Turbulence transition in pipe flow: some open questions

Bruno Eckhardt

Fachbereich Physik, Philipps-Universität Marburg, D-35032 Marburg, Germany

Abstract. The transition to turbulence in pipe flow is a long standing problem in fluid dynamics. In contrast to many other transitions it is not connected with linear instabilities of the laminar profile and hence follows a different route. Experimental and numerical studies within the last few years have revealed many unexpected connections to the nonlinear dynamics of strange saddles and have considerably improved our understanding of this transition. The text summarizes some of these insights and points to a few outstanding problems in areas where nonlinear dynamics can be expected to provide useful insights.

1. Introduction

The equations of fluid flow come naturally with a built in nonlinearity in the form of the convective derivative and, hence, constitute a prominent playground for applications of nonlinear dynamical systems theory. This mutually beneficial relationship has figured prominently in bifurcation theory, the routes to chaos and the development of nonlinear dynamics and fluid mechanics in general (see e.g. (Chandrasekhar 1961, Drazin & Reid 1981, Koschmieder 1993). It is therefore appropriate to commemorate the 20th anniversary of Nonlinearity by highlighting recent developments and open problems in an area that also has a reason to celebrate an anniversary: 2008 marks the 125th anniversary of Osborne Reynolds's papers (Reynolds 1883*a*, Reynolds 1883*b*) on the 'Conditions which determine whether the flow of a fluid is sinuous' in which he describes his observations on the intermitten transition to turbulence in circular pipes. Despite its long tradition and obvious practical relevance in many engineering situations there are many aspects of this transition that have puzzled scientists. For instance, one might expect that after more than a century the 'critical flow rate', measured by the dimensionless Reynolds number Re, for the transition to turbulence in pipe flow should be firmly established. Instead, one finds in the literature numbers which range between Re near 1000 (Prandtl & Tietjens 1931) and more than 3000. It turns out that this wide range is a natural consequence of the intrinsic properties of the system, and directly linked to the presence of a chaotic saddle. In the following, I will summarize the work that has led to this observation as well as several other key developements, and describe a few open questions that have come out of these investigations. More background information as well as more details may be found in recent reviews ((Kerswell 2005, Eckhardt, Schneider, Hof & Westerweel 2007) or the proceedings (Mullin & (eds) 2004).

The outline is as follows. In section 2 I will summarize a few experimental and theoretical facts about pipe flow. Section 3 then deals with coherent structures and section 4 with their connections in state space. Section 5 discusses the issues connected with the observed transience of turbulence. In section 6 we focus on the edge of chaos and in section 7 on the minimal perturbations needed to trigger turbulence. The global dynamics in relation to the localization of the turbulence in puffs is discussed in section 8. In the concluding section 9 we briefly outline connections to other shear flows.

2. Observations and elementary properties

Pressure driven flow down a smooth circular pipe develops a parabolic velocity profile sufficiently far from the inlet. In the usual dimensionless units one measures length in units of the diameter and velocities in units of the mean velocity. From these units and the viscosity of the fluid one can form a dimensionless number, the Reynolds number $Re = UD/\nu$. Hydrodynamic similarity theory states that all flows with the same Reynolds number, independent of flow speed or diameter of the pipe behave in the same manner if the Reynolds numbers agree.

The unusual properties of the transition to turbulence in pipe flow are causally connected to the fact that the parabolic profile is linearly stable agains infinitesimal perturbations (Salwen et al. 1980, Brosa 1986, Meseguer & Trefethen 2003). Experimentally, the laminar flow has been maintained for Reynolds numbers as high as 100.000 (Pfenniger 1961). The non-normality of the linear operator then shows that perturbations can temporarily extract energy from the laminar flow and give rise to transient amplification before the final decay (Boberg & Brosa 1988, Trefethen et al. 1993, Reddy et al. 1993, Grossmann 2000, Schmid & Henningson 1999, Waleffe 1995, Henningson 1996, Kim & Moehlis 2006, Eckhardt, Dietrich, Jachens & Schumacher 2007). This has led Meseguer and Trefethen (Meseguer & Trefethen 2003) to argue that for Reynolds numbers in excess of 10^7 the transient amplification is so strong that both experimentally and numerically it becomes practically impossible to control the perturbations and to prevent the transition.

In view of the linear stability, the transition to turbulence requires finite amplitude perturbations. These can derive from perturbations swept from the reservoir into the pipe or from perturbations originating in the inflow region. A typical experiment then shows an intermittent variation between laminar and turbulent domains which move downstream (for experimental demonstrations on the original apparatus used by Reynolds, see the movies provided with (Homsy et al. 2004) or (Eckhardt, Schneider, Hof & Westerweel 2007); for time traces, see (Rotta 1956)). For controlled experiments one turns to controlled perturbations, such as jets of fluids injected into the pipe (Wygnanski & Champagne 1973, Wygnanski et al. 1975, Darbyshire & Mullin 1995) or devices such as the iris diaphragm (Durst & Ünsal 2006) that temporarily block the flow. For sufficiently low Reynolds numbers these perturbations decay as they are swept downstream and do not recover. For Reynolds numbers up to about 2400, the perturbations develop into localized patches of about 30D lengths which move downstream with a speed close to but not identical to the mean velocity: this implies that there is a continuous flux of liquid through the patch. Interestingly, these patches keep their length. For higher Reynolds number, the upstream and downstream fronts move with different speeds and the localized patches spread out along the pipe axis. These structures are called puffs and slugs, respectively, and are discussed extensively in (Wygnanski & Champagne 1973, Wygnanski et al. 1975). Almost all the discussions here concern Reynolds numbers below about 3000 and dynamics in puffs.

From a mathematical perspective, the problem is the characterization of an initial value problem for a nonlinear, nonlocal partial differential equation, the Navier-Stokes equation. The temporal evolution of a velocity field $\mathbf{u}(\mathbf{x}, t)$ obeys

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\partial p + \nu \Delta \mathbf{u} \tag{1}$$

together with the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

and appropriate boundary conditions. Taking the divergence of the Navier-Stokes equation (1) gives a Poisson equation for the pressure,

$$\Delta \mathbf{u} = -\nabla \cdot ((\mathbf{u} \cdot \nabla)\mathbf{u}) \tag{3}$$

which results in a non-local dependence on the velocity gradients. The boundary conditions are that the fluid velocity vanishes at the walls. In the axial direction very often periodic boundary conditions are used. A first question, worthy of a million USD in bounty, concerns the smoothness of solutions starting from smooth initial conditions for all times (see (Feffermann 2000) for the prize question and (Doering & Gibbon 1995) for some background information). Should it be possible to arrive at singularities in finite times, then numerical representations on finite-dimensional spaces of basis function become dubious. In the absence of any positive evidence for singularities we will assume that the numerical representations are acceptable (modulo the usual resolution problems).

3. Lowest Reynolds number for coherent structures

A persistent turbulent dynamics requires the presence of persistent structures in state space other than the laminar profile. While there are examples of dynamical systems without periodic orbits, the most likely candidates for such persistent structures are some form of periodic motions. Perhaps the simples form are travelling waves, where a certain velocity field flows downstream without changing its form, $\mathbf{u}_{TW}(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x} - c\mathbf{e}_z t)$. More complicated ones have a non-trivial time-dependence and are periodic in time or come in the form of helical waves where a translation in time and a translation

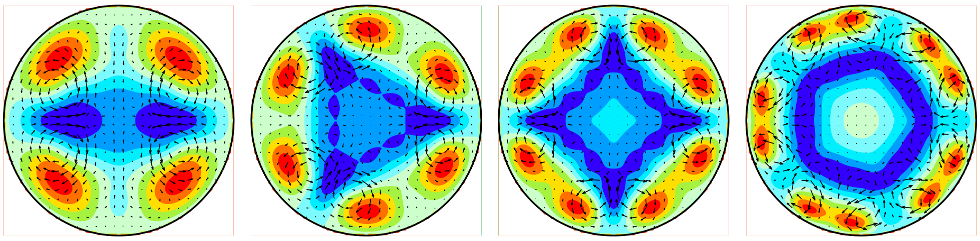

Figure 1. Examples of coherent structures in pipe flow. The profiles are obtained by averaging along the axis in order to highlight the symmetry. The arrows indicate the velocity in the cross section and the color code the downstream velocity relative to the parabolic profile. In the red regions the flow is faster than the parabola with the same mean flux, in the blue regions it is slower. From (Faisst & Eckhardt 2003).

in downstream or azimuthal direction are coupled (so-called relative periodic states). The method of choice for converging such states from appropriately identified initial conditions is the Newton method, combined with various methods for obtaining good initial conditions (see (Faisst & Eckhardt 2003, Wedin & Kerswell 2004) for pipe flow and (Waleffe 1998, Waleffe 2001, Waleffe 2003*a*, Viswanath 2007*b*) for other flows).

For pipe flow, the structures that have first been identified are families of coherent structures with symmetric arrangements of vortices (Faisst & Eckhardt 2003, Wedin & Kerswell 2004) (Fig. 1). More recently, asymmetric states, still of travelling wave type, have been identified (Pringle & Kerswell 2007). Secondary bifurcations of Hopf type will then lead to the creation of periodic orbits, as mentioned earlier. The critical Reynolds numbers at which the first symmetric structures appear is around 1250, that for the asymmetric states around 770. The questions that derives from this observation is:

Question: Are there any persistent coherent states, of travelling wave or more complicated types, with Reynolds numbers below 770?

That the lowest coherent states can be more complicated than fixed points can be seen in studies of low-dimensional dynamical system for shear flows, where the states that extend to the lowest Reynolds numbers are indeed periodic ones, with fixed points appearing at much higher Reynolds numbers only (Moehlis et al. 2004, Moehlis et al. 2005). On the other hand, this may be a resolution effect, since the low-dimensional model is most closely related to plane Couette flow, and there the lowest lying states are, as far as we know, again of fixed point type (Nagata 1990, Clever & Busse 1997, Waleffe 2003*b*).

We do know from the study of the energy balance that for Reynolds numbers below about 80, all perturbations decay monotonically in energy (Joseph 1975). This then provides a lower bound. While it may be difficult to prove the existence of states, it might be possible to prove the absence of any by establishing an asymptotic decay for higher Reynolds numbers, following ideas from control theory (Hinrichsen et al. 2004). The inverse question then is:

Question: What is the maximal Reynolds number below which no coherent states can exist?

As a warmup to this problem, it might be useful to address the related question in low-dimensional models (Eckhardt & Mersmann 1999, Moehlis et al. 2004), where the problem is one of ordinary differential equations with quadratic nonlinearitis, and where perhaps methods like Groebner bases can be put to good use.

4. Restructuring state space

Independent of where the first coherent states come into being at a Reynolds number of 770 or lower, it is an intriguing fact that experimental and numerical observations fail to come up with a somewhat longer lived turbulent dynamics for Reynolds numbers below about 1700 (Darbyshire & Mullin 1995, Peixinho & Mullin 2006, Mullin & Peixinho 2006). Thus, while all the prerequisites for turbulence exist, the flow fails to show turbulence in any substantial manner. This suggests:

Question: What happens in state space between the appearance of the first coherent structures and the experimental observation of turbulence?

The problem could be a qualitative one, in that the first states that appear are isolated and do not connect sufficiently to maintain turbulent dynamics, so that observing turbulence requires a global bifurcation which then links different structures to form a noticable attractor. The problem could also be simply a quantitative one, in that the structures form soon after the first bifurcations, and in a localized region in state space, and then simply grow and spread in until they are sufficiently volume filling to be noticable. Most likely, the answer to the problems involves a combination of both. Some progress towards tracking the manifolds is described in (Gibson et al. 2007).

An aspect of the restructuring is the observation that all states found so far are unstable, which means that they cannot be observed as permanent structures. However, it is possible to observe these states transiently in the experiment and numerics (Hof et al. 2004, Schneider, Eckhardt & Vollmer 2007, Kerswell & Tutty 2007, Eckhardt et al. 2005). The theory of chaotic systems holds that the frequency with which these states appears is related to their instability (Cvitanovic & Eckhardt 1991, Eckhardt & Ott 1994).

Question: Can the frequency with which travelling waves appear be related to their stability?

The studies (Schneider, Eckhardt & Vollmer 2007, Kerswell & Tutty 2007) are a first step in this direction, but cannot establish the quantitative relations between the indicator and the stability of the identified objects.

5. Transient turbulence

Very often, turbulent dynamics is associated with the formation of an attractor: then the dynamics would be persistent, chaotic, with an invariant measure and many other nice features. Long ago, (Brosa 1989) noticed that all his numerical runs returned to the

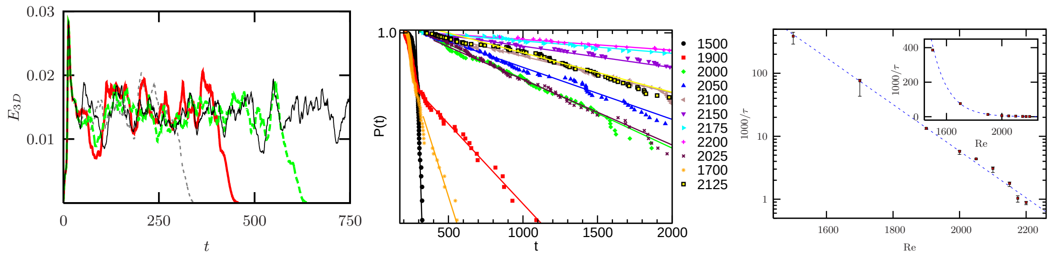

Figure 2. The energy traces in a turbulent section as a function of time show turbulent dynamics followed by a decay back to the laminar profile. When collected over many initial conditions, exponential distributions of life times result (middle). The characteristic lifetimes increase rapidly with Reynolds number (right).

laminar profile when followed long enough, and took the bold step to conclude that this was not specific to his numerics but an intrinsic property of turbulence: he suggested that all turbulence in pipe flow was transient (Fig. reffig:lifetimesa).

Interestingly, all numerical and experimental studies so far (Faisst & Eckhardt 2004, Hof et al. 2006, Mullin & Peixinho 2005, Mullin & Peixinho 2006, Peixinho & Mullin 2006, Peixinho & Mullin 2007, Willis & Kerswell 2007) agree that the lifetimes of turbulent pipe flow are exponentially distributed, so that the probability to still be turbulent after a time t is $P(t) \sim \exp(t/\tau)$ with a characteristic time $\tau(Re)$ (Fig. reffig:lifetimesb). The similarity sign indicates that this is only true for long times, and that certain short time transients have to be excluded (Schneider 2007, Hof et al. 2007). Such a behaviour is consistent with the one expected for a chaotic saddle (Kadanoff & Tang 1984, Kantz & Grassberger 1985, Tél 1991).

The variation of the characteristic time with Reynolds number is a hotly debated subject. A transition from a chaotic saddle to a turbulent attractor would require some form of boundary crisis (Grebogi et al. 1983) at which point $\tau(Re)$ diverges. Some measurements and numerical simulations are in agreement with this expectation (Faisst & Eckhardt 2004, Mullin & Peixinho 2006, Peixinho & Mullin 2006, Willis & Kerswell 2007). Other studies (Hof et al. 2006, Schneider 2007) suggest that $\tau(Re)$ increases rapidly and present evidence that $\tau(Re)$ increases exponentially (Fig. 2c). These latter studies are based on several different experimental set ups and are consistent with numerical studies for long pipes and long observation times (Schneider 2007). The very fact that there can be a dispute about the variations of lifetimes with Reynolds number shows that the presence (or absence) of an attractor in pipe flow at sufficiently high Reynolds numbers cannot be answered as easily as one might have expected.

Question: Is turbulence in pipe flow transient for all Reynolds numbers or is there a crisis bifurcation to an attractor? If there is no crisis, how rapidly does the lifetime increase with Reynolds number?

More computer power and additional experimental efforts will most likely reduce the statistical uncertainty, and provide better data for the Reynolds number dependence $\tau(Re)$ (other dynamical systems show a variety of dependencies on parameters, see e.g. (Kaneda 1990, Crutchfield & Kaneko 1988, Lai & Winslow 1995, Braun & Feudel 1996, Goren et al. 1998, Rempel & Chian 2003)

It would also be helpful to find other indicators in the boundary dynamics, in the fluctuations or any other quantity directly accessible from the turbulent dynamics that might point to the existence or absence of a bifurcation. However, a definite answer to the appearence of an attractor requires constructive criteria for the identification of the necessary global bifurcation. But even then the turbulence need not be persistent, as spontaneous, noise-induced transitions between the two coexisting attractors could cause a relaminarization (Lagha & Manneville 2007, Schoepe 2004).

6. Edge of chaos

In the case of two coexisting attractors, there are basins and boundaries which separate the basins. If the turbulence is only transient, the turbulent dynamics connects to the laminar one, and there can be no basin boundary dividing state space into a turbulent and a laminar region. Hence, the concept of the dividing surface between the two regions has to be reconsidered and suitably generalized (Skufca et al. 2006, Schneider & Eckhardt 2006, Schneider, Eckhardt & Yorke 2007, Vollmer et al. 2007)) Starting from the properties of the saddle state in a saddle-node bifurcation, one could use the lifetimes of perturbations, i.e. the time it takes to relax to the laminar profile, as an indicator: approaching the stable manifold from the laminar side of the saddle, the lifetime increases, reaches infinity and stays there, if the state on the other side is attracting. If the state on the other side is transient, there will be wild variations in lifteimes from one initial condition to a nearby one. Therefore, we denoted this point the edge of chaos. All edge points seem to be connected in state space. A second observation concerns these connections and the dynamics in the edge of chaos: numerical studies show an evolution towards a relative attractor (see (Skufca et al. 2006) for a model study, (Schneider & Eckhardt 2006, Schneider, Eckhardt & Vollmer 2007) for pipe flow and (Itano & Toh 2000, Toh & Itano 2003, Viswanath 2007*a*) for other flows). We have used a two-dimensional map to characterize some of these properties, including possible bifurcations in the edge state and the appearance of chaos (Vollmer et al. 2007). Most intriguingly, there is a possibility that this boundary can be fractal or not, depending on the ratio of two Lyapunov exponents.

Question: What are the properties of the edge of chaos and the invariant state within the edge? Is the edge of chaos a global relative attractor or are there additional attractors in the edge of chaos?

The models in (Vollmer et al. 2007) can easily be extended to cover multiple attractors in the edge, as in the model studied in (Skufca et al. 2006), but it would be nice to have an example in a realistic flow.

7. Minimal perturbations

In cases in which the laminar profile is stable, a finite perturbation is require to trigger turbulence. Experiments and the observations on non-normal amplification of perturbations show that the flow becomes increasingly sensitive to perturbations as the Reynolds number increases (Hof et al. 2003). This suggests that the diameter of the basin of attraction of the laminar profile decreases with increasing Reynolds number. While there are results that suggest that the boundary contracts like $1/Re$ ((Hof et al. 2003) and, for low Re, Fig. 3), some experiments and numerics find steeper decays (Mellibovsky & Meseguer 2005, Mellibovsky & Meseguer 2007, Peixinho & Mullin 2007, Philip et al. 2007).

Question: How does the amplitude of the minimal perturbation required to trigger turbulence decay with Reynolds number?

To approach this question, one may look for minimal variations in the laminar profile which turn it unstable (Gavarini et al. 2004), or for optimal 3-d perturbations which triggering turbulence: (Ben-Dov & Cohen 2007) find a global minimum in energy norm for triggering secondary instabilites. It is also possible to study the scaling of the invariant state in the edge or perhaps of other relevant coherent states with Reynolds number. This is facilitated by the observation that the states do not become more complex with Re (Wang et al. 2007, Schneider 2007). In simple models they can dominate the size of the basins (Eckhardt & Lathrop 2006), but here there is evidence that they maintain a finite distance from the laminar profile as $Re \to \infty$ (Wang et al. 2007):

Question: Can one characterize the $Re \to \infty$ limit of coherent structures? Do the solutions approach the laminar profile or do some of them keep a finite distance for large Re?

If the states keep a finite distance from the laminar profile then the reduction in the basin of attraction of the laminar profile has to come from a scaling of the stable manifolds. Such an connection would also fit with the observations of kinks and folds in the edge (Fig. 3), which are typical for stable manifolds.

8. Turbulent spot dynamics

The coherent structures described earlier provide a conveninent means for describing the dynamics in a finite section of the pipe with periodic boundary conditions. This, however, covers only one aspect of the dynamics, in that the experimentally induced turbulence is usually a localized one, confined to a domain of about 30D length (Wygnanski & Champagne 1973, Wygnanski et al. 1975) (see Fig. 4). If a shorter turbulent region is induced, it grows until it reaches this length, if a longer one is induced it either shrinks or breaks up into two shorter ones which again grow to be 30D long. Therefore, there is considerable robustness in the turbulent dynamics of spots:

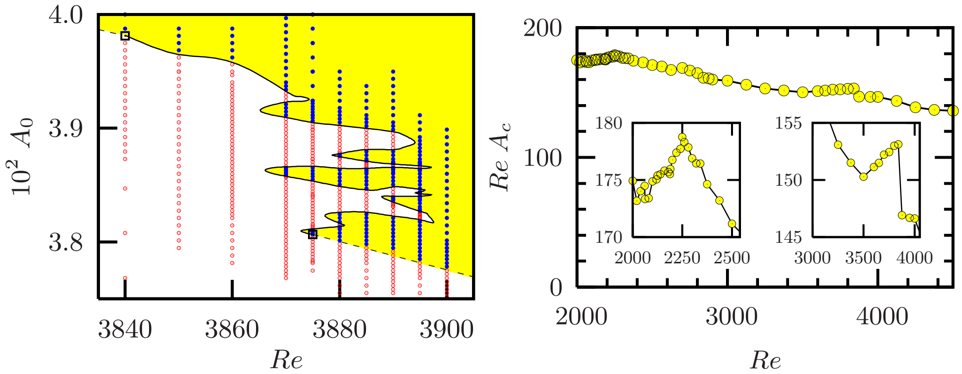

Figure 3. The boundary between laminar and turbulent (left) for a specific perturbation and the scaling of the boundary with Reynolds number. The insets on the right show modulations in the scaling curve connected with the folds on the left. From (Schneider, Eckhardt & Yorke 2007).

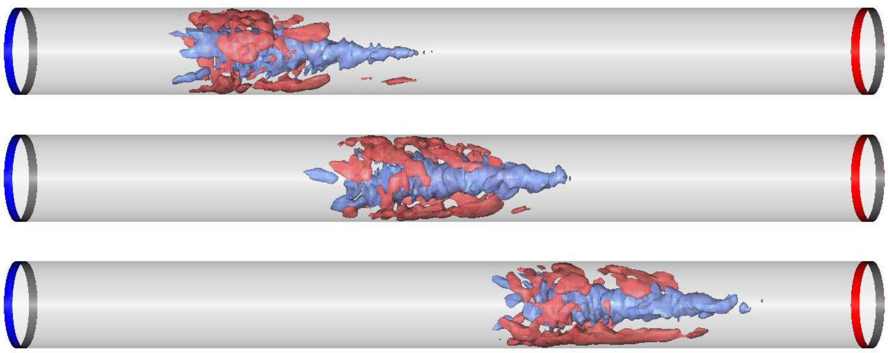

Figure 4. Three snapshots of a turbulent flow in a pipe at $Re = 1825$, moving from left to right. The snapshots are separated in time by $20D/u_{cl}$ with u_{cl} the centerline velocity. The red and blue regions indicate isosurfaces of downstream velocities somewhat faster or slower than the parabolic laminar profile. The aspect ratio is not shown to scale: the pipe is 50 diameters long. From (Eckhardt & Schneider 2008).

Question: For Reynolds numbers below about 2700, the turbulence in a long pipe comes in localized puffs. How is the dynamics of the puffs, their length selection and their boundary dynamics connected to the periodic coherent structures?

In the simplest form one can imagine some sort of Ginzburg-Landau type model for the dynamics of an envelope of the turbulent region (as in (Prigent et al. 2002)), but this is unsatisfactory unless the equations and their coefficient can be derived from the underlying Navier-Stokes dynamics. It is interesting to note that a simliar localization phenomenon can be studied in plane Couette flow (Barkley & Tuckerman 2005).

9. Final remarks

The discussion in the preceeding section has focussed on pipe flow, but there are several other related problems. One is plane Couette flow between parallel plates in relative motion, where the laminar profile is also linearly stable for all Reynolds numbers (Dauchot & Daviaud 1994, Dauchot & Daviaud 1995, Bottin et al. 1998, Dauchot & Vioujard 2000). Pressure driven flow between parallel plates, plane Poiseuille flow, has a parabolic profile and a curious linear instability at Reynolds numbers of about 5772. However, turbulence is observed at Reynolds numbers of about 1000, and hence can follow a similar mechanism to the case described here. Finally, there is also a regime in flow between rotating concentric cylinders, Taylor-Couette flow, where a stable laminar profile and a turbulent dynamics coexist (Faisst & Eckhardt 2000, Hristova et al. 2002). At present there seem to be many similiarities and connections between these flows, suggesting that the transition to turbulence in these flows follows an independent route with many as yet unexplored dynamical properties.

The ultimate challenge is to understand the full hydrodynamic flow and hence the properties of the Navier-Stokes equation. But it is good to know that for the development and test of ideas there are models on all levels of complexity, from ordinary differential equations with varying numbers of degrees of freedom to simplified partial differential equations, see e.g. (Waleffe 1995, Eckhardt & Mersmann 1999, Moehlis et al. 2004, Moehlis et al. 2005, Brosa & Grossmann 1999, Smith et al. 2005, Manneville & Locher 2000, Lagha & Manneville 2007).

Osborne Reynolds motivated his study not only with the obvious practical relevance of pipe flow, but also with his interest in the nature of the transition. He could not have anticipated that while he was among the first to describe a transition in detail, his example would be among the last among to be explained. But in hindsight it is clear that any serious explanation of the transition requires quite a bit of nonlinear dynamics. Fortunately, the transition in pipe flow is not only at the receiving end: the complexity of the edge state and the intriguing possibilites for the connections between the different states in the high-dimensional state space can be expected to stimulate nonlinear dynamics as well.

Acknowledgements

It is a pleasure to thank Holger Faisst and Tobias M. Schneider for their untiring efforts and original contributions to elucidating the properties of transitional pipe flow, and to Jerry Westerweel and Björn Hof for sharing their experiments and many ideas. This work was supported by Deutsche Forschungsgemeinschaft.

References

Barkley D & Tuckerman L S 2005 *Phys. Rev. Lett.* **94**, 014502.
Ben-Dov G & Cohen J 2007 *Physical Review Letters* **98**(6), 064503–+.

Boberg L & Brosa U 1988 *Z. Naturforsch.* **43a**, 697–726.
Bottin S, Daviaud F, Manneville P & Dauchot O 1998 *Europhys. Lett.* **43**(2), 171–176.
Braun R & Feudel F 1996 *Phys. Rev. E* **53**, 6562–6565.
Brosa U 1986 *Zeitschrift für Naturforschung* **41a**, 1141.
Brosa U 1989 *J. Stat. Phys.* **55**, 1303–1312.
Brosa U & Grossmann S 1999 *Eur. Phys. J. B* **9**, 343–354.
Chandrasekhar S 1961 *Hydrodynamic and hydromagnetic stability* Oxford University Press Oxford.
Clever R & Busse F H 1997 *Journal of Fluid Mechanics* **344**, 137–153.
Crutchfield J P & Kaneko K 1988 *Phys. Rev. Lett.* **60**, 2715–2718.
Cvitanovic P & Eckhardt B 1991 *Journal of Physics A: Mathematical and General* **24**, L237–L241.
Darbyshire A G & Mullin T 1995 *J. Fluid Mech.* **289**, 83–114.
Dauchot O & Daviaud F 1994 *Europhys. Lett.* **28**, 225–230.
Dauchot O & Daviaud F 1995 *Phys. Fluids* **7**, 335–343.
Dauchot O & Vioujard N 2000 *Eur. Phys. J. B* **14**, 377.
Doering C R & Gibbon J 1995 *Applied analysis of the Navier-Stokes equation* Cambridge University Press Cambridge.
Drazin P G & Reid W H 1981 *Hydrodynamic Stability* Cambridge University Press Cambridge.
Durst F & Ünsal B 2006 *Journal of Fluid Mechanics* **560**, 449–464.
Eckhardt B, Dietrich A, Jachens A & Schumacher J 2007 *in* J Peinke, ed., 'Progress in Turbulence II' Springer.
Eckhardt B, Faisst H, Schmiegel A & Schneider T M 2005 *Phil. Trans. R Soc (London)* p. submitted.
Eckhardt B & Lathrop D P 2006 *Nonlinear Phenomena and Complex Systems* **9**, 133–140.
Eckhardt B & Mersmann A 1999 *Physical Review E* **60**, 509–517.
Eckhardt B & Ott G 1994 *Zeitschrift für Physik B* **93**, 259–266.
Eckhardt B & Schneider T M 2008 *European Physical Journal B* p. in press.
Eckhardt B, Schneider T M, Hof B & Westerweel J 2007 *Annual Review of Fluid Mechanics* **39**, 447–468.
Faisst H & Eckhardt B 2000 *Physical Review E* **61**, 7227–7230.
Faisst H & Eckhardt B 2003 *Physical Review Letters* **91**, 224502 (4 pages).
Faisst H & Eckhardt B 2004 *Journal of Fluid Mechanics* **504**, 343–352.
Feffermann C L 2000 Existence and smoothness of the Navier-Stokes equation Technical report http://www.claymath.org/millennium/Navier-Stokes_Equations/navierstokes.pdf.
Gavarini M I, Bottaro A & Nieuwstadt F T M 2004 *Journal of Fluid Mechanics* **517**, 131–165.
Gibson J F, Halcrow J & Cvitanovic P 2007 *arxiv* **0705.3957**, (31 pages).
Goren G, Eckmann J P & Procaccia I 1998 *Phys. Rev. E* **57**, 4106–4134.
Grebogi C, Ott E & Yorke J A 1983 *Physica D* **7**, 181–200.
Grossmann S 2000 *Rev. Mod. Phys.* **72**, 603–618.
Henningson D 1996 *Physics of Fluids* **8**, 2257–2258.
Hinrichsen D, Plischke E & Wirth F 2004 *in* V. D Blondel & A Megretski, eds, 'Unsolved problems in mathematical systems theory' V D Blondel and A Megretski (eds), Princeton University Press pp. 197–202.
Hof B, Juel A & Mullin T 2003 *Phys. Rev. Lett.* p. 244502.
Hof B, van Doorne C W H, Westerveel J, Nieuwstadt F T M, Faisst H, Eckhardt B, Wedin H, Kerswell R R & Waleffe F 2004 *Science* **305**, 1594–1598.
Hof B, Westerweel J, Schneider T & Eckhardt B 2007 *arxiv* p. 0707.2642 (1 page).
Hof B, Westerweel J, Schneider T M & Eckhardt B 2006 *Nature* **443**, 60–64.
Homsy G M, Aref H, Breuer G S & et al. 2004 *Multi Media Fluid Mechanics* Cambridge University Press Cambridge.
Hristova H, Roch S, Schmid P J & Tuckerman L S 2002 *Physics of Fluids* **14**, 3475–3484.
Itano T & Toh S 2000 *Journal Physical Society of Japan* **???**, ???
Joseph D D 1975 *Stability of Fluid Motions, vol I and II* Springer.
Kadanoff L P & Tang C 1984 *Proc. Natl. Acad. Sci. USA* **81**, 1276.

Kaneda K 1990 *Phys. Lett. A* **149**, 105–112.
Kantz H & Grassberger P 1985 *Physica D* **17**, 75–86.
Kerswell R R 2005 *Nonlinearity* **18**, R17–R44.
Kerswell R R & Tutty O R 2007 *Journal of Fluid Mechanics* **584**, 69–102.
Kim L & Moehlis J 2006 *Physics Letters A* **358**, 431–437.
Koschmieder E L 1993 *Bénard Cells and Taylor Vortices* Cambridge University Press.
Lagha M & Manneville P 2007 *European Physical Journal B* **58**, 433–447.
Lai Y C & Winslow R L 1995 *Phys. Rev. Lett.* **74**, 5208–5211.
Manneville P & Locher F 2000 *Comptes Rendus de l'Academie des Sciences Series IIB Mechanics Physics Astronomy* **328**, 159–164.
Mellibovsky F & Meseguer A 2005 *Journal of Physics Conference Series* **14**, 192–205.
Mellibovsky F & Meseguer A 2007 *Physics of Fluids* **???**, submitted.
Meseguer A & Trefethen L 2003 *Journal of Computational Physics* **186**, 178–197.
Moehlis J, Faisst H & Eckhardt B 2004 *New Journal of Physics* **6**, nr. 56 (17 pages).
Moehlis J, Faisst H & Eckhardt B 2005 *SIAM Journal of Applied Dynamical Systems* **4**, 352–376.
Mullin T & (eds) R K 2004 *Laminar-turbulent transition and finite amplitude solutions* Springer Dordrecht.
Mullin T & Peixinho J 2005 *in* Mullin & ???, eds, 'IUTAM Symposium transition in shear flows' Springer.
Mullin T & Peixinho J 2006 *Journal of Low Temperature Physics* **145**, 75–88.
Nagata M 1990 *Journal of Fluid Mechanics* **217**, 519–527.
Peixinho J & Mullin T 2006 *Physical Review Letters* **96**, 094501 (4 pages).
Peixinho J & Mullin T 2007 *Journal of Fluid Mechanics* **582**, 169–178.
Pfenniger W 1961 *in* G Lachman, ed., 'Boundary Layer and Flow Control' Pergamon pp. 970–980.
Philip J, Svizher A & Cohen J 2007 *Physical Review Letters* **98**(15), 154502–+.
Prandtl L & Tietjens O 1931 *Hydro- und Aeromechanik* Julius Springer Berlin.
Prigent A, Gregoire G, Chate H, Dauchot O & van Saarloos W 2002 *Phys. Rev. Lett.* **89**, 014501.
Pringle C & Kerswell R R 2007 *Physical Review Letters* **99**, 074502.
Reddy S C, Schmid P J & Henningson D S 1993 *SIAM Journal of Applied Mathematics* **53**, 15–47.
Rempel E L & Chian A C L 2003 *Phys. Lett. A* **319**, 104–109.
Reynolds O 1883*a Philosophical Transactions Royal Society (London)* **174**, 935–982 + 3 plates.
Reynolds O 1883*b Proceedings Royal Society (London)* **35**, 84–99.
Rotta J 1956 *Ingenieur-Archiv* **24**, 258–281.
Salwen H, Cotton F W & Grosch C E 1980 *J. Fluid Mech.* **98**, 273–284.
Schmid P J & Henningson D S 1999 *Stability and Transition of Shear Flows* Springer New York.
Schneider T M 2007 State space properties of transitional pipe flow PhD thesis Philipps Universität Marburg.
Schneider T M & Eckhardt B 2006 *Chaos* **16**, 020604.
Schneider T M, Eckhardt B & Vollmer J 2007 *Physical Review E* **75**, 066313.
Schneider T M, Eckhardt B & Yorke J A 2007 *Physical Review Letters* **99**, 034502.
Schoepe W 2004 *Phys. Rev. Lett.* **92**, 095301.
Skufca J D, Yorke J A & Eckhardt B 2006 *Physical Review Letters* **96**, 174101 (4 pages).
Smith T R, Moehlis J & Holmes P 2005 *Journal of Fluid Mechanics* **538**, 71–110.
Tél T 1991 *in* H Bai-Lin, D Feng & J Yuan, eds, 'Directions in Chaos' Vol. 3 World Scientific, Singapore p. 149.
Toh S & Itano T 2003 *Journal of Fluid Mechanics* **481**, 67–76.
Trefethen L N, Trefethen A E, Reddy S C & Driscoll T A 1993 *Science* **261**, 578–584.
Viswanath D 2007*a arxiv* **0701337**, 15 pages.
Viswanath D 2007*b Journal of Fluid Mechanics* **580**, 339–358.
Vollmer J, Schneider T & Eckhardt B 2007 *Nonlinearity* **to be submitted**.
Waleffe F 1995 *Phys. Fluids* **7**, 3060–3066.

Waleffe F 1998 *Physical Review Letters* **81**, 4140–4143.
Waleffe F 2001 *Journal of Fluid Mechanics* **435**, 93–102.
Waleffe F 2003*a* *Phys. Fluids* **15**, 1517–1534.
Waleffe F 2003*b* *Physics of Fluids* **15**, 1517–1534.
Wang J, Gibson J & Waleffe F 2007 *Physical Review Letters* **98**, 204501.
Wedin H & Kerswell R R 2004 *Journal of Fluid Mechanics* **508**, 333–371.
Willis A P & Kerswell R R 2007 *Physical Review Letters* **98**, 014501.
Wygnanski I J & Champagne F H 1973 *Journal of Fluid Mechanics* **59**, 281–335.
Wygnanski I, Sokolov M & Friedman D 1975 *Journal of Fluid Mechanics* **69**, 283–304.