrm{SL}(n, \mathbb{Z})$ is then A commutes with $-I_{\mathbb{R}^n}$ and so the resulting map $\hat{A}$ induced on $\mathbb{T}^n$ is equivariant with respect to the induced $\mathbb{Z}_2$-action. If A is a hyperbolic matrix then $\hat{A}$ will be a $\mathbb{Z}_2$-Anosov. This is already interesting when $n = 2$ and $A(x, y) = (2x + y, x + y)$ is the 'cat' map of the 2-torus. The map induced by $\hat{A}$ on the orbit space $\mathbb{T}^2/\mathbb{Z}_2 \approx S^2$ is a *pseudo-Anosov diffeomorphism* (for an introduction to pseudo-Anosov diffeomorphisms see Boyland [**21**, section 7]).

(2) If $A \in \mathrm{SL}(n, \mathbb{Z})$ is hyperbolic and $k \geq 1$, then $A^k \in \mathrm{SL}(nk, \mathbb{Z})$ induces an $\mathbb{Z}_2^k \rtimes S_k$-equivariant Anosov diffeomorphism of $\mathbb{T}^{nk}$. S_k acts on $(\mathbb{R}^n)^k$ by permuting $\mathbb{R}^n$-factors and $\mathbb{Z}_2^k$ acts as multiplication by ± 1 on each factor.
(3) If H is a finite subgroup of the compact Lie group G, M is an H-manifold, $f \in \mathrm{Diff}_H(M)$ is H-Anosov and $\chi : M \to G$ is a smooth skew H-equivariant map then $f_\chi \in \mathrm{Diff}_G(G \times_H M)$ is G-Anosov. In combination with (2), this gives examples of G-Anosov maps where the G-action is not free and G-orbits are not finite.
(4) Let M be a compact Riemannian G-manifold with unit tangent bundle $T_1(M)$. Suppose that all isotropy groups for the action of G on M are finite. Let $T_1^G(M) = \{v \in T_1(M) \mid v \perp G(\tau_M v)\}$. The smooth G- vector bundle $T_1^G(M)$ is invariant under the geodesic flow on $T_1(M)$ (a geodesic which is normal to some G-orbit is normal to all G-orbits it meets as it (locally) minimizes distance between G-orbits). Suppose that the sectional curvatures are strictly negative for all 2-planes defined by tangent vectors in $T_1^G(M)$. Just as in the non-equivariant case, the induced geodesic flow on $T_1^G(M)$ is G-Anosov. Specific examples may be constructed using the twisted product constructions described above. However, it would be interesting to find examples which were not globally twisted products. In other words, is it possible to give a classification of G-manifolds which are negatively curved transverse to the G-action?

9.8. Notes on chapter 9

Equivariant handlebundle theory was originally developed by Wassermann in his work on equivariant differential topology [**176**]. Previously, a version of Morse theory that allowed for critical manifolds (as opposed to critical points) had been developed by Bott [**23**]. In later work, Atiyah and Bott developed a natural equivariant version of Morse theory with connections to equivariant cohomology – a nice introduction is in the article by Bott [**24**]. It would be interesting if there were connections between intersection theory on G-manifolds (G-transversality) and equivariant cohomology theory (algebraic structure). A few speculations appear at the end of [**56**]. The work we describe on G-handlebundle decompositions appeared in [**53**] and was motivated by Smale's work on handlebody decompositions (hence the requirement that stable and unstable manifolds of critical G-orbits are transverse). Everything we say breaks down (badly) if intersections are G-transverse but not transverse. The representability, density and isotopy theorems on G-subshifts of finite type given in section 9.3 originally appeared in [**56**]. The existence results on symmetric hyperbolic attractors for equivariant flows and diffeomorphisms are mainly taken from the paper [**64**]. The main theorems in [**64**] are sharp and, in sufficiently high dimensions, give necessary and sufficient conditions for the existence of hyperbolic attractors with specified symmetry. Previously, Melbourne et al. [**125, 7**] had proved general results on the existence and structure of symmetric attractors for non-invertible equivariant maps. The results on G-Anosov diffeomorphisms are all taken from [**55**].

In this chapter we emphasized global results where intersections of invariant manifolds were *transverse* – as opposed to G-transverse. In chapters 4, 5 we gave many examples where there were homoclinic and heteroclinic cycles which are necessarily G-transverse non-transverse intersections. Of course, a striking feature of general G-transverse intersections is the possibility of stably singular intersections. However, we know of no example where singular G-transverse intersections occur in a codimension 1 steady state bifurcation. Nor, at this time, are we aware of any large classes of naturally defined equivariant vector fields which possess stably singular G-transverse intersections. However, this phenomenom is more likely to appear in higher dimensional problems and perhaps also if there is additional structure (such as reversibility or Hamiltonian dynamics).

CHAPTER 10

Applications of G-transversality to bifurcation theory II

In this chapter we extend the bifurcation theory developed in chapter 7 to include general compact Lie groups, branches of relative equilibria and the equivariant Hopf bifurcation. We also consider bifurcation theory for equivariant maps as well as applications to bifurcations of relative equilibria and relative periodic orbits.

10.1. Technical preliminaries and basic notations

The emphasis in chapter 7 was on bifurcation theory for absolutely irreducible representations of finite groups. In this chapter we will allow for compact groups and irreducible representations over the complex numbers. We start with some preliminaries on complex representations and complex structures that will enable us to give a unified treatment of bifurcation on real and complex representations – including bifurcation to relative equilibria and relative periodic orbits (equivariant Hopf bifurcation). We conclude with a review of the terminology we need for our later work on branching, stability and determinacy.

10.1.1. Complex representations and complex structures. Suppose that (W, G) is a real representation of the compact Lie group G. The action of G on W extends to a $\mathbb{C}$-linear action on the complexification $V = W \otimes_{\mathbb{R}} \mathbb{C}$ of W. The representation (V, G) is irreducible as a complex representation if and only if (W, G) is absolutely irreducible.

We recall the following basic result on complex representations (see [**30**] or [**2**]).

LEMMA 10.1.1. *Let (V, G) be an irreducible complex representation. Then one of the three following mutually exclusive possibilities must occur.*

(R) *(V, G) is the complexification of an absolutely irreducible representation.*
(C) *If we regard (V, G) as a real representation, then (V, G) is of complex type.*
(Q) *If we regard (V, G) as a real representation, then (V, G) is of quaternionic type.*

REMARKS 10.1.2. (1) If (V, G) is an irreducible complex representation, we say (V, G) is of real, complex or quaternionic type according to whether (R), (C) or (Q) holds. Note that this is the same terminology that we used earlier for $\mathbb{R}$-representations. The context will always make the intended meaning clear.

(2) If (V,G) is the complexification of the absolutely irreducible representation (W,G), then the isotypic decomposition of V as a real representation is W^2.

EXAMPLE 10.1.3. The standard complex irreducible representation of $\mathbf{D}_n$ on $\mathbb{C}^2$ is of real type. Every non-trivial action of SO(2) on $\mathbb{C}$ is irreducible of complex type. The standard action of SU(2) on $\mathbb{C}^2$ is of quaternionic type.

A *complex structure* J on a (real) vector space V is a linear endomorphism of J of V such that $J^2 = -I_V$. If J is a complex structure on V, then we may give V the structure of a complex vector space by defining

$$(a + \imath b)v = av + bJv, \ (v \in V, a, b \in \mathbb{R}).$$

If (V,G) is an $\mathbb{R}$-representation and J is a G-equivariant complex structure on V, then (V,G) will be a $\mathbb{C}$-representation with respect to the complex structure on V defined by J.

REMARK 10.1.4. Suppose that (V,G) is a real representation (not necessarily irreducible), $X \in C^\infty_G(V,V)$ and that $DX(0)$ has eigenvalues $\pm\imath a$, $a \neq 0$, and is diagonalizable over $\mathbb{C}$. Then $J_V = \frac{1}{a}DX(0)$ defines a complex structure on V ($DX(0)^2 = -a^2 I_V$). Since J_V is G-equivariant, (V,G) is a complex representation with respect to this complex structure. Now suppose $X_\lambda \in C^\infty_G(V,V)$ is a family such that $DX_0(0)$ has nonzero eigenvalues on the the imaginary axis. Generically, we can expect the eigenvalues to equal $\pm\imath a$, $a \neq 0$, and $DX_0(0)$ to be diagonalizable (semisimple). The eigenspace W corresponding to $\pm\imath a$ will inherit a complex structure from $DX_0(0)|W$. Generically, (W,G) will be irreducible as a complex representation. Via a usual centre manifold reduction, we may therefore reduce to the study of families defined on an irreducible complex representation.

If (V,G) is a complex representation, let $L_G(V,V)$ denote the space of $\mathbb{R}$-linear equivariant endomorphisms of V.

LEMMA 10.1.5. *Suppose that (V,G) is an irreducible complex representation. Let $X \in C^\infty_G(V \times \mathbb{R}, V)$ and suppose that $DX_0(0) \in L_G(V,V)$ has nonzero eigenvalues lying on the imaginary axis. Then there exists $\lambda_0 > 0$, a complex structure $J_0 \in L_G(V,V)$ and a smooth family of invertible maps $P_\lambda \in L_G(V,V)$, $\lambda \in (-\lambda_0, \lambda_0)$, such that if we define $\tilde{X} \in C^\infty_G(V \times (-\lambda_0,\lambda_0), V)$ by*

$$\tilde{X}(x,\lambda) = P_\lambda X(P_\lambda^{-1}x, \lambda), \ \ (x,\lambda) \in V \times (-\lambda_0, \lambda_0),$$

then $D\tilde{X}_\lambda(0) \in L_G(V,V)$ is $\mathbb{C}$-linear with respect to J_0, for all $\lambda \in (-\lambda_0,\lambda_0)$. If (V,G) is of complex type, we may take $\lambda_0 = +\infty$, $P_\lambda = I_V$ and J_0 equal to the given complex structure on V.

PROOF. The proof is trivial if (V,G) is irreducible of complex type. We prove for (V,G) irreducible of real type and leave the quaternionic case to the reader. If (V,G) is irreducible of real type, then $L_G(V,V) \cong M(2,2;\mathbb{R})$, where $M(2,2;\mathbb{R})$ is the space of 2×2 real matrices. If V is the complexification of the absolutely irreducible representation (W, G), then we may write $V = W \oplus \imath W$. Relative to this decomposition of V, the matrix $[a_{ij}] \in M(2, 2; \mathbb{R})$ corresponds to $A \in L_G(V, V)$ where

$$A = \begin{pmatrix} a_{11} I_W & a_{12} I_W \\ a_{21} I_W & a_{22} I_W \end{pmatrix}$$

(The map A is $\mathbb{C}$-linear if and only if $a_{11} = a_{22}$ and $a_{12} = -a_{21}$.) Set $A_\lambda = DX_\lambda(0)$. We are given that A_0 has a pair of nonzero eigenvalues, say $\pm\imath\omega$. Replacing X by X/ω, it is no loss of generality to assume that A_0 has eigenvalues $\pm\imath$. Since A_0 is semisimple, $A_0^2 = -I_V$ and so A_0 defines a complex structure $J_0 \in L_G(V, V)$. Using J_0, we define a new complex structure on V and corresponding decomposition $V = W \oplus \imath W$ such that A_0 corresponds to $[J_0] = \begin{pmatrix} 0 & -I_W \\ +I_W & 0 \end{pmatrix}$. Choose $\lambda_0 > 0$, so that A_λ is semisimple, $\lambda \in (-\lambda_0, \lambda_0)$ (it suffices that the eigenvalues of A_λ have nonzero imaginary part). Applying the real Jordan normal form theorem to $A_\lambda \in M(2, 2; \mathbb{R})$, we may find a smooth family $P_\lambda \in L_G(V, V)$ of non-singular maps such that $P_0 = I_V$ and $\tilde{A}_\lambda = P_\lambda A_\lambda P_\lambda^{-1}$ has matrix form $\tilde{A}_\lambda = \begin{pmatrix} \alpha(\lambda) I_W & -\beta(\lambda) I_W \\ \beta(\lambda) I_W & \alpha(\lambda) I_W \end{pmatrix}$, where the eigenvalues of A_λ are $\alpha(\lambda) \pm \imath\beta(\lambda)$ and $\tilde{A}_\lambda$ commutes with J_0. If we define $\tilde{X}(x, \lambda) = P_\lambda X(P_\lambda^{-1} x, \lambda)$, then $D\tilde{X}_\lambda(0)$ commutes with J_0 and so $D\tilde{X}_\lambda(0) \in L_G(V, V)$ is $\mathbb{C}$-linear with respect to J_0, for all $\lambda \in (-\lambda_0, \lambda_0)$. $\square$

The next lemma – the proof of which is immediate – shows that stabilities are unchanged under the transformations given by lemma 10.1.5.

LEMMA 10.1.6. *Let (V, G) be a representation (real or complex), $P \in L_G(V, V)$ be non singular and $X \in C_G^\infty(V, V)$. Set $\tilde{X} = PXP^{-1}$. Then $X(x) = 0$ if and only if $\tilde{X}(P(x)) = 0$. The stabilities (spectrum) of the linearizations of X at x and $\tilde{X}$ at $P(x)$ are the same.*

10.1.2. Normalized families on a complex representation. Let (V, G) be an irreducible complex representation. It follows from lemmas 10.1.5, 10.1.6 that, as far as the generic codimension 1 local bifurcation theory is concerned, it is no loss of generality to restrict attention to families $X \in C_G^\infty(V \times \mathbb{R}, V)$ satisfying

$$X(x, \lambda) = \sigma(\lambda) I_V + g(x, \lambda),$$

where $\sigma : \mathbb{R} \to \mathbb{C}$ satisfies $\sigma(0) = \imath$, $(\mathrm{Re}(\sigma))'(0) \neq 0$, and $g(x, \lambda) = O(\|x\|^2)$. Just as in section 5.6, we may rescale time and change parameter so that $\sigma(\lambda) = \lambda + \imath$ for λ near zero. Thus we restrict attention to the space $\mathcal{V}_0(V, G) = \mathcal{V}_0$ of normalized vector fields defined by

$$\mathcal{V}_0(V, G) = \{X \in C_G^\infty(V \times \mathbb{R}, V) \mid DX_\lambda(0) = (\lambda + \imath) I_V\}.$$

Let $S^1 \subset \mathbb{C}$ denote the group of complex numbers of unit modulus and let S^1 act on V as scalar multiplication. Since (V, G) is a complex representation, we have an action of $G \times S^1$ on V and corresponding irreducible complex representation $(V, G \times S^1)$. The representation $(V, G \times S^1)$ is always irreducible of complex type.

DEFINITION 10.1.7. Let (V, G) be a complex representation which is irreducible of complex type. If $G = K \times S^1$, where S^1 acts on V by scalar multiplication, we say that (V, G) is *irreducible of* $\mathbb{C}$*-normal type.*

We recall the strategy outlined in section 5.6 for the analysis of the Hopf bifurcation on a complex representation. Suppose $X \in \mathcal{V}_0(V, G)$. Since $DX_0(0) = \imath I_V$, it follows from the theory of equivariant normal forms (for example, [**84**, Chapter XVI, §5]) that we can make a smooth λ-dependent G-equivariant change of coordinates on a neighbourhood of the origin of $V \times \mathbb{R}$ so that the Taylor series of X_λ at $x = 0$ is $G \times S^1$-equivariant. Next we analyse bifurcations for generic families $X \in \mathcal{V}_0(V, G \times S^1)$. After finding branches of limit cycles, relative equilibria and relative periodic orbits for $X \in \mathcal{V}_0(V, G \times S^1)$, there remains the problem of proving that branches and stabilities persist when we take account of the G-equivariant, but not $G \times S^1$-equivariant, tail. It is shown in [**60, 62**] that (generically) branches and their stabilities persist when we allow for the tail. We say a little more about these "strong determinacy" results in section 10.6.7 when we study the trickier problem of normal forms and strong determinacy for maps. We also caution the reader that while this approach shows that branches of limit cycles persist when we break normal form symmetry it does not, however, prove that no new branches of limit cycles appear. There is also the often difficult problem of determining the dynamics for the original problem.

EXERCISE 10.1.8. Investigate what happens if (V, G) is irreducible of quaternionic type, $X \in C^\infty_G(V \times \mathbb{R}, V)$ and the eigenvalues of $DX_0(0)$ have zero real part and nonzero imaginary part. In particular, show that the normal form reduction still applies and that $(V, G \times S^1)$ is always irreducible of complex type. Using invariant sphere theorem techniques, describe what happens if $V = \mathbb{C}^2$ and G is the quaternionic group $\{\pm I, \pm\imath, \pm\jmath, \pm k\}$ (see also example 10.6.51).

10.1.3. Branches of relative equilibria. In this section, we extend the formalism of branching patterns and stability developed in chapters 4, 7 to allow for branches of relative equilibria. Most of what we do is very close to section 7.6 of chapter 7.

Suppose that (V, G) is a representation which is either an absolutely irreducible orthogonal representation or an irreducible unitary complex representation of complex type. If $X \in \mathcal{V}_0(V, G)$, then $DX_\lambda(0) = \lambda I_V$ (if (V, G) is absolutely irreducible) or $DX_\lambda(0) = (\lambda + \imath)I_V$ (if (V, G) is an irreducible complex representation of complex type).

DEFINITION 10.1.9. Given $X \in \mathcal{V}_0(V, G)$, define G-invariant closed subsets of $V \times \mathbb{R}$ by

$$\begin{aligned} \mathbf{Z}(X) &= \{(x, \lambda) \mid X_\lambda(x) = 0\}, \\ \mathbf{I}(X) &= \{(x, \lambda) \mid X_\lambda(x) \in T_x Gx\}. \end{aligned}$$

Note that $\mathbf{I}(X)$ is the set of relative equilibria of X and that if G is finite, $\mathbf{I}(X) = \mathbf{Z}(X)$. If G is the product of a finite group with S^1, then $\mathbf{I}(X)$ will consist of limit cycles.

For every $\tau \in \mathcal{O}(V, G)$, choose $H \in \tau$ and set $\Delta_\tau = G/H$. Every G-orbit of isotropy type τ is G-equivariantly diffeomorphic to Δ_τ.

DEFINITION 10.1.10. Let $X \in \mathcal{V}_0(V, G)$ and $\tau \in \mathcal{O}(V, G)$. A branch of relative equilibria of isotropy type τ for X at zero consists of a C^1 G-equivariant map

$$\phi = (\mathbf{x}, \lambda) : [0, \delta] \times \Delta_\tau \to V \times \mathbb{R}$$

such that

(1) $\phi(0, u) = (0, 0)$, for all $u \in \Delta_\tau$.
(2) For all $s \in (0, \delta]$, $\alpha_s = \mathbf{x}(s, \Delta_\tau)$ is a relative equilibrium of $X_{\lambda(s)}$ and α_s is of isotropy type τ.
(3) For every $u \in \Delta_\tau$, the map $\phi_u : [0, \delta] \to V \times \mathbb{R}$, $s \mapsto \phi(s, u)$, is a C^1-embedding.

If, in addition, we can choose $\delta > 0$ so that

(4) For all $s \in (0, \delta]$, α_s is a normally hyperbolic relative equilibrium $X_{\lambda(s)}$.

ϕ is a *branch of normally hyperbolic relative equilibria for* X at zero.

REMARK 10.1.11. A branch of relative equilibria is a G-invariant subset of $V \times \mathbb{R}$. This should be contrasted with the definition we gave for a branch of equilibria in section 4.2 where the branch only defined a G-invariant subset when the branch was of trivial isotropy. Of course, the G-orbit of a branch of equilibria will define a branch of relative equilibria in the sense defined above.

Suppose that $\phi_i = (\mathbf{x}_i, \lambda_i) : [0, \delta_i] \times \Delta_\tau \to V \times \mathbb{R}$, $i = 1, 2$, are branches of relative equilibria for f. The branches are *equivalent* if we can find $\delta_i' \in (0, \delta_i]$, $i = 1, 2$, and a C^1 G-equivariant diffeomorphism $H : [0, \delta_1'] \times \Delta_\tau \to [0, \delta_2'] \times \Delta_\tau$, such that $\phi_1 = \phi_2 \circ H$ on $[0, \delta_1'] \times \Delta_\tau$. Note that this definition allows for translation in Δ_τ.

We let $[\phi]$ denote the equivalence class of the branch ϕ. Typically, we identify $[\phi]$ with the germ of the image of ϕ in $V \times \mathbb{R}$.

EXAMPLE 10.1.12. Define $c^\pm : [0, \infty) \times \Delta_{(G)} = [0, \infty) \to V \times \mathbb{R}$ by $c^\pm(s) = (0, \pm s)$, $s \in [0, \infty)$. Then $c^\pm$ define the two trivial branches of relative equilibria for any $X \in \mathcal{V}_0(V, G)$.

Let $\phi = (\mathbf{x}, \lambda) : [0, \delta] \times \Delta_\tau \to V \times \mathbb{R}$ be a branch of relative equilibria for X. Let $S(V)$ denote the unit sphere of V. We define the direction of branching set $\mathcal{D}(\phi) \subset S(V)$ by

$$\mathcal{D}(\phi) = \{\mathbf{x}'_u(0)/\|\mathbf{x}'_u(0)\| \mid u \in \Delta_\tau\}.$$

It is straightforward to check that $\mathcal{D}(\phi) \subset S(V)$ is a G-orbit of isotropy type $\mu \geq \tau$ and that $\mathcal{D}(\phi)$ depends only on the equivalence class of ϕ (see the proof of lemma 4.2.2).

A branch $\phi = (\mathbf{x}, \lambda)$ of relative equilibria is *supercritical* (respectively, *subcritical*) if $\lambda > 0$ (respectively, $\lambda < 0$) on an interval $(0, \delta)$. Supercritical and subcritical depend only on the equivalence class of ϕ.

LEMMA 10.1.13. *A branch of normally hyperbolic relative equilibria is either supercritical or subcritical.*

PROOF. Let $X \in \mathcal{V}_0(V, G)$ and suppose that $\phi = (\mathbf{x}, \lambda) : [0, \delta] \times \Delta_\tau \to V \times \mathbb{R}$ is a branch of normally hyperbolic relative equilibria for X. It suffices to prove that $\lambda' \neq 0$ on $(0, \delta]$. Suppose the contrary. Then there exists $\tilde{s} \in (0, \delta]$ such that $\lambda'(\tilde{s}) = 0$. Choose $\varepsilon > 0$ so that $I = [\tilde{s} - \varepsilon, \tilde{s}] \subset (0, \delta]$. For each $s \in I$, set $\alpha_s = \rho(s, \Delta_\tau)$. It follows from lemma 8.4.6 that we may choose a smooth family $\{\tilde{X}_s \in C^\infty_G(V, V) \mid s \in I\}$ such that $\tilde{X}_s$ is tangent to G orbits and $X_{\lambda(s)} - \tilde{X}_s | \alpha_s \equiv 0$. Hence α_s is a normally hyperbolic equilibrium orbit for $Z_s = X_{\lambda(s)} - \tilde{X}_s$, $s \in I$. We have

$$Z(\mathbf{x}(s, u), s) \equiv 0, \ (x, u) \in I \times \Delta_\tau.$$

Differentiating with respect to s at $s = \tilde{s}$, and using $\lambda'(\tilde{s}) = 0$, we find that

$$D_1 Z(\mathbf{x}(\tilde{s}, u), \tilde{s}) \frac{\partial \mathbf{x}}{\partial s}(\tilde{s}, u) \in T_{\mathbf{x}(\tilde{s}, u)} \alpha_{\tilde{s}}, \ \ (u \in \Delta_\tau).$$

Since ϕ_u is a C^1-embedding, $\frac{\partial \mathbf{x}}{\partial s}(\tilde{s}, u) \notin T_{\mathbf{x}(\tilde{s}, u)} \alpha_s$, for all $u \in \Delta_\tau$. Hence we have $D_1 Z(\mathbf{x}(\tilde{s}, u), \tilde{s}) \frac{\partial \mathbf{x}}{\partial s}(\tilde{s}, u) = 0$ and so the multiplicity of $0 \in \mathbf{vspec}(\alpha_{\tilde{s}}, Z_{\tilde{s}})$ is at least $g_\tau + 1$, contradicting the genericity of $\alpha_{\tilde{s}}$. $\square$

Suppose that ϕ is a branch of normally hyperbolic relative equilibria for X at zero. We let $\mathrm{ind}(\phi)$ denote the index of relative equilibria along the branch. That is, $\mathrm{ind}(\phi)$ is the dimension of the stable manifold of $\phi(s, \Delta_\tau)$, $s > 0$. The index depends only on the equivalence class of ϕ. We define the sign of the branch, $\mathrm{sgn}(\phi)$ to be $+1$ if the branch is supercritical and -1 if the branch is subcritical. The sign function depends only on the equivalence class of ϕ.

The branching pattern and branching conditions.

DEFINITION 10.1.14. Given $X \in \mathcal{V}_0(V, G)$, let $\mathcal{B}(X)$ be the set of equivalence classes of nontrivial branches of relative equilibria. We call $\mathcal{B}(X)$ the *branching pattern* of X.

REMARK 10.1.15. If G is finite then $\Sigma(X)$ and $\mathcal{B}(X)$ define the same germs of subsets of $V \times \mathbb{R}$. However, we do not give $\mathcal{B}(X)$ the structure of a G-set. (If we wished we could give $\mathcal{B}(X)$ the structure of a G/G_0-set and then $\mathcal{B}(X)$ would be a natural extension of our definition of the branching pattern for finite groups G.)

Following section 4.2, we consider the following *branching conditions* on $X \in \mathcal{V}_0(V,G)$:

B1* There is a finite set $\phi_1, \dots, \phi_{r+2}$ of branches of relative equilibria for X, with images $C_1, \dots, C_{r+2}$, such that

(1) $\mathcal{B}(X) = \{[\phi_1], \dots, [\phi_r]\}$, $[\phi_{r+1}] = [c^+]$, $[\phi_{r+2}] = [c^-]$.

(2) There is a neighborhood N of $(0,0)$ in $V \times \mathbb{R}$ such that if $(x,\lambda) \in N$ and α is a relative equilibrium of X_λ then

$$\alpha \times \{\lambda\} \subset \cup_{j=1}^{r+2} C_j$$

(3) If $i \neq j$, then $C_i \cap C_j = \{(0,0)\}$.

B2* Every $[\phi] \in \mathcal{B}(X)$ is a branch of normally hyperbolic relative equilibria.

DEFINITION 10.1.16. A family $X \in \mathcal{V}_0(V,G)$ is *weakly regular* if X satisfies the branching condition **B1***. If, in addition, X satisfies the branching condition **B2***, we say that X is *regular*.

REMARK 10.1.17. If X is regular then $(0,0) \in V \times \mathbb{R}$ is an *isolated* bifurcation point of X.

DEFINITION 10.1.18. Suppose $X \in \mathcal{V}_0(V,G)$ is regular. The *signed indexed branching pattern* $\mathcal{B}^\star(X)$ of X consists of the set $\mathcal{B}(X)$, labelled by isotropy types, together with the maps $\mathrm{ind} : \mathcal{B}(X) \to \mathbb{N}$ and $\mathrm{sgn} : \mathcal{B}(X) \to \{\pm 1\}$.

If X is regular then X has a well-defined signed indexed branching pattern $\mathcal{B}^\star(X)$ which describes the stabilities of all the relative equilibria on some neighborhood of zero.

DEFINITION 10.1.19. If $X, Y \in \mathcal{V}_0(V,G)$ are weakly regular, then $\mathcal{B}(X)$ is isomorphic to $\mathcal{B}(Y)$ if there is a bijection between $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ preserving isotropy type.

If X, Y, $\mathcal{B}^\star(X)$ is isomorphic to $\mathcal{B}^\star(Y)$ if $\mathcal{B}(X)$ is isomorphic to $\mathcal{B}(Y)$ by an isomorphism preserving the sign and index functions and isotropy type.

10.1.4. Stability.

DEFINITION 10.1.20. Let $X \in \mathcal{V}_0(V,G)$.

(W) X is *weakly stable* and X has a *stable branching pattern* if
  (1) X satisfies the branching condition **B1***.
  (2) There exists an open neighbourhood $\mathcal{U}$ of X in $\mathcal{V}_0$ such that all $Y \in \mathcal{U}$ satisfy the branching condition **B1*** and the isomorphism class of $\mathcal{B}(Y)$ is constant on $\mathcal{U}$.

(S) X is *stable* and X has a *stable signed indexed branching pattern* if
  (1) X satisfies the branching conditions **B1*** and **B2***.

(2) There exists an open neighbourhood $\mathcal{U}$ of X in $\mathcal{V}_0$ such that all $Y \in \mathcal{U}$ satisfy the branching conditions **B1***, **B2*** and the isomorphism class of $\mathcal{B}^\star(Y)$ is constant on $\mathcal{U}$.

Let $\mathcal{S}^\star(V,G)$ (respectively, $\mathcal{S}^\star_w(V,G)$) denote the open subset of $\mathcal{V}_0(V,G)$ consisting of stable families (respectively, weakly stable families). We show later in this chapter that $\mathcal{S}^\star(V,G)$ is a dense subset of $\mathcal{V}_0(V,G)$. This gives a generalization of theorem 7.4.2 to general compact Lie groups and branches of relative equilibria.

10.1.5. Determinacy.

DEFINITION 10.1.21. G-equivariant bifurcation problems on (V,G) are *finitely determined* if there exist $q \in \mathbb{N}$ and an open dense semialgebraic subset $\mathcal{R}(q)$ of $P^{(q)}_G(V,V)$ such that if $X \in \mathcal{V}_0(V,G)$ and $j^qX_0(0) = j^q_1X(0,0) \in \mathcal{R}(q)$, then X is stable. Similarly, we define weak finite determinacy by requiring that there exists $q_w \in \mathbb{N}$ and an open dense semialgebraic subset $\mathcal{R}_w(q_w)$ of $P^{(q_w)}_G(V,V)$ such that if $X \in \mathcal{V}_0(V,G)$ and $j^{q_w}X_0(0) \in \mathcal{R}_w(q_w)$, then X is weakly stable.

REMARKS 10.1.22. (1) We say that G-equivariant bifurcation problems on V are q-determined if q is the smallest positive integer for which we can find $\mathcal{R}(q)$ satisfying the conditions of Definition 10.1.21. We say that X is *q-determined* if $j^qX_0(0) \in \mathcal{R}(q)$.
(2) Let $X \in \mathcal{V}_0(V,G)$ be q-determined and set $Q = j^qX_0(0)$. If we define $J^Q \in \mathcal{V}_0(V,G)$ by

$$J^Q(x,\lambda) = \lambda x + Q(x).$$

then $\mathcal{B}^\star(X) \cong \mathcal{B}^\star(J^Q)$. (Note that if (V,G) irreducible of complex type then $j^1Q(0) = \imath I_V$.) Conversely, if $X \in \mathcal{S}^\star(V,G)$, then there exists $Q \in \mathcal{R}(q)$ such that $\mathcal{B}^\star(X)$ is isomorphic to $\mathcal{B}^\star(J^Q)$ (same proof as that of lemma 4.4.6).

EXAMPLE 10.1.23. Let G be one of the classical simple Lie groups $\mathrm{SO}(n)$, $n > 2$, $\mathrm{SU}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$ or a simple compact Lie group of type F_4 or G_2. The representation space for the adjoint action of G is $\mathfrak{g}$, the Lie algebra of G, and the action of G on $\mathfrak{g}$ is absolutely irreducible. Let W denote the Weyl group of G and $\mathbf{T}$ denote a maximal torus for G. Each G-orbit meets $\mathfrak{t}$ (the Lie algebra of $\mathbf{T}$) in a W-orbit [**70**, 11.1]. The invariants and equivariants for the adjoint action of G on $\mathfrak{g}$ are obtained by extension of the corresponding invariants and equivariants for the action of W on $\mathfrak{t} \subset \mathfrak{g}$ (see [**70**, section 11]). Consequently, if W-equivariant bifurcation problems on $\mathfrak{t}$ are d-determined then so are G-equivariant bifurcation problems on $\mathfrak{g}$. We know from chapter 4 and [**70, 73**] that W-equivariant bifurcation problems on $\mathfrak{t}$ are finitely determined. Hence, the same is true for G-equivariant bifurcation problems on $\mathfrak{g}$. In particular, there is an open and dense subset $\mathcal{S}^\star(\mathfrak{g},G)$ of $\mathcal{V}_0(\mathfrak{g}\times\mathbb{R},\mathfrak{g})$ such that if $X \in \mathcal{S}^\star(\mathfrak{g},G)$ then

(1) $\mathcal{B}^\star(X)$ is a finite union of germs of branches of normally hyperbolic relative equilibria.
(2) $\mathcal{B}^\star(X)$ contains a branch of relative equilibria of isotropy type τ, whenever τ is maximal and, if $G \cong \mathrm{SO}(2k), k \geq 4$, whenever $\tau = (S_{k-1})$.

REMARK 10.1.24. For the adjoint representations considered above, it is known that if τ is an isotropy type satisfying the hypotheses of (2), then $n_\tau = 0$ and so the branches given by (1) are all branches of normally hyperbolic equilibrium G-orbits (see [**84**, Theorem 8.2] for the case of SO(3)). If $n_\tau > 0$ then if follows by proposition 7.6.15 that we can always perturb X within $\mathcal{V}_0(V, G)$ so that the corresponding branches are branches of relative equilibria, but not equilibria. It was asked in [**84**, page 136] if $n_\tau = 0$ for all maximal isotropy types when (V, G) is absolutely irreducible. Subsequently, Melbourne [**124**] found examples of absolutely irreducible representations (V, G) and maximal isotropy types $\tau \in \mathcal{O}(V, G)$ for which $n_\tau = 1$ and either $N(H)/H \cong S^1$ or $N(H)/H \cong \mathrm{SU}(2)$, $H \in \tau$.

10.2. The universal variety for relative equilibria

We assume that (V, G) is a representation which is either an absolutely irreducible orthogonal representation or an irreducible unitary complex representation of complex type. In the complex case the real inner product $(\ ,\ )$ associated to the Hermitian structure on V satisfies

$$(v, J_V v) = 0, \quad \text{for all } v \in V \tag{10.1}$$

(J_V denote the complex structure on V.)

Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a minimal set of homogeneous polynomial generators for the $P(V)^G$-module $P_G(V, V)$. As usual, we suppose that $F_1 = I_V$ and $1 \leq d_2 \leq \dots \leq d_k$, where $\deg(F_i) = d_i$. If (V, G) is absolutely irreducible, $d_2 > 1$. If (V, G) is of complex type, we take $F_2 = J$ (where $J = J_V$ is the complex structure on V) and then $1 < d_3$. If $G = K \times S^1$ (the representation is of $\mathbb{C}$-normal type), then d_i is odd, for all $i \geq 1$. Let $\mathcal{P} = \{p_1, \dots, p_\ell\}$ be a minimal set of homogeneous generators for the $\mathbb{R}$-algebra $P(V)^G$. We set $P = (p_1, \dots, p_\ell) : V \to \mathbb{R}^\ell$ and recall that $P : V \to P/G \subset \mathbb{R}^\ell$ may be regarded as the orbit map.

10.2.1. The variety $\Sigma^\star$. Recall that $\boldsymbol{\vartheta} : V \times \mathbb{R}^k \to V$ is defined by $\boldsymbol{\vartheta}(x, t) = \sum_{i=1}^k t_i F_i(x)$. Let

$$\Sigma^\star = \{(x, t) \mid \boldsymbol{\vartheta}(x, t) \in T_x N(G_x)x\}.$$

If $n_\tau = 0$, all $\tau \in \mathcal{O}(V, G)$, then $\Sigma^\star = \Sigma$. Otherwise, $\Sigma^\star \supsetneq \Sigma$. Given $\tau \in \mathcal{O}(V, G)$, let $\Sigma^\star_\tau$ denote the subset of $\Sigma^\star$ consisting of points of isotropy type τ. Clearly, $\Sigma^\star = \cup_{\tau \in \mathcal{O}(V,G)} \Sigma^\star_\tau$.

EXAMPLE 10.2.1. If we take the standard representation of SO(2) on $\mathbb{C}$, then $\mathcal{F} = \{z, \imath z\}$ $(k = 2)$, and

$$\boldsymbol{\vartheta}(z, (t_1, t_2)) = t_1 z + \imath t_2 z.$$

We have $\Sigma = \{(z,(t_1,t_2) \mid z = 0 \text{ or } t_1,t_2 = 0\}$ and $\Sigma^\star = \{(z,(t_1,t_2) \mid z = 0 \text{ or } t_1 = 0\}$. For example, if we take $\tau = (e)$, we find that

$$\begin{aligned}\Sigma_{(e)} &= \{(z,(0,0)) \mid z \in \mathbb{C}^\star\},\\ \Sigma^\star_{(e)} &= \{(z,(0,t)) \mid t \in \mathbb{R}, z \in \mathbb{C}^\star\}.\end{aligned}$$

Hence $\operatorname{codim}(\Sigma_{(e)}) = 2$ and $\operatorname{codim}(\Sigma^\star_{(e)}) = 1$.

Lemma 10.2.2. *$\Sigma^\star$ is a G-invariant* algebraic *subset of $V \times \mathbb{R}^k$.*

Proof. Define the polynomial map $\mathbf{P} : V \times \mathbb{R}^k \to \mathbb{R}^\ell$ by

$$\mathbf{P}(x,t) = DP(x)(\boldsymbol{\vartheta}(x,t)),\ ((x,t) \in V \times \mathbb{R}^k).$$

Then $\mathbf{P}(x,t) = 0$ if and only if $\boldsymbol{\vartheta}(x,t) \in T_xGx$. Since $\boldsymbol{\vartheta}$ is G equivariant, $\boldsymbol{\vartheta}(x,t) \in T_xGx$ if and only if $\boldsymbol{\vartheta}(x,t) \in T_xN(G_x)x$ and so $\mathbf{P}^{-1}(0) = \Sigma^\star$. ☐

Lemma 10.2.3. *For each $\tau \in \mathcal{O}(V,G)$, $\Sigma^\star_\tau$ is a G-invariant semialgebraic smooth $k+g_\tau$-dimensional submanifold of $V \times \mathbb{R}^k$.*

Proof. Since $\Sigma^\star_\tau = \Sigma^\star \cap (V \times \mathbb{R}^k)_\tau$, it follows from lemmas 6.8.3, 10.2.2 that $\Sigma^\star_\tau$ is a G-invariant semialgebraic subset of $V \times \mathbb{R}^k$. It remains to prove that $\Sigma^\star_\tau$ is a smooth submanifold of $V \times \mathbb{R}^k$ of dimension $k + g_\tau$. Fix $(x,t) \in \Sigma^\star_\tau$ and set $G_x = H$. For each $y \in V^H_\tau$, $\{F_1(y),\dots,F_k(y)\}$ spans V^H. Given $y \in V^H_\tau$, set $N(y) = T_yN(H)y$. Note that $N(y) \subset V^H$ and $\dim(N(y)) = n_\tau$. Let $\pi_y : V^H \to V^H$ denote the projection of V^H on the orthogonal complement of $N(y)$ in V^H. Clearly π_y depends smoothly on $y \in V^H_\tau$ (as a map into the space of linear endomorphisms of V^H). Let $S \subset V \times \mathbb{R}^k$ be a differentiable slice for the action of G on $V \times \mathbb{R}^k$ at (x,t). (For S we may take a sufficiently small Euclidean ball centered at (x,t) in $(x,t) + (T_xG(x,t))^\perp$.) Let $S_\tau \subset S$ be the set of points of isotropy type τ and note that $S_\tau \subset V^H \times \mathbb{R}^k$ and $\dim(S_\tau) = \dim(V^H) - n_\tau + k$. Consider the smooth map $K : S_\tau \to V^H$ defined by

$$K(y,s) = \pi_y \boldsymbol{\vartheta}(y,s)$$

A straightforward computation shows at all points $(y,s) \in \Sigma^\star_\tau \cap S_\tau$, $DK(y,s)$ has constant rank equal to $\dim(V^H) - n_\tau$ and kernel of dimension k. Noting that $\Sigma^\star_\tau \cap S_\tau = K^{-1}(0)$ it follows by the rank theorem that $\Sigma^\star_\tau \cap S_\tau$ is smooth of dimension k and hence that $\Sigma^\star_\tau \cap GS_\tau$ is smooth of dimension equal to $k + g_\tau$. ☐

Remark 10.2.4. It may be shown that $\Sigma^\star_\tau$ has the structure of a smooth real affine algebraic subvariety of $V \times \mathbb{R}^{k+1}$ (see [**60**, Lemma 10.6.1]).

Lemma 10.2.5. *Let $\gamma, \tau \in \mathcal{O}(V,G)$. Then*

(1) $\Sigma^\star_\tau \cap \overline{\Sigma}^\star_\gamma = \emptyset$ *if $\gamma > \tau$.*
(2) $\dim(\Sigma^\star_\tau \cap \overline{\Sigma}^\star_\gamma) < g_\tau + k$, *if $\gamma < \tau$.*

PROOF. Statement (1) follows since $(V\times\mathbb{R}^k)_\tau \cap \overline{(V\times\mathbb{R}^k)_\gamma} = \emptyset$ if $\gamma > \tau$. The proof of (2) is similar to that of lemma 6.9.6. Specifically, let $q_1,\ldots,q_s$ be a minimal set of homogeneous generators for the $\mathbb{R}$-algebra $P(V\times\mathbb{R}^k)^G$ and let $Q : V\times\mathbb{R}^k \to \mathbb{R}^s$ denote the corresponding orbit map. For all $\gamma\in\mathcal{O}$, Q maps $(V\times\mathbb{R}^k)_\gamma$ submersively onto the smooth submanifold $(V\times\mathbb{R}^k)_\gamma/G$ of $\mathbb{R}^s$ (see section 6.6.4). By lemma 10.2.3, $Q(\Sigma^\star_\gamma)$ is a k-dimensional smooth submanifold of $\mathbb{R}^s$. Since Q is proper, $\overline{Q(\Sigma^\star_\gamma)} = Q(\overline{\Sigma}^\star_\gamma)$. Hence for all $\tau,\gamma\in\mathcal{O}$, $\tau>\gamma$, we have

$$\begin{aligned} Q(\Sigma^\star_\tau \cap \overline{\Sigma}^\star_\gamma) &= Q(\Sigma^\star_\tau)\cap\overline{Q(\Sigma^\star_\gamma)},\\ &= Q(\Sigma^\star_\tau)\cap\partial Q(\Sigma^\star_\gamma). \end{aligned}$$

Now $\dim(\partial Q(\Sigma^\star_\gamma)) < \dim(Q(\Sigma^\star_\gamma)) < k$ and so $\dim(Q(\Sigma^\star_\tau\cap\overline{\Sigma}^\star_\gamma) < k$. Therefore, $\dim(\Sigma^\star_\tau\cap\overline{\Sigma}^\star_\gamma) = \dim(Q^{-1}Q(\Sigma^\star_\tau\cap\overline{\Sigma}^\star_\gamma)) < k+g_\tau$. ☐

LEMMA 10.2.6. *For all $\tau\in\mathcal{O}^\star(V,G)$,*

$$\overline{\Sigma}^\star_\tau \cap \mathbb{R}^k \subset \{t\in\mathbb{R}^k \mid t_1 = 0\}.$$

PROOF. We have already shown this when G is finite and (V,G) is absolutely irreducible. We give the proof when (V,G) is irreducible of complex type. Let $t\in\mathbb{R}^k$ and suppose that $(0,t)\in\overline{\Sigma}_\tau\cap\mathbb{R}^k$. Choose a sequence (x^n,t^n) in $\Sigma^\star_\tau$ converging to $(0,t)$. Since $\boldsymbol{\vartheta}(x^n,t^n)$ is tangent to Gx^n, $(\boldsymbol{\vartheta}(x^n,t^n),x^n)=0$ for all n. Now $\boldsymbol{\vartheta}(x^n,t^n)$ is the sum of $t_1^n x^n + t_2^n J_V x^n$ and terms of order at least two in x^n. Noting (10.1), we see that $0 = (\boldsymbol{\vartheta}(x^n,t^n),x^n) = t_1^n\|x^n\|^2 + O(\|x^n\|^3)$, for all n. Dividing by $\|x^n\|^2$ and letting $n\to\infty$, it follows that $t_1=0$. ☐

Suppose that (V,G) is irreducible of $\mathbb{C}$-normal type (and so $G = K\times S^1$). Let $\mathbf{L}_2\subset\mathbb{R}^k\subset V\times\mathbb{R}^k$ denote the t_2-axis and $\mathcal{T}_2$ denote the group of translations of $V\times\mathbb{R}^k$ parallel to $\mathbf{L}_2$. That is, if $T\in\mathcal{T}_2$, then $T(x,t) = (x,t)+\mathbf{u}$, where $\mathbf{u}\in\mathbf{L}_2$.

LEMMA 10.2.7. *Suppose (V,G) is irreducible of $\mathbb{C}$-normal type. Then $\Sigma^\star$ is $\mathcal{T}_2$-invariant. In particular, for all $\tau\in\mathcal{O}^\star(V,G)$, $\overline{\Sigma}^\star_\tau\cap\mathbb{R}^k$ is $\mathcal{T}_2$-invariant.*

PROOF. By definition, $(x,t)\in\Sigma^\star$ if and only if $\boldsymbol{\vartheta}(x,t)$ is tangent to Gx. Hence $(x,t)\in\Sigma^\star$ if

$$t_1x + t_2Jx + \sum_{j=3}^k t_jF_j(x) \in T_xGx = T_xKx + \mathbb{R}Jx$$

But then $t_1x + aJx + \sum_{j=3}^k t_jF_j(x) \in T_xKx + \mathbb{R}Jx = T_xGx$ for all $a\in\mathbb{R}$. ☐

Let $\mathcal{S}^\star$ denote the canonical semialgebraic stratification of $\Sigma^\star$.

THEOREM 10.2.8. (1) *The stratification $\mathcal{S}^\star$ induces a semialgebraic Whitney regular stratification $\mathcal{S}^\star_\tau$ of $\Sigma^\star_\tau$, for all $\tau\in\mathcal{O}(V,G)$. In particular, each $\Sigma^\star_\tau$ is a union of $\mathcal{S}^\star$-strata.*

(2) *If (V,G) is irreducible of $\mathbb{C}$-normal type, then the stratifications $\mathcal{S}^\star$, $\mathcal{S}^\star_\tau$ are $\mathcal{T}_2$-invariant. In particular, if $S\in\mathcal{S}^\star$, then $S = S^\star\times\mathbf{L}_2$, where $S^\star = S\cap\mathbf{L}_2^\perp$.*

PROOF. Part (1) of the proof is exactly the same as that of theorem 6.10.1. The final statement is immediate from lemma 10.2.7. □

For the remainder of the section we shall assume that if (V, G) is not absolutely irreducible then (V, G) is irreducible of $\mathbb{C}$-normal type. Let $\mathbf{R}_2$ denote the orthogonal complement of $\mathbf{L}_2$ in $\mathbb{R}^k$.

Given $\tau \in \mathcal{O}(V, G)$, define $R_\tau = \mathbb{R}^k \cap \overline{\Sigma}^\star_\tau$. We also define

$$\begin{aligned} R^\star_\tau &= R_\tau, \text{ if } (V,G) \text{ is absolutely irreducible,} \\ &= \mathbf{R}_2 \cap R_\tau, \text{ if } (V,G) \text{ is of } \mathbb{C}\text{-normal type.} \end{aligned}$$

We remark that if (V, G) is irreducible of $\mathbb{C}$-normal type then $R_\tau = R^\star_\tau \times \mathbf{L}_2$ and $R^\star_\tau \subset \mathbb{R}^{k-2} = \{t \in \mathbb{R}^k \mid t_1, t_2 = 0\}$. We let $r_\tau = \dim(R_\tau)$, $\tau \neq (G)$.

Recall that $d_\tau = \dim(V^H_\tau)$, $H \in \tau$.

LEMMA 10.2.9. *If $\tau \in \mathcal{O}^\star(V, G)$, then*

(1) *R_τ is a closed semialgebraic subset of $\mathbb{R}^{k-1}$.*
(2) *$k - d_\tau + n_\tau \leq r_\tau \leq k - 1$.*

PROOF. (1) Similar to that of lemma 7.1.1 and omitted. (2) By lemma 10.2.6 we have $\dim(R_\tau) \leq k - 1$. It remains to prove the first inequality of (2). Let $x \in V_\tau$. Then $T(x) = \{t \in \mathbb{R}^k \mid \boldsymbol{\vartheta}(t, x) \in T_x Gx\}$ is a linear subspace of $\mathbb{R}^k$ of dimension $k - d_\tau + n_\tau$. Consider $T(x)$ as a point in the Grassmann variety $\mathbf{Gr}_p(\mathbb{R}^k)$ of $p = k - d_\tau + n_\tau$-dimensional subspaces of $\mathbb{R}^k$. Replacing x by λx and letting $\lambda \to 0$, we see that each limit point of $T(\lambda x)$ is a $k - d_\tau + n_\tau$-dimensional subspace of $\mathbb{R}^k$ contained in R_τ. Hence $r_\tau \geq k - d_\tau + n_\tau$. (See also lemma 6.15.2 and examples 6.15.3(1).) □

EXAMPLE 10.2.10. Let (V, G) be an absolutely irreducible representation and τ be a maximal isotropy type. Suppose that $d_\tau = 1 + n_\tau$ and let $H \in \tau$. By lemma 10.2.9, $\dim(R_\tau) = \mathbb{R}^{k-1}$ and the proof of lemma 10.2.9 shows that we have $R_\tau = \mathbb{R}^{k-1}$ (and so τ is generically symmetry breaking – see definition 4.3.5). If $n_\tau = 0$, this is just a restatement of the equivariant branching lemma and $N(H)/H$ is either trivial or isomorphic to $\mathbb{Z}_2$ (see also examples 6.15.3(3)). If $n_\tau \neq 0$ and $d_\tau = 1 + n_\tau$, then the identity component of $N(H)/H$ is isomorphic to either S^1 or $\mathrm{SU}(2)$ and the associated action on V^H is irreducible and free [**26**, Chapter III, Theorem 8.5]. Melbourne [**124**] shows that both these possibilities can occur (see also remark 10.1.24).

From now on we let $\mathcal{A}^\star$ denote the stratification induced on $\Sigma^\star_{(G)} = \mathbb{R}^k$ by $\mathcal{S}^\star$. Denote the union of the i-dimensional strata of $\mathcal{A}^\star$ by $A^\star_i$, $i \geq 0$. If (V, G) is irreducible of $\mathbb{C}$-normal type, $A_i^\star = \mathbf{L}_2 \times A_{i-1}^{\star\star}$, where $A_i^{\star\star} \subset \mathbf{R}_2 \subset \mathbb{R}^k$. We have

$$\begin{aligned} A_k^\star &= \mathbb{R}^k \setminus \bigcup_{\tau \neq (G)} R_\tau, \\ A_k^{\star\star} &= \mathbf{R}_2 \setminus \bigcup_{\tau \neq (G)} R_\tau^\star, \\ A_i^\star &\subset \mathbb{R}^{k-1},\ i < k, \\ A_i^{\star\star} &\subset \mathbb{R}^{k-2},\ i < k-1, \text{ if } (V,G) \text{ is of } \mathbb{C}\text{-normal type.} \end{aligned}$$

If (V,G) is irreducible of $\mathbb{C}$-normal type, we let $\mathcal{A}^{\star\star}$ denote the Whitney stratification of $\mathbf{R}_2$ with strata given by $A_i^{\star\star}$, $i \leq k-1$.

Remark 10.2.11. $A_{k-1}^\star$ is an open semialgebraic subset of $\mathbb{R}^{k-1}$. If (V,G) is absolutely irreducible we follow the convention of remark 7.1.2 and always assume $A_{k-1}^\star$ is dense in $\mathbb{R}^{k-1}$. If (V,G) is irreducible of $\mathbb{C}$-normal type, $A_{k-1}^{\star\star}$ is always dense in $\mathbb{R}^{k-2}$.

Suppose that $X \in \mathcal{V}_0(V,G)$ and write $X(x,t) = \sum_{i=1}^k f_i(x,t)F_i(x)$, where f_i are smooth invariants. Exactly as in chapter 6, we may factorize X as $X = \boldsymbol{\vartheta} \circ \Gamma_X$, where $\Gamma_X : V \times \mathbb{R} \to V \times \mathbb{R}^k$ is the graph map

$$\Gamma_X(x,\lambda) = (x, (f_1(x,\lambda), \dots, f_k(x,\lambda))), \ \ (x,\lambda) \in V \times \mathbb{R}.$$

Define $\gamma(X) \in C^\infty(\mathbb{R}, \mathbb{R}^k)$ by $\gamma(X)(\lambda) = (\lambda, f_2(0,\lambda), \dots, f_k(0,\lambda))$. If (V,G) is irreducible of $\mathbb{C}$-normal type, we define $\gamma^\star(X) \in C^\infty(\mathbb{R}, \mathbf{R}_2)$ by $\gamma^\star(X)(\lambda) = (\lambda, f_3(0,\lambda), \dots, f_k(0,\lambda))$.

Lemma 10.2.12. *Let $X \in \mathcal{V}_0(V,G)$. Then $\Gamma_X \pitchfork \Sigma^\star$ at $(x,\lambda) = (0,0)$ if and only if $\gamma(X) \pitchfork \mathcal{A}^\star$ at $\lambda = 0$. If (V,G) is irreducible of $\mathbb{C}$-normal type, $\Gamma_X \pitchfork \Sigma^\star$ at $(x,\lambda) = (0,0)$ if and only if $\gamma^\star(X) \pitchfork \mathcal{A}^{\star\star}$ at $\lambda = 0$.*

Proof. The result follows from theorem 10.2.8. □

Remarks 10.2.13. (1) Lemma 10.2.12 allows us to prove weak stability and determinacy results for branches of relative equilibria along very similar lines to what we did in chapter 7 for branches of equilibria when G was finite. We give the main results in the next section. Note that this approach does not depend on the detailed structure of the *invariants* or reduction to the orbit space.
(2) The stratifications $\mathcal{A}^\star$, $\mathcal{A}^{\star\star}$ may be proved to be induced from a stratification of $\mathbb{U}$ defined independently of the choice of $\mathcal{F}$. It is also possible to prove invariance results along the lines of section 6.11. Some details and results may be found in [**60**, section 4].

10.3. Weak stability and determinacy

We continue to assume (V,G) is a representation which is either absolutely irreducible or irreducible of $\mathbb{C}$-normal type.

Define

$$\mathcal{K}^\star_G(V) = \{X \in \mathcal{V}_0(V,G) \mid \Gamma_X \pitchfork \Sigma^\star \text{ at } (0,0)\}.$$

THEOREM 10.3.1. (1) *$\mathcal{K}^\star_G(V)$ is an open and dense subset of $\mathcal{V}_0(V,G)$.*

(2) *$\mathcal{K}^\star_G(V) \subset \mathcal{S}^\star_w(V,G)$.*

(3) *Let $X \in \mathcal{K}^\star_G(V)$. We may find an open neighborhood $\mathcal{U}$ of X in $\mathcal{V}_0$ such that if $\{X_t \mid t \in [0,1]\}$ is any continuous path in $\mathcal{U}$ with $X_0 = X$, there is an open neighborhood W of $(0,0)$ in $V \times \mathbb{R}$ and an (equivariant) isotopy $\{K_t : W \to V \times \mathbb{R} \mid t \in [0,1]\}$ of (continuous) embeddings satisfying*
   (a) *K_0 is the inclusion of W in $V \times \mathbb{R}$.*
   (b) *$K_t(W \cap \mathbf{I}(X)) = \mathbf{I}(X_t) \cap K_t(W)$, all $t \in [0,1]$.*

PROOF. Similar to the proof of theorem 7.1.5. □

10.3.1. Symmetry breaking isotropy types. Let $\tau \in \mathcal{O}^\star(V,G)$. We extend our previous definitions of symmetry breaking isotropy types to allow for branches of relative equilibria. Specifically, we say that τ is *symmetry breaking* (respectively, *generically symmetry breaking*) if there exists a nonempty open (respectively, open and dense) subset $\mathcal{U}$ of $\mathcal{V}_0(V,G)$ such that for every $X \in \mathcal{U}$, the germ of $\mathbf{I}(X)$ at zero contains points of isotropy type τ.

LEMMA 10.3.2. *For all $\tau \in \mathcal{O}^\star(V,G)$, R_τ inherits a semialgebraic Whitney stratification $\mathcal{A}^\star_\tau$ from $\mathcal{A}^\star$. (We assume here that $A_k = \mathbb{R}^k \backslash \mathbb{R}^{k-1}$, see remark 7.1.2 and below for the case when (V,G) is irreducible of $\mathbb{C}$-normal type.)*

PROOF. Immediate from theorem 6.10.1 and the definition of $\mathcal{A}^\star$. □

PROPOSITION 10.3.3. *(Notation as above.) Let $X \in \mathcal{K}^\star_G(V)$ and $\tau \in \mathcal{O}^\star(V,G)$. Then*

(1) *The map $\gamma(X) : \mathbb{R} \to \mathbb{R}^k$ is transverse to $\mathcal{A}^\star_\tau$.*
(2) *If $r_\tau < k-1$, then $\mathcal{B}(X)$ contains no branches of relative equilibria of isotropy type τ.*
(3) *If $r_\tau = k-1$ and $\gamma(X)(0) \in R_\tau$, then there is a branch of relative equilibria of isotropy type τ in $\mathcal{B}(X)$.*

PROOF. The proof, using lemma 10.3.2, is the same as that of proposition 7.1.7. □

COROLLARY 10.3.4. *(Notation as above.) Let $\tau \in \mathcal{O}^\star(V,G)$.*

(1) *τ is a symmetry breaking isotropy type if and only if $r_\tau = k-1$.*
(2) *τ is generically symmetry breaking if and only if $R_\tau = \mathbb{R}^{k-1}$.*

EXAMPLE 10.3.5. Let (V,G) be irreducible of $\mathbb{C}$-normal type. If $H \in \tau \in \mathcal{O}(V,G)$, then V^H is a $\mathbb{C}$-linear subspace of V. If $\dim_{\mathbb{C}}(V^H) = 1$, then $R^\star_\tau = \mathbb{R}^{k-2}$ (trivially) and so τ is generically symmetry breaking. This is the complex version of the equivariant branching lemma (see [**84**, Chapter XVI]).

10.3.2. Weak determinacy. Set $d = d_k$ and define $P_G^{(d)}(V,V)_0 = \{P \in P_G^{(d)}(V,V) \mid DP(0) = 0\}$. Let $\Pi_0 : P_G^{(d)}(V,V)_0 \to \mathbb{R}^k$ be the restriction of the projection $\Pi : P_G(V,V) \to P_G(V,V)/\mathfrak{M}P_G(V,V) \cong \mathbb{R}^k$ to $P_G^{(d)}(V,V)_0$. Note that Π_0 maps $P_G^{(d)}(V,V)_0$ onto $\mathbb{R}^{k-1}$ if (V,G) is absolutely irreducible and maps onto $\mathbb{R}^{k-2}$ otherwise. If (V,G) is absolutely irreducible, we define

$$\mathcal{R}_w^\star(d) = \{P \in P_G^{(d)}(V,V)_0 \mid \Pi_0(P) \in A_{k-1}^\star\}.$$

If (V,G) is irreducible of $\mathbb{C}$-normal type, we define

$$\mathcal{R}_w^\star(d) = \{P \in P_G^{(d)}(V,V)_0 \mid \Pi_0(P) \in A_{k-1}^{\star\star}\}.$$

Since $A_{k-1}^\star$ is an open and dense semialgebraic subset of $\mathbb{R}^{k-1}$, $\mathcal{R}_w^\star(d)$ is an open and dense semialgebraic subset of $P_G^{(d)}(V,V)_0$. If $X \in C_G^\infty(V\times\mathbb{R},V)$, let $J_0^d(X) \in P_G^{(d)}(V,V)_0$ be the d-jet in V-variables of X_0, omitting the linear and constant terms. That is, $J_0^d(X)(x) = \sum_{j=2}^d D^jX(0)(x^j)/j!$.

LEMMA 10.3.6. *We have*

$$\mathcal{K}_G^\star(V) = \{X \in \mathcal{V}_0 \mid J_0^d(X) \in \mathcal{R}_w^\star(d)\}.$$

In particular, G-equivariant bifurcation problems on (V,G) are weakly d_w-determined, where $d_w \le d_k$, and $X \in \mathcal{S}_w^\star(V,G)$ if $J_0^d(X) \in \mathcal{R}_w^\star(d)$.

PROOF. Immediate from the definitions and lemma 10.2.12. □

THEOREM 10.3.7. *Let (V,G) be irreducible of $\mathbb{C}$-normal type. Every maximal isotropy type is generically symmetry breaking and the corresponding branch is a branch of* limit cycles. *In particular, if $X \in \mathcal{S}_w^\star(V,G)$, then $\mathcal{B}(X) \neq \emptyset$.*

PROOF. Let $X \in \mathcal{S}_w^\star(V,G)$. Without changing $\mathcal{B}(X)$, we may perturb X so that $X \in \mathcal{K}_G^\star(V)$. By weak determinacy we have $X^a = X + a\|x\|^2x \in \mathcal{K}_G^\star(V)$, for all $a \in \mathbb{C}$. In particular, we may choose $a \in \mathbb{R}$ so that X^a satisfies the conditions of the invariant sphere theorem (theorem 5.6.24). Consequently, there exists $\lambda_0 > 0$ and a branch $S(\lambda)$ of G-invariant, attracting and flow-invariant spheres $S(\lambda)$ for X_λ^a, $\lambda \in (0,\lambda_0)$. Let $H \in \tau \in \mathcal{O}^\star(V,G)$ be a maximal isotropy type. Then $S(\lambda) \cap V^H$ will be flow-invariant and so, quotienting by the S^1-action, we obtain a flow on $\mathbb{P}^s(\mathbb{C})$, where $s = \dim_{\mathbb{C}}(V^H) - 1$. Since the Euler characteristic $\chi(\mathbb{P}^s(\mathbb{C})) = s+1 > 0$, there is a least one zero for the induced flow on $\mathbb{P}^s(\mathbb{C})$ and so X^a has at least one branch of limit cycles of isotropy type τ. □

REMARKS 10.3.8. (1) Theorem 10.3.7 was originally proved by Fiedler [**47**] and is referred to in [**84**] (Theorem 4.5, Chapter XVI). Theorem 10.3.7 can be strengthened to take account of stabilities and also allow for breaking the normal form symmetries [**60, 62**]. We show shortly how this gives a nice lower bound on the number of branches of limit cycles.
(2) A consequence of theorem 10.3.7 is that $A_{k-1}^\star$ is always dense in $\mathbb{R}^{k-1}$ (cf remark 7.1.2).

(3) Suppose that (V,G) is irreducible of $\mathbb{C}$-normal type. Let $X \in C^\infty_G(V\times\mathbb{R},V)$ and suppose that $DX_0(0)=0$. If $\gamma(X) \pitchfork \mathbb{R}^{k-1}$, and τ is a symmetry breaking isotropy type with $n_\tau = 1$, then branches of isotropy type τ will generically be branches of limit cycles. The period of the limits cycles $\to \infty$ as $\lambda \to 0$.

EXAMPLES 10.3.9. (1) Take a nontrivial representation of $\mathrm{O}(2)$ on $\mathbb{R}^2=\mathbb{C}$. Then $(\mathbb{R}^2,\mathrm{O}(2))$ is absolutely irreducible. It is easy to verify that $\mathrm{O}(2)$-equivariant bifurcation problems are weakly 2-determined (if $X \in \mathcal{V}_0(\mathbb{R}^2,\mathrm{O}(2))$, X is already weakly stable). In this case, the single branch is a branch of equilibrium $\mathrm{O}(2)$-orbits.
(2) Let $M = \mathrm{O}(2)\times_{\mathbb{Z}_2}\mathbb{R}$, where $\mathbb{Z}_2$ is generated by $\kappa(x,y)=(x,-y)$ and acts on $\mathbb{R}$ as multiplication by -1. Let $\gamma = \mathrm{O}(2)\bar{0}$, where $\bar{0}=[e,0]\in\mathrm{O}(2)\times_{\mathbb{Z}_2}\mathbb{R}$. In this case generic bifurcation off the equilibrium orbit γ leads to a pair of contra-rotating periodic orbits $\gamma^\pm_\lambda$ with equal periods $p_\lambda\to\infty$, as $\lambda\to 0$. All of this follows easily by writing $\mathrm{O}(2)$-equivariant vector fields X on M in tangent and normal form, $X = X_T + X_N$, and noting that X_N and X_T are $\mathbb{Z}_2$-equivariant.

10.4. Weak stability and determinacy for reversible systems

We indicate how the results of the previous section generalize to equivariant reversible systems. (The results presented in this section are not used elsewhere in the chapter and the reader may pass safely to the next section where we start our investigation of stabilities of branches of relative equilibria.)

Throughout this section we assume that V is a finite dimensional real inner product space and G is a compact Lie group.

Suppose we are given a pair of orthogonal representations $\rho,\sigma : G\to\mathrm{O}(V)$. Let ${}_\rho V$ denote V with the action on V determined by ρ. We similarly define ${}_\sigma V$. We always assume that

$$T_x\rho(G)x = T_x\sigma(G)x, \text{ for all } x\in V. \tag{10.2}$$

We remark that this is no restriction if G is finite and that (10.2) also holds if the two actions have the same G-orbits (cf lemma 8.3.42).

We consider G-equivariant vector fields $f : {}_\rho V \to {}_\sigma V$. An important class where $\rho\neq\sigma$ and (10.2) holds is given by the *reversible equivariant vector fields*. For this class, we fix an index two subgroup K of G and consider representations ρ,σ which restrict to the same representation of K but differ by a sign on $G\setminus K$. That is, $\rho(g) = -\sigma(g)$, $g\in G\setminus K$.

More formally, we shall assume that there is a homomorphism $r: G\to\mathbb{Z}_2\subset\mathrm{O}(V)$, where $\mathbb{Z}_2$ is generated by $-I_V$. We set $K=\ker(r)$. If r is trivial, $K=G$ (and so $\rho=\sigma$). If r is non-trivial, then K is an index 2 subgroup of G (the group of spatial symmetries).. Set $G^R = G\setminus K$. Elements of G^R are called *time reversing symmetries*. We may define the representation σ in terms of ρ and r by

$$\sigma(g) = r(g)\rho(g),\ g\in G. \tag{10.3}$$

Usually, we just write gx instead of $\rho(g)x$, and $r(g)gx$ instead of $\sigma(g)x$, $x \in V$. It follows from this description of σ that the representations ρ, σ satisfy (10.2).

Henceforth, assume the representations ρ, σ satisfy (10.3) where $\rho(g) = -\sigma(g)$, $g \in G \setminus K$, and that r is non-trivial. We refer to G-equivariant vector fields $X : {}_\rho V \to {}_\sigma V$ as *reversible G-equivariant vector fields.*

EXERCISE 10.4.1. (1) Show that the composition of two elements of G^R lies in K.
(2) Show that $\dim(K) = \dim(G)$ and that K is a union of connected components of G. In particular, K/G_0 is an index two subgroup of G/G_0.

LEMMA 10.4.2. *Let $({}_\rho V, G)$, $({}_\sigma V, G)$ be representations of G satisfying (10.3). If $X : {}_\rho V \to {}_\sigma V$ is a smooth reversible G-equivariant vector field, then the flow ϕ of X satisfies*

$$\phi_{gx}(t) = g\phi_x(r(g)t),$$

where the G-action on domain and range is defined by ρ, and ϕ_x denotes the integral curve of X through $x \in V$.

PROOF. We have $X(gx) = r(g)gX(x)$, $g \in G$. We claim that if $\phi_x(t)$ is the integral curve of X through $x \in V$, then $\psi(t) = g\phi_x(r(g)t)$ is the integral curve through gx. Differentiating ψ with respect to t we have

$$\begin{aligned} \psi'(t) &= r(g)g\phi_x'(r(g)t), \\ &= r(g)gX(\phi_x(r(g)t)), \text{ since } \phi_x \text{ is an integral curve,} \\ &= X(g\phi_x(r(g)t)), \text{ by equivariance,} \\ &= X(\psi(t)). \end{aligned}$$

Hence, by uniqueness of integral curves, $\psi(t)$ is the integral curve of X through $\psi(0) = gx$. □

LEMMA 10.4.3. *Let $X : {}_\rho V \to {}_\sigma V$ be a smooth reversible G-equivariant vector field. If $X(x) \in T_xGx$ at some point $x \in V$, then $X(x) \in {}_\sigma V^{G_x} \cap T_xGx$ and*

$$X(x) \in T_xN(K_x)x \cap {}_\sigma V^{G_x}.$$

PROOF. Let $x \in {}_\rho V$ and $X(x) \in T_xGx$. For all $g \in G_x$, we have $X(x) = X(gx) = \sigma(g)X(x)$ and so, by (10.2) $X(x) \in {}_\sigma V^{G_x} \cap T_xGx$. The final statement uses the K-equivariance of X. □

DEFINITION 10.4.4. Let $X : {}_\rho V \to {}_\sigma V$ be a smooth G-equivariant vector field. If the G-orbit Gx is invariant by the flow of X (equivalently, if X is tangent to Gx), then Gx is called a *relative equilibrium* of X.

REMARK 10.4.5. An orbit Gx is a relative equilibrium for $X : {}_\rho V \to {}_\sigma V$ if and only if there exists at least one point $y \in Gx$ such that $X(y) \in T_yGx$.

EXAMPLE 10.4.6. Let ${}_\rho\mathbb{C}$ denote the standard representation of $\mathrm{O}(2)$ on $\mathbb{C}$. Define the representation ${}_\sigma\mathbb{C}$ by $\sigma(g) = \det(\rho(g))\rho(g)$. Let $z \neq 0$. If $\mathrm{O}(2)z$ is a relative equilibrium of $X : {}_\rho\mathbb{C} \to {}_\sigma\mathbb{C}$, then the only restriction on $X(z)$ is that $X(z)$ is tangent to $\mathrm{O}(2)z$. In particular, relative equilibria will typically be periodic orbits.

10.4.1. The universal variety. In what follows we assume that ${}_\rho V^G = \{0\}$ (it need not be the case that ${}_\sigma V^G = \{0\}$). Let $\mathcal{F} = \{F_1, \dots, F_k\}$ denote a minimal set of homogeneous generators for the $P({}_\rho V)^G$-module $P_G({}_\rho V, {}_\sigma V)$. Let $p_1, \dots, p_\ell$ be a minimal set of homogeneous generators for the $\mathbb{R}$-algebra $P({}_\rho V)^G$ and $q_1, \dots, q_m$ be a minimal set of homogeneous generators for the $\mathbb{R}$-algebra $P({}_\sigma V)^G$.

As usual we define $\boldsymbol{\vartheta} : {}_\rho V \times \mathbb{R}^k \to {}_\sigma V$ by $\boldsymbol{\vartheta}(x,t) = \sum_{j=1}^k t_j F_j(x)$. We let

$$\boldsymbol{\Sigma}^\star = \{(x,t) \mid \boldsymbol{\vartheta}(x,t) \in T_x Gx\}.$$

For each $\tau \in \mathcal{O}({}_\rho V, G)$, let $\boldsymbol{\Sigma}^\star_\tau$ denote the subset of $\boldsymbol{\Sigma}^\star$ consisting of points of isotropy type τ. Clearly $\boldsymbol{\Sigma}^\star$ is the disjoint union over $\mathcal{O}({}_\rho V, G)$ of the sets $\boldsymbol{\Sigma}^\star_\tau$.

LEMMA 10.4.7. *$\boldsymbol{\Sigma}^\star$ is a G-invariant* algebraic *subset of $V \times \mathbb{R}^k$.*

PROOF. Let $Q : V \to \mathbb{R}^m$ be the orbit map determined by $q_1, \dots, q_m$. If we define the polynomial map $T : V \times \mathbb{R}^k \to \mathbb{R}^\ell$ by

$$T(x,t) = DQ(x)(\boldsymbol{\vartheta}(x,t)), \ \ ((x,t) \in V \times \mathbb{R}^k),$$

then $\boldsymbol{\Sigma}^\star$ is the zero set of T. $\square$

REMARK 10.4.8. Noting (10.2), we might just as well have defined the map T of lemma 10.4.7 using the orbit map $P : V \to \mathbb{R}^\ell$.

Before stating the next lemma, we need to review some notation. Let $H \in \tau \in \mathcal{O}({}_\rho V, G)$ and suppose $x \in {}_\rho V_\tau{}^H$. We define $g_\tau = \dim(Gx)$. We set $n_\tau(\rho) = \dim(N(H)/H) = \dim(N(H)x)$ and $n_\tau(\sigma) = \dim(T_x Gx \cap {}_\sigma V^H)$. We define

$$N_\tau = n_\tau(\sigma) - n_\tau(\rho).$$

Finally, we recall from section 6.9.1 that $d_\tau = d_\tau({}_\rho V) = \dim({}_\rho V_\tau{}^H)$, $e_\tau = e_\tau({}_\sigma V) = \dim({}_\sigma V^H)$, and $i_\tau = i_\tau({}_\rho V, {}_\sigma V) = d_\tau - e_\tau$. (If $\rho = \sigma$, then $n_\tau(\sigma) = n_\tau(\rho)$ and $i_\tau = 0$.)

LEMMA 10.4.9. *For each $\tau \in \mathcal{O}({}_\rho V, G)$, $\boldsymbol{\Sigma}^\star_\tau$ is a Γ-invariant smooth semi-algebraic submanifold of $V \times \mathbb{R}^k$. We have*

$$\dim(\boldsymbol{\Sigma}^\star_\tau) = k + g_\tau + N_\tau + i_\tau.$$

PROOF. Since $\boldsymbol{\Sigma}^\star_\tau = \boldsymbol{\Sigma}^\star \cap (V \times \mathbb{R}^k)_\tau$, it follows from lemma 10.4.7 that $\boldsymbol{\Sigma}^\star_\tau$ is a G-invariant semialgebraic subset of $V \times \mathbb{R}^k$. It remains to prove that $\boldsymbol{\Sigma}^\star_\tau$ is a smooth submanifold of $V \times \mathbb{R}^k$ of the specified dimension. Fix $(x,t) \in \boldsymbol{\Sigma}^\star_\tau$ and set $G_x = H$. For each $y \in {}_\rho V_\tau{}^H$, $\{F_1(y), \dots, F_k(y)\}$ spans ${}_\sigma V^H$. Set $N(y) = T_y Gy \cap$

${}_\sigma V^H$. Note that $N(y)$ is a linear subspace of ${}_\sigma V^H$ and that $\dim(N(y)) = n_\tau(\sigma)$. Let $\pi_y : V^H \to V^H$ denote the projection of ${}_\sigma V^H$ on the orthogonal complement of $N(y)$ in ${}_\sigma V^H$. Clearly π_y depends smoothly on $y \in {}_\rho {V_\tau}^H$ (as a map into the space of linear endomorphisms of ${}_\sigma V^H$). Let $S \subset {}_\rho V \times \mathbb{R}^k$ be a differentiable slice for the action of G on ${}_\rho V \times \mathbb{R}^k$ at (x,t). (For S we may take a sufficiently small Euclidean ball centered at (x,t) in $(x,t) + (T_x G(x,t))^\perp$.) Let $S_\tau \subset S$ be the set of points of isotropy type τ and note that $S_\tau \subset {}_\rho {V_\tau}^H \times \mathbb{R}^k$ and $\dim(S_\tau) = d_\tau - n_\tau(\rho) + k$. Consider the smooth map $K : S_\tau \to {}_\sigma V^H$ defined by

$$K(y,s) = \pi_y \boldsymbol{\vartheta}(y,s)$$

At all points $(y,s) \in \boldsymbol{\Sigma}_\tau^\star \cap S_\tau$, $DK(y,s)$ is surjective of rank $\dim({}_\sigma V^H) - n_\tau(\sigma)$ and kernel of dimension $k + N_\tau + i_\tau$. Since $\boldsymbol{\Sigma}_\tau^\star \cap S_\tau = K^{-1}(0)$, we may use the rank theorem to deduce that $\boldsymbol{\Sigma}_\tau^\star \cap S_\tau$ is smooth of dimension $k + N_\tau + i_\tau$ and hence that $\boldsymbol{\Sigma}_\tau^\star \cap GS_\tau$ is smooth of dimension equal to $k + g_\tau + N_\tau + i_\tau$. $\square$

Take the canonical stratification $\mathcal{S}^\star$ of $\boldsymbol{\Sigma}^\star$ and let $\mathcal{S}_\tau^\star$ denote the stratifications induced on $\boldsymbol{\Sigma}_\tau^\star$. Note that it will *not* generally be the case that $\mathcal{S}_\tau^\star$ is a union of $\mathcal{S}^\star$-strata. Set $\mathcal{A}^\star = \mathcal{S}_{(G)}^\star$. We denote the i-dimensional stratum of $\mathcal{A}^\star$ by $A_i^\star$, $0 \le i \le k$.

Set $\mathcal{V} = C_G^\infty({}_\rho V \times \mathbb{R}, {}_\sigma V)$. If $X \in \mathcal{V}$, $DX_\lambda(0) \in L_G({}_\rho V, {}_\sigma V)$. Unless ${}_\rho V$ and ${}_\sigma V$ are isomorphic G-representations, $DX_\lambda(0)$ will be singular. Roughly speaking, a bifurcation point of the family X will be a value of λ for which the topological type of the germ of $\mathbf{I}(X_\lambda)$ at zero changes. In the usual way, every $X \in \mathcal{V}$ factorizes as $X = \boldsymbol{\vartheta} \circ \Gamma_X$, where $\Gamma_X(x,\lambda) = (x, (f_1(x,\lambda), \ldots, f_k(x,\lambda))$. We define $\gamma(X) \in C^\infty(\mathbb{R}, \mathbb{R}^k)$ by $\gamma(X)(\lambda) = (f_1(0,\lambda), \ldots, f_k(0,\lambda))$.

Definition 10.4.10. We say $\lambda_0 \in \mathbb{R}$ is a *regular* point for the family $X_\lambda \in \mathcal{V}$ if $\gamma(X)(\lambda_0) \notin \cup_{i=0}^{k-1} A_i^\star$. If $\gamma(X)(\lambda_0) \in \cup_{i=0}^{k-1} A_i^\star$, we refer to λ_0 as a *bifurcation point*.

Let $\mathcal{V}_0$ be the subset of $\mathcal{V}$ for which there is a bifurcation point at $\lambda = 0$. Define

$$\mathcal{K}_G^\star({}_\rho V, {}_\sigma V) = \{X \in \mathcal{V}_0 \mid \gamma(X) \pitchfork \mathcal{A}^\star \text{ at } (0,0)\}.$$

The proofs of the remaining lemmas are by now routine and are omitted.

Lemma 10.4.11. (1) *$X \in \mathcal{K}_G^\star({}_\rho V, {}_\sigma V)$ if and only if $\gamma(X)(0) \in A_{k-1}^\star$ and $\gamma(X) \pitchfork A_{k-1}^\star$ at $\lambda = 0$. (We allow $A_{k-1}^\star = \emptyset$.)*
(2) *If $X \in \mathcal{K}_G^\star({}_\rho V, {}_\sigma V)$, then $\lambda = 0$ is an isolated bifurcation point of X.*
(3) *$\mathcal{K}_G^\star({}_\rho V, {}_\sigma V)$ is an open and dense subset of $\mathcal{V}_0$.*

Lemma 10.4.12. *If $X \in \mathcal{K}_G^\star({}_\rho V, {}_\sigma V)$, we may choose an open neighborhood $\mathcal{U}$ of X in $\mathcal{K}_G^\star({}_\rho V, {}_\sigma V)$ such that if $\{X_t \mid t \in [0,1]\}$ is any continuous path in $\mathcal{U}$ with $X_0 = X$, there is an open neighborhood W of $(0,0)$ in ${}_\rho V \times \mathbb{R}$ and a G-equivariant isotopy $\{K_t : W \to {}_\rho V \times \mathbb{R} \mid t \in [0,1]\}$ of (continuous) embeddings satisfying*

(a) *K_0 is the inclusion of W in ${}_\rho V \times \mathbb{R}$.*

(b) $K_t(W \cap \mathbf{I}(X)) = \mathbf{I}(X_t) \cap K_t(W)$, *all* $t \in [0,1]$.

For $\tau \in \mathcal{O}({}_\rho V, G)$, $\tau \neq (G)$, define $R^\star_\tau = \mathbb{R}^k \cap \mathbf{\Sigma}^\star_\tau$ and $r^\star_\tau = \dim(R^\star_\tau)$.

LEMMA 10.4.13. *Let* $\tau \in \mathcal{O}({}_\rho V, G)$, $\tau \neq (G)$. *Then*

$$k - e_\tau + n_\tau(\sigma) \leq r^\star_\tau \leq \min\{k, k + N_\tau + i_\tau - 1\}.$$

COROLLARY 10.4.14. *Let* $X \in \mathcal{K}^\star_G({}_\rho V, {}_\sigma V)$. *If* $k - e_\tau + n_\tau(\sigma) = k$, *then the germ at zero of* $\mathbf{I}(X))$ *will always contain points of isotropy type* τ. *If* $k+N_\tau+i_\tau < k$, *the germ at zero of* $\mathbf{I}(X))$ *contains no points of isotropy type* τ.

Let $d = d_k$ and $\Pi_0 : P^{(d)}_G({}_\rho V, {}_\sigma V) \to \mathbb{R}^k$ denote the restriction of $\Pi : P_G({}_\rho V, {}_\sigma V) \to P_G({}_\rho V, {}_\sigma V)/\mathfrak{M}P_G({}_\rho V, {}_\sigma V)$ to $P^{(d)}_G({}_\rho V, {}_\sigma V)$. We define

$$\mathcal{R}^\star_w(d) = \{P \in P^{(d)}_G({}_\rho V, {}_\sigma V) \mid \Pi_0(P) \in A^\star_{k-1}\}$$

PROPOSITION 10.4.15 (Finite determinacy). *Let* $X \in \mathcal{V}_0$. *Then* $X \in \mathcal{K}^\star_G({}_\rho V, {}_\sigma V)$ *if and only if*

(1) $\gamma(X)(0) \pitchfork A^\star_{k-1}$ *(a condition on* $j^1_2 X(0)$*).*
(2) $J^d(X) \in \mathcal{R}^\star_w(d)$ *(a condition on* $j^d_1 X(0)$*).*

10.5. Stability and determinacy

In this section we generalize the stability and determinacy theorems of chapter 7 to non-finite compact groups and branches of relative equilibria.

Throughout this section (V, G) will denote a representation of the compact Lie group G which is either absolutely irreducible or irreducible of $\mathbb{C}$-normal type.

Let $TV = V \times V$ denote the tangent bundle of V. Define the bundle of tangent vectors to G-orbits by

$$T_G V = \{(x, X) \in TV \mid X \in T_x Gx\}.$$

Although $T_G V$ is a G-invariant subset of TV, it is neither closed nor a smooth subbundle of TV except in the case when G_0 acts trivially on V. We may write $T_G V$ as the disjoint union over $\mathcal{O}(V, G)$ of the sets $T_G V_\tau = T_G V \cap (V_\tau \times V)$.

EXERCISE 10.5.1. Take the standard irreducible representation $(\mathbb{C}, \mathrm{SO}(2))$. What are (a) $T_{\mathrm{SO}(2)}\mathbb{C}$, (b) $\overline{T_{\mathrm{SO}(2)}\mathbb{C}}$?

LEMMA 10.5.2. *For all* $\tau \in \mathcal{O}(V, G)$, $T_G V_\tau$ *is a semialgebraic subset of* TV. *In particular,* $T_G V$ *is a semialgebraic subset of* TV.

PROOF. Let $\tau \in \mathcal{O}(V, \Gamma)$. Pick a minimal set of homogeneous generators for $P(V)^G$ and let $P : V \to \mathbb{R}^\ell$ denote the corresponding orbit map. If we let $P_\tau = P|V_\tau$, then P_τ maps V_τ submersively onto $V_\tau/G \subset \mathbb{R}^\ell$, V_τ/G is a semialgebraic smooth submanifold of $\mathbb{R}^\ell$ and the fibers of the map P_τ are G-orbits. We have $TP_\tau : TV_\tau \to T(V_\tau/G) \subset T\mathbb{R}^\ell$. Since the fibers of P_τ are G-orbits, $TP_\tau(x, X) = (P(x), 0)$ if and only if $X \in T_x Gx$. We have $T_G V_\tau = (TP_\tau)^{-1}(T_0(V_\tau/G))$, where

$T_0(V_\tau/G)$ is the zero-section of $T(V_\tau/G)$. Since $T_0(V_\tau/G) \subset T\mathbb{R}^\ell$ is semialgebraic and TP_τ is a polynomial map, $T_G V_\tau$ is a semialgebraic subset of TV. $\square$

For $\tau \in \mathcal{O}(V,G)$, define

$$\begin{aligned} Z_0(\tau) &= \{(x,X) \in V \times V \mid x \in V_\tau,\ X \in T_x N(G_x)x\}, \\ Z_0 &= \text{closure}(\bigcup_{\tau \in \mathcal{O}} Z_0(\tau)). \end{aligned}$$

LEMMA 10.5.3. *For every $\tau \in \mathcal{O}(V,G)$, $Z_0(\tau)$ is a semialgebraic G-invariant submanifold of $V \times V$. In particular, Z_0 is semialgebraic.*

PROOF. Obviously, $Z_0(\tau)$ is G-invariant and it is easy to show that $Z_0(\tau)$ is a smooth submanifold of $V \times V$. Since $Z_0(\tau) = T_G V_\tau \cap (TV)_\tau$, it follows from lemma 10.5.2 that $Z_0(\tau)$ is semialgebraic. $\square$

We need the following extension of lemma 8.4.2.

LEMMA 10.5.4. *Let $\tau \in \mathcal{O}(V,G)$. There exists a continuous map $\Xi : Z_0(\tau) \to C^\infty_G(V,V)$ satisfying*

1. *For all $(x,X) \in Z_0(\tau)$, $\Xi(x,X)$ is everywhere tangent to G-orbits.*
2. *$\Xi(x,X)(x) = X$, for all $(x,X) \in Z_0(\tau)$.*
3. *$\Xi(x,0) \equiv 0$, all $(x,0) \in Z_0(\tau)$.*
4. *The map $\mathcal{J} : Z_0(\tau) \to L(V,V)$ defined by*

$$\mathcal{J}(x,X) = D\Xi(x,X)(x)$$

is smooth and semialgebraic[1].

Before we prove lemma 10.5.4 we need some preliminaries. Let (V,G) be an orthogonal representation. Given $x \in V_\tau$, let $L(x) = x + (T_x Gx)^\perp \subset T_x V$. For $r > 0$, let $S_x(r) = \{v \in L(x) \mid \|v - x\| < r\}$. For sufficiently small $r > 0$ (depending on Gx), $S_x(r)$ is a differentiable slice for the action of G on V at x. Using an equivariant partition of unity, we may construct a smooth strictly positive function $\eta \in C^\infty(V_\tau)^G$ such that $S_x(\eta(x))$ is a slice for all $x \in V_\tau$ ($\eta(x) \to 0$ as $x \to \partial V_\tau$).

Quotienting out by the kernel of the representation $G \to \mathrm{O}(V)$, it is no loss of generality to assume that $G \subset \mathrm{O}(V)$. Fix $H \in \tau$ and define

$$Z_0(H) = \{(x,X) \in Z_0(\tau) \mid G_x = H\} = Z_0(\tau)^H.$$

Let $\mathfrak{o}_V$ denote the lie algebra of $\mathrm{O}(V)$ and identify $\mathfrak{o}_V$ with the subspace of $L(V,V)$ consisting of skew-symmetric maps. Let $\mathfrak{c} \subset \mathfrak{o}_V \subset L(V,V)$ denote the Lie algebra of $C_G(H)$. Choose $c_1, \ldots, c_p \in \mathfrak{c}$ which project to a vector space basis of $\mathfrak{c}/(\mathfrak{h} \cap \mathfrak{c})$ – the Lie algebra of $C_G(H)/(H \cap C_G(H))$. We have

$$\frac{d}{dt}(\exp(tc_i)(x))|_{t=0} = c_i(x),\ 1 \le i \le p,$$

[1] That is, the *graph* is semialgebraic.

and so $\{c_1(x), \dots, c_p(x)\}$ defines a vector space basis of $T_x N(H)x$ for all $x \in V_\tau^H$. Consequently, for every $(x, X) \in Z_0(H)$, there exists a unique $\phi(x, X) = (\phi_1(x, X), \dots, \phi_p(x, X)) \in \mathbb{R}^p$ such that

$$X = \sum_{i=1}^{p} \phi_i(x, X) c_i(x).$$

Lemma 10.5.5. *The map $\phi : Z_0(H) \to \mathbb{R}^p$ is smooth and semialgebraic.*

Proof. Since

$$\mathrm{graph}(\phi) = \{((x, X), a) \in Z_0(H) \times \mathbb{R}^p \mid \sum_{i=1}^{p} a_i c_i(x) = X\},$$

is semialgebraic, by definition ϕ is semialgebraic. The proof that ϕ is smooth is routine. □

Proof of lemma 10.5.4. Fix $H \in \tau$ and choose $c_1, \dots, c_p \in \mathfrak{c}$ as described above. Set $s = \dim(V) - g_\tau$ and note that $\dim(L(x)) = s$, all $x \in V_\tau^H$. The map $V_\tau^H \to \mathbf{Gr}_s(V)$, $x \mapsto L(x) - x$, is smooth and so

$$\Pi = \{((x, X), y) \mid (x, X) \in Z_0(H),\ y \in L(x)\}$$

is a smooth submanifold of $Z_0(H) \times V$. Define $\Omega : \Pi \to V$ by

$$\Omega((x, X), y) = \sum_{i=1}^{p} \phi_i(x, X) c_i(y)$$

The map Ω is smooth by lemma 10.5.5. By definition of ϕ, we have $\Omega((x, X), x) = X$, for all $(x, X) \in Z_0(H)$. For $(x, X) \in Z_0(H)$, define the smooth map $\Omega_{x,X} : L(x) \to V$ by $\Omega_{x,X}(y) = \Omega((x, X), y)$, $y \in L(x)$. Since $h \exp(c) h^{-1} = \exp(c)$, for all $h \in H$, $c \in \mathfrak{c}$, $\Omega_{x,X}$ is H-equivariant. Obviously, for all $y \in L(x)$, $(x, X) \in Z_0(H)$, we have

$$\Omega_{x,X}(y) \in T_y G y$$

Choose a smooth function $\psi : \mathbb{R} \to [0, 1]$ satisfying: (1) $\psi(t) = 1$, $|t| \leq 1$, and (2) $\psi(t) = 0$, $|t| \geq 3/2$. Let $\eta \in C^\infty(V_\tau)^G$ be as defined above.

For every $(x, X) \in Z_0(H)$, define $\hat{\Omega}_{x,X} : L(x) \to V$ by

$$\begin{aligned} \hat{\Omega}_{x,X}(y) &= \psi(\tfrac{|y-x|}{2}) \Omega_{x,X}(y),\ y \in S_x(\eta(x)) \\ &= 0, \text{ if } y \in L(x) \setminus S_x(\eta(x)) \end{aligned}$$

Extend $\hat{\Omega}_{x,X}$ by G-equivariance to a smooth G-equivariant vector field $\Xi(x, X)$ on V. Note that $\Xi(x, X)$ is everywhere tangent to G-orbits and is identically zero outside of a neighborhood of Gx in V. Restricted to Gx, $\Xi(x, X)$ is equal to the G-equivariant extension of X to Gx.

Our constructions define a continuous map $\Xi : Z_0(H) \to C^\infty_G(V, V)$. The map Ξ clearly extends to all of $Z_0(\tau)$ to give a map satisfying (1,2,3) of lemma 10.5.4.

It remains to show that the derivative map $\mathcal{J}$ satisfies (4). For this it suffices to compute the derivative of $\Xi(x, X)$ restricted to $L(x)$ and to T_xGx and verify that both maps are semialgebraic. Computing, we find that for all $(x, X) \in Z_0(H)$, we have

$$\begin{aligned} D\Xi(x,X)(x)|L(x) &= \sum_{i=1}^{p} \phi_i(x,X)c_i \\ D\Xi(x,X)(x)(c_j(x)) &= c_j(X),\ 1 \leq j \leq p. \end{aligned}$$

Hence $\mathcal{J}$ is semialgebraic. □

For $\tau \in \mathcal{O}(V, G)$, let $L_\tau(V, V)$ denote the subset of $L(V, V)$ consisting of maps that have at least $g_\tau + 1$ eigenvalues on the imaginary axis (counting multiplicities). We define the subset $Z_1(\tau)$ of $J^1(V, V) = V \times V \times L(V, V)$ by

$$Z_1(\tau) = \{((x, X), A) \in Z_0(\tau) \times L(V, V) \mid A - \mathcal{J}(x, X) \in L_\tau(V, V)\}$$

LEMMA 10.5.6. (1) *Let $x \in V_\tau, A \in L(V, V)$. Then $((x, 0), A) \in Z_1(\tau)$ if and only if $A \in L_\tau(V, V)$.*

(2) *Suppose $X \in C^\infty_G(TV)$ has relative equilibrium α of isotropy type τ. Then α is generic if and only if $((x, X(x)), DX(x)) \notin Z_1(\tau)$, $x \in \alpha$.*

PROOF. (1) is trivial since $\mathcal{J}(x, 0) = 0$. (2) follows from lemma 8.4.4 and the definition of genericity. □

REMARK 10.5.7. In general the set $Z_1(\tau)$ depends on $\mathcal{J}$. However, this does not affect our intended applications to generic relative equilibria on account of (2) of lemma 10.5.6. Alternatively, we may give an invariant definition of the *equivariant* singular 1-jets by defining

$$\tilde{Z}_1(\tau) = \{(x, X), A) \in Z_1(\tau) \mid A \in L_{G_x}(V, V)\}.$$

If we take the G-action on $Z_1(\tau)$ defined by $g((x, X), A) = ((gx, gX), gAg^{-1})$, then $\tilde{Z}_1(\tau) = Z_1(\tau)_\tau$ and $\tilde{Z}_1(\tau)^H = Z_1(\tau)^H$. Lemma 10.5.6 continues to hold with $Z_1(\tau)$ replaced by $\tilde{Z}_1(\tau)$.

LEMMA 10.5.8. *For all $\tau \in \mathcal{O}(V, G)$, $Z_1(\tau)$ is a G-invariant semialgebraic subset of $J^1(V, V)$. (G acts on $J^1(V, V)$ as $((x, X), A) \mapsto (gx, gX), gAg^{-1})$.)*

PROOF. Using the method of [**1**, §30], $L_\tau(V, V)$ is a closed semialgebraic subset of $L(V, V)$. Since $A \in L_\tau(V, V)$ if and only if $gAg^{-1} \in L_\tau(V, V)$, $L_\tau(V, V)$ a G-invariant subset of $L(V, V)$. Now apply lemma 10.5.4(4). □

If we define

$$Z_1 = \text{closure}\left(\bigcup_{\tau \in \mathcal{O}} Z_1(\tau)\right),$$

then, by lemma 10.5.8, Z_1 is a closed G-invariant semialgebraic subset of $J^1(V, V)$.

LEMMA 10.5.9. *Let $f \in C^\infty_G(V, V)$ and suppose that $\alpha \subset V$ is a relative equilibrium of f. Then α is generic if and only if $j^1 f(\alpha) \cap Z_1 = \emptyset$.*

PROOF. Let $\alpha \subset V_\tau$ and suppose that $j^1 f(\alpha) \cap Z_1 = \emptyset$. Necessarily, $j^1 f(\alpha) \cap Z_1(\tau) = \emptyset$ and so by lemma 10.5.6(2) α is generic. Conversely, if $\alpha \subset V_\tau$ is generic, then $j^1 f(\alpha) \cap Z_1(\tau) = \emptyset$. If $j^1 f(\alpha) \cap Z_1 \neq \emptyset$, then by G-invariance $j^1 f(\alpha) \subset Z_1$ and there exists $\mu < \tau$ such that $j^1 f(\alpha) \subset \overline{Z}_1(\mu)$. Since $\mu < \tau$, $g_\mu \geq g_\tau$. Choosing a sequence in $Z_1(\mu)$ converging to $j^1 f(\alpha)$, we see that $\mathbf{vspec}(\alpha, f) \geq g_\tau + 1$, contradicting the genericity of α. Hence $j^1 f(\alpha) \cap Z_1 = \emptyset$. $\square$

In order to prove stability and determinacy theorems for families we follow the strategy of chapter 7. We briefly review the setup for equivariant jet transversality.

Fix minimal homogeneous sets of generators $\mathcal{P} = \{p_1, \dots, p_\ell\}$ for $P(V)^G$ and $\mathcal{F} = \{F_1, \dots, F_k\}$ for $P_G(V, V)$.

Recall that for $s \geq 1$, there is a natural projection map $\pi_1 : J^1(V \times \mathbb{R}^s, V) \to J^1(V, V)$ defined by restriction (if $A \in L(V \times \mathbb{R}^s, V)$, then $\pi_1(A) \in L(V, V)$ is the map defined by $\pi_1(A)(v) = A(v, 0)$). Given $f \in C^\infty_G(V \times \mathbb{R}^s, V)$, let $j^1_1 f : V \times \mathbb{R}^s \to J^1(V, V)$ be defined by

$$j^1_1 f(x, \boldsymbol{\lambda}) = j^1 f_{\boldsymbol{\lambda}}(x),\ (x, \boldsymbol{\lambda}) \in V \times \mathbb{R}^s.$$

For all $f \in C^\infty_G(V \times \mathbb{R}^s, V)$, we have

$$\pi_1 \circ j^1 f = j^1_1 f.$$

We may factorize $j^1_1 f : V \times \mathbb{R}^s \to J^1(V, V)$ as $j^1_1 f = \tilde{U}_1 \circ (\tilde{I}, \tilde{H}_1)$, where

$$\begin{aligned} \tilde{I} : V \times \mathbb{R}^s &\to V,\ (x, \boldsymbol{\lambda}) \mapsto x, \\ \tilde{H}_1 : V \times \mathbb{R}^s &\to \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k), \\ \tilde{U}_1 : V \times \mathbf{P}_1(\mathbb{R}^\ell, \mathbb{R}^k) &\to J^1(V, V). \end{aligned}$$

The definitions of of $\tilde{H}_1$ and $\tilde{U}_1$ are given in section 7.3. Let Q be a closed semialgebraic subset of $J^1(V, V)$ and A be a closed subset of $V \times \mathbb{R}^s$. Recall that if $f \in C^\infty_G(V \times \mathbb{R}^s, V)$, then $j^1_1 f \pitchfork_G Q$ if $(I, \tilde{H}_1)$ is transverse to the canonical stratification of $\tilde{U}_1^{-1}(Q)$ along A.

We define

$$\begin{aligned} \mathcal{K}^{1,\star}_G(V) &= \{f \in \mathcal{K}^\star_G(V) \mid j^1_1 f \pitchfork_G Z_1 \text{ at } (0,0) \in V \times \mathbb{R}\} \\ &= \{f \in \mathcal{V}_0(V, G) \mid j^0 f \pitchfork_G \Sigma^\star, j^1_1 f \pitchfork_G Z_1 \text{ at } (0,0) \in V \times \mathbb{R}\}. \end{aligned}$$

LEMMA 10.5.10. (1) *$\mathcal{K}^{1,\star}_G(V)$ is an open and dense subset of $\mathcal{V}_0(V, G)$.*
(2) *If $f \in \mathcal{K}^{1,\star}_G(V)$, then f satisfies the branching conditions ($\mathbf{B1}^\star$, $\mathbf{B2}^\star$).*

PROOF. Similar to that of lemma 7.4.1. $\square$

THEOREM 10.5.11. *We have $\mathcal{K}^{1,\star}_G(V) \subset \mathcal{S}^\star(V, G)$ (stable families). In particular, $\mathcal{S}^\star(V, G)$ is an open and dense subset of $\mathcal{V}_0(V, G)$.*

PROOF. Similar to that of theorem 7.4.2. $\square$

Let D_V denote the sum of the maximum degrees of polynomials in $\mathcal{F}$ and $\mathcal{P}$.

THEOREM 10.5.12. *Equivariant bifurcation problems on* (V, G) *are* d*-determined, where* $d \leq D_V$.

PROOF. Similar to that of theorem 7.4.3. □

EXAMPLE 10.5.13. Let (V, G) be irreducible of $\mathbb{C}$-normal type. Denote the maximal isotropy types for the action of G on V by $\tau_1, \ldots, \tau_q$. Choose $H_j \in \tau_j$ and define $m_j = \dim(V^{H_j})$, $1 \leq j \leq q$. If $X \in \mathcal{K}_G^{1,\star}(V)$, then $\mathcal{B}^\star(X)$ contains at least $\sum_{j=1}^q m_j$ branches of (G-orbits) limit cycles. This is just the proof of theorem 10.3.7 together with the fact that $\chi(\mathbb{P}^{m_j-1}(\mathbb{C})) = m_j$. The branches of G-orbits will be normally hyperbolic. In particular, if $G_0 = S^1$, each branch will be a branch of hyperbolic limit cycles.

10.5.1. Bifurcation from relative equilibria. Using the tangent and normal form for vector fields in a neighbourhood of a relative equilibrium, our genericity results apply straightforwardly to bifurcation from relative equilibria. We indicate the general approach, one or two of the main results and refer the reader to Krupa's paper [**105**] for more details, examples and applications.

Reduction. Let M be a smooth G-manifold and let X_λ, $\lambda \in \mathbb{R}$, be a smooth family of G-equivariant vector fields on M. We assume that there is a smooth curve α_λ of relative equilibria for X_λ (the relative equilibria will be of constant isotropy type). We further assume that α_λ is generic for $\lambda \neq 0$ and that α_0 is not generic. Equivariantly isotoping the family X_λ, we may require that α_λ is a fixed G-orbit α, for all $\lambda \in \mathbb{R}$, and that α is generic for X_λ if and only if $\lambda = 0$. Working on a tubular neighbourhood of α, applying a centre manifold reduction and making the usual genericity and normalization assumptions, we reduce to a smooth family X_λ of vector fields defined on the twisted product $G \times_H V$, where

(a) $\alpha \cong G/H$.
(b) (V, H) is either absolutely irreducible or complex irreducible.
(c) If we write X_λ in tangent and normal form as $X_\lambda = X_\lambda^T + X_\lambda^N$, then $X^N \in \mathcal{V}_0(V, H)$.

Steady state bifurcation. In the steady state case, we have $X_\lambda^N(x) = \lambda x + F(x, \lambda)$ on V, where $F(x, \lambda) = O(\|x\|^2)$. We can apply generic bifurcation theory for the representation (V, H). By theorem 10.5.11, there will be an open dense subset of $\mathcal{V}_0(V, H)$ consisting of stable families and bifurcation problems on (V, H) will be finitely determined.

EXAMPLES 10.5.14. (1) Let $G = \mathbf{D}_4 \times \mathrm{SO}(2)$ and $H = \mathbf{D}_4 \times \langle e^{\imath\pi} \rangle \subset G$. Take the representation of H on $V = \mathbb{C}$ obtained by mapping H into $\mathrm{O}(2)$ as $\langle \mathbf{D}_4, e^{\imath\pi} \rangle$ and set $M = G \times_H V$. Necessarily, $\alpha = G/H \subset M$ is a relative equilibrium of every $X \in C_G^\infty(TM)$. Suppose we are given a family $X_\lambda \in C_G^\infty(TM)$ and that (c) is satisfied. For generic families $X^N \in \mathcal{V}_0(V, H)$, the signed indexed branching pattern $\Sigma^\star(X^N)$ will consist of two branches of hyperbolic equilibria with isotropy types (κ) and $(\rho\kappa)$, where $\kappa z = \bar{z}$ and $\rho(z) = \imath z$. We may translate these results

back to the original family X_λ. Generically, α will be a periodic orbit for X_λ. Denote the period of α for X_λ by p_λ and assume $p_0 \neq 0$ (this is equivalent to $X_0^T|\alpha \neq 0$). Each non-trivial equilibrium of X^N will generate a (hyperbolic) periodic orbit of X_λ of period approximately $2p_\lambda$. The factor 2 appears because points $v \in V$ are identified with $-v \in V$ when we form the twisted product. In particular, the SO(2)-action on M is free off $\alpha = G \times_H (V \setminus \{0\})$. For this example, bifurcation gives rise to period doubling. If instead we had chosen $H = \mathbf{D}_4 \times \{e_{\mathrm{SO}(2)}\}$, there would have been no period doubling. In either case, the only condition we impose on the family X^T is that X_0^T is non-zero on α.
(2) Let $G = \mathrm{O}(2)$ and $H = \langle\kappa\rangle \cong \mathbb{Z}_2$. Take the representation of H on $\mathbb{R}$ defined by $\kappa(x) = -x$ and define $M = G \times_H \mathbb{R}$. In this case, $\alpha = G \times_H \{0\}$ is an equilibrium G-orbit for every $X \in C^\infty(TM)$. Suppose $X_\lambda \in C^\infty_G(TM)$ satisfies (c). We have $X_\lambda^T|\alpha \equiv 0$. For generic families $X^N \in \mathcal{V}_0(\mathbb{R}, H)$, the signed indexed branching pattern $\Sigma^\star(X^N)$ will a branch of hyperbolic equilibria with isotropy type (e). The corresponding branches of X_λ will consist of a pair of (contra-rotating) hyperbolic periodic orbits. Perturbing X^T, we can assume that (for λ sufficiently small) none of these periodic orbits are equilibrium orbits. As $\lambda \to 0$, the period of the orbits $\to \infty$.

Hopf bifurcation. In the Hopf case, (V, H) will be a complex irreducible representation. In the standard way, we do a normal form analysis and start by assuming that the family X_λ^N is $H \times S^1$-equivariant and $(V, H \times S^1)$ is irreducible of complex type. We illustrate the method with two examples.

EXAMPLES 10.5.15. (1) Let $G = \mathbf{D}_4 \times \mathrm{SO}(2)$ and $H = \mathbf{D}_4 \subset G$. Take the standard complex representation of H on $V = \mathbb{C}^2$ (see example 5.6.5). Set $M = G \times_H V$. Necessarily, $\alpha = G/H \subset M$ is a relative equilibrium of every $X \in C^\infty_G(TM)$. Suppose we are given a family $X_\lambda \in C^\infty_G(TM)$ and that (c) is satisfied. We start by assuming X^N is in normal form, that is $X^N \in \mathcal{V}_0(V, H \times S^1)$. For generic families, there will be branches of hyperbolic limit cycles of isotropy type $(\mathbb{Z}_4)$, $(\langle\kappa\rangle)$, $(\langle\rho\kappa\rangle)$ (see the table for example 5.6.16). There may also exist (stably) submaximal branches of limit cycles and branches of normally hyperbolic 2-tori supporting quasi-periodic flow (see example 5.6.27).

When we add back in the tangential vector field, the branches of limit cycles become normally hyperbolic branches of invariant 2-tori which are orbits of the $\mathbb{T}^2$-action defined by the product of SO(2) with the normal form symmetries S^1. When we break normal form symmetries, these normally hyperbolic branches persist as SO(2)-invariant branches. These branches will be branches of *relative periodic orbits* for the $\mathbf{D}_4 \times \mathrm{SO}(2)$-equivariant family. The SO(2)-invariance implies that the 2-tori are foliated either by limit cycles or are quasi-periodic for the flow (no phase-locking occurs). With a little more work, it is not hard to show that X_T can be chosen so that for almost all λ in some neighbourhood of 0, we have quasi-periodic flow.

On the other hand, the branches of normally hyperbolic 2-tori for X^N will become normally hyperbolic branches of invariant 3-tori for the original flow. When we break normal form symmetry, then the determination of possible dynamics on the invariant 3-tori (or the 2-tori for X^N) is much more delicate.

(2) (Krupa [**105**, Example 5.3]) Regard $\mathrm{O}(2) = \langle \mathrm{SO}(2), \kappa \rangle$ and suppose $\langle \kappa \rangle = \mathbb{Z}_2$ acts on $\mathbb{R}^2$ as multiplication by ± 1. Let $M = \mathrm{O}(2) \times_{\mathbb{Z}_2} \mathbb{R}^2$. The zero section of M will be diffeomorphic to S^1 and will be an equilibrium $\mathrm{O}(2)$-orbit for every $X \in C^\infty_{\mathrm{O}(2)}(TM)$. Suppose we are given a family $X_\lambda \in C^\infty_{\mathrm{O}(2)}(TM)$, such that $DX_0^N(0)$ has non-zero imaginary eigenvalues and (c) is satisfied. By the Hopf bifurcation theorem, we obtain for generic X^N a branch β_λ of hyperbolic limit cycles. Denote the flow of X_λ^N on $\mathbb{R}^2$ by Φ_t^λ. If β_λ has period T_λ then, on account of the $\mathbb{Z}_2$-equivariance, we have the relation $\Phi^\lambda(x, t + T_\lambda/2) = \kappa \Phi^\lambda(x, t)$, for all $x \in \beta_\lambda$. Add back in the tangential field and let Ψ_t^λ denote the corresponding flow. By lemma 8.4.6, we may write $\Psi^\lambda(x, t) = \gamma(x, t)\Phi^\lambda(x, t)$, where $\gamma(x, t) \in \mathrm{SO}(2)$, $t \in \mathbb{R}$. Given $x \in \beta_\lambda$, we have

$$\begin{aligned}
\Psi^\lambda(x, T_\lambda) &= \Psi^\lambda_{T_\lambda/2}(\Psi^\lambda(x, T_\lambda/2)), \\
&= \Psi^\lambda_{T_\lambda/2}(\gamma(x, T_\lambda/2)\kappa x)), \\
&= (\gamma(x, T_\lambda/2)\kappa)\Psi^\lambda_{T_\lambda/2}(x), \\
&= (\gamma(x, T_\lambda/2)\kappa)^2 x, \\
&= x,
\end{aligned}$$

since $(g\kappa)^2 = e$, for all $g \in \mathrm{SO}(2)$. Hence the trajectory of X_λ through any $x \in \beta_\lambda$ has period T_λ. This is a simple example of a relative periodic orbit for which the corresponding Cartan subgroup of 0(2) is zero dimensional (that is $s = 0$ in lemma 8.4.12).

Additional results on drift dynamics. We review some genericity theorems proved by Krupa.

Steady state bifurcation. Assume (V, H) is absolutely irreducible. Suppose that $X_\lambda = X_\lambda^N + X_\lambda^T$ and that $X^N \in \mathcal{S}^\star(V, H)$. Let $C^\infty_H(V, H)_T$ denote the space of tangential vector fields (restricted to V). Each $X^T \in C^\infty_H(V, H)_T$ will determine a flow along group orbits which satisfies the conditions of lemma 8.4.6. Suppose that β is a relative equilibrium for $X_\lambda = X_\lambda^T + X_\lambda^N$. Then, by proposition 8.4.1, there is a foliation of β by tori with quasi-periodic flow. Let $d(\beta)$ denote the dimension of one of these tori and recall that $d(\beta) \leq \mathrm{rk}(N(J)/J)$ where (J) is the isotropy type of β. Krupa [**105**, Theorems 4.1, Proposition 4.10] proves that there exists a residual subset $B = B(X^N)$ of $C^\infty_H(V, H)_T$ such that if $X^T \in B$, then for each branch β_λ of non-trivial relative periodic orbits of $X = X^T + X^N$

(1) $d(\beta_\lambda) = \mathrm{rk}(N(J)/J)$ except for at most countably many values of λ.
(2) $d(\beta_\lambda) \geq \mathrm{rk}(N(J)/J) - 1$.

(If $\mathrm{rk}(N(J)/J) = 1$, then generically we have $d(\beta_\lambda) = 1$ – this is i either proposition 7.6.15 (G/H is zero-dimensional) or the obvious argument on X^T (if H is finite and $N(H)/H$ is zero-dimensional.)

Hopf bifurcation. The results are similar to those for relative equilibria except that the dimensions of maximal tori are now framed in terms of Cartan subgroups. We refer to [**105**, Theorem 5.2] for a precise statement.

Finally, we remark that we can also approach these problems using the skew product formulation described in remarks 8.4.7(3). See also [**112**] for more recent developments as well as background and references.

10.6. Stability and determinacy for maps

It is straightforward to extend our genericity results from vector fields to smooth equivariant diffeomorphisms and maps. We indicate the basic ideas and results in this section. For the most part we omit proofs as they are very similar to those we gave earlier for vector fields (more details and proofs may be found in [**62**]). Of special interest are applications to equivariant bifurcation from relative equilibria and relative periodic orbits. This topic has been intensively studied by Lamb, Melbourne, Wulff and others (see for example [**111, 115, 113, 183, 112**]). Their theory applies to quite general bifurcations of equivariant and reversible equivariant vector fields and allows for proper noncompact group actions and the detailed study of drift along group orbits. In the final section of the chapter we give a partial introduction to some aspects of this theory and indicate applications of our genericity theorems for maps.

Let (V, G) be a real representation. We shall assume that (V, G) is irreducible of real, complex or quaternionic type. If (V, G) is irreducible of complex type, we assume that V is given a complex structure J with respect to which (V, G) is irreducible as a complex representation (J is unique up to multiplication by ± 1). We will not give any details for the quaternionic case as, in our applications, we typically enlarge the group G and thereby obtain a representation which is irreducible of complex type.

Let $\mathcal{F} = \{F_1, \dots, F_k\}$ be a minimal set of homogeneous generators for the $P(V)^G$-module $P_G(V, V)$. As usual, we set $\deg(F_i) = d_i$ and label the F_j so that $1 \leq d_1 \leq \ldots \leq d_k$. Obviously the set of generators $\mathcal{F}$ is linearly independent (over $\mathbb{R}$). Let $V_{\mathcal{F}}$ denote the $\mathbb{R}$-vector subspace of $P_G(V, V)$ with basis $\mathcal{F}$.

LEMMA 10.6.1. *Suppose that (V, G) is a irreducible of complex type. We may choose $\mathcal{F}$ so that $V_{\mathcal{F}}$ has the structure of a complex vector subspace of $P_G(V, V)$.*

PROOF. Choose a minimal set $\mathcal{H}$ of homogeneous polynomial generators for $P_G(V, V)$ regarded as a module over the complex valued G-invariants. Then $\mathcal{H} \cup \imath\mathcal{H}$ is a minimal set of homogeneous polynomial generators for $P_G(V, V)$ over $P(V)^G$ (use the complex version of lemma 6.6.4). $\square$

REMARK 10.6.2. When (V, G) is a irreducible of complex type, we always choose $F_1 = I_V$ and $F_2 = J$ (J a choice of complex structure for V).

10.6.1. Branches of relative fixed points. Let $f \in C^\infty_G(V \times \mathbb{R}, V)$. Clearly $x = 0$ is a fixed point of $f(x, \lambda) = x$. We refer to $x = 0$ as the *trivial branch* of fixed points of f.

LEMMA 10.6.3. *Let (V, G) be absolutely irreducible or irreducible of complex type. Suppose that $f \in C^\infty_G(V \times \mathbb{R}, V)$, $\lambda_0 \in \mathbb{R}$ and $Df_{\lambda_0}(0)$ has no eigenvalues of unit modulus. We may choose a neighborhood U of $(0, \lambda_0)$ in $V \times \mathbb{R}$, such that if $(x, \lambda) \in U$ and $f_\lambda(x) \in Gx$ then $x = 0$.*

PROOF. If $Df_{\lambda_0}(0)$ has no eigenvalues of unit modulus, then 0 is a hyperbolic fixed point of f_λ for λ close to λ_0. It follows that for x sufficiently close to $0 \in V$, either $f^n(x) \to 0$ or $\|f^n_\lambda(x)\| \to \infty$, as $n \to +\infty$. Either circumstance excludes $f_\lambda(x) \in Gx$ as then $\|f_\lambda(x)\| = \|x\|$. □

Lemma 10.6.3 implies that bifurcations of the trivial branch of fixed points only occur when $Df_\lambda(0)$ has an eigenvalue on the unit circle. Our interest will be in finding branches of relative fixed points rather than just fixed points. Later, we describe techniques that also identify branches of relative periodic points.

As usual we restrict attention to families f that have a non-degenerate change of stability of the trivial branch of fixed points at $\lambda = 0$. That is, if we write $Df_\lambda(0) = \sigma_f(\lambda) I_V$, where $\sigma_f : \mathbb{R} \to \mathbb{C}$, then

$$|\sigma_f(0)| = 1, \quad |\sigma'_f(0)| \neq 0 \tag{10.4}$$

If (V, G) is irreducible of complex type, we reparameterize the bifurcation variable λ, and restrict attention to the space

$$\mathcal{M}(V, G) = \{f \in C^\infty_G(V \times \mathbb{R}, V) \mid \sigma_f(\lambda) = \exp(\imath\omega_f(\lambda))(1 + \lambda)\},$$

where $\omega_f : \mathbb{R} \to \mathbb{R}$ is a smooth map. If (V, G) is absolutely irreducible, we replace the term $\exp(\imath\omega(\lambda))(1+\lambda)$ by $\pm(1+\lambda)$. (These conditions force loss of stability of the trivial branch at $\lambda = 0$.) Note that $\mathcal{M}(V, G)$ allows for any choice of smooth map ω_f, or choice of sign when (V, G) is absolutely irreducible.

We refer to elements of the spaces $\mathcal{M}(V, G)$ as *normalized families*. If $\theta \in [0, 2\pi)$, we define $\mathcal{M}^\theta(V, G) = \{f \in \mathcal{M}(V, G) \mid \omega(0) = \theta\}$. If (V, G) is absolutely irreducible, we let $\mathcal{M}^+(V, G)$, $\mathcal{M}^-(V, G)$ denote the subspaces of $\mathcal{M}(V, G)$ corresponding to $\sigma_f(0) = 1$, $\sigma_f(0) = -1$ respectively.

For $f \in \mathcal{M}(V, G)$, let $\mathbf{F}(f)$ denote the set of relative fixed points of f:

$$\mathbf{F}(f) = \{(x, \lambda) \mid f_\lambda(x) \in Gx\}.$$

Clearly $\mathbf{F}(f)$ is a closed G-invariant subset of $V \times \mathbb{R}$.

Just as for vector fields, we may define symmetry breaking isotropy types.

DEFINITION 10.6.4. Let $\tau \in \mathcal{O}(V, G)$. If $\sigma \in \{+, -\}$ and (V, G) is absolutely irreducible, τ is *σ-symmetry breaking* (respectively, *generically σ-symmetry breaking*) if there exists a non-empty open1 (respectively, open and dense) subset $\mathcal{U}$

of $\mathcal{M}^\sigma(V,G)$ such that for all $f \in \mathcal{U}$, the germ of $\mathbf{F}(f)$ at zero contains points of isotropy type τ. If (V,G) is irreducible of complex type, τ is *symmetry breaking* (respectively, *generically symmetry breaking*) if there is a non-empty open (respectively, open and dense) subset $\mathcal{U}$ of $\mathcal{M}(V,G)$ such that for all $f \in \mathcal{U}$, the germ of $\mathbf{F}(f)$ at zero contains points of isotropy type τ.

We briefly describe the straightforward definition and properties of branches of relative fixed points for maps. (All of this is very close to the definitions we gave earlier for branches of relative equilibria.)

For $\tau \in \mathcal{O}(V,G)$, choose $H \in \tau$ and set $\Delta_\tau = G/H$.

DEFINITION 10.6.5. Let $f \in \mathcal{M}(V,G)$ and $\tau \in \mathcal{O}(V,G)$. A branch of relative fixed points of isotropy type τ for f consists of a C^1 G-equivariant map

$$\phi = (\mathbf{x}, \lambda) : [0,\delta] \times \Delta_\tau \to V \times \mathbb{R}$$

such that λ is independent of $u \in \Delta_\tau$ and

(1) $\phi(0,u) = (0,0)$, all $u \in \Delta_\tau$.
(2) For all $s \in (0,\delta]$, $\alpha_s = \mathbf{x}(s,\Delta_\tau)$ is a relative fixed set of $f_{\lambda(s)}$ and α_s is of isotropy type τ.
(3) For every $u \in \Delta_\tau$, the map $\phi_u : [0,\delta] \to V \times \mathbb{R}$, $s \mapsto \phi(s,u)$, is a C^1-embedding.

If, in addition, we can choose $\delta > 0$ so that

(4) For all $s \in (0,\delta]$, $f_{\lambda(s)}$ is normally hyperbolic at α_s,

we refer to ϕ as a branch of *normally hyperbolic relative fixed sets* for f at zero.

In the usual way, we define equivalence of branches.

DEFINITION 10.6.6. Let $f \in \mathcal{M}(V,G)$. The *branching pattern* $\mathbf{B}(f)$ of f is the set of all equivalence classes of *non-trivial* branches of relative fixed sets for f. Each point in $\mathbf{B}(f)$ is labelled with the isotropy type of the associated branch.

Associated to every branch ϕ of relative fixed points of f, we may define the *direction of branching set* $\mathcal{D}(\phi) \subset S(V)$. The set $\mathcal{D}(\phi)$ will be a G-orbit. The isotropy type of $\mathcal{D}(\phi)$ is greater than or equal to that of the branch ϕ.

We may also refine the definition of branching pattern to take account of stabilities and direction of branching. Every branch of normally hyperbolic relative fixed sets is either a supercritical or subcritical branch. For maps all of whose branches are normally hyperbolic, we may define the signed indexed branching pattern $\mathbf{B}^\star(f)$. We may also define the class $\mathbf{S}_w(V,G)$ of weakly stable families, the class $\mathbf{S}(V,G)$ of stable families and define the associated concepts of determinacy. In the case when (V,G) is absolutely irreducible, we may refine our definitions in the obvious way to allow for$+/-$-determinacy etc.

10.6.2. The varieties $\Xi, \Xi^\star$. Let $\mathcal{P} = \{p_1, \ldots, p_\ell\}$ be a minimal set of homogeneous generators for the $\mathbb{R}$-algebra $P(V)^G$ and let $P : V \to \mathbb{R}^\ell$ denote the corresponding orbit map.

Define $\nabla : V \times \mathbb{R}^k \to V$ by $\nabla(x,t) = \sum_{i=1}^k t_i F_i(x)$ and let

$$\begin{aligned} \Xi &= \{(x,t) \in V \times \mathbb{R}^k \mid \nabla(x,t) = x\}, \\ \Xi^\star &= \{(x,t) \in V \times \mathbb{R}^k \mid P(x) = P(\nabla(x,t))\} \end{aligned}$$

EXAMPLES 10.6.7. (1) Suppose that $G = \mathbb{Z}_2$ acts non-trivially on $V = \mathbb{R}$. Then $P(x) = x^2$ and $\nabla(x,t) = tx$. We see that $\Xi^\star$ is the zero variety of $x^2(t^2-1)$ while Ξ is the zero variety of $x(t-1)$. In particular, $\Xi^\star \supsetneq \Xi$.
(2) Let $G = \mathrm{SO}(2)$ act in the standard way on $V = \mathbb{C}$. Then $P_{\mathrm{SO}(2)}(\mathbb{C},\mathbb{C})$ is generated by $\{I, \imath I\}$ and $P(z) = |z|^2$. Hence Ξ is the union of $z = 0$ and $t_1 = 1$ and $\Xi^\star$ is the union of $z = 0$ and $t_1^2 + t_2^2 = 1$.

LEMMA 10.6.8. (1) *$\Xi \subset \Xi^\star$.*
(2) *Ξ and $\Xi^\star$ are G-invariant algebraic subsets of $V \times \mathbb{R}^k$.*
(3) *If $(x,t) \in \Xi^\star$, then $\nabla(x,t) \in V^{G_x} \cap Gx$.*

10.6.3. Geometric properties of $\Xi, \Xi^\star$.

LEMMA 10.6.9. *If $\tau \in \mathcal{O}(V,G)$, then $\Xi_\tau, \Xi_\tau^\star$ are semialgebraic smooth submanifolds of $V \times \mathbb{R}^k$ and*
(1) $\dim(\Xi_\tau) = k + g_\tau - n_\tau$.
(2) $\dim(\Xi_\tau^\star) = k + g_\tau$.

EXAMPLE 10.6.10. Take the standard representation of SO(2) on $\mathbb{C}$. If $\tau = (e)$, then $g_\tau = n_\tau = 1$, $k = 2$ and $\dim(\Xi_\tau) = 2$, $\dim(\Xi_\tau^\star) = 3$ which is consistent with our explicit computations of Ξ, $\Xi^\star$ given in the previous example.

LEMMA 10.6.11. *Let $\tau, \mu \in \mathcal{O}(V,G)$. Then*
(1) *$\Xi_\mu \cap \overline{\Xi}_\tau$, $\Xi_\mu^\star \cap \overline{\Xi}_\tau^\star = \emptyset$ if $\tau > \mu$.*
(2) *$\dim(\Xi_\mu \cap \overline{\Xi}_\tau) < k + g_\mu - n_\mu$ and $\dim(\Xi_\mu^\star \cap \overline{\Xi}_\tau^\star) < k + g_\mu$, if $\tau < \mu$.*

As usual we regard $\mathbb{R}^k$ as embedded in $V \times \mathbb{R}^k$ as $\{0\} \times \mathbb{R}^k$. We let $\mathbb{R}^{k-1}$ and $\mathbb{R}^{k-2}$ be the subspaces of $\mathbb{R}^k$ defined by $t_1 = 0$ and $t_1 = t_2 = 0$ respectively.

Suppose that (V,G) is absolutely irreducible. Let $\mathbf{C}_+, \mathbf{C}_- \subset \mathbb{R}^k$ denote the affine hyperplanes defined by $t_1 = +1$ and $t_1 = -1$ respectively. Set $\mathbf{C} = \mathbf{C}_+ \cup \mathbf{C}_-$.

If (V,G) is irreducible of complex type, let $\mathbf{C} \subset \mathbb{R}^k$ denote the cylinder $t_1^2 + t_2^2 = 1$. For $\theta \in [0, 2\pi)$, let $\mathbf{C}_\theta = \{(\cos\theta, \sin\theta)\} \times \mathbb{R}^{k-2}$ – so $\mathbf{C} = \cup_\theta \mathbf{C}_\theta$.

LEMMA 10.6.12. *Let (V,G) be either absolutely irreducible or irreducible of complex type. Then for all $\tau \in \mathcal{O}^\star(V,G)$, $\partial\Xi_\tau^\star \cap \mathbb{R}^k \subset \mathbf{C}$.*

PROOF. We prove when (V,G) is irreducible of complex type. We may assume that (V,G) is unitary. Let $\|\ \|$ denote the G-invariant Euclidean norm associated to the Hermitian structure on V. Set $(h_1, \ldots, h_\ell)(x,t) = P(\nabla(x,t)) - P(x)$.

Since we may take $F_2 = \imath I_V$, $\nabla(x,t) = (t_1 + \imath t_2)I_V(x) + O(\|x\|^2)$. Since the lowest degree invariant p_1 may be taken to be the square of the Euclidean norm on V we have

$$h_1(x,t) = (t_1^2 + t_2^2 - 1)\|x\|^2 + O(\|x\|^3), \ \ (x,t) \in V \times \mathbb{R}^k). \tag{10.5}$$

Suppose that (x^n, t^n) is a sequence of points of $\Xi_\tau^\star$ converging to the point $(0,t) \in V \times \mathbb{R}^k$. Substituting in (10.5), dividing by $\|x^n\|^2$, and letting $n \to \infty$, we see that $t_1^2 + t_2^2 = 1$. □

EXAMPLE 10.6.13. Let $G = \mathbf{D}_3$ act in the standard way on $\mathbb{C} = \mathbb{R}^2$. As basis for $P_G(\mathbb{C},\mathbb{C})$ take $\{|z|^2, \mathrm{Re}(z^3)\}$. The action of G on $\mathbb{C}$ has three isotropy types: $\tau_0 = (G)$, $\tau_1 = (\mathbb{Z}_2)$ and $\tau_2 = (e)$. It is easy to verify directly that $\overline{\Xi}_{\tau_1}^\star$ meets $\mathbb{R}^2$ along the line $t_1 = 1$. On the other hand $\overline{\Xi}_{\tau_2}^\star \cap \mathbb{R}^2$ consists of the line $t_1 = -1$ together with the isolated point $(1,0)$. To see this, suppose $t_2 \neq 0$, and define $z(\rho) = \imath\rho\exp(\imath\rho)R(\rho)$, where $R(\rho) = \sin(2\rho)/(\rho t_2\cos(3\rho))$, $\rho \in (-\pi/6, \pi/6)$. By direct computation, we find that $\nabla(t_1(\rho), t_2, z(\rho)) = \overline{z(\rho)}$ if $t_1(\rho) = -\cos(2\rho) + \tan(3\rho)\sin(2\rho)$, $\rho \in (-\pi/6, \pi/6)$. Hence $(-1, t_2) \in \overline{\Xi}_{\tau_2}^\star$ for all $t_2 \in \mathbb{R}$. Note that $z(\rho)$ defines a curve of points of period two for the map $f_\rho(z) = t_1(\rho)z + t_2\bar{z}^2$ and that $z(\rho)$ is tangent at $\rho = 0$ to the line in $\mathbb{C}$ on which $z \mapsto \bar{z}$ acts as minus the identity.

Given $\tau \in \mathcal{O}$, define $C_\tau = \mathbb{R}^k \cap \overline{\Xi}_\tau$ and $C_\tau^\star = \mathbb{R}^k \cap \overline{\Xi}_\tau^\star$. Here we emphasize the sets $C_\tau^\star$; similar results hold for C_τ. Clearly $C_{(G)}^\star = \mathbb{R}^k$. If $\tau \neq (G)$, it follows from lemma 10.6.12 that $C_\tau^\star \subset \mathbf{C}$. If (V,G) is absolutely irreducible, we define $C_\tau^{+\star} = C_\tau^\star \cap \mathbf{C}_+$ and $C_\tau^{-\star} = C_\tau^\star \cap \mathbf{C}_-$. (Note that $C_\tau^{-\star}$ may be empty – example 10.6.13 – but $C_\tau^{+\star}$ always contains $(1,0,\ldots,0)$). If (V,G) is irreducible of complex type, then for each point θ of $t_1^2 + t_2^2 = 1$, we define $C_\tau^{\theta\star} = \mathbf{C}_\theta \cap C_\tau^\star$.

LEMMA 10.6.14. *Let (V,G) be absolutely irreducible. Let $H \in \tau \in \mathcal{O}$.*

(1) *If $dim(V^H) = 1$, then $C_\tau^{+\star}$ is the hyperplane $t_1 = 1$.*
(2) *If there exist $\gamma \in N(H)$ and $\mathbf{u} \in V_\tau^H$ such that the fixed point space of the map $-\gamma : V^H \to V^H$ is the line $\mathbb{R}\mathbf{u}$, then C_τ^- is the hyperplane $t_1 = -1$.*

PROOF. (1) is well-known and follows from the equivariant branching lemma (see examples 6.15.3(3)). For (2), observe that if $\gamma u = -u$ then $\gamma^2 u = u$. Since $u \in V_\tau^H$ and $N(H)/H$ acts freely on V_τ^H, $\gamma^2 = I$ on V^H. If we let W denote the orthogonal complement of $\mathbb{R}\mathbf{u}$ in V^H, then $\gamma|W = I$. Using the implicit function theorem, it is now straightforward to construct a smooth solution $(x(s), t(s))$ to $\nabla(x,t) = \gamma x$ such that $x(s) = (sq(s)\mathbf{u}, s^b\hat{w}(s)) \in \mathbb{R}\mathbf{u} \oplus W$, and $t(0) = (-1, t_2, \ldots, t_k)$, where $(t_2, \ldots, t_k)$ is a general point in $\mathbb{R}^{k-1}$. We refer to [**62**, section 4] for complete details. □

REMARKS 10.6.15. (1) Lemma 10.6.14(2) is easy if H is maximal since we then have $V^H = \mathbb{R}u$ and $-I \in N(H)/H$. In this case, the branch we obtain lies in $\mathbb{R}u$. When H is not maximal, the branch given by lemma 10.6.14(2) will

generally not lie in $\mathbb{R}u$ – it will be tangent to $\mathbb{R}u$ at the origin.
(2) Results similar to lemma 10.6.14(2) have been obtained previously by Vanderbauwhede [**174**] and Peckham & Kevrekidis [**141**]. See also [**80**, Lecture 2].

LEMMA 10.6.16. *Let $H \in \tau \in \mathcal{O}(V, G)$.*

(1) *Suppose (V, G) is absolutely irreducible and $-I \in N(H)/H \subset O(V^H)$, then $C_\tau^{-\star} = -C_\tau^{+\star}$. In particular, if $-I \in G \subset O(V)$, then $C_\tau^{-\star} = -C_\tau^{+\star}$, for all $\tau \in \mathcal{O}(V, G)$.*
(2) *Suppose (V, G) is irreducible of complex type. If $\exp(\imath\theta)I \in N(H)/H \subset O(V^H)$, then $\exp(\imath\theta)C_\tau^{0\star} = C_\tau^{\theta\star}$. In particular, if $S^1 \subset N(H)/H$, then S^1 acts freely on $C_\tau^\star$ and $\exp(\imath\theta)C_\tau^{\phi\star} = C_\tau^{\theta+\phi\star}$, all $\theta, \phi \in [0, 2\pi)$. If S^1 (respectively $\mathbb{Z}_p \subset S^1$) is a subgroup of $G \subset O(V)$, then $\mathbf{C}$ is S^1-invariant (respectively, $\mathbb{Z}_p$-invariant).*

PROOF. We prove (1); (2) is similar. If $\tilde{t} = (1, t_2, \ldots, t_k) \in C_\tau^{+\star}$, then by the curve selection lemma there exists a C^1-curve $\phi = (s, t) : [0, \delta] \to V^H \times \mathbb{R}^k$ such that $\nabla(x(s), t(s)) = x(s)$, $s \in [0, \delta]$, and $x(0) = 0$, $t(0) = \tilde{t}$. Obviously, $\nabla(x(s), -t(s)) = -x(s)$. If $-I \in N(H)/H$, then $-x(s) \in Gx(s)$ and so $-t(0) \in C_\tau^{-1\star}$, □

LEMMA 10.6.17. *Suppose that ϕ is a branch of relative fixed points of isotropy type τ for $f \in \mathcal{M}(V, G)$. Then*

$$Df_0(0)(\mathcal{D}(\phi)) = \mathcal{D}(\phi).$$

PROOF. The result follows easily from the definition of $\mathcal{D}(\phi)$. □

REMARK 10.6.18. Lemma 10.6.17 can impose strong restrictions on possible branches of relative fixed points. For example, let ϕ be a branch of relative fixed points of isotropy type τ. Fix $H \in \tau$ and let $\Delta_\tau^H = N(H)/H$. If we define $\phi^H : [0, \delta] \times \Delta_\tau^H \to V^H \times \mathbb{R}$ by $\phi^H = \phi|[0, \delta] \times \Delta_\tau^H$, then ϕ^H defines the intersection of the original branch ϕ with $V^H \times \mathbb{R}$. The group $N(H)/H$ acts on V^H and restricts to a free action on V_τ^H. Let $u \in \mathcal{D}(\phi) \cap S(V^H)$. Then $Df_0(0)(u) \in Gu$. If (V, G) is absolutely irreducible, $Df_0(0) = \pm I_V$. If $Df_0(0) = I_V$, there are no restrictions. However, if $Df_0(0) = -I_V$, then $Df_0(0)(u) = -u \in Gu$ and this can only occur if $\exists \gamma \in N(H)/H$ such that $\gamma u = -u$. Similar observations can be made when (V, G) is irreducible of complex type.

10.6.4. Stratification of $\Xi^\star$. Let $\mathcal{S}^\star$ denote the canonical (minimal) semialgebraic stratification of $\Xi^\star$.

THEOREM 10.6.19. *Let $\tau \in \mathcal{O}(V, G)$. The stratification $\mathcal{S}^\star$ induces a semialgebraic Whitney stratification $\mathcal{S}_\tau^\star$ of $\Xi_\tau^\star$. In particular, $\mathcal{S}_\tau^\star$ is a union of $\mathcal{S}^\star$-strata.*

As a corollary of theorem 10.6.19 and the definition of $C_\tau^\star$, we have

PROPOSITION 10.6.20. *For each $\tau \in \mathcal{O}(V, G)$, $C_\tau^\star$ inherits a Whitney regular stratification $\mathcal{C}_\tau^\star$ from $\mathcal{S}^\star$.*

We denote the stratification $\mathcal{C}^\star_{(G)}$ of $\Xi^\star_{(G)} = \mathbb{R}^k$ by $\mathcal{B}^\star$. Let $B^\star_i$ denote the union of the i-dimensional strata of $\mathcal{B}^\star$. We have

$$B^\star_k = \mathbb{R}^k \setminus \bigcup_{\tau\neq(G)} C^\star_\tau \tag{10.6}$$

$$B^\star_i \subset \mathbf{C},\ i < k \tag{10.7}$$

There is a close relationship between the stratification $\mathcal{B}^\star$ of $\Xi^\star_{(G)}$ and the stratification $\mathcal{A}^\star$ of $\Sigma^\star_{(G)}$ constructed earlier in the chapter. The following proposition details this relationship in the most interesting cases.

PROPOSITION 10.6.21. *Let $H \in \tau \in \mathcal{O}^\star(V,G)$.*

(1) *If (V,G) is absolutely irreducible, then $C^{+\star}_\tau = R^\star_\tau + 1$. In particular, $k - d_\tau + n_\tau \leq \dim(C^{+\star}_\tau) \leq k-1$. If $-I \in N(H)/H$, then $C^{-\star}_\tau = R^\star_\tau - 1$.*
(2) *If (V,G) is irreducible of complex type and $\exp(\imath\theta) \in G$, then $C^{\theta\star}_\tau = \exp(\imath\theta)R^\star_\tau$. In particular, if $S^1 \subset G$ then $C^\star_\tau = S^1 R^\star_\tau$.*

PROOF. We refer to [**62**, Proposition 4.4.11] for details which depend on approximating maps by time-one maps of a vector field. (Note that (1,2) are elementary if G is finite). $\square$

10.6.5. Stability theorems. Suppose that $f \in \mathcal{M}(V,G)$ and write $f(x,t) = \sum_{i=1}^k f_i(x,t)F_i(x)$, where f_i are smooth invariants. Exactly as in chapter 6, we may factorize f as $f = \nabla \circ \Gamma_f$, where $\Gamma_f : V \times \mathbb{R} \to V \times \mathbb{R}^k$ is the graph map

$$\Gamma_f(x,\lambda) = (x, (f_1(x,\lambda), \dots, f_k(x,\lambda))),\ (x,\lambda) \in V \times \mathbb{R}.$$

We define $\gamma(f) \in C^\infty(\mathbb{R},\mathbb{R}^k)$ by $\gamma(f)(\lambda) = (f_1(0,\lambda), f_2(0,\lambda), \dots, f_k(0,\lambda))$. If (V,G) is absolutely irreducible, we have $f_1(0,\lambda) = \pm(1+\lambda)$, where we take the positive or negative sign according to whether $f \in \mathcal{M}^+(V,G)$ or $f \in \mathcal{M}^-(V,G)$. If (V,G) is irreducible of complex type, we have

$$f_1(0,\lambda) + \imath f_2(0,\lambda) = \exp(\imath\omega(\lambda))(1+\lambda).$$

LEMMA 10.6.22. *Let $f \in \mathcal{M}(V,G)$. Then $\Gamma_f \pitchfork \Xi^\star$ at $(x,\lambda) = (0,0)$ if and only if $\gamma(f) \pitchfork \mathcal{B}^\star$ at $\lambda = 0$.*

Define

$$\begin{aligned}
\mathcal{L}_G(V) &= \{f \in \mathcal{M}(V,G) \mid \Gamma_f \pitchfork \Xi^\star \text{ at } (0,0)\},\\
\mathcal{L}^\pm_G(V) &= \{f \in \mathcal{L}_G(V) \mid f \in \mathcal{M}^\pm(V,G)\},\ (V,G) \text{ absolutely irreducible},\\
\mathcal{L}^\theta_G(V) &= \{f \in \mathcal{L}_G(V) \mid f \in \mathcal{M}^\theta(V,G)\},\ (V,G) \text{ irreducible of complex type}
\end{aligned}$$

THEOREM 10.6.23. (1) *$\mathcal{L}_G(V)$ is an open and dense subset of $\mathcal{M}(V,G)$.*
(2) *$\mathcal{L}_G(V) \subset \mathbf{S}^\star_w(V,G)$.*
(3) *Let $f \in \mathcal{L}_G(V)$. We may find an open neighborhood $\mathcal{U}$ of f in $\mathcal{M}(V,G)$ such that if $\{g_t \mid t \in [0,1]\}$ is any continuous path in $\mathcal{U}$ with $g_0 = f$, there is an open neighborhood W of $(0,0)$ in $V \times \mathbb{R}$ and an (equivariant) isotopy $\{K_t : W \to V \times \mathbb{R} \mid t \in [0,1]\}$ of (continuous) embeddings satisfying*

(a) *K_0 is the inclusion of W in $V \times \mathbb{R}$.*
(b) *$K_t(W \cap \mathbf{F}(f)) = \mathbf{F}(g_t) \cap K_t(W)$, all $t \in [0,1]$.*

(4) *Weakly stable mappings are weakly d_w-determined where $d_w \leq d_k$.*

THEOREM 10.6.24. *Let $f \in \mathcal{L}_G(V)$. Then*

(1) *If codim$(C^\star_\tau) \geq 2$, the germ of $\mathbf{F}(f)$ at zero contains no points of isotropy type τ.*
(2) *If codim$(C^\star_\tau) = 1$ and $\gamma_f(0) \in C^\star_\tau$, there is a branch of invariant group orbits of isotropy type τ for f at zero.*
(3) *The map $\gamma(f) : \mathbb{R} \to \mathbb{R}^k$ is transverse to the canonical stratification of $C^\star_\tau$ for all $\tau \in \mathcal{O}^\star$.*

Similar results hold if we replace $C^\star_\tau$ by $C^{\pm\star}_\tau$. If (V,G) is irreducible of complex type and $C^{\theta\star}_\tau \neq \emptyset$ for only finitely many values of θ, then codim$(C^\star_\tau) \geq 2$ and the germ of $\mathbf{F}(f)$ at zero contains no points of isotropy type τ.

REMARK 10.6.25. If (V,G) is absolutely irreducible then theorems 10.6.23, 10.6.24 have little to say about maps in $\mathcal{M}^-(V,G)$ (or $\mathcal{L}^-_G(V)$) *unless* $-I \in G$ (or, more generally, $-I \in N(H)/H$, $H \in \tau$). Similarly, if (V,G) is irreducible of complex type and G is finite or $G \not\supset S^1$, then the theorems give no information at all as codim$(C^\star_\tau) \geq 2$, $\tau \neq (G)$. Similar remarks apply to stable mappings (see below). We resolve this difficulty in exactly the same way we handled the Hopf bifurcation. We use the theory of normal forms in combination with a stability theorem allows us to break normal form symmetry but not destroy branches that we have found using normal form symmetry.

THEOREM 10.6.26. (1) *The space $\mathbf{S}(V,G)$ of stable mappings is an open and dense subset of $\mathcal{M}(V,G)$.*
(2) *Stable mappings are d-determined where $d \leq d_k + d_\ell$.*

PROOF. The proof is similar to that of theorems 7.4.2, 10.5.11. Details may be found in [**62**, section 4.6]. □

10.6.6. Examples with G finite. In this section we look at a number of examples for which G is finite and (V,G) is absolutely irreducible. None of these examples depends on methods using normal forms.

As we did in chapter 4 we consider for $n \geq 2$ the class $\mathcal{W}_n$ of representations $(\mathbb{R}^n, G)$ where G is a subgroup of the hyperoctahedral groups H_k and

(IR) $(\mathbb{R}^k, G)$ is absolutely irreducible.
(C) $P^2_G(\mathbb{R}^k, \mathbb{R}^k) = \{0\}$.

Following chapter 4, let $\mathcal{E}$ denote the set of non-zero vectors $\varepsilon \in \mathbb{R}^n$ such that $\varepsilon_i \in \{0, +1, -1\}$, $1 \leq i \leq n$. If $(\mathbb{R}^n, G) \in \mathcal{W}_n$, let $\mathcal{O}_S = \{\iota(\varepsilon) \mid \varepsilon \in \mathcal{E}\}$. By theorem 4.5.11, the isotropy type τ is symmetry breaking for 1-parameter families of vector fields if and only if $\tau \in \mathcal{O}_S$. Combining this result with proposition 10.6.21(1) (or directly), we have a simple characterization of +-symmetry breaking isotropy types for representations in the class $\mathcal{W}_n$.

PROPOSITION 10.6.27. *Let* $(\mathbb{R}^n, G) \in \mathcal{W}_n$. *Then* $\tau \in \mathcal{O}(V, G)$ *is* $+$*-symmetry breaking if and only if* $\tau \in \mathcal{O}_S$.

As a simple and direct consequence of the determinacy results of chapter 4, we have a determinacy result for representations in the class $\mathcal{W}_n$.

LEMMA 10.6.28. *There is an open dense semialgebraic subset* $\mathcal{R}$ *of* $P^3_G(\mathbb{R}^n, \mathbb{R}^n)$ *such that*

(1) *if* $f \in \mathcal{M}^+(\mathbb{R}^n, G)$ *and* $D^3 f_0(0) \in \mathcal{R}$, *then* f *is stable.*
(2) $\mu\mathcal{R} = \mathcal{R}$, *for all* $\mu \in \mathbb{R}$, $\mu \neq 0$.

Next we turn our attention to branches of relative fixed points which consist of period two points.

LEMMA 10.6.29. *Let* $(\mathbb{R}^n, G) \in \mathcal{W}_n$.

(1) *Let* $H \in \tau \in \mathcal{O}_S$. *Then* τ *is* $-$*-symmetry breaking if* $-I \in N(H)/H$.
(2) *If* $-I \in G$, *then* $\tau \in \mathcal{O}(\mathbb{R}^n, G)$ *is* $-$*-symmetry breaking if and only if* $\tau \in \mathcal{O}_S$.

PROOF. Lemma 10.6.16. □

LEMMA 10.6.30. *Let* $(\mathbb{R}^n, G) \in \mathcal{W}_n$ *and suppose* $-I \in G$. *Let* $\mathcal{R} \subset P^3_G(\mathbb{R}^n, \mathbb{R}^n)$ *be the subset given by lemma 10.6.28. If* $f \in \mathcal{M}(V, G)$ *and* $D^3 f_0(0) \in \mathcal{R}$, *then* f *is stable.*

PROOF. It suffices to show that if $f \in \mathcal{M}^-(\mathbb{R}^n, G)$ and $D^3 f_0(0) \in \mathcal{R}$, then f is stable. Suppose that $F_2, \dots, F_r$ are the cubic equivariants in $\mathcal{F}$ and set $F_{r+1}(x) = |x|^2 x$. Then $F_2, \dots, F_{r+1}$ define a basis for $P^3_G(\mathbb{R}^n, \mathbb{R}^n)$. Suppose $f \in \mathcal{M}^-(\mathbb{R}^n, G)$. Then

$$\frac{1}{3!} D^3 f_0(0)(x) = \sum_{i=2}^{r+1} a_i F_i(x),$$

where $a_2, \dots, a_{r+1} \in \mathbb{R}$. A simple computation verifies that

$$\frac{1}{3!} D^3 f_0^2(0)(x) = \sum_{i=2}^{r+1} -2a_i F_i(x).$$

Hence, by lemma 10.6.28(2), $D^3 f_0^2(0) \in \mathcal{R}$ if and only if $D^3 f_0(0) \in \mathcal{R}$. □

EXAMPLE 10.6.31. Let $(\mathbb{R}^n, G) \in \mathcal{W}_n$ and suppose $-I \in G$. If $\tau \in \mathcal{O}_S$ then, by lemma 10.6.30, there exists a nonempty open subset of $\mathcal{M}^-(\mathbb{R}^n, G)$ consisting of stable families which have a hyperbolic branch of period two points of isotropy type τ. For example, if $G = \Delta_3 \rtimes \mathbb{Z}_3 \subset H_3$, then generic families $f \in \mathcal{M}^-(\mathbb{R}^3, G)$ will have branches of period two points of isotropy type $G_{(1,0,0)}$ and $G_{(1,1,1)}$. The isotropy type $G_{(1,1,0)}$ will be symmetry breaking but not generically symmetry breaking.

We may extend our results to include the case where $-I \notin G$, $(\mathbb{R}^n, G) \in \mathcal{W}_n$. Let $\mathbb{Z}_2 \subset \mathrm{O}(n)$ be the subgroup generated by $-I$ and set $\tilde{G} = G \times \mathbb{Z}_2$. (If $-I \in G$, then $(-I, -I) \in \tilde{G}$ fixes every point of $\mathbb{R}^n$.) Identifying $\tilde{G}$ with its image in $\mathrm{O}(n)$, we have $(\mathbb{R}^n, \tilde{G}) \in \mathcal{W}_n$. Since $(\mathbb{R}^n, G) \in \mathcal{W}_n$, $P^3_G(\mathbb{R}^n, \mathbb{R}^n) = P^3_{\tilde{G}}(\mathbb{R}^n, \mathbb{R}^n)$ and so the determinacy set $\mathcal{R}$ is the same for both representations.

Let $\tilde{\mathcal{O}}_S$ be the set of $+$-symmetry breaking isotropy types for $\tilde{G}$. If $\tilde{H} \in \tau \in \tilde{\mathcal{O}}_S$, then $(\tilde{H} \cap G) \in \mathcal{O}_S$. Since $g\tilde{H}g^{-1} \cap G = g(\tilde{H} \cap G)g^{-1}$, $g \in \tilde{G}$, this construction defines a natural surjective map $\Pi : \tilde{\mathcal{O}}_S \to \mathcal{O}_S$. If $\tau \in \tilde{\mathcal{O}}_S$ then either $\exists \tilde{H} \in \tau$ such that $\tilde{H} \subset G$ or for all $\tilde{H} \in \tau$, $\tilde{H} \not\subset G$. In the first case, $\tilde{H} \subset G$ for all $\tilde{H} \in \tau$ and we write $\tau \in \mathcal{O}_S$. Otherwise, we write $\tau \notin \mathcal{O}_S$. If x has $\tilde{G}$-isotropy τ, then $\Pi(\tau)$ is the G-isotropy of x.

Lemma 10.6.32. *Let $\tau \in \tilde{\mathcal{O}}_S$. Either $\tau \in \mathcal{O}_S$ or $\tau \notin \mathcal{O}_S$. If $\tau \notin \mathcal{O}_S$, $\tilde{H} \in \tau$ and we set $H = \tilde{H} \cap G$, then there exists $g \in G \setminus H$ such that $g^2 \in H$ and*

$$\tilde{H} = \langle H, -g \rangle.$$

Proof. We leave this to the reader. □

Proposition 10.6.33. *Let $\tau \in \tilde{\mathcal{O}}_S$. We have the following possibilities.*

(a) *$\tau \in \mathcal{O}_S$ and then τ is $-$-symmetry breaking if and only if $-I \in G$.*
(b) *$\tau \notin \mathcal{O}_S$ and then $\Pi(\tau)$ is $-$-symmetry breaking for $(\mathbb{R}^n, G)$ if and only if τ is $-$-symmetry breaking for $(\mathbb{R}^n, \tilde{G})$.*

Proof. If $-I \in G$, only (a) occurs and the result is just lemma 10.6.29. Henceforth we assume $-I \notin G$. Suppose $\tau \notin \mathcal{O}_S$. Set $\mathbb{R}^n = V$ and write $V^H = V^{\tilde{H}} \oplus W$, where W is the orthogonal complement of $V^{\tilde{H}}$ in V^H. By lemma 10.6.32, there exists $g \in H$ such that $\tilde{H} = \langle H, -g \rangle$. Hence $g|V^{\tilde{H}} = -I$, $g|W = I$. Suppose that $f \in \mathcal{M}^-(V, \tilde{G})$, $D^3 f_0(0) \in \mathcal{R}$ and f has a curve $\phi = (\mathbf{x}, \lambda)$ of $\tilde{G}$-invariant points of prime period two and isotropy group $\tilde{H}$. Denote the initial direction $\mathbf{x}'(0)$ of the curve by $u \in S(V)$. Since f is 3-determined, the initial direction u depends only on $D^3 f_0(0)$. Consequently, if we take any G-equivariant perturbation $\hat{f}$ of f by terms of order at least four, the resulting perturbed curve $\hat{\phi}$ of period two points will have the *same* initial direction u. The V-component of $\hat{\phi}$ will be contained in V^H. By G-equivariance, $g\hat{\phi}$ must also be a branch of points of period two. Since $g\hat{\phi}$ has, up to sign, the same initial direction u as $\hat{\phi}$, it follows that $g\hat{\phi} = \hat{\phi}$ (with reverse parameterization). But this implies that the period-two points on $\hat{\phi}$ are related by symmetry. In more detail, if we write $\hat{\phi}(s) = (\mathbf{x}(s), \lambda(s))$, then $\mathbf{x}(s)$, $\mathbf{y}(s) = \hat{f}_{\lambda(s)}(\mathbf{x}(s))$ are the unique period two points for $\hat{f}_{\lambda(s)}$ on the branch. By equivariance, $g\mathbf{x}(s)$ is also a period 2 point on the branch. Hence $g\mathbf{x}(s) = \mathbf{y}(s)$. There remains the case $\tau \in \mathcal{O}_S$ and $-I \notin G$. We now have $V^H = V^{\tilde{H}}$ and there does not exist $g \in G \setminus H$, $x \in V^H$ such that $gx = -x$. Using this, it is now relatively straightforward to perturb a family $f \in \mathcal{M}^-(V, \tilde{G})$ with stable branch ϕ in V^H to a family $\hat{f} \in \mathcal{M}^-(V, G)$

such that the perturbed curve $\hat{\phi}$ of hyperbolic period two points is asymmetric – there exists no $g \in G \setminus H$ such that $g\hat{\phi} = \hat{\phi}$ (we perturb f so that ϕ changes but $-\phi$ is unchanged). Consequently, τ cannot be a $-$-symmetry breaking isotropy type. $\square$

EXAMPLE 10.6.34. Let $(\mathbb{R}^5, G) \in \mathcal{W}_5$, where $G = \Delta_5' \rtimes \mathbb{Z}_5$. Example 4.5.19 and proposition 10.6.27 imply that all isotropy types in $\mathcal{O}^\star(\mathbb{R}^5, G)$ are $+$-symmetry breaking. By proposition 10.6.33 (or direct computation) the maximal isotropy types $\iota(1, \ldots, 1, \pm 1)$ are *not* $-$-symmetry breaking. On the other hand, the trivial isotropy type $\iota(1, 1, 1, 1, 0)$ is $-$-symmetry breaking. In this case, branches of invariant orbits will be tangent to the G-orbit of the plane $x_5 = 0$. All of the remaining isotropy types satisfy the conditions of proposition 10.6.33 and so are $-$-symmetry breaking.

REMARK 10.6.35. Even though an isotropy type $\tau \in \mathcal{O}_S$ may not be $-$-symmetry breaking it will be the case that there exist nonempty open subsets of $\mathcal{M}^-(\mathbb{R}^n, G)$ consisting of families which have branches of hyperbolic period 2 points of isotropy τ. However, these points are not symmetry related. It follows from our results that this happens whenever τ is $+$-symmetry breaking, that is $\tau \in \mathcal{O}_S$. Thus, in the the previous example, even though there are no *symmetric* branches of period 2 points with isotropy type $(\mathbb{Z}_5)$, there are nonetheless generically always branches of period two points of isotropy type $\mathbb{Z}_5$. The periodic points are just not on the same group orbit.

EXERCISE 10.6.36. Using the results of sections 4.10, 4.11, investigate the bifurcation theory of maps $f : \mathbb{R}^n \to \mathbb{R}^n$ with S_{n+1} and $S_{n+1} \times \mathbb{Z}_2$ symmetry (see also [**5**]).

10.6.7. Strong determinacy. In this section we shall give some more refined definitions of stability and determinacy that allow for perturbations by maps which are only equivariant to some finite order. Most of what we say here is based on [**62**] and applies to families of (sufficiently) smooth equivariant maps. Similar results hold for families of equivariant vector fields and we refer to [**60**] for more details and proofs.

Suppose that (V, G) is a G-representation (real or complex). Let H be a closed subgroup of G and H act on $V \times \mathbb{R}$ and V by restriction of the action of G.

DEFINITION 10.6.37. Let Δ be a smooth compact H-manifold and $f \in C^\infty_H(V \times \mathbb{R}, V)$, where $Df_0(0)$ has eigenvalues of modulus 1. Suppose $1 \leq r \leq \infty$. A branch of normally hyperbolic C^r-submanifolds of type Δ for f consists of a C^1 H-equivariant map $\phi = (\mathbf{x}, \lambda) : [0, \delta] \times \Delta \to V \times \mathbb{R}$ satisfying the following conditions:

(1) $\phi(0, x) = (0, 0)$, all $x \in \Delta$.
(2) The map $\lambda : [0, \delta] \times \Delta \to \mathbb{R}$ depends only on $s \in [0, \delta]$.

(3) For each $s \in (0, \delta]$, $\mathbf{x}(\Delta, s) = \Delta_s$ is a normally hyperbolic C^r-submanifold of V for $f_{\lambda(s)}$.
(4) $\phi|(0, \delta] \times \Delta$ is a C^r H-equivariant embedding and for all $x \in \Delta$, $\phi_x : [0, \delta] \to V$ is a C^1-embedding.

REMARKS 10.6.38. (1) We emphasize that the submanifolds Δ_s in Definition 10.6.37 are only required to be C^r: in general, Δ_s will not be a group orbit.
(2) Branches as *equivalent* if they differ by a local C^r reparameterization.
(3) For possibly smaller $\delta > 0$, ϕ is either sub- or supercritical.

Given $f \in C^\infty(V \times \mathbb{R}, V)$ and $d \geq 1$, let $f^{[d]} = j^d f(0, 0) \in P^{(d)}(V \times \mathbb{R}, V)$. For $d \geq 1$, define

$$\mathcal{M}^{[d]}[G\!:\!H](V) = \{f \in C_H^\infty(V \times \mathbb{R}, V) \mid f^{[d]} \in \mathcal{M}(V, G)\}$$

We set $\mathcal{M}^{[d]}[G : \{e\}](V) = \mathcal{M}^{[d]}[G](V)$. If $f \in \mathcal{M}^{[d]}[G : H](V)$, then $f^{[d]}$ is a normalized family in $C_G^\infty(V \times \mathbb{R}, V)$.

Suppose that (V, G) is either absolutely irreducible or irreducible of complex type. Let $f \in \mathbf{S}(V, G)$. Choose a G-invariant neighborhood U of the origin in $V \times \mathbb{R}$ such that

$$U \cap \mathbf{F}(f) = \bigcup_{i \in I} E_i$$

where each E_i is a (the image of) branch of normally hyperbolic relative fixed points. Set $\mathcal{E} = \{E_i \mid i \in I\}$ – we refer to $\mathcal{E}$ as a *local representation of* $\mathbf{F}(f)$ *at zero*. Let $\tau(E)$ denote the isotropy of $E \in \mathcal{E}$.

DEFINITION 10.6.39. Let $f \in \mathbf{S}(V, G)$ and $\mathcal{E}$ be a local representation of $\mathbf{F}(f)$ at zero. Let H be a closed subgroup of G and $d \in \mathbb{N}$. We say f is (d, H)-*stable* if there exists an open neighborhood $\mathcal{U}$ of f in $\mathcal{M}^{[d]}[G\!:\!H](V)$ such that for every continuous path $\{f_t \mid t \in [0, 1]\}$ in $\mathcal{U}$ with $f_0 = f$, there exists an H-invariant compact neighborhood A of zero in $V \times \mathbb{R}$ and a continuous H-equivariant isotopy $K : A \times [0, 1] \to V \times \mathbb{R}$ of embeddings such that
(1) $K_0 = I_A$.
(2) For every $E \in \mathcal{E}$, $t \in [0, 1]$, $K_t(A \cap E)$ is a branch of normally hyperbolic submanifolds of type $\Delta_{\tau(E)}$ for f_t.

REMARKS 10.6.40. (1) If $H = \{e\}$ in definition 10.6.39, we say f is *strongly* d-*stable*.
(2) In (2) of Definition 10.6.39, we implicitly assume that the branch is C^r for some $r \geq 1$. The differentiability class does not play a major role in our results and the strong determinacy theorem we discuss holds for $r \geq 1$.

DEFINITION 10.6.41. We say G-equivariant bifurcation problems on V are (generically) *strongly determined* if there exist $d \in \mathbb{N}$ and an open dense semianalytic subset $\mathcal{R}(d) \subset P_G^{(d)}(V, V)$ such that if $f \in \mathcal{M}(V, G)$ and $j^d f_0(0) \in \mathcal{R}(d)$ then f is strongly d-stable.

REMARKS 10.6.42. (1) We say that G-equivariant bifurcation problems on V are (generically) strongly d-determined if d is the smallest positive integer for which we can find $\mathcal{R}(d)$ satisfying the conditions of definition 10.6.41. For this value of d, we let $\mathcal{N}(d)$ denote the maximal semianalytic open subset of $P_G^{(d)}(V,V)$ satisfying the conditions of definition 10.6.41. We say that f is *strongly d-determined* if $j^d f_0(0) \in \mathcal{N}(d)$.
(2) Let H a closed subgroup of G. We say that G-equivariant bifurcation problems on V are (generically) strongly H-determined if there exist $d \in \mathbb{N}$ and an open and dense semianalytic subset $\mathcal{R}(d) \subset P_G^{(d)}(V,V)_0$ such that if $f \in \mathcal{M}(V,G)$ and $j^d f_0(0) \in \mathcal{R}(d)$ then f is (d,H)-stable. Modulo statements about H-equivariance of isotopies (see definition 10.6.39), it is clear that strong determinacy implies strong H-determinacy for all closed subgroups H of G.
(3) It is simplest to explain strong H-determinacy when G is a finite group. Suppose that $f \in \mathbf{S}(V,G)$ is (d,H)-stable and that ϕ is a branch of relative fixed points of f. If we perturb f to an H-equivariant family f' so that $j^d(f-f')(0,0) \in P_G^{(d)}(V\times\mathbb{R},V)$, then the branch ϕ will typically break into a finite set of branches of H-orbits. Some of these branches may be branches of relative fixed points of f'. Other branches may be permuted by f' and correspond, for example, to branches of hyperbolic points of prime period 2 (this would be expected if (V,G) is absolutely irreducible).

EXAMPLE 10.6.43. If $(\mathbb{R}^n, G) \in \mathcal{W}_n$, $n \geq 2$, then G-equivariant bifurcation problems on $\mathbb{R}^n$ are strongly 3-determined.

THEOREM 10.6.44 ([**62**, Theorem 6.6.1]). *Let (V,G) be either absolutely irreducible or irreducible of complex type. Then G-equivariant bifurcation problems on V are strongly determined. In particular, there exists $d \in \mathbb{N}$ and an open and dense semianalytic subset $\mathcal{N}(d)$ of $P_G^{(d)}(V,V)$ such that if $f \in \mathcal{M}(V,G)$ and $j^d f_0(0) \in \mathcal{N}(d)$ then*

(1) *f is strongly determined.*
(2) *If H is a closed subgroup of G then f is (d,H)-stable.*

PROOF. As the proof of theorem 10.6.44 is similar to that of the corresponding result for vector fields which is given in [**60**], we shall only outline the main ideas (see also [**62**, section 6.2]). We start by restricting to the set $\mathcal{M}_\omega(V,G) \subset \mathcal{M}(V,G)$ of real analytic families. Using methods based on resolution of singularities, it can be shown that we can find $d, N \in \mathbb{N}$, and an open and dense semialgebraic subset $\mathcal{R}^1$ of $P_G^{(d)}(V,V)$ such that if we define

$$\mathcal{M}_\omega^\star(V,G) = \{f \in \mathcal{M}_\omega(V,G) \mid j^d f_0(0) \in \mathcal{R}^1\}$$

then, for all $p \in \mathbb{N}$, the p-jet at zero of solution branches of $f \in \mathcal{M}_\omega^\star(V,G)$ depends analytically on $j^{p+N} f(0,0)$. (Full details of this construction are given in [**60**, §10].) If G is finite (and therefore (V,G) may be assumed absolutely irreducible by remark 10.6.25), we may use this parameterization theorem, in combination with methods based on Newton-Puiseux series, to obtain estimates on eigenvalues of the linearization along branches of invariant group orbits. A routine application of Tougeron's implicit function theorem [**170**, Chapter 3, theorem 3.2] then yields strong determinacy for smooth maps. If G is not finite, we have to work a little harder. First of all we blow-up along orbit strata using results of Schwarz on the coherence of the orbit stratification (see [**156**] and [**60**, §9]). In this way, we desingularize the branch. Next we use the tangential and normal form for the family and apply the same arguments used for the G-finite case to the normal component to obtain eigenvalue estimates along the branch. □

REMARKS 10.6.45. (1) We emphasize that if (V, G) is irreducible of complex type and G is finite or, more generally, if the centre of G does not contain S^1, then there may be no nontrivial branches of relative fixed points for generic $f \in \mathcal{M}(V, G)$. Unlike what happens in the Hopf bifurcation for vector fields, arithmetic properties of the spectrum of the derivative $Df_0(0)$ play a crucial role in the analysis of bifurcations of elements of $f \in \mathcal{M}(V, G)$ when (V, G) is irreducible of complex type. Practically, we start by assuming f is in normal form – which will depend on $Df_0(0)$ – and then apply theorem 10.6.44. We illustrate with some simple examples in the next section.
(2) The strong determinacy theorem continues to hold provided that families are sufficiently differentiable. The same proof works except that a C^r (and easy to prove) version of Tougeron's implicit function theorem is used when making the transition from real analytic to C^r-maps. This remark has a number of important consequences: (a) We do not have to develop a C^r-version of equivariant transversality in order to prove genericity theorems theorems in equivariant bifurcation theory, (b) Centre manifold arguments which typically do not allow the assumption of smoothness apply, and (c) at least for codimension 1 equivariant bifurcation theory, we do not need to develop C^r versions of invariant theory.

10.6.8. Normal form theorems.

Absolutely irreducible representations. Suppose that (V, G) is an absolutely irreducible orthogonal representation. We allow G to be a general compact Lie group. Our interest lies in examples where $-I_V \notin G \subset \mathrm{O}(V)$. Let $\tilde{G} = \langle G, -I_V \rangle \cong G \times \mathbb{Z}_2$.

It follows by strong determinacy (theorem 10.6.44) that there exists $d \geq 3$ such that $\tilde{G}$-equivariant bifurcation problems on V are strongly (d, G)-determined. Suppose that $f \in \mathcal{M}^-(V, \tilde{G})$ and f is (d, G)-stable. Let $f' \in \mathcal{M}^-(V, G)$ satisfy $j^d f'(0, 0) = j^d f(0, 0)$. We regard f' as a perturbation of f breaking symmetry from $\tilde{G}$ to G. Applying theorem 10.6.44, each branch of normally hyperbolic invariant $\tilde{G}$-orbits in $\mathbf{F}(f)$ will persist as a branch of G-invariant normally hyperbolic submanifolds for f'. Typically, some of these branches will be branches of relative fixed points (and so will appear in $\mathbf{F}(f')$), others will not be branches of G-orbits. If G is finite, each branch for f' which is not in $\mathbf{F}(f')$ will consist of hyperbolic points which are of prime period two (but not symmetry related).

EXAMPLE 10.6.46. Let $G = \Delta_5' \rtimes \mathbb{Z}_5$ (example 10.6.34). Then $\tilde{G}$-equivariant bifurcation problems are strongly $(3, G)$-determined. It is easy to verify directly that if $f \in \mathcal{L}_G^-(\mathbb{R}^5)$, then f has branches of points of prime period two contained in the axes $\mathbb{R}(1,1,1,1,\pm 1)$. However, the period two points are not related by G-symmetries (that is, by multiplication by ± 1). In this example, $(\mathbb{R}^5, G)$, $(\mathbb{R}^5, \tilde{G})$ have the same cubic equivariants and quartic equivariants are required to break symmetry from $\tilde{G}$ to G.

Let $P_G^{(d)}(V \times \mathbb{R}, V)_0$ denote the subset of $P_G^{(d)}(V \times \mathbb{R}, V)$ consisting of polynomial maps with linear term $-(1+\lambda)I_V$. We similarly define $P_{\tilde{G}}^{(d)}(V \times \mathbb{R}, V)_0$. The next result uses the theory of equivariant normal forms [**84**, Chapter XVI, §5] (see also the proof of [**60**, Lemma 9.18.3]).

LEMMA 10.6.47. *Let $d \in \mathbb{N}$. There is a polynomial submersion*

$$N_d : P_G^{(d)}(V \times \mathbb{R}, V)_0 \to P_{\tilde{G}}^{(d)}(V \times \mathbb{R}, V)_0,$$

such that if $f \in \mathcal{M}^-(V, G)$ then $N_d(j^d f(0,0))$ is the $\tilde{G}$-equivariant normal form of f to order d. Moreover, if $p > d$, $N_d(N_p(j^p f(0,0))) = N_d(j^d f(0,0))$. In particular, N_d restricts to the identity map on $P_{\tilde{G}}^{(d)}(V \times \mathbb{R}, V)_0 \subset P_G^{(d)}(V \times \mathbb{R}, V)_0$.

Suppose that G-equivariant bifurcation problems on V are p-determined, $\tilde{G}$-equivariant bifurcation problems on V are q-determined and $\tilde{G}$-equivariant bifurcation problems on V are strongly (d, G)-determined. We have $d \geq p, q$.

THEOREM 10.6.48. *There exists an open and dense semianalytic subset $\mathcal{N}$ of $P_G^{(d)}(V \times \mathbb{R}, V)$ such that if $f \in \mathcal{M}^-(V, G)$ and $j^d f(0,0) \in \mathcal{N}$ then*

(1) *$f \in \mathbf{S}(V, G)$.*
(2) *$\tilde{f} = N_d(j^d f(0,0)) \in \mathbf{S}(V, \tilde{G})$.*
(3) *Every branch of normally hyperbolic relative fixed points for $\tilde{f}$ persists as one or two branches of normally hyperbolic invariant G-orbits for f, each of which is a branch of relative fixed points for f^2. Conversely, every branch of relative fixed points for f^2 arises via such a perturbation.*

PROOF. Let $\mathcal{R}, \tilde{\mathcal{R}}$ be the open and dense semialgebraic subsets of $P_G^{(d)}(V \times \mathbb{R}, V)$, $P_{\tilde{G}}^{(d)}(V \times \mathbb{R}, V)$ that respectively determine stable maps for G- and $\tilde{G}$-equivariant bifurcation problems on V. Let $\mathcal{D}$ be the semianalytic subset of $P_{\tilde{G}}^{(d)}(V \times \mathbb{R}, V)$ that determines the strongly (d, G)-stable mappings in $\mathcal{M}^-(V, \tilde{G})$. Define

$$\mathcal{N} = \mathcal{R} \cap N_d^{-1}(\tilde{\mathcal{R}} \cap \mathcal{D}).$$

Since N_d is a polynomial submersion, $\mathcal{N}$ is an open and dense semianalytic subset of $P_G^{(d)}(V \times \mathbb{R}, V)$. The theorem follows. $\square$

Irreducible unitary representations. Suppose that (V,G) is an irreducible unitary representation.

Let $\theta \in [0,2\pi)$ and $d \in \mathbb{N}$. We let $\mathcal{P}_G^{(d)}(V \times \mathbb{R}, V)_\theta$ denote the subset of $\mathcal{M}^\theta(V,G)$ consisting of families f which can be written in the form

$$f(x,\lambda) = \exp(\imath\omega(\lambda))(1+\lambda)x + \sum_{i=1}^{d} a_i(\lambda)P_i(x),$$

where $a_i \in C^\infty(\mathbb{R})$, $P_i \in P_G^i(V,V)$, $1 \le i \le d$, and $\omega(0) = \theta$. Let $K = \langle \exp(\imath\theta)\rangle$. Either $K = \mathbb{Z}_m$ ($\exp(\imath\theta)$ is an mth root of unity) or $K = S^1$. Suppose that $K = \mathbb{Z}_m$. If (V,G) is irreducible of complex representation there are no restrictions on m. If (V,G) is the complexification of an absolutely irreducible representation then we can assume $m \ge 3$ since the cases $m = 1,2$ would not be generic in our setup and we could reduce to the case of bifurcation on an absolutely irreducible representation. Similar comments hold in the quaternionic case: generically we can assume there are no real eigenvalues.

Let $\tilde{G} = \langle G, K\rangle \subset \mathrm{O}(V)$. Since $K \cap G \subset Z(G)$, we must just as well have defined $\tilde{G} = G \times K$ as both groups define the same subgroup of $\mathrm{O}(V)$. Because of our assumption son θ, the representation $(V,\tilde{G})$ is irreducible of complex type. Let $\mathcal{P}_{\tilde{G}}^{(d)}(V \times \mathbb{R}, V)_\theta$ the corresponding subspace of $\mathcal{M}^\theta(V,\tilde{G})$.

PROPOSITION 10.6.49. *Let $d \in \mathbb{N}$. There is a polynomial submersion*

$$N_d : \mathcal{P}_G^{(d)}(V \times \mathbb{R}, V)_\theta \to \mathcal{P}_{\tilde{G}}^{(d)}(V \times \mathbb{R}, V)_\theta$$

such that if $f \in \mathcal{M}^\theta(V,G)$ then $N_d(j^d f(0,0))$ is the $\tilde{G}$-equivariant normal form of f to order d. Moreover, if $p > d$, $N_d(N_p(j^p f(0,0))) = N_d(j^d f(0,0))$. In particular, N_d restricts to the identity map on $\mathcal{P}_G^{(d)}(V \times \mathbb{R}, V)_\theta \subset \mathcal{P}_{\tilde{G}}^{(d)}(V \times \mathbb{R}, V)_\theta$.

PROOF. This follows by the standard iterated coordinate change proof and we refer to [**84**, Chapter XVI, §5] for details. $\square$

EXAMPLE 10.6.50. Let $(\mathbb{C}^2, \mathbf{D}_3)$ be the standard complex irreducible representation of $\mathbf{D}_3$. We investigate the bifurcation theory of maps $f \in \mathcal{M}^\theta(\mathbb{C}^2, \mathbf{D}_3)$ for various values of θ – more precisely, for various choices of $K = \langle \exp \imath\theta\rangle$.

We start by taking $K = S^1$. In this case one can show (see example 5.6.33) that $(\mathbb{C}^2, \mathbf{D}_3 \times S^1)$ is 5-determined, strongly $(5,\mathbf{D}_3)$-determined and there exists an open and dense semialgebraic subset $\mathcal{R}$ of $P^{(5)}_{\mathbf{D}_3 \times S^1}(\mathbb{C}^2 \times \mathbb{R}, \mathbb{C}^2)_0$ which determines stability and strong $(5,\mathbf{D}_3)$-stability. If $F \in \mathcal{M}^\theta(\mathbb{C}^2, \mathbf{D}_3 \times S^1)$ and $j^5F(0,0) \in \mathcal{R}$, then F will have three normally hyperbolic branches of relative fixed points with isotropy types

$$(\tilde{\mathbb{Z}}_3),\ (\langle\kappa\rangle),\ (\langle -\kappa\rangle).$$

(See table 1, chapter 5.) Each branch will consist of F-invariant circles. In all, there will be eight branches of normally hyperbolic invariant circles – two of type $(\tilde{\mathbb{Z}}_3)$ and three each of types $(\langle\kappa\rangle)$ and $(\langle-\kappa\rangle)$. If $f \in \mathcal{M}^\theta(\mathbb{C}^2, \mathbf{D}_3)$ is such that the 5th order normal form $N_5(j^5 f(0,0)) \in \mathcal{R}$, then f will have eight branches of normally hyperbolic invariant circles. The five branches that come from branches of type $(\tilde{\mathbb{Z}}_3)$ and type $(\langle-\kappa\rangle)$ will have no $\mathbf{D}_3$-symmetry. On the other hand, the three branches that are associated to the branches of isotropy type $(\langle\kappa\rangle)$ will continue to be κ-invariant. Notwithstanding these results, the reader is cautioned that in breaking symmetry from $\mathbf{D}_3 \times S^1$ to $\mathbf{D}_3$ we have not excluded the possibility of new branches of relative fixed sets appearing.

Suppose next that $\langle \exp \imath\theta\rangle = \mathbb{Z}_q$, $q \geq 3$. If $q \geq 7$, then

$$P^j_{\mathbf{D}_3\times S^1}(\mathbb{C}^2, \mathbb{C}^2) = P^j_{\mathbf{D}_3\times \mathbb{Z}_q}(\mathbb{C}^2, \mathbb{C}^2),\ j \leq 5,$$

and $(\mathbb{C}^2, \mathbf{D}_3 \times \mathbb{Z}_q)$ is 5-determined and strongly $(5, \mathbf{D}_3)$-determined. The results for $(\mathbb{C}^2, \mathbf{D}_3 \times S^1)$ continue to apply (this situation is one of 'weak' resonance). On the other hand if $3 \leq q \leq 6$, then $P^{q-1}_{\mathbf{D}_3\times\mathbb{Z}_q}(\mathbb{C}^2, \mathbb{C}^2) \neq P^{q-1}_{\mathbf{D}_3\times S^1}(\mathbb{C}^2, \mathbb{C}^2)$. This is the regime of strong resonance and each case requires separate investigation.

We conclude with an example where (V, G) is quaternionic.

EXAMPLE 10.6.51. Let $Q = \langle i, j, k\rangle \subset H_4 \subset \mathrm{O}(4)$ be the group of unit quaternions. We have $|Q| = 8$ and the representation $(\mathbb{R}^4, Q)$ is irreducible of quaternionic type. Suppose that $f \in C^\infty_Q(\mathbb{R}^4 \times \mathbb{R}, \mathbb{R}^4)$. We may write $Df_\lambda(0) = \sigma(\lambda)A_\lambda$, where $\sigma : \mathbb{R} \to \mathbb{R}$ is smooth and A_λ is Q-equivariant and orthogonal (that is, $A_\lambda \in L_{\mathrm{SU}(2)}(\mathbb{C}^2, \mathbb{C}^2) \cap \mathrm{SO}(4)$). Relative to the standard basis of $\mathbb{R}^4 \approx \mathbb{C}^2$, we have

$$A_\lambda = \begin{pmatrix} E & F \\ -F & E \end{pmatrix},\ \text{where } E = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},\ F = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix},$$

and $\alpha, \beta, \gamma, \delta : \mathbb{R} \to \mathbb{R}$. Note that A_λ is *not* $\mathbb{C}$-linear with respect to the standard $\mathbb{C}$-structure on $\mathbb{C}^2 \approx \mathbb{R}^4$. The orthogonality of A_λ is equivalent to $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$. The eigenvalues of A_λ are

$$\alpha \pm \imath\sqrt{\beta^2 + \gamma^2 + \delta^2}.$$

Bifurcations will occur when $\sigma(\lambda) = \pm 1$. Suppose there is a bifurcation at $\lambda = 0$. Since $-I$, it is no loss of generality to assume $\sigma(0) = 1$. Generically, we may suppose that $\sigma'(0) \neq 0$. After a change of parameter, we may write f in the normal form

$$f(X, \lambda) = (1 + \lambda)\begin{pmatrix} E(\lambda) & F(\lambda) \\ -F(\lambda) & E(\lambda) \end{pmatrix}(X) + O(\|X\|^2),$$

where $X \in \mathbb{R}^4$. Set $\alpha(0) = \alpha_0$, $\ldots$, $\delta(0) = \delta_0$ and define $\exp(\imath\theta) = \alpha_0 + \imath(\beta_0^2 + \gamma_0^2 + \delta_0^2)^{\frac{1}{2}}$. Generically, we may assume that $\langle \exp(\imath\theta)\rangle = S^1$. Let $d \in \mathbb{N}$. Using the standard normal form argument, we may make a local smooth Q-equivariant change of coordinates at the origin so that $j^d f(0,0)$ is $Q \times S^1$-equivariant. In particular, this defines a complex structure on $\mathbb{R}^4$ with respect to which Q acts $\mathbb{C}$-linearly. The representation $(\mathbb{R}^4, Q \times S^1)$ is easily shown to be irreducible of complex type (it is not quaternionic). Applying our determinacy results, $(\mathbb{R}^4, Q \times S^1)$ will be strongly (d, Q)-determined for some $d \in \mathbb{N}$. In this way by analysing bifurcations on the complex representation $(\mathbb{R}^4, Q \times S^1)$ we may deduce results for the original representation $(\mathbb{R}^4, Q)$. Even if $\langle \exp(\imath\theta) \rangle = \mathbb{Z}_q$, $q \geq 3$, it will still be the case that $(\mathbb{R}^4, Q \times \mathbb{Z}_q)$ is a complex (non-quaternionic) representation. We remark that this approach applies to vector fields and, using the methods of section 5.6, we may show that generic $f \in \mathcal{V}(\mathbb{R}^4 \times \mathbb{R}, \mathbb{R}^4, Q)$ will have at least six branches of limit cycles (the induced Q-action on $P^1(\mathbb{C})$ has six points of maximal isotropy which lie on three distinct Q-orbits).

10.7. Relative periodic orbits

In this section we develop methods for the study of generic bifurcations from relative periodic orbits (for flows). Much of what we say is closely based on the article by Lamb and Melbourne on bifurcation from discrete rotating waves [**111**] and we refer the reader to that paper for background as well as an extensive set of examples. More refined and generally applicable theories are developed by Lamb, Melbourne and Wulff in [**183, 112, 113**]. In particular, these works give results on proper actions by non-compact Lie groups as well as a careful analysis of drift dynamics along group orbits and the effects of resonances when breaking normal form symmetry. See also the notes at the end of the chapter.

We start with a rough description of our approach. Suppose that Σ is a relative periodic orbit (not a relative equilibrium) of the smooth G-equivariant vector field X. Let the prime period of the limit cycle of the vector field $X^\star$ induced on $\Sigma/G \approx S^1$ by X be $T > 0$. Choose a smooth isotopy of equivariant vector fields X_t on Σ such that $X_0 = X$, X_t induces the vector field $X^\star$ on Σ/G, all $t \in [0, 1]$, and Σ is foliated by periodic orbits of $X_1 = Y$ of period qT, where $q \geq 1$ is minimal. We may think of Y as a normal form for X (no information is lost as we can recover X from Y by reversing the original isotopy[2]. Each periodic orbit of Y will be a *discrete rotating wave*. All this works perfectly well for families of vector fields. The next step is to study the bifurcation theory of the Poincaré map of a periodic orbit for Y in Λ. For this we use the theory we have developed for families of equivariant maps. Finally, we translate back to the original vector field X by substituting back the original drifts. In practice it is useful to take advantage of normal form symmetries. This can be done either by looking at normal forms for the Poincaré map or by viewing the Poincaré map as the time-one map of a vector field (modulo high order terms). Both approaches are successful in determining branches of solutions but also lose information about the detailed structure of the dynamics (and possibly also miss branches of relative periodic orbits).

[2]The isotopy can be used to define a moving coordinate frame on the relative periodic orbit – this is the approach used in [**183**].

Our first step in the analysis of a relative periodic orbit Σ is to reduce to an analysis of the Poincaré map of Σ. More specifically, suppose that Σ is a relative periodic orbit of the G-equivariant flow $\Psi_t : M \to M$. Let $\alpha = G/H \subset \Sigma$ and choose a Poincaré system $P : D' \subset D \to D$ for the flow near Σ (see section 8.4.5 and note that we assume $D' \pitchfork \Sigma$, $D' \cap \Sigma = \alpha$). Let $T > 0$ denote the time of first return to α (so α is a relative fixed set for Ψ_T). Working locally, we may suppose that P is a G-equivariant diffeomorphism of a G-vector bundle $p : E \to \alpha$ and that $P(\alpha) = \alpha$, where we have identified the zero section E_0 of E with α. It follows from lemmas 8.3.26, 8.3.28 that – up to composition with a isotopically trivial skew-equivariant map – we may assume that (a) $P : E \to E$ is fibre preserving, and (b) there exists a smallest integer $m \geq 1$ such that if $f = P|\alpha$, then $f^m = I_\alpha$. In fact P can be viewed as the Poincaré map of the (same) relative periodic orbit Σ but with respect to a new flow Φ_t on M related to the original flow Ψ_t by

$$\Phi_t(x) = \chi_t(x)\Psi_t(x),$$

where $\chi_t : M \to G$ is skew-equivariant and satisfies $\chi_{t+s}(x) = \chi_t(x)\chi_s(\Psi_t(x))$ for all $t, s \in \mathbb{R}$, $x \in M$. We may assume χ_t is constant, equal to the identity, outside some preassigned G-invariant neighbourhood of Σ in M.

The Poincaré maps for Ψ_t and Φ_t differ only in dynamics along group orbits. Both maps induce the same map on orbit space and relative fixed points and periodic orbits of the maps have the same stabilities. Note that since $f^m = I_\alpha$, the Φ_t-orbit through any point of $x \in \alpha$ will be a periodic orbit γ of Φ_t of prime period mT. In particular, $P^m : E_{[H]} \to E_{[H]}$ will be a Poincaré map for γ.

We simplify matters a little by noting that $E(H) = E|\alpha^H$ is an $N(H)$-vector bundle over α^H and P restricts to an $N(H)$-equivariant bundle map $P : E(H) \to E(H)$. Results we obtain for $P|E(H)$ extend immediately to E and so to the original Poincaré map P. Thus it will be no loss of generality to assume $G = N(H)$ or, equivalently, $H \lhd G$.

Overall then we look at families $P_\lambda \in \mathrm{Diff}_G(E)$ which cover a fixed f and preserve the zero section. If $1 \in \mathbf{spec}(P_0, \alpha)$, then we can expect branches of relative fixed points to appear as λ passes through zero. These correspond to bifurcations of the periodic orbit γ and may be shown to define branches of relative periodic orbits. Just as in the previous section, we get far stronger results by using methods based on normal forms and most of what we do will be angled towards providing a framework where we can apply a normal form theorem – in this case, the normal form theorem of Lamb for twisted equivariant maps [**110**]. All of this requires some serious preliminaries starting with an investigation of the class of G-vector bundle maps which cover a finite order equivariant diffeomorphism of a group orbit. The results we obtain will apply to the linearization of the Poincaré map along α.

10.7.1. G-vector bundles and maps over a group orbit. Let H be a closed normal subgroup of the compact Lie group G and set $\alpha = G/H$. Let

(V, H) be a finite dimensional H-representation. We form the twisted product $E = G \times_H V$ and recall that E has the natural structure of a G-vector bundle $p : E \to \alpha$. Since $H \lhd G$, each fibre E_x has the structure of an H-representation. Let $f \in \mathrm{Diff}_G(\alpha)$, $F \in \mathrm{Diff}_G(E)$ be such that the diagram

$$\begin{array}{ccc} E = G \times_H V & \xrightarrow{F} & G \times_H V = E \\ \downarrow p & & \downarrow p \\ \alpha & \xrightarrow{f} & \alpha \end{array}$$

commutes and F is a vector bundle map ($F : E_x \to E_{f(x)}$ is an H-linear isomorphism for all $x \in \alpha$).

Fix a G-invariant Riemannian structure on $p : E \to \alpha$. For each $x \in \alpha$, we have an orthogonal representation $\rho_x : H \to \mathrm{O}(E_x)$. Since the Riemannian structure on E is G-invariant, $g : E_x \to E_{gx}$ is an isometry for all $g \in G, x \in \alpha$. Fix the base point $[H] \in \alpha$ and denote the corresponding orthogonal representation of H by $\rho : H \to \mathrm{O}(E_{[H]})$. The representation ρ equals the given representation (V, H) since $E_{[H]} \approx V$. In what follows we usually identify $E_{[H]}$ with V. In general, if $x, y \in \alpha$, the H-representations ρ_x, ρ_y may not be isomorphic. However, since $F|E_x$ intertwines the representations (E_x, ρ_x), $(E_{f(x)}, \rho_{f(x)})$, the representations $\rho_x, \rho_{f(x)}$ are isomorphic for all $x \in \alpha$.

LEMMA 10.7.1. *(Assumptions as above). Let f correspond to $\tilde{n} \in G/H \approx \mathrm{Diff}_G(G/H)$. If we define $K = \{\gamma \in G/H \mid \rho_\gamma \cong \rho\}$ then K is an open and closed subgroup of G/H and*

$$K \supset \langle (G/H)_0, \tilde{n} \rangle$$

PROOF. Clearly K is a closed subgroup of G/H. It follows from corollary 3.10.3 that $(G/H)_0 \approx C_G(H)_0/Z(H)_0$ and so $K \supset (G/H)_0$ and is therefore open. Finally, as noted before the statement of the lemma, ρ and $\rho_{\tilde{n}}$ are isomorphic and consequently $\tilde{n} \in K$. $\square$

We shall assume that there exists a smallest $m \geq 1$ such that $f^m = I$. Set $V_0 = V$, $V_1 = E_{f([H])}$, $\ldots$, $V_{m-1} = E_{f^{m-1}([H])}$, $V_m = V_0$. Let $A_j = F|V_j : V_j \to V_{j+1}$, (where $j + 1$ is computed modulo m). We often write $A_0 = A$.

Choose $n \in G$ such that $f([H]) = n[H]$. Since $f^m = I$, $n^m[H] = [H]$ and so $n^m \in H$. Let C be a Cartan subgroup of $\mathcal{H} = \cup_{j=0}^{m-1} n^j H \subset G$ which contains n. Since $C \cong \mathbb{T}^s \times \mathbb{Z}_r$, there exists $\delta \in nH$ of finite order. Consequently, we may find the minimal $k \geq 1$ and a corresponding $\delta \in nH$ such that $\delta^k \in C_G(H)$. Define $L = \delta^{-1} A \in L(V, V)$ and note that L is non-singular.

From now on m, k, δ will remain fixed. We remark that if G is Abelian then $k = 1 \leq m$. On the other hand if G is not Abelian we may have $k > m$ (see exercise 8.3.16 and later in this section).

Define the representation $\sigma : H \to \mathrm{O}(V)$ by

$$\sigma(h)(v) = \delta^{-1} \rho_{f([H])}(h) \delta(v), \quad (v \in V, h \in H).$$

As we did earlier, we use the notation ${}_\rho V$ (respectively, ${}_\sigma V$) to denote V together with the action on V given by ρ (respectively σ). With these conventions, we have

$$L \in L_H({}_\rho V, {}_\sigma V).$$

LEMMA 10.7.2. (a1) $({}_\rho V, H)$ *and* $({}_\sigma V, H)$ *are isomorphic representations.*
(a2) $\rho(H) = \sigma(H) \subset O(V)$. *In particular, ρ and σ have the same H-orbits.*
(a3) *kernel*$(\rho) =$ *kernel*(σ).
(a4) *If $k > 1$, then $\pm I \notin L_H({}_\rho V, {}_\sigma V)$.*
(a5) $C^\infty_H({}_\rho V, {}_\rho V) \approx C^\infty_H({}_\sigma V, {}_\sigma V)$ *(we denote either space by $C^\infty_H(V, V)$). In particular, $L_H({}_\rho V, {}_\rho V) \approx L_H({}_\sigma V, {}_\sigma V)$ (we denote either space by $L_H(V, V)$).*
(a6) $L^k \in L_H(V, V)$.
(a7) *If $\tau \in \mathcal{O}({}_\rho V, H)$, then ${}_\rho V_\tau = {}_\sigma V_\tau \overset{def}{=} V_\tau$.*

PROOF. We prove (a1,3,4) and leave the remaining statements to the reader. (a1) is immediate since L intertwines the representations $({}_\rho V, H)$ and $({}_\sigma V, H)$. Since $({}_\rho V, H)$ and $({}_\sigma V, H)$ are isomorphic, we have kernel$(\rho) =$ kernel(σ), proving (a3). Finally, if $I \in L_H({}_\rho V, {}_\sigma V)$, then $h = Ih = \delta^{-1}h\delta I = \delta^{-1}h\delta$ for all $h \in H$. Hence $\delta \in C_G(H)$, contradicting our assumption that $k > 1$. $\square$

EXERCISE 10.7.3. (1) Suppose that ρ is irreducible. Show that if $k > 1$ then $L_H({}_\rho V, {}_\rho V) \cap L_H({}_\rho V, {}_\sigma V) = \{0\}$.
(2) Let K be as defined in lemma 10.7.1 and let $\tilde{G}$ be the subgroup of G such that $\tilde{G}/H = K$. Show that kernel$(\rho_x) \lhd \tilde{G}$ for all $x \in K$ and hence deduce that if ρ, σ are not faithful then it is no loss of generality to replace H by $H/$kernel(ρ), G by $\tilde{G}/$kernel(ρ) and assume ρ and σ are faithful.

Noting lemma 10.7.2(a3) and exercise 10.7.3(2), we assume for the remainder of the section that $\rho : H \to \mathrm{O}(V)$ is a faithful representation of H.

REMARK 10.7.4. Lemma 10.7.2(a2) implies that ρ, σ have the same invariants and so we may write $P(V)^H$ for the algebra of invariants of either ${}_\rho V$ or ${}_\sigma V$. By lemma 10.7.2(a7), we have $\mathcal{O}({}_\rho V, H) = \mathcal{O}({}_\sigma V, H)$ and so we may unambiguously write $\mathcal{O}(V, H)$ for the set of isotropy types for either action of H on V.

EXAMPLE 10.7.5. For $p \in \mathbb{Z}$, $p \neq 0$, define the irreducible action of SO(2) on $\mathbb{C}$ by $R_p(\theta) = e^{\imath p\theta}z$, $\theta \in [0, 2\pi]/(0 = 2\pi)$. Set $\rho = R_p$, $\sigma = R_q$, $q \neq 0$. Then ρ, σ satisfy (a2). If $p = \pm q$, ρ, σ satisfy (a1–8).

LEMMA 10.7.6. (1) *There exists an orthogonal map $J \in L_H({}_\rho V, {}_\sigma V)$ ($\|Jx\| = \|x\|$, all $x \in V$).*
(2) *If $L \in L_H({}_\rho V, {}_\sigma V)$ is orthogonal and $k > 1$, then $L \notin H$.*

PROOF. (1) Suppose first that ρ, σ are real absolutely irreducible (or quaternionic) orthogonal H-representations. Denote the H-invariant inner product on V by $(\ ,\ )$. Since (V, H) is absolutely irreducible, every H-invariant inner product on V is a non-zero positive multiple of $(\ ,\ )$. In particular, if $J \in L_H({}_\rho V, {}_\sigma V)$ and we define the H-invariant inner product $(\ ,\ )^\star$ on V by $(x,y)^\star = (Jx, Jy)$, we have $(x,y)^\star = a^2(x,y)$, for some $a > 0$. Consequently $\frac{1}{a}J : V \to V$ is orthogonal. The same proof works if ρ, σ are irreducible unitary representations. The general case follows straightforwardly using the isotypic decomposition. (2) If $h \in H$, then $Lh = \delta^{-1}h\delta L$. Hence $(\delta L)h = h(\delta L)$, for all $h \in H$ and so $\delta L \in C_{\mathrm{O}(V)}(H)$. If $L \in H$, then $\delta L \in C_G(H)$, contradicting the assumption that $k > 1$ (replace δ by δL to get $k = 1$). ◻

EXAMPLE 10.7.7. Suppose that ρ, σ are irreducible and $L \in L_H({}_\rho V, {}_\sigma V)$ is orthogonal. If ρ is absolutely irreducible, then $\pm L$ are the unique orthogonal maps intertwining ρ, σ. If ρ is irreducible of complex type, then $\{e^{\imath\theta}L \mid e^{\imath\theta} \in S^1\}$ are the unique orthogonal maps intertwining ρ, σ. It turns out that the key invariant of L is $\langle L \rangle \subset \mathrm{O}(V)$. We show some of the possibilities in the following examples.

(1) Let $\rho : \mathrm{O}(2) \to \mathrm{O}(\mathbb{C})$ denote the standard representation of $\mathrm{O}(2)$ and define the representation σ by $\sigma(g) = L\rho(g)L^{-1}$, $g \in \mathrm{O}(2)$. Define $L : \mathbb{C} \to \mathbb{C}$ by $L(z) = e^{\imath\theta}\bar{z}$. Obviously, $L \in L_{\mathrm{O}(2)}({}_\rho V, {}_\sigma V)$ is orthogonal. The group $\langle L \rangle \subset \mathrm{O}(2)$ is isomorphic to $\mathbb{Z}_2$. If instead we had defined $Lz = e^{\imath\theta}z$, then $\langle L \rangle \cong \mathbb{Z}_q$ if $\theta/2\pi = p/q$, $(p,q) = 1$, and is otherwise isomorphic to $S^1 = \mathrm{SO}(2) \subset \mathrm{O}(2)$.
(2) Let $\rho : \mathbf{D}_4 \to \mathrm{O}(\mathbb{C})$ be the standard representation of $\mathbf{D}_4$ and define the representation σ by $\sigma(g) = -\imath\rho(g)\imath$, $g \in \mathbf{D}_4$. If we let $Lz = \imath z$, then $L \in L_{\mathbf{D}_4}(\mathbb{C}, \mathbb{C})$ and $\langle L \rangle \cong \mathbb{Z}_4$. It is also possible to choose L so that $\langle L \rangle \cong \mathbb{Z}_1, \mathbb{Z}_2$. For this representation of $\mathbf{D}_4$, $\mathbb{Z}_1, \mathbb{Z}_2, \mathbb{Z}_4$ are the only possibilities for $\langle L \rangle$.

LEMMA 10.7.8. *Suppose that ρ, σ are irreducible as orthogonal $\mathbb{R}$-representations. Then there exists $J \in L_H({}_\rho V, {}_\sigma V)$, $\|J\| = 1$, such that*

(1) *If $({}_\rho V, H)$ is absolutely irreducible, then $L_H({}_\rho V, {}_\sigma V) = \{aJ \mid a \in \mathbb{R}\}$ and $\|aJ\| = |a|$, $a \in \mathbb{R}$.*
(2) *If $({}_\rho V, H)$ is irreducible of complex type, then*
$$L_H({}_\rho V, {}_\sigma V) = \{(a + \imath b)J \mid a, b \in \mathbb{R}\}$$
and $\|(a+\imath b)J\| = |a + \imath b|$, $a + \imath b \in \mathbb{C}$.
(3) *If $({}_\rho V, H)$ is irreducible of quaternionic type, then*
$$L_H({}_\rho V, {}_\sigma V) = \{(a + \imath b + \jmath c + kd)J \mid a, b, c, d \in \mathbb{R}\},$$
where $\imath, \jmath, k$ are the unit quaternions and $\|(a + \imath b + \jmath c + kd)J\| = |a + \imath b + \jmath c + kd|$, $a + \imath b + \jmath c + kd \in \mathbb{H}$.

PROOF. It follows from the previous lemma that there exists an isometry $J \in L_H({}_\rho V, {}_\sigma V)$. Since $J^{-1}L_H({}_\rho V, {}_\sigma V) = L_H({}_\rho V, {}_\rho V)$ is a real associative division algebra, the result follows from theorem 2.7.14. ◻

REMARK 10.7.9. The map $J \in L_H({}_\rho V, {}_\sigma V)$ given by lemma 3.5.1 is only determined up to multiplication by unit scalars. For example, J is determined up to multiplication by $\pm I_V$ if ${}_\rho V$ is absolutely irreducible.

Let $\mathrm{GL}({}_\rho V, {}_\sigma V)$ denote the open and dense subset of $L_H({}_\rho V, {}_\sigma V)$ consisting of invertible maps. Define

$$\mathbf{sing}({}_\rho V, {}_\sigma V) = \{L \in \mathrm{GL}({}_\rho V, {}_\sigma V) \mid 1 \in \mathrm{spec}(L)\}$$

and note that $\mathbf{sing}({}_\rho V, {}_\sigma V)$ is a closed subset of $\mathrm{GL}({}_\rho V, {}_\sigma V)$. If $L \in \mathbf{sing}({}_\rho V, {}_\sigma V)$, let $\mathbb{E}(L)$ denote the generalized eigenspace of $L : V \to V$ corresponding to $1 \in \mathrm{spec}(L)$. Obviously $\mathbb{E}(L)$ is L-invariant. Since ρ, σ are orthogonal representations, $\|L^n hv\| = \|L^n v\|$ for all $v \in \mathbb{E}(L)$, $h \in H$, $n \geq 1$, and so $\mathbb{E}(L)$ is H-invariant. Hence we have *isomorphic* H-representations ${}_\rho\mathbb{E}(L)$ and ${}_\sigma\mathbb{E}(L)$. We say that $\mathbb{E}(L)$ is irreducible if there are no proper real L- and H-invariant subspaces of $\mathbb{E}(L)$.

LEMMA 10.7.10 ([**111**, Theorem 3.2]). *The set $\mathbf{sing}_0({}_\rho V, {}_\sigma V)$ of L such that $\mathbb{E}(L)$ is irreducible is an open and dense subset of $\mathbf{sing}({}_\rho V, {}_\sigma V)$.*

PROOF. Let $L \in \mathbf{sing}_0({}_\rho V, {}_\sigma V)$, and suppose $\mathbb{E}(L)$ is not irreducible. Choose a minimal proper L- and H-invariant space U of $\mathbb{E}(L)$. Since $\mathbb{E}(L)$ is an H-representation, we may write $\mathbb{E}(L) = U \oplus W$, where W is an H-invariant subspace of $\mathbb{E}(L)$. Since $L(U) = U$, $L : U \oplus W \to U \oplus W$ has block matrix form

$$L = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

where $D : {}_\rho W \to {}_\sigma W$ is H-equivariant. If we perturb D to $D_\varepsilon = (1+\varepsilon)D$ then, for the corresponding map $L_\varepsilon : U \oplus W \to U \oplus W$, we have $\mathbb{E}(L_\varepsilon) = U$ and obviously $L_\varepsilon | U$ is irreducible. $\square$

PROPOSITION 10.7.11 ([**111**, Theorem 3.4]). *Suppose that $L \in \mathbf{sing}_0({}_\rho V, {}_\sigma V)$ and that $\mathbb{E}(L) = V$. Then*

(1) *$L^k = \alpha I_V$, where $|\alpha| = 1$.*
(2) *$\langle H, L\rangle$ is a compact subgroup of $\mathrm{GL}(V)$.*
(3) *We may choose an H-invariant inner product on V such that $\langle H, L\rangle \subset \mathrm{O}(V)$. (Equivalently, there is a G-invariant Riemannian structure on $p : E \to \alpha$ relative to which $\langle H, L\rangle \subset \mathrm{O}(V)$.)*
(4) *$H \lhd \langle H, L\rangle$.*
(5) *Generically, $(V, \langle H, L\rangle)$ is absolutely irreducible or irreducible of complex type.*

PROOF. By lemma 10.7.2(a6), $L^k \in L_H(V, V)$ and so eigenspaces of L^k are H- and L-invariant. Since V is assumed $\langle L\rangle$-irreducible, it follows that L^k must be a real or complex multiple α of the identity. Since $\mathrm{spec}(L) = \{1\}$, $|\alpha| = 1$, proving (1). As a consequence of (1), $\langle L^k\rangle$ is a compact subgroup of $\mathrm{GL}(V)$ and so since $\langle H, L^k\rangle$ is compact. Since $\langle H, L\rangle / \langle H, L^k\rangle$ is finite, $\langle H, L\rangle \subset \mathrm{GL}(V)$ is compact, proving (2). Average the given H-invariant inner product on V over $\langle L\rangle$ to obtain a $\langle H, L\rangle$-invariant inner product on V such that $\langle H, L\rangle \subset \mathrm{O}(V)$. Since $L \in L_H({}_\rho V, {}_\sigma V)$, $LhL^{-1} = \delta h \delta^{-1} \in H$, $h \in H$, and so $H \lhd \langle H, L\rangle$. Finally, suppose that $(V, \langle H, L\rangle)$ is irreducible of quaternionic type and so $L_{\langle H,L\rangle}(V, V) \approx \mathbb{H}$. Now $L^k \in L_{\langle H,L\rangle}(V, V)$ and so $L^k = hI$, some $h \in \mathbb{H}$. Since $L^k \in \langle H, L\rangle$, it follows by equivariance that L^k must commute with all quaternions. The only way this can happen is if h is real. But we can always perturb L within $\mathbf{sing}_0({}_\rho V, {}_\sigma V)$ so that L^k has complex eigenvalues and so force $h \in \mathbb{H} \setminus \mathbb{R}$. $\square$

For the remainder of this subsection we assume that $L \in \mathbf{sing}_0({}_\rho V, {}_\sigma V)$, $\mathbb{E}(L) = V$ and we have taken a Riemannian structure on E for which $\langle H, L\rangle \subset \mathrm{O}(V)$.

PROPOSITION 10.7.12. *Let $p \geq 1$ be the minimal positive integer such that $L^p \in H$. If $L^j \notin H$ for all $j \geq 1$, set $p = 0$.*

(1) *If $(V, \langle H, L\rangle)$ is absolutely irreducible, then*
  (a) *$\langle L^k\rangle \cong \mathbb{Z}_q$, where $q = 1$ or 2. If $q = 1$, $L^k = I_V$ and $\langle L\rangle \cong \mathbb{Z}_k$, and if $q = 2$, $L^k = -I_V$ and $\langle L\rangle \cong \mathbb{Z}_{2k}$. In either case, $p|qk$.*
  (b) *$\langle H, L\rangle / H \cong \mathbb{Z}_p$ and we have the short exact sequence of groups*
  $$1 \to \mathbb{Z}_{qk/p} \to H \rtimes \langle L\rangle \to \langle H, L\rangle \to 1,$$
  If $p = qk$, then $\langle H, L\rangle \cong H \rtimes \langle L\rangle$.

(2) *If $(V, \langle H, L\rangle)$ is irreducible of complex type, then*
  (a) *$\langle L^k\rangle \subset S^1 \subset \mathrm{O}(V)$, where S^1 acts on V as scalar multiplication by complex numbers of unit modulus.*
  (b) *If $p = 0$ ($L^j \notin H$, for all $j \geq 1$), then there exists $q \geq 1$ such that $\langle L\rangle = S^1 \times \mathbb{Z}_q$. We have $\langle H, L\rangle = H \rtimes (S^1 \times \mathbb{Z}_q)$.*
  (c) *If $p \geq 1$ and $\langle L\rangle = S^1 \times \mathbb{Z}_q$, then $p|q$, $\langle L^k\rangle = S^1$, $H \supset S^1$, $\langle H, L\rangle / H \cong \mathbb{Z}_p$ and we have the short exact sequence of groups*
  $$1 \to \mathbb{Z}_{q/p} \to H \rtimes \langle L\rangle \to \langle H, L\rangle \to 1,$$
  (d) *If $\langle L^k\rangle = \mathbb{Z}_q$ then $p|kq$, $\langle H, L\rangle / H \cong \mathbb{Z}_p$ and we have the short exact sequence of groups*
  $$1 \to \mathbb{Z}_{kq/p} \to H \rtimes \langle L\rangle \to \langle H, L\rangle \to 1,$$

PROOF. Suppose that $(V, \langle H, L\rangle)$ is absolutely irreducible. It follows from proposition 10.7.11 that $L^k \in L_{\langle H,L\rangle}(V, V) \cap \mathrm{O}(V)$ and so $L^k = \pm I_V$. If $L^k = I_V$, then $L^k \in H$ and so $p|k$. If $L^k = -I_V$, then $L^{2k} \in H$ and so $p|2k$. Since $H \lhd \langle H, L\rangle$, $\langle H, L\rangle / H \cong \mathbb{Z}_p$. The short exact sequence statement holds since the natural map $H \rtimes \langle L\rangle \to \langle H, L\rangle$ is onto and has kernel $H \cap \langle L\rangle \cong \mathbb{Z}_{K/p}$. The proofs of the corresponding statements when $(V, \langle H, L\rangle)$ is irreducible of complex type are similar. $\square$

10.7.2. Normal form theory. We continue with the assumptions and notational conventions of the previous section. In particular, let $E = G \times_H V$ be a G-vector bundle over $G/H = \alpha$ associated to the H-representation (V, H) and let $E_0 \approx \alpha$ denote the zero section of E. The positive integers m, k, p, ℓ will be as previously defined (always minimal with respect to their defining property).

Let $P_\lambda \in \mathrm{Diff}_G(E)$ be a smooth family of maps covering $f \in \mathrm{Diff}_G(\alpha)$, where $f^m = I_\alpha$. We assume that $P_\lambda(\alpha) = \alpha$ for all $\lambda \in \mathbb{R}$. Let $F_\lambda : E \to E$ denote the G-vector bundle map covering f defined by $F_\lambda | E_x = T_x P_\lambda | E_x$ (here $x \in \alpha \approx E_0$ and we have used the natural identification between $T_x E$ and $E_x \oplus T_x\alpha$, $x \in E_0$).

Identifying $E_{[H]}$ with V, we set $F_\lambda | V = A_\lambda$ and $L_\lambda = \delta^{-1} A_\lambda$, where $\delta^k \in C_H(G)$. When $\lambda = 0$, we often drop the subscript "0". Assume $1 \in \mathrm{spec}(A)$, $\mathbb{E}(L) = V$ is irreducible and $(V, \langle H, L\rangle)$ is either absolutely irreducible or irreducible of complex type (see proposition 10.7.11). Define $\nu(\lambda) = \mathrm{spec}(A_\lambda)$. We make the generic assumption that $\nu'(0) \neq 0$. Reparameterizing, it is no loss of generality to assume that $\nu(\lambda) = 1 + \lambda$, $|\lambda| < 1/2$. Set $\delta^{-1} P_\lambda | V = Q_\lambda$. Then $Q_\lambda : {}_\rho V \to {}_\sigma V$ is H-equivariant and we may write

$$Q_\lambda = L k_\lambda,$$

where $k_\lambda \in \mathrm{Diff}_H(V)$, $Dk_0(0) = I_V$, $DQ_0(0) = L$ and $\mathrm{spec}(Dk_\lambda(0)) = 1 + \lambda$. Let $\mathcal{V}_L(V, H) \subset C^\infty_H({}_\rho V \times \mathbb{R}, {}_\sigma V)$ denote the space of smooth families $q : V \times \mathbb{R} \to V$ such that for $|\lambda| < 1/2$ we can write

$$q(x, \lambda) = L h(x, \lambda),$$

where

(a) $Dh_0(0) = I_V$ (and so $Dq_0(0) = L$).
(b) $\mathrm{spec}(Dh_\lambda(0)) = \mathrm{spec}(Dq_\lambda(0)) = 1 + \lambda$.
(c) $\mathbb{E}((1 + \lambda)^{-1} Dq_\lambda(0)) = V$ and is irreducible.
(d) h is H-equivariant.

Clearly, $Q_\lambda \in \mathcal{V}_L(V, H)$. For $d \geq 1$, let $\mathcal{P}^{(d)}_H({}_\rho V \times \mathbb{R}, {}_\sigma V)_L$ denote the space of smooth families of H-equivariant polynomial maps $q_\lambda : {}_\rho V \to {}_\sigma V$ such that $Dq_\lambda(0)$ satisfies (a,b,c) above. We let $\mathcal{P}^{(d)}_{\langle H,L\rangle}({}_\rho V \times \mathbb{R}, {}_\sigma V)_L \subset \mathcal{P}^{(d)}_H({}_\rho V \times \mathbb{R}, {}_\sigma V)_L$ denote the subspace consisting of $\langle H, L\rangle$-equivariant families. Finally, we Let $\mathcal{P}^{(d)}_{\langle H,L\rangle}(V \times \mathbb{R}, V)_0$ denote the space of $\langle H, L\rangle$-equivariant polynomial families $p_\lambda : V \to V$ such that $Lp_\lambda \in \mathcal{P}^{(d)}_{\langle H,L\rangle}({}_\rho V \times \mathbb{R}, {}_\sigma V)_L$. We have $j^d Q_\lambda(0) \in \mathcal{P}^{(d)}_H({}_\rho V \times \mathbb{R}, {}_\sigma V)_L$ and $j^d k_\lambda(0) \in \mathcal{P}^{(d)}_H(V \times \mathbb{R}, V)_0$.

THEOREM 10.7.13 (Lamb [**110**]). *Let $d \geq 2$. There is a polynomial submersion*

$$N_d : \mathcal{P}^{(d)}_H({}_\rho V \times \mathbb{R}, {}_\sigma V)_L \to \mathcal{P}^{(d)}_{\langle H,L\rangle}({}_\rho V \times \mathbb{R}, {}_\sigma V)_L$$

such that if $f \in \mathcal{V}_L(V, H)$ then $N_d(j^d f(0, 0))$ is the $\langle H, L\rangle$-equivariant normal form of f to order d. Moreover, if $p > d$, $N_d(N_p(j^p f(0, 0))) = N_d(j^d f(0, 0))$.

PROOF. This result is proved in Lamb [**110**]. We remark that the normal form is achieved by making a succession of polynomial H-equivariant coordinate changes on V. This procedure is consistent and natural since H-equivariant maps on source and target spaces are the *same* – see lemma 10.7.2(a5). $\square$

We now follow the strategy described in [**111**]. If $(V, \langle H, L\rangle)$ is absolutely irreducible, set $S = L$. If $(V, \langle H, L\rangle)$ is irreducible of complex type, set $S = \alpha^{-1/k} I_V$, where $L^k = \alpha I_V$. In either case we may write $\delta^{-1}P_0 = Q_0 = Sh$, where $h \in \mathrm{Diff}_H(V)$ and $Dh(0) = I_V$ in the real case and $\alpha^{1/k} I_V$ in the complex case (and so $\langle Dh(0)\rangle \subset S^1$ which would not have been true had we written $Q = Lk$). If Q commutes with L up to terms of order N then, since S commutes with L, it follows that h is $\langle H, L\rangle$-equivariant to order N.

Matters now proceed much as in section 10.6. We assume that $P_\lambda : G\times_H V \to G \times_H V$ is a smooth G-equivariant family covering $f \in \mathrm{Diff}_G(\alpha)$, where $f^m = I_\alpha$. Choose $\delta \in G$ such that $\delta^{-1}P_\lambda = Q_\lambda : {}_\rho V \to {}_\sigma V$ where $\delta^k \in C_G(H)$. Let $L \in \mathrm{O}({}_\rho V, {}_\sigma V)$ denote the derivative of $Q_\lambda | V$ at zero. Write $Q_\lambda = L k_\lambda$, where $k_\lambda : V \to V$ is $\langle H, L\rangle$-equivariant. We consider the cases where $(V, \langle H, L\rangle)$ is either absolutely irreducible or irreducible of complex type. If $(V, \langle H, L\rangle)$ is irreducible of complex type, we assume for the present that $\langle L\rangle \cong S^1 \times \mathbb{Z}_q$. That is, if $L^k = \alpha I$, α is not a complex root of unity.

It follows by G-equivariance that for all $j \in \mathbb{Z}$ we have

$$P_\lambda^j = \delta^j L^j k_\lambda^j.$$

Since $1 \in \mathbf{spec}(P_0^m, \alpha)$, and $Dk_0(0) = I_V$, we have

$$\begin{aligned} \delta^m L^m &= \pm I, \ (\text{if } (V, \langle H, L\rangle) \text{ is absolutely irreducible}), \\ &= \beta I, \ (\text{otherwise, where } \beta \in \mathbb{C}) \end{aligned}$$

10.7.3. $(V, \langle H, L\rangle)$ **absolutely irreducible.** In the absolutely irreducible case, $L^m = \pm\delta^{-m}$ and so $P_\lambda^m = \pm k_\lambda^m$. Hence, every branch of relative fixed points of k_λ determines a branch of relative periodic orbits of P_λ^m. In particular, every branch of fixed points of k_λ determines a branch of fixed points or a branch of period two points of P_λ^m. Applying the general theory given in the previous section, we see that branches of (relative) fixed points of k_λ will generically be normally hyperbolic. We also have stability, finiteness, determinacy and strong determinacy results.

The conditions $\delta^m L^m = \pm I$ and $L^k = \pm I$ impose additional constraints on the type of bifurcations that can occur. For a period doubling bifurcation, we have the linearization of $P_0^m | V$ to be $-I$ and so $\delta^m L^m = -I$. If $k = 1$ (for example, if G is Abelian), we have $L = \pm I$. Consequently if m is even then a necessary condition for a period doubling bifurcation is that δ^m is of even order. Thus there can be no period doubling bifurcation if δ^m is of odd order. A simple example is given by $G = H \rtimes \mathbb{Z}_m$. In this case $\delta^m = 1$ so if $m|k$ is even there is no period doubling bifurcation. If $L = -I$ ($k = 1$), $\delta^m = -I$ and m is odd, then

$\delta^m L^m = I$ and so there can be no period doubling. We refer the reader to [**111**] for more general results and examples (see also later in this section).

10.7.4. $(V, \langle H, L\rangle)$ **irreducible of complex type.** In this case, we have by S^1-equivariance:

$$k_\lambda(x) = (1+\lambda)\exp(\imath\omega(\lambda))x + \mathrm{O}(\|x\|^3)$$

If H is finite, we expect to see branches of normally hyperbolic invariant circles appear at the bifurcation. These will correspond flow-invariant two-tori for the original flow. Just as in the absolutely irreducible case, branches of (relative) fixed points of k_λ will be generically normally hyperbolic and we have stability, finiteness, determinacy and strong determinacy results.

10.7.5. Breaking normal form symmetry. We start with the easiest case when $(V, \langle H, L\rangle)$ is absolutely irreducible and H is finite. It follows – just as in the previous section – that when we break normal form symmetry at sufficiently high order, all hyperbolic branches persist. However, some of the branches will now only be approximately symmetric. If H is not finite, then branches persist as branches of normally hyperbolic invariant sets. Determination of the dynamics on each branch may require further – possibly delicate – analysis.

Suppose $(V, \langle H, L\rangle)$ is irreducible of complex type and that $L^k = \alpha I$. Fix $d \geq 2$. Providing we exclude finitely many values of α, we can require that the $\langle H, L\rangle$-equivariants to order d coincide with the equivariants in the case when α is not a complex root of unity (see [**111**, Theorem 6.4]). Consequently, the *normal form* analysis and determinacy statements continue to hold provided we avoid finitely many values of α, all complex roots of unity. Breaking the normal form symmetry at sufficiently high order will then typically lead to quasi-periodic, phase-locked periodic or chaotic solutions on normally hyperbolic invariant tori. Since we allow H to be a general compact Lie group, the flow invariant tori that appear can be of dimension strictly greater than two. We refer to [**183, 113**] for more details and examples (including for the case of proper group actions). We give a simple example at the end of the section.

10.7.6. Representation theory. We now sketch some of the representation theory of $(V, \langle H, L\rangle)$. Our results our based on those given in [**111**] and the reader is referred to [**111**] for a discussion of cases we omit.

Suppose that $(V, \langle H, L\rangle)$ is irreducible of complex type. Let $L^k = \alpha L$ and set $S = \alpha^{-1/k}L$ (there are k choices for S). Clearly $\langle S\rangle \cong \mathbb{Z}_k$.

PROPOSITION 10.7.14 ([**111**, Proposition 6.2]). *Suppose that $(V, \langle H, L\rangle)$ is irreducible of complex type, Then $(V, \langle H, S\rangle)$ is either the sum of two isomorphic absolutely irreducible representations, neither of which is L-invariant, or is irreducible but not absolutely irreducible.*

PROOF. The map S (and therefore $(V, \langle H, S\rangle)$) is unchanged if we multiply L by a complex number of unit modulus. Consequently, we may assume that $L^k = \alpha I_V$, where α is not a complex root of unity. We then have $\langle L^k\rangle \cong S^1$. The action of S^1 commutes with action of $\langle H, S\rangle$ on V (since this is true for the generator L^k). It follows that if $(V, \langle H, S\rangle)$ has isotypic decomposition $\oplus_{j=1}^r W_j$, then each factor W_j is S^1-invariant and so $\langle H, L\rangle$-invariant. Hence there is just one term in the isotypic decomposition, say W^s, $s \geq 1$, and W^s has a complex structure inherited from the action of S^1. We have $L^{\mathbb{C}}_{\langle H,L\rangle}(W^s, W^s) \cong \mathbb{C}$. If $(W, \langle H, S\rangle)$ is absolutely irreducible, then $s = 2$, otherwise $s = 1$. □

If $(V, \langle H, L\rangle)$ is absolutely irreducible, $\ell \geq 1$ will always denote the order of L (thus $\ell = k$ or $2k$). If $(V, \langle H, L\rangle)$ is irreducible of complex type, we let $\ell = k$ (the order of $S = \alpha^{-1/k}L$). We have the following corollary of proposition 10.7.14.

COROLLARY 10.7.15. *Suppose that $(V, \langle H, L\rangle)$ is irreducible of complex type and that $L^k = \alpha I_V$ where α is not a complex root of unity.*

(1) $\langle H, S\rangle \subset \langle H, L\rangle$.
(2) *Either $(V, \langle H, L\rangle)$ is isomorphic to the complexification of an absolutely irreducible representation $(W, \langle H, S\rangle)$, $W \subset V$, or $(V, \langle H, S\rangle)$ is irreducible of either complex or quaternionic type.*

Using the methods of [**111**], we give the structure of $\langle H, S\rangle$-irreducible representations of V. In the complex case, we assume α is not a complex root of unity. Using normal form and strong determinacy theory (see the previous subsection) we may describe the generic (open and dense) $\langle H, L\rangle$-bifurcations of 1-parameter families.

Throughout we assume that $S \in \mathrm{O}(V)$ and $\langle S\rangle \cong \mathbb{Z}_\ell$. When $(V, \langle H, L\rangle)$ is absolutely irreducible, $\ell \geq 1$ will always denote the order of L and we take $S = L$. If $(V, \langle H, L\rangle)$ is irreducible of complex type, we let $\ell = k$, take $S = \alpha^{-1/k}L$ and note that $S \in L_H({}_\rho V, {}_\sigma V)$.

Suppose that $\bar{V} \subset V$ is irreducible as an $\langle H, S\rangle$-representation. Let W be an H-isotypic component of $\bar{V}$. For $s \geq 1$, define $W^s = \oplus_{j=0}^{s-1} W$, $s \geq 1$. For $j \in \mathbb{Z}$, define the H-representation $\kappa_j : H \to \mathrm{O}(W)$ by

$$\kappa_j(h)(w) = \delta^j h \delta^{-j}(w), \ \ (h \in H, w \in W).$$

Let $s \geq 1$ be the smallest value of j for which $\kappa_j \cong \kappa_0$. We say that (W, H) has *order s*.

Since $\kappa_k = \kappa_0 = \rho|W$, s divides k and therefore ℓ. We define the H-representation $\boldsymbol{\rho} : H \to \mathrm{O}(W^s)$ by

$$h(w_0, \dots, w_{s-1}) = (\kappa_0(h)(w_0), \dots, \kappa_{s-1}(h)(w_{s-1})), \ ((w_0, \dots, w_{s-1}) \in W^s, h \in H).$$

Let $\boldsymbol{\sigma} : H \to \mathrm{O}(W^s)$ be the representation defined by

$$\boldsymbol{\sigma}(h) = \boldsymbol{\rho}(\kappa_{-1}(h)).$$

Let ${}_{\boldsymbol{\rho}}W^s$ denote W^s with the action $\boldsymbol{\rho}$ and ${}_{\boldsymbol{\sigma}}W^s$ denote W^s with the action $\boldsymbol{\sigma}$. If we define $\mathbf{S} : {}_{\boldsymbol{\rho}}W^s \to {}_{\boldsymbol{\sigma}}W^s$ by

$$\mathbf{S}(w_0, \dots, w_{s-1}) = (S^s w_{s-1}, w_0, \dots, w_{s-2}),$$

then $\mathbf{S}$ is an H-equivariant orthogonal map and $\langle \mathbf{S} \rangle \cong \mathbb{Z}_\ell$ (since $(\mathbf{S}^s)^{l/s} = (S^\ell, \dots, S^\ell) = I_{W^s}$).

Define the linear map $\mathcal{E} : W^s \to V$ by

$$\mathcal{E}(w_0, \dots, w_{s-1}) = \sum_{j=0}^{s-1} S^j w_j.$$

LEMMA 10.7.16.

(1) $\mathcal{E} : {}_{\boldsymbol{\rho}}W^s \to \bar{V}$ *is an H-equivariant linear isomorphism.*
(2) $\mathcal{E} \circ \mathbf{S} = S \circ \mathcal{E}$.
(3) $(\bar{V}, \langle H, S \rangle) \approx (W^s, \langle H, \mathbf{S} \rangle)$, *where we take the $\boldsymbol{\rho}$-action of H on W^s.*

PROOF. Since $S \in L_H({}_\rho V, {}_\sigma V)$, we have $S^j(\delta^j h \delta^{-j} w_j) = h S^j w_j$, $j \geq 0$. Hence $\mathcal{E}$ is H-equivariant. Moreover, $\mathcal{E}$ restricted to each factor of W^s is an embedding and, since the factors are non-isomorphic H-representations, it follows that $\mathcal{E}$ is an H-equivariant embedding onto the corresponding isomorphic factors in V. Obviously, $\mathcal{E}(W^s)$ is S- and H-invariant and so $\mathcal{E}(W^s) = \bar{V}$ by the $\langle H, S \rangle$-irreducibility of $\bar{V}$. The remaining statements are routine computations which we leave to the reader. $\square$

Granted lemma 10.7.16, it is not difficult to figure out the irreducible representations of $(\bar{V}, \langle H, S \rangle) \approx (W^s, \langle H, \mathbf{S} \rangle)$ that can arise in our situation (a general classification is given in [**111**, section 7] but the approach there is more general and not directly tied to the operators S).

Elementary irreducible representations of $\langle H, \mathbf{S} \rangle$. We say that the representation $(W^s, \langle H, \mathbf{S} \rangle)$ (or $(\bar{V}, \langle H, S \rangle)$) is *elementary* if the representation $\kappa_0 : H \to \mathrm{O}(W)$ is irreducible. In this case, each κ_j is irreducible. Since $\langle H, \mathbf{S}^s \rangle$ preserves the factors of W^s, each factor will be an irreducible $\langle H, \mathbf{S}^s \rangle$-representation (since it is already irreducible as an H-representation). If (W, H) is absolutely irreducible then it is not hard to see that $(W^s, \langle H, \mathbf{S} \rangle)$ is also absolutely irreducible. If (W, H) is irreducible of complex type then $(W^s, \langle H, \mathbf{S} \rangle)$ may be absolutely irreducible or irreducible of complex type. Finally, if (W, H) is irreducible of quaternionic type, then $(W^s, \langle H, \mathbf{S} \rangle)$ may be irreducible of complex type or quaternionic type. (We indicate how to establish the results for (W, H) irreducible of complex or quaternionic type in example 10.7.22 below.)

It is possible to obtain results on the number of different representations of each type. This can be done by making appropriate changes to L (the general theory is sketched in the next section, see also [**111**]). Here we give some partial results for the easiest case when (W, H) is absolutely irreducible and $S = L$. Suppose first that ℓ/s is odd. Since $L = \delta^{-1} A$, if we change A to $-A$ then L changes to $-L$. If s is even then a case by case check shows that if $\ell = k$ or

$\ell = 2k$ and k is even, then the representation $(W^s, \langle H, \mathbf{S}\rangle)$ is unchanged. In all other cases, the order of $\langle L\rangle$ changes when we switch L to $-L$. This is generally true: if ℓ/s is odd there is exactly one elementary representation $(W^s, \langle H, \mathbf{S}\rangle)$. The case ℓ/s even is a little trickier. If s is odd, then we can find two non-isomorphic representations just by changing L to $-L$ (or A to $-A$). Generally, it can be shown that if ℓ/s is even there are either two elementary representations $(W^s, \langle H, \mathbf{S}\rangle)$ or none.

EXERCISE 10.7.17. Show that if $(W^s, \langle H, \mathbf{S}\rangle)$ is irreducible, (W, H) is irreducible of complex type and L is $\mathbb{C}$-linear, then by varying A (equivalently, $F : E \to E$) we can find ℓ/s distinct representations of $\langle H, \mathbf{S}\rangle$ on W^s which are irreducible of complex type. In all cases we will have $\langle \mathbf{S}\rangle \cong \mathbb{Z}_\ell$, $\ell = k$.

Non-elementary representations of $\langle H, \mathbf{S}\rangle$. We continue with the notational conventions of the previous subsection. Given the irreducible representation $\kappa_0 : H \to \mathrm{O}(W)$, we let ${}_jW$ denote the space W together with the (orthogonal) action defined by κ_j, $j \in \mathbb{Z}$. In particular, ${}_0W$ will denote the space W together with the action defined by κ_0. We remark that for all $j \in \mathbb{Z}$ we have

$$L_H({}_jW, {}_jW) = L_H({}_0W, {}_0W), \tag{10.8}$$

$$L_H({}_0W, {}_sW) = L_H({}_jW, {}_{j+s}W). \tag{10.9}$$

In view of (10.8), we may let $L_H(W, W)$ unambiguously denote $L_H({}_jW, {}_jW)$ for all $j \in \mathbb{Z}$. If $K \in L_H({}_0W, {}_sW)$, then $BK, KB \in L_H({}_0W, {}_sW)$ for all $B \in L_H(W, W)$. Let $\mathrm{O}({}_0W, {}_sW)$ denote the subset of $L_H({}_0W, {}_sW)$ consisting of orthogonal maps ($\mathrm{O}({}_0W, {}_sW)$ is non-empty by lemma 10.7.6). Suppose $K \in \mathrm{O}({}_0W, {}_sW)$. We define $K^\star : L_H(W, W) \to L_H(W, W)$ by

$$K^\star(B) = KBK^{-1},\ B \in L_H(W, W).$$

Since (W, H) is irreducible, $L_H(W, W) = \mathcal{D}$ is a real division algebra isomorphic to one of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (theorem 2.7.14). The morphism $K^\star$ is an automorphism of $\mathcal{D}$. Since $K^\star(1) = 1$, it follows that if (W, H) is absolutely irreducible ($\mathcal{D} = \mathbb{R}$) then $K^\star$ is the identity. If (W, H) is irreducible of complex type, then $K^\star|\mathbb{R}$ is the identity. Hence $K^\star(\imath) = \pm\imath$ since $K^\star(\imath^2) = -1$. It follows there are two possibilities: either $K^\star$ is the identity or $K^\star$ is complex conjugation. Finally, if (W, H) is irreducible of quaternionic type then $K^\star$ is conjugation by a quaternion. In this case, we can always compose K with an element A of $L_H(W, W) \cong \mathbb{H}$ so that $(KA)^\star = 1$. We sum up these arguments in the following lemma.

LEMMA 10.7.18.

(1) *If (W, H) is absolutely irreducible and $K \in \mathrm{O}({}_0W, {}_sW)$, then $BK = KB$ for all $B \in L_H(W, W)$.*
(2) *If (W, H) is irreducible of complex type and $K \in \mathrm{O}({}_0W, {}_sW)$ then either*
    (a) *$BK = KB$ for all $B \in L_H(W, W)$ ("(W, H) is of inner complex type"), or*

(b) $\bar{B}K = KB$ *for all* $B \in L_H(W,W)$ *("(W,H) is of outer complex type").*

(3) *If* (W,H) *is irreducible of quaternionic type, then there exists* $K \in \mathrm{O}({}_0W, {}_sW)$ *such that* $BK = KB$ *for all* $B \in L_H(W,W)$.

If $K \in \mathrm{O}({}_0W, {}_sW)$ satisfies the commutativity conditions of lemma 10.7.18, we call K a *central element* (of $\mathrm{O}({}_0W, {}_sW)$).

LEMMA 10.7.19. *Suppose that* $K \in \mathrm{O}({}_0W, {}_sW)$ *is a central element. Then* K *is a central element of* $\mathrm{O}({}_jW, {}_{j+s}W)$ *for all* $j \in \mathbb{Z}$.

PROOF. Immediate from (10.8,10.9). ☐

Let (U,H) be the sum of p-copies of (W,H), $p \geq 1$. We say that $\mathbf{K} \in \mathrm{O}(U^s, U^s) \subset L_H({}_0U^s, {}_sU^s)$ is a central element if $\mathbf{KB} = \mathbf{BK}$ (respectively, $\mathbf{KB} = \overline{\mathbf{B}}\mathbf{K}$) when (W,H) is not of outer complex type (respectively, is of outer complex type) for all $\mathbf{B} \in L_H(U^s, U^s)$. An an immediate consequence of our definitions we have

LEMMA 10.7.20. *If* $K \in \mathrm{O}({}_0W, {}_sW)$ *is a central element,* $U = W^p$, *and we define* $\mathbf{K} = \oplus_{j=0}^{s-1} K^p \in \mathrm{O}(U^s, U^s)$, *then* $\mathbf{K}$ *is a central element.*

If $K \in \mathrm{O}({}_0W, {}_sW)$ then $K^{k/s} \in \mathrm{O}({}_0W, {}_0W) \subset L_H(W,W) \approx \mathcal{D}$. We may use the equivariance of $K^{k/s}$ to show that we may require central elements to be of order either k/s or $2k/s$.

LEMMA 10.7.21. *Let* $K \in \mathrm{O}({}_0W, {}_sW)$ *be a central element.*

(1) *If* (W,H) *is absolutely irreducible, then* $K^{k/s} = \pm I_W$. *If* k/s *is odd, we can choose* K *so that* $K^{k/s} = I_W$.

(2) *If* (W,H) *is irreducible of complex type, then*

(a) *If* (W,H) *is of inner complex type, we can choose* K *so that* $K^{k/s} = I_W$.

(b) *If* (W,H) *is of outer complex type, then* k/s *is even and* $K^{k/s} = \pm I_W$.

(3) *If* (W,H) *is quaternionic, then* $K^{k/s} = \pm I_W$.

PROOF. The first part of (1) is obvious since $K^{k/s} \in L_H(W,W)$ is orthogonal. If k/s is odd and $K^{k/s} = -I_W$, then $(-K)^{k/s} = I_W$. (2) If (W,H) is of inner complex type then $K^{k/s} = \alpha I_W$, where $|\alpha| = 1$. Replace K by the central element uK where $\alpha u^{k/s} = 1$. If (W,H) is of outer complex type, then k/s is even since $K^{k/s} \in L_H(W,W)$ is $\mathbb{C}$-linear and K is conjugate complex linear. Since K is conjugate complex linear it follows that $K^{k/s} = \alpha I_W$, $\alpha \in \mathbb{R}$ and so $K^{k/s} = \pm I_W$. Finally, if (W,H) is quaternionic, $K^{k/s} = \alpha I_W$ commutes with elements of $L_H(V,V) \approx \mathbb{H}$ and so $\alpha \in \mathbb{R}$. Hence $K^{k/s} = \pm I_W$. ☐

EXAMPLE 10.7.22. Suppose that $(W^s, \langle H, \mathbf{S} \rangle)$ is an elementary irreducible representation and (W,H) is irreducible of complex type. Set $\mathbf{K} = \mathbf{S}^s|_0W$. If

$\mathbf{K}$ is $\mathbb{C}$-linear, then (W, H) is of inner complex type and $\mathbf{K}$ is a central element. Hence $(W, \langle H, \mathbf{K}\rangle)$ is irreducible of complex type and it is easy to see that $(W^s, \langle H, \mathbf{S}\rangle)$ is irreducible of complex type. If $\mathbf{K}$ is not $\mathbb{C}$-linear, then it follows from lemma 10.7.18 that (W, H) is of outer complex type and $\mathbf{K}$ is conjugate complex linear and a central element. Hence $(W, \langle H, \mathbf{K}\rangle)$ is absolutely irreducible and we deduce that $(W^s, \langle H, \mathbf{S}\rangle)$ is absolutely irreducible.

If (W, H) is irreducible of quaternionic type, then $(W, \langle H, \mathbf{K}\rangle)$ is irreducible of quaternionic type if and only if $\mathbf{K}$ is central. As we have already shown (proposition 10.7.11), this is a non-generic situation which corresponds to the automorphism of $\mathbb{H}$ determined by $\mathbf{K}$ being the identity. If $\mathbf{K}$ is not central, then $(W, \langle H, \mathbf{K}\rangle)$ is irreducible of complex type and so also is $(W^s, \langle H, \mathbf{L}\rangle)$.

EXERCISE 10.7.23. Use lemma 10.7.21 to complete the classification of elementary irreducible representations. In particular show that (a) if (W, H) is absolutely irreducible then if ℓ/s is even then the number of associated elementary representations is either 2 or 0, and (b) if (W, H) is of quaternionic type there are $[\ell/2s]$ distinct associated elementary representations.

THEOREM 10.7.24 ([**111**, Theorem 7.10]). *Suppose that $(\bar{V}, \langle H, S\rangle)$ is irreducible and not an elementary representation. If (W, H) is an irreducible subrepresentation of $(\bar{V}, H)$ of order s, we have*

(1) $(\bar{V}, \langle H, S\rangle) \cong (W^s, \langle H, \mathbf{S}\rangle) \oplus (W^s, \langle H, \mathbf{S}\rangle)$,
(2) (W, H) *is either absolutely irreducible or of outer complex type and*
  (a) *If (W, H) is absolutely irreducible, then $(\bar{V}, \langle H, L\rangle)$ is irreducible of complex type.*
  (b) *If (W, H) is of outer complex type then $(\bar{V}, \langle H, S\rangle)$ is irreducible of complex or quaternionic type.*

PROOF. Let U be the isotypic component of $(\bar{V}, H)$ containing W. If the representation of H on W is defined by $\kappa_0 : H \to \mathrm{O}(W)$, let ${}_jU$ denote the representation on U determined by κ_j, $j \in \mathbb{Z}$. Applying lemma 10.7.16 we have

$$(\bar{V}, \langle H, S\rangle) \cong (U^s, \langle H, \mathbf{S}\rangle).$$

Since $(U^s, \langle H, \mathbf{S}\rangle)$ is irreducible, $(U, \langle H, \mathbf{S}^s\rangle)$ is irreducible (and conversely). In particular, if $(U^s, \langle H, \mathbf{S}\rangle)$ is irreducible and $U^\star \subset U$ is a nontrivial H- and $\mathbf{S}^s$-invariant subspace, then $U^\star = U$.

If (W, H) is of inner complex type, then $\mathbf{S}^s : U \to U$ is $\mathbb{C}$-linear. By lemma 10.7.20, we may choose a central element $\mathbf{K} \in \mathrm{O}(U, U)$. Write $\mathbf{S}^s = \mathbf{K}\mathbf{B}$, where $\mathbf{B} = \mathbf{K}^{-1}\mathbf{S}^s \in L_H(U, U)$. Let $\alpha \in \mathbb{C}$ be an eigenvalue of $\mathbf{B}$ and let $E \subset U$ denote the corresponding eigenspace. Since maps and spaces are complex, E will be a $\mathbb{C}$-vector subspace of U. Since $\mathbf{B}$ is H-equivariant, E will be H-invariant and therefore an H-subrepresentation of U. Let $E^\star \subset E$ be H-irreducible and set $U^\star = \mathbf{K}(E^\star)$. We have

$$\mathbf{B}(U^\star) = \mathbf{B}\mathbf{K}(E^\star) = \mathbf{K}\mathbf{B}(E^\star) = \mathbf{K}(E^\star) = U^\star.$$

Hence $U^\star$ is $\mathbf{B}$-invariant and so has the structure of an irreducible H-representation isomorphic to (W, H). Since $\mathbf{S}^s(U^\star) = U^\star$, it follows by irreducibility of $(U, \langle H, \mathbf{S}^s\rangle)$ that $U = U^\star$ and so $(U, \langle H, \mathbf{S}^s\rangle)$ is elementary.

The same argument applies to show that if (W, H) is quaternionic and $(\bar{V}, \langle H, S\rangle)$ is irreducible, then $(\bar{V}, \langle H, S\rangle)$ is elementary.

Suppose next that (W, H) is absolutely irreducible. Following the previous argument, if $\mathbf{B}$ has a real eigenvalue then $(U, \langle H, \mathbf{S}^s\rangle)$ is elementary. On the other hand if $\mathbf{B}$ has a complex eigenvalue, $\mathbf{B}$ has an H-invariant subspace E of dimension $2\dim(W)$. Up to a choice of sign, E carries a natural complex structure inherited from the complex eigenvalue with respect to which $\mathbf{B}|E$ is complex scalar multiplication. Setting $U^\star = \mathbf{K}(E)$, the resulting H-representation is isomorphic to the complexification of (W, H) and $(U^\star, \langle H, \mathbf{S}^s\rangle) \cong (W^s, \langle H, \mathbf{S}\rangle) \oplus (W^s, \langle H, \mathbf{S}\rangle)$ is irreducible of complex type.

There remains the case when (W, H) is of outer complex type. Since $\mathbf{S}^s : U \to U$ is conjugate complex linear, $\mathbf{S}^{2s} : U \to U$ is $\mathbb{C}$-linear. Just as in the case when (W, H) is of inner complex type, we may find an irreducible H-subrepresentation $U^\star \subset U$ which is $\mathbf{S}^{2s}$-invariant. Either $\mathbf{S}^s(U^\star) = U^\star$ or $\mathbf{S}^s(U^\star) \neq U^\star$. In the first case, $(U^\star, H)$ determines an elementary representation (which is absolutely irreducible). If $\mathbf{S}^s(U^\star) \neq \mathbf{S}^s(U^\star)$, then $U^\star + \mathbf{S}^s(U^\star)$ must be of dimension twice that of W and $U^\star + \mathbf{S}^s(U^\star) = U$ by irreducibility. □

REMARKS 10.7.25. (1) If (W, H) is absolutely irreducible or of outer complex type then $(W^s, \langle H, \mathbf{S}\rangle) \oplus (W^s, \langle H, \mathbf{S}\rangle)$ need not be irreducible. In the case when (W, H) is absolutely irreducible, we obtain irreducibility of the sum if and only if the matrix $\mathbf{B}$ of the proof of theorem 10.7.24 has complex eigenvalues. In the outer complex case, the representation may be irreducible of complex or quaternionic type. It is also possible that $(W^s, \langle H, \mathbf{S}\rangle)$ decomposes into the sum of two irreducible representations of complex type. Indeed, this possibility is suggested by the proof of theorem 10.7.24. We refer to [**111**, Remark 7.13] for details on this point.
(2) It follows from the proof of theorem 10.7.24 that if $(\bar{V}, \langle H, S\rangle)$ is not elementary then the representation is not absolutely irreducible. As indicated in the previous remark, the representation can be irreducible of complex type or quaternionic.

EXAMPLES 10.7.26. (1) We look at possibilities that can occur when $H = \mathbf{D}_n$, $G = \mathbf{D}_{2n}$ and $n \geq 2$ is *even* (this, and the next example, are based on [**111**, section 8]). We emphasize in what follows that we regard irreducible representations $\langle H, L\rangle$, $\langle H, L'\rangle$ on V as distinct if either $(V, \langle H, L\rangle)$ and $(V, \langle H, L'\rangle)$ are not isomorphic or $(V, \langle H, L\rangle)$ and $(V, \langle H, L'\rangle)$ are isomorphic but $\langle L\rangle$ and $\langle L'\rangle$ are not isomorphic.

For $\theta \in \mathbb{R}$, we let $R_\theta \in \mathrm{SO}(2)$ denote rotation through θ and $\kappa \in \mathrm{O}(2)$ denote reflection in the x-axis (or complex conjugation). We recall that $\kappa R_\theta = R_{-\theta}\kappa$ for all $\theta \in \mathbb{R}$.

If we embed $\mathbf{D}_n$ in the standard way in O(2), then generators for $\mathbf{D}_n$ are given by $R_{2\pi/n}$ and κ.

There are precisely $\frac{n}{2}+3$ irreducible real representations of $\mathbf{D}_n$ and they are all absolutely irreducible. Four of the representations are one dimensional. These are the representations $\nu_0, \nu_1, \nu_a, \nu_b$ defined by

$$\begin{aligned} \nu_0(R_{2\pi/n}) &= 1, \quad & \nu_1(\kappa) &= 1, \text{ (trivial representation)} \\ \nu_1(R_{2\pi/n}) &= 1, \quad & \nu_1(\kappa) &= -1, \\ \nu_a(R_{2\pi/n}) &= -1, \quad & \nu_a(\kappa) &= 1, \\ \nu_b(R_{2\pi/n}) &= -1, \quad & \nu_b(\kappa) &= -1. \end{aligned}$$

(If n is odd, we do not get the representations ν_a, ν_b.) The $\frac{n}{2}-1$ two-dimensional absolutely irreducible representations ξ_j of $\mathbf{D}_n$ are defined by

$$\xi_j(R_{2\pi/n}) = R_{2\pi j/n},\ \xi_j(\kappa) = \kappa,\ 1 \le j < n/2.$$

If $\delta \in \mathbf{D}_{2n} \setminus \mathbf{D}_n$, then $\delta \notin C_{\mathbb{D}_n}(\mathbb{D}_{2n})$ (this uses n even, if n is odd we may take $\delta = R_\pi = -I$). On the other hand, $\delta^2 \in C_{\mathbf{D}_n}(\mathbf{D}_{2n})$ provided that δ is a reflection. Hence $k = 2$. For δ we may take any reflection from $\mathbf{D}_{2n}$ which does not lie in $\mathbf{D}_n$. We choose $\delta = R_{\pi/n}\kappa$ and note that $\delta = \delta^{-1}$.

Next we look at the representations $\langle \mathbf{D}_n, L\rangle$ on $\nu_1, \dots, \xi_{n/2-1}$ associated to δ. We start with the two-dimensional representations ξ_j, $1 \le j \le n/2$. We claim that the order of each of these representations is one. This amounts to showing that there exists a nonzero $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$L(R_{2\pi/n}v) = \delta R_{2\pi/n}\delta^{-1}Lv,\ L(\kappa v) = \delta\kappa\delta^{-1}L(v),\ \ (v \in \mathbb{R}^2). \tag{10.10}$$

Using the relation $\kappa R_\theta = R_{-\theta}\kappa$, we find that $L = \pm R_{\pi/n}\kappa = \pm\delta$ are the unique orthogonal maps satisfying (10.10). All of this translates immediately to the twisted product $\mathbf{D}_{2n} \times_{\mathbf{D}_n} \mathbb{R}^2$. In particular, the group $\langle \mathbf{D}_n, L\rangle$ is isomorphic to $\mathbf{D}_n \rtimes \mathbb{Z}_2 \cong \mathbf{D}_{2n}$ if $L = R_{\pi/n}\kappa$ or n is even. (If n is odd and $L = -R_{\pi/n}\kappa$ then $L \in \mathbf{D}_n$ and so $\langle \mathbf{D}_n, L\rangle = \mathbf{D}_n$.) Since $(\pm L)^2 = I_{\mathbb{R}^2}$, $\ell = 2$. We have constructed $n-2$ distinct elementary absolutely irreducible representations of $\langle \mathbf{D}_n, L\rangle$ on $\mathbb{R}^2$.

Similarly, the one dimensional representations ν_0, ν_1 have order one and, with $L = \pm 1$, we obtain four distinct elementary absolutely irreducible representations of $\mathbf{D}_n \rtimes \mathbb{Z}_\ell$ on $\mathbb{R}$, where $\ell = 2$ if $L = -1$. In these cases, the action of $\mathbf{D}_n$ is not faithful and so, viewed as matrix representations, $k = 1$ and nothing really changes unless the representation is ν_0 and $L = -1$.

Finally, we consider the representations ν_a, ν_b. We find that ν_a and ν_b both have order two and $\delta\nu_a\delta = \nu_b$ (since $\delta R_{2\pi/n}\delta = R_{2\pi/n}^{-1}$ and $\delta\kappa\delta = R_{2\pi/n}\kappa$). Since $\delta\nu_a = \nu_b\delta$, we can take $L = (\delta, \pm\delta)$. In matrix form, on $\nu_a \oplus \nu_b$,

$$L = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}.$$

We have $\langle \mathbf{D}_n, L\rangle \cong \mathbf{D}_4$ and acting on $\mathbb{R}^2$ in the standard way. If we take $L = (\delta, \delta)$, then $\ell = 2$, otherwise $\ell = 4$ ($L^2 = -I$). We count two distinct

irreducible representations. This gives us a total of $n+4$ absolutely irreducible representations of $\langle \mathbf{D}_n, L\rangle$.
(2) We look at possibilities that can occur when $H = \mathbf{Z}_n$, $G = \mathbf{D}_n$. We follow the notation of the previous example and let $R_{2\pi/n}$ denote the generator for $\mathbf{Z}_n \subset \mathrm{SO}(2)$.

If n is odd. there are $\frac{(n+1)}{2}$ irreducible representations ν_0, $\xi_1, \ldots, \xi_{(n+1)/2}$ of $\mathbb{Z}_n$. Here ν_0 is the trivial 1-dimensional representation and the representations ξ_j are two-dimensional of complex type and defined by

$$\xi_j(R_{2\pi/n}) = R_{2\pi j/n},\ 1 \leq j < n/2. \tag{10.11}$$

If n is even, we have two one-dimensional absolutely irreducible representations ν_0 (the trivial representation) and ν_1, defined by $\nu_1(R_{2\pi/n}) = -1$. There are $\frac{n}{2} - 1$ two-dimensional irreducible representations ξ_j of complex type, defined by (10.11).

If $\delta \in \mathbf{D}_n \setminus \mathbf{Z}_n$ then $\delta \notin C_{\mathbf{Z}_n}(\mathbf{D}_n)$ and $\delta^2 = I$, Hence $k = 2$ and we may take $\delta = \kappa$ (complex conjugation). The complex representations ξ_j are all of order one and of outer complex type. We may take $L = \delta$, so $L^2 = I$, and $\ell = k = 2$. Summarizing, each irreducible representation ξ_j determines a two-dimensional absolutely irreducible representation of $\langle \mathbb{Z}_n, L\rangle \approx \mathbf{D}_n$. It remains to consider the one dimensional representations of $\mathbb{Z}_n$. In this case we obtain two distinct irreducible one-dimensional representations of $\langle \mathbb{Z}_n, L\rangle$. The first will be the trivial representation with ν_0 and $L = I$ (or ν_1 and $L = -1$), the second will be with ν_0 and $L = -1$ (or ν_1 and $L = 1$). Note that in this example it is not possible to have $\ell = 4$.

We conclude with two examples showing how we may apply these ideas to bifurcation of relative periodic orbits. Again these examples are closely based on the article by Lamb and Melbourne [**111**] and the reader should consult that work for a more complete set of examples.

EXAMPLES 10.7.27. (1) Take $G = \mathbf{D}_{2n}$, $H = \mathbf{D}_n$ and assume n is even. We follow the notation of examples 10.7.26(1). Let $(V, H) = \nu_a \oplus \nu_b$. We have $\langle H, L\rangle \cong \mathbf{D}_4$, where $L = (\delta, \pm\delta)$. Suppose that $P_\lambda = Lh_\lambda$, where $h_\lambda \in \mathrm{Diff}_{\mathbf{D}_4}(V, V)$ is a normalized family. Thus $Dh_0 = I_V$. We apply the results of the previous section to deduce that there exists an open and dense subset $\mathbf{S}(V, \mathbf{D}_4)$ consisting of stable families. Each $h_\lambda \in \mathbf{S}(V, \mathbf{D}_4)$ will be 3-determined and have a signed indexed branching pattern $\mathbf{B}(h)$. In figure 1 we indicate fixed points of h_λ together with their isotropy groups (with respect to G).

If $L = (\delta, \delta)$, then $L^2 = I_V$ and so all the non-trivial periodic points of Lh_λ will be points of prime period two, that is fixed points of $L^2 h_\lambda^2 = P_\lambda$ – the associated Poincaré map. Consequently, when we break the normal form symmetry $\langle L\rangle$, we will see bifurcation to periodic orbits of approximately the same period as the original periodic orbit. The isotropy of the new periodic orbits will be as labelled

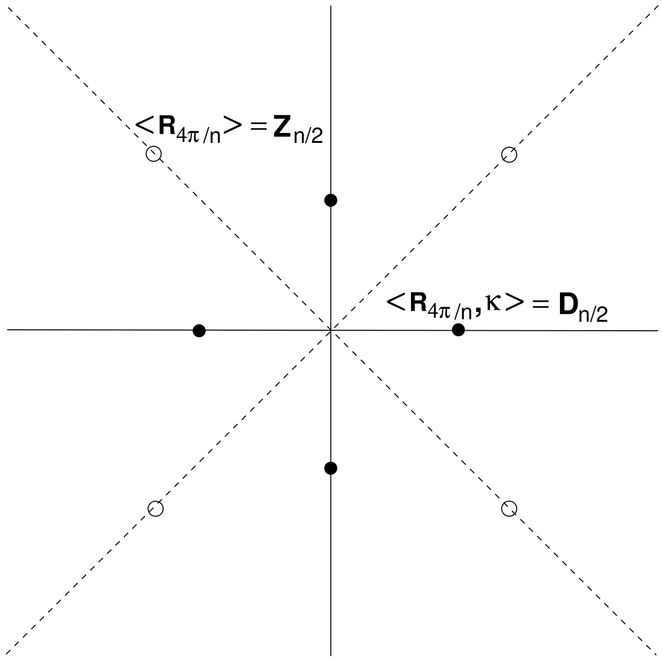


FIGURE 1. Fixed points of h_λ

in figure 1. If $L = (\delta, -\delta)$, then $L^2 = -I_V$ and so $-h_\lambda^2 = P_\lambda$ and we get period doubling.

In all cases, it is easy to work out the (spatiotemporal) symmetry group G_γ of the periodic orbits γ. If $L^2 = I_V$ and points on γ have isotropy $\mathbf{D}_{n/2}$ (respectively, $\mathbb{Z}_{n/2}$), then $G_\gamma = \mathbf{D}_{n/2}$ (respectively, $\mathbf{D}_{n/2}$). If $L^2 = -I_V$, and points on γ have isotropy $\mathbf{D}_{n/2}$ (respectively, $\mathbb{Z}_{n/2}$), then $G_\gamma = \mathbf{D}_n$ (respectively, $\mathbb{Z}_{2n}$).
(2) We look at an example where there is a Hopf bifurcation of the Poincaré map. Suppose that $G = \langle R_{\pi/2}, \kappa\rangle = \mathbf{D}_4$, $H = \langle R_\pi, \kappa\rangle = \mathbf{D}_2$ and $H \subset G \subset \mathrm{O}(2)$ in the standard way. Let $\delta = R_{\pi/2}\kappa$. If we take the standard reducible complex representation of $\mathbf{D}_2$ on $V = \mathbb{C}^2$, then $L^2 = -I_V$ and $S^2 = I_V$. We find that as generators for the action of $\langle \mathbf{D}_2, S\rangle$ on $V = \mathbb{C}^2$, we may take

$$R_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Obviously $\langle \mathbf{D}_2, S\rangle \cong \mathbf{D}_4$. Assuming no resonances, we start by analysing families $h_\lambda \in \mathrm{Diff}_{\mathbf{D}_4\times S^1}(V)$ with $dh_0(0) = \imath I_V$. Applying the results of the previous section there exists an open and dense subset $\mathbf{S}(V, \mathbf{D}_4)$ consisting of stable families. Each $h_\lambda \in \mathbf{S}(V, \mathbf{D}_4)$ will be d-determined ($d \geq 3$ – example 5.6.27). If $h_\lambda \in \mathbf{S}(V, \mathbf{D}_4)$ there will be three three branches of normally hyperbolic invariant circle (corresponding to maximal isotropy groups for the action of $\mathbf{D}_4$ on V. There is also the possibility of submaximal branches as well as branches of normally hyperbolic invariant 2-tori (see example 5.6.27 for the vector field case). It is not hard to compute the isotropy groups of the branches (we refer the reader to [**111**, Example 6.5] for details). When we go to the Poincaré map – still keeping the S^1-normal form symmetry – we will see branches of normally hyperbolic 2-tori (and 3-tori when there are submaximal branches). When we break normal form symmetries at high order, the branches will persist as branches of group invariant normally hyperbolic submanifolds (this uses strong determinacy). However, finding the detailed structure of dynamics of flow invariant group orbits can be challenging – there are possibilities of phase locking, quasi-periodic flow or even chaotic dynamics [**167**].

10.8. Notes for chapter 10

The extension of equivariant transversality arguments to allow for bifurcation to relative equilibria (as opposed to branches of equilibrium G-orbits) first appeared in [**57**]. A more complete presentation, which includes a proof of the strong determinacy theorem and allows for complex representations is in [**60**]. After Ruelle's pioneering work [**151**], bifurcation from relative equilibria was first systematically studied by Krupa [**105**] who introduced the 'tangent and normal form' for a vector field. The extension of equivariant transversality arguments to the bifurcation theory of maps first appears in [**62**]. Applications are given to the equivariant Hopf bifurcation and normal forms (strong determinacy). Lamb, Melbourne and Wulff [**111, 115, 113, 183, 112**] have recently made significant progress in the understanding of bifurcation from relative periodic orbits, including the case of proper G-actions. In our brief introduction to some of their work, we downplay the study of drift dynamics and reduce to the analysis of the Poincaré map of a discrete rotating wave (as in [**111**]). However, it is possible to avoid discussion of bifurcation of maps and instead reduce to problems about bifurcation from relative equilibria and skew product dynamics. The reduction to vector fields due to Takens [**168**] and has been adopted by Lamb, Melbourne and Wulff in their more recent work (see [**112**] for a good description of alternative approaches).

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Index of Notational Conventions

$\mathbb{Z}$ integers, $\mathbb{N}$ set of nonnegative integers, $\mathbb{N}^+ = \{n \in \mathbb{Z} \mid n > 0\}$
$\mathbb{R}$ real numbers, $\mathbb{R}^+$ strictly positive real numbers
$\mathbb{R}^\star$ multiplicative group of nonzero real numbers
$\mathbb{C}$ complex numbers, $\mathbb{C}^\star$ multiplicative group of nonzero complex numbers
$\mathbf{n}$ the set $\{1, \ldots, n\}$
$\mathbb{H}$ quaternions
$|G|$ order of a finite group
G_0 identity component of topogical group G
(H) congugacy class of subgroup H
(X, G) G-space X
X^H fixed point space of H, $H \subset G$
(V, G) G-representation V
X_τ points in X with isotropy type τ
$\iota(x)$ isotropy type of x
$\mathcal{O}(X, G)$ the set of isotropy types of (X, G)
$\mathcal{O}^\star(X, G)$ the set of proper isotropy types of (X, G) (page 79)
$Z(G)$ center of G
$N_G(H)$ (or $N(H)$) normalizer of H in G
$C_G(H)$ centralizer of H in G
$H \lhd G$ H is a normal subgroup of G
$H \rtimes J$ semidirect product of H and J ($H \lhd H \rtimes J$)
$\mathrm{Aut}(G)$ automorphism group of G
L_g left translation
R_h right translation
$\mathfrak{g}$ Lie algebra of G
L_X Lie derivative
$[X, Y]$ Lie bracket of vector fields X and Y
$\mathbb{Z}_n$ cyclic group of order $n \geq 1$
$\mathbf{D}_n$ dihedral group of of order $2n$
$\mathcal{B}(X)$ group of bijections of X
$\mathrm{Iso}(X)$ isometry group of X
S_n symmetric group of degree n
A_n alternating group of degree n
$\mathrm{GL}(V, \mathbb{F})$ general linear group (field $\mathbb{F}$)
$\mathrm{GL}(V)$ general linear group (field $\mathbb{R}$ or $\mathbb{C}$)
$\mathrm{GL}(n, \mathbb{R})$ general linear group $\mathrm{GL}(\mathbb{R}^n)$
$\mathrm{GL}(n, \mathbb{C})$ general linear group $\mathrm{GL}(\mathbb{C}^n)$
$M(d, d)$ space of $d \times d$ matrices
$M(p, q; \mathbb{F})$ space of $p \times q$ matrices over $\mathbb{F}$
S^1 group of complex numbers of unit modulus ($\approx \mathrm{SO}(2)$)

$\mathbb{T}^n$ n-dimensional torus
G' determinant one subgroup of $G \subset \mathrm{GL}(n,\mathbb{R})$ $\mathrm{SO}(n)$ special orthogonal group of degree n
$\mathrm{O}(n)$ orthogonal group of degree n
$\mathrm{SU}(n)$ special unitary group of degree n
$\mathrm{U}(n)$ unitary group of degree n
$\mathbf{E}(n)$ Euclidean group of $\mathbb{R}^n$
$\mathbf{SE}(n)$ special Euclidean group of $\mathbb{R}^n$
$\mathrm{SL}(V)$ special linear group
$\mathrm{SL}(n,\mathbb{R})$ special linear group of degree n
$\mathrm{SL}(n,\mathbb{Z})$ special linear group of degree n, components in $\mathbb{Z}$
H_n hyperoctahedral group, group of signed permutation matrices
$\mathcal{E}$ parameter set for axes of symmetry of H_n
Δ_n group of orthogonal diagonal matrices (entries ± 1)
$\mathrm{PGL}(n,\mathbb{C})$ projective linear group
$\mathrm{PGL}(n,\mathbb{R})$ real projective linear group
$\mathrm{GL}(n,\mathbb{F}_{p^n})$ group of invertible $n \times n$ with entries in $\mathbb{F}_{p^n}$ (field with p^n elements)
$\mathrm{Aff}_1(\mathbb{F})$ group of affine isomorphisms of field $\mathbb{F}$
$\langle g_1, \dots, g_k \rangle$ subgroup generated by $g_1, \dots, g_k \in G$
$L_G^{\mathbb{F}}(V,W)$ G-equivariant $\mathbb{F}$-linear maps from V to W
$L_G(V,W)$ G-equivariant $\mathbb{R}$-linear maps from V to W
$P^d(V,V)$ homogeneous polynomial maps of degree d
$P^{(d)}(V,V)$ polynomial maps of degree d
$d(v,G)$ critical degree
$P_G(V,W)$ space of G-equivariant polynomials from V to W
$P(V)^G$ $\mathbb{R}$-algebra of polynomial G-invariants on V
$J^d(X)$ the d jet of X at $x=0$
$C_G^\infty(V,W)$ space of smooth (C^∞) G-equivariant maps from V to W
$C^\infty(V)^G$ space of smooth (C^∞) G-invariants on V
$C_G^\infty(V \times \mathbb{R})$ smooth 1-parameter families of G-equivariant vector fields on V
(V,G) $C_G^\infty(V \times \mathbb{R})$
$\mathcal{V}_0(V,G)$ normalized families of vector fields on V
Gx G-orbit of a point
GA G-orbit of set A
M/G orbit space of G-space M
$G \times_H X$ twisted product
$i_H^G X$ induced G-space
$S(V)$ unit sphere in V
S^n unit sphere in $\mathbb{R}^{n+1}$
$\mathcal{P}_Q$ phase vector field
$[\gamma]$ equivalence class of branch
sgn sign function (of branch)
ind index function (of branch)
$\Sigma(X)$ branching pattern of X
$\Sigma^\star(X)$ signed indexed branching pattern of X
$\mathbf{Z}(X)$ zero set of vector field X
$\Gamma(k+1)$ section 4.11.1
$D(p,q,r)$ section 4.11.1
$\mathcal{F}$ equivariant generating set (Section 6.6)
$\pitchfork$ transversality symbol (page 68,173)

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