Lyapunov exponents and all that

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Alexandr Mikhailovich Lyapunov

* 1857 Jaroslavl

1870–1876 Nizhni Novgorod

1876–1882 St Petersburg University

1884 Magister Thesis (superwiser P. L. Tschebyschev)

1885–1902 Kharkov University

1892 Doctor Thesis (“A general problem of stability of motion”)

1902–1917 St. Petersburg University

† 1918 Odessa

Elementary one-dimensional dynamics

One-dimensional map and its linearization

$$x_{n+1} = F(x_n) \qquad \delta x_{n+1} = F'(x_n)\delta x_n$$

$$\ln|\delta x_{n+1}| = \ln|F'(x_n)| + \ln|\delta x_n|$$

$$\ln|\delta x_T| - \ln|\delta x_0| = \sum_{k=0}^{T-1} \ln|F'(x_n)|$$

$$\lim_{T\to\infty} \frac{\ln|\delta x_T| - \ln|\delta x_0|}{T} = \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} \ln|F'(x_n)| = \langle \ln|F'(x_n)| \rangle = \lambda$$

Matrix products and Oseledets theorem

In dimensions larger than one we have a product of matrices

$$\delta\mathbf{x}_{n+1} = M_n \delta\mathbf{x}_n$$

Oseledets multiplicative ergodic theorem:

Lyapunov exponents = exponential asymptotic growth rates of vectors

$$\chi^+(\mathbf{v}) = \lim_{T\to\infty} \frac{1}{T} \ln \left\| \prod_{k=0}^{T-1} M_k \mathbf{v} \right\|$$

are well defined a.e. and attain at most $\dim M$ different values. Furthermore, there is a Lyapunov decomposition into subspaces corresponding to the vatious Lyapunov exponents, whose dimension defines the multiplicity of the corresponding exponent.

Comparing to other characteristics of stability

Lyapunov exponents heavily rely on the existense of ergodic measure, they describe stability in a statistical sense.

Remarkable: LEs are not realy needed for mathematical theory of hyperbolic systems
(in the book of Katok and Hasselblatt they appear only in an Appendix devoted to non-uniform hyperbolicity)

LEs as a main tool of exploring chaos

- sensitive dependence on initial conditions = positive Lyapunov exponent
- KS-entropy = sum of positive LEs (Pesin theorem)
- independent of the metric used (but this is not true in infinite-dimensional case!)
- invariant to smooth transformations of variables (diffeomorphisms) (but not to general transformations – homeomorphisms!)

Geometric picture

Lyapunov exponents are associated with Lyapunov vectors and with stable and unstable manifolds

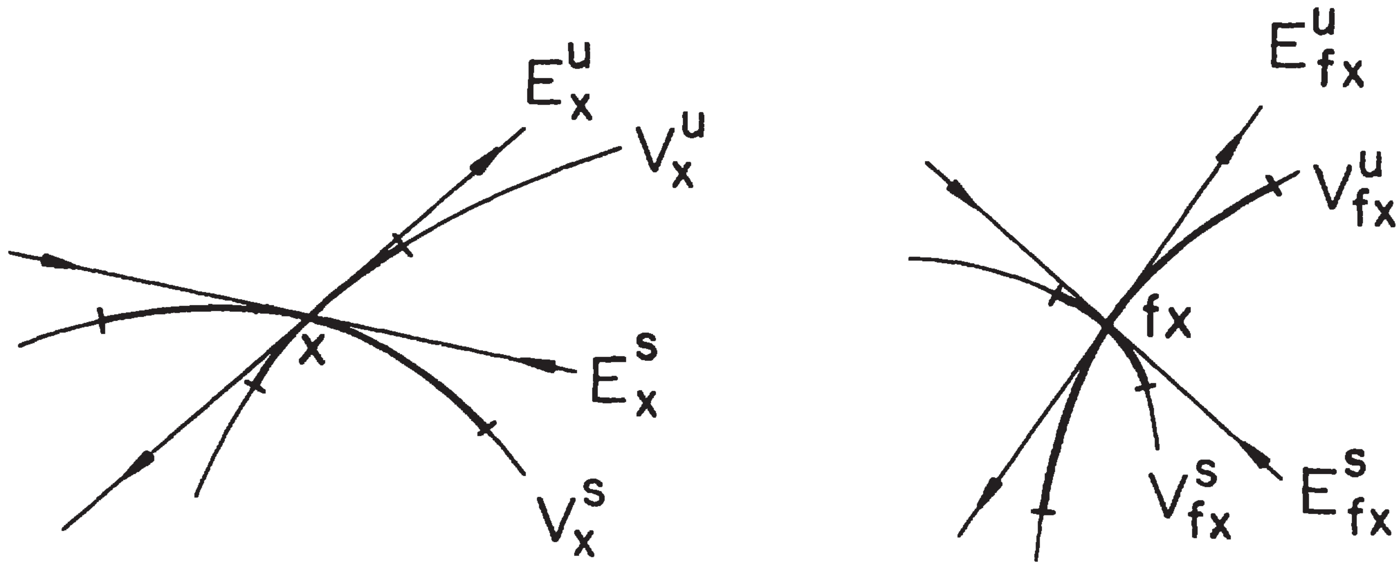


FIG. 13. Stable and unstable manifolds can be defined for points that are neither fixed nor periodic. The stable and unstable directions E_x^s and E_x^u are tangent to the stable and unstable manifolds V_x^s and V_x^u, respectively. They are mapped by f onto the corresponding objects at fx.

Numerical implementation

- Bennetin, Galgani, Giorgilli and Strelcyn (1980) algorithm: Gram-Schmidt orthogonalization
- Eckmann and Ruelle (1985): QR-decomposition (may be numerically preferable)
- Greene and Kim (1987): Calculation of LEs and Lyapunov directions
- Bridges and Reich (2001): A stable variant of continuous-time decomposition

Finite-time fluctuations of LEs

Calculation of LEs over a time interval T gives a distribution $P_T(\Lambda)$ for which one assumes an ansatz

$$P_T(\Lambda) \propto e^{TS(\Lambda)}$$

with an entropy function $S(\Lambda) \leq 0$ that reaches the maximum at $\Lambda = \lambda$.

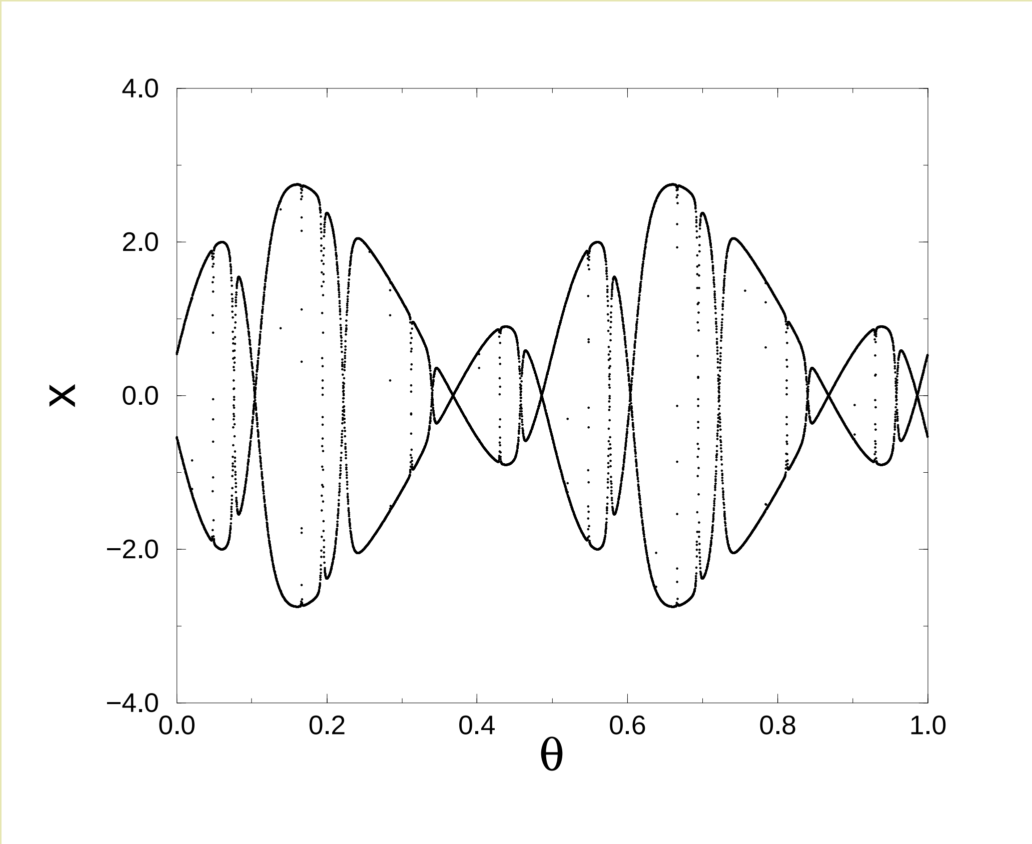
Another representation – via generalized exponents

$$L(q) = \lim_{T\to\infty} \frac{\ln\langle \|\mathbf{v}\|^q \rangle}{T} \qquad \lambda = \left.\frac{dL(q)}{dq}\right|_{q=0}$$

Flinite-time fluctuations of negative LE and SNA

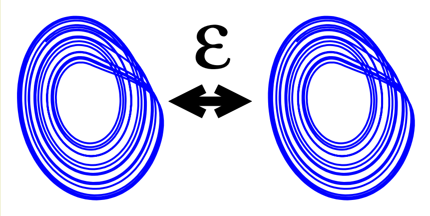
If the largest LE is negative but the entropy function $S(\Lambda)$ has a tail at positive Λ:

Stability in average but unstable trajectories are inserted in the attractor

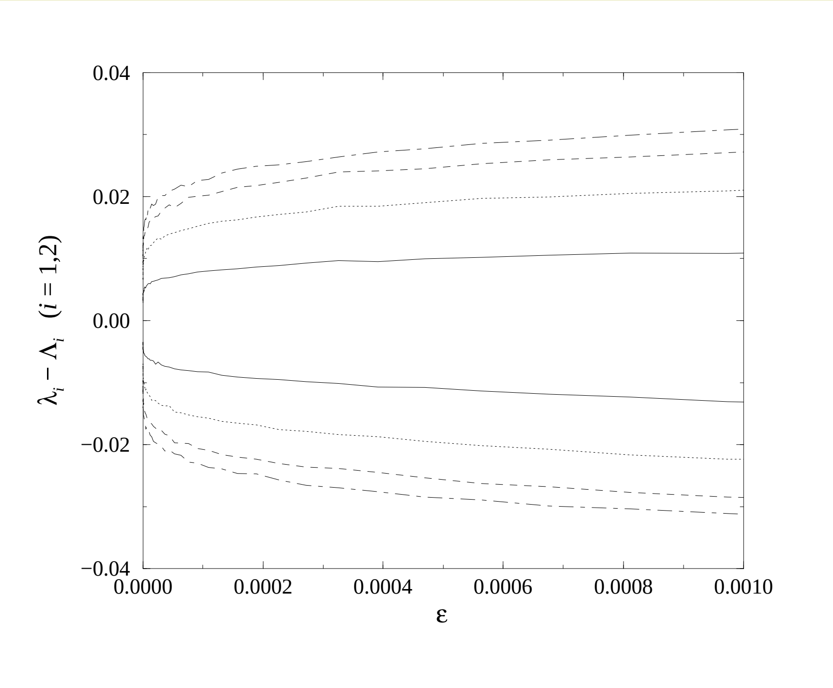


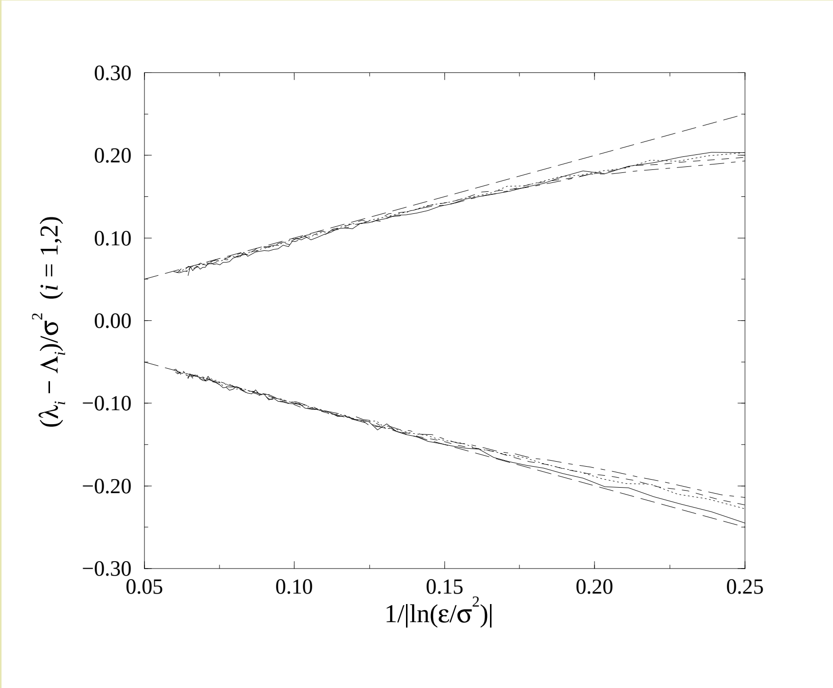
Coupling sensitivity of chaos

Lyapunov exponents in weakly coupled systems depend on the coupling in a singular way

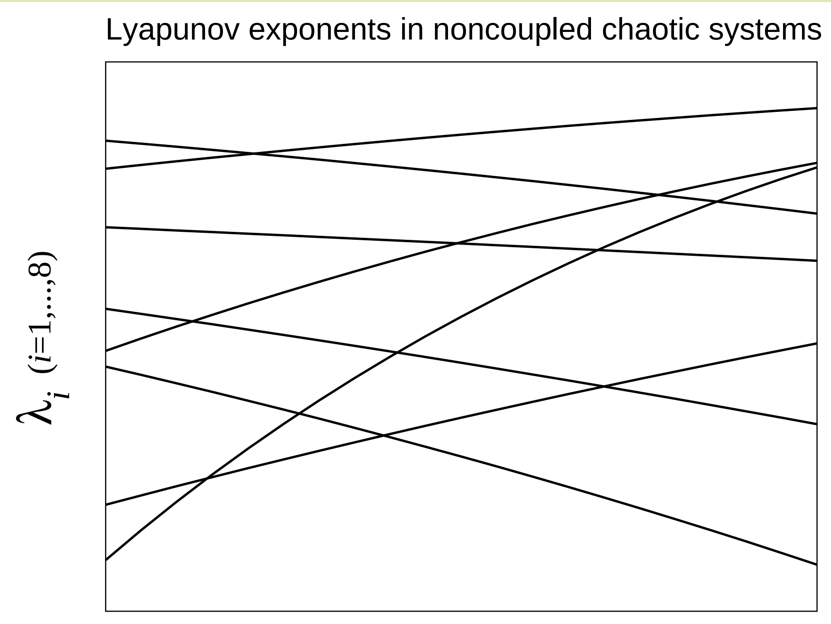


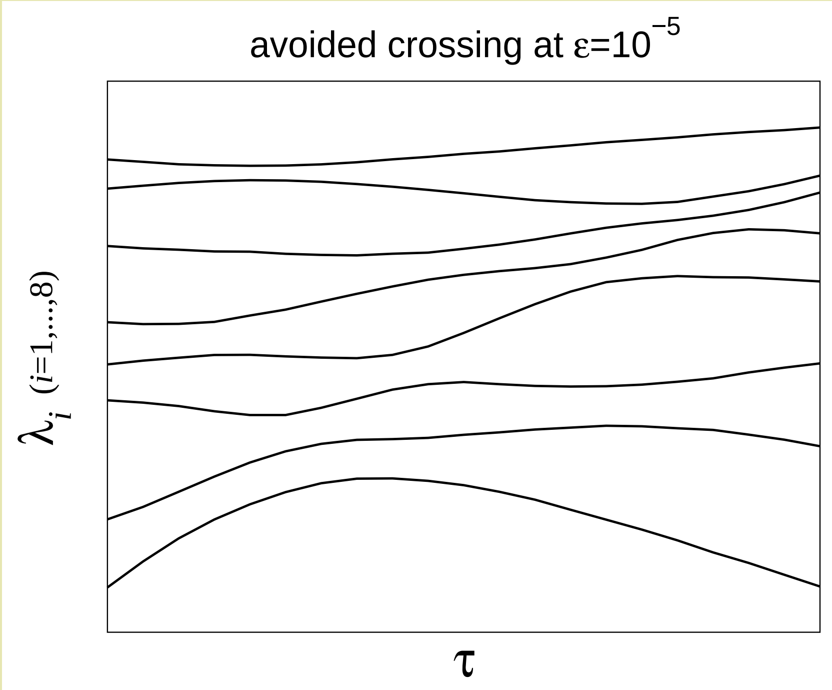
$$\Delta\lambda \sim \frac{1}{|\log \varepsilon|}$$



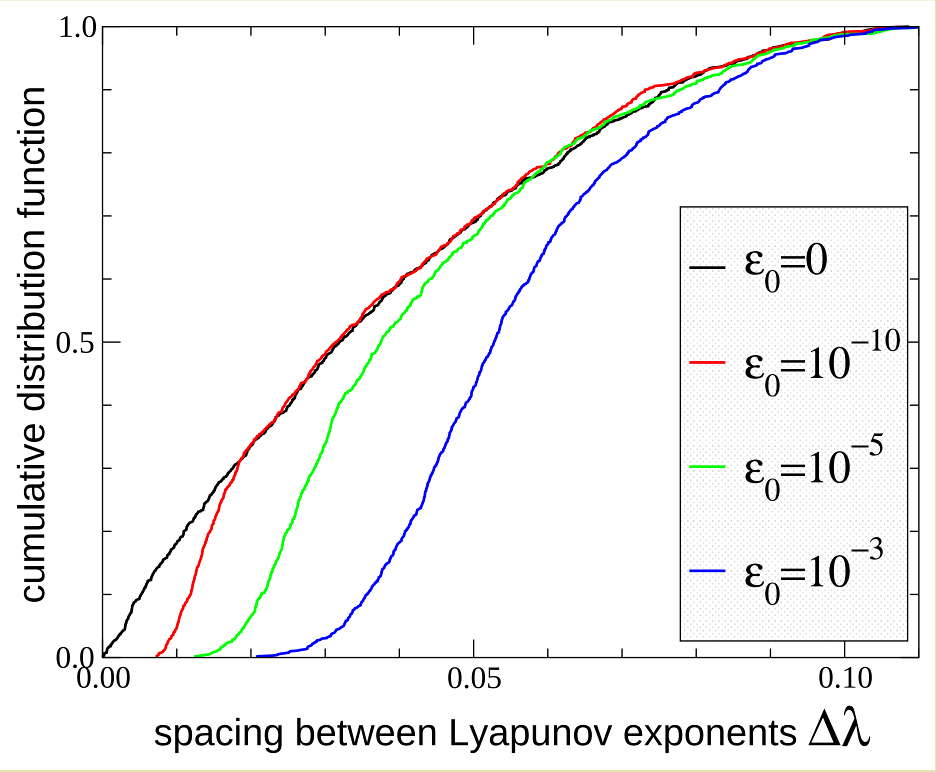


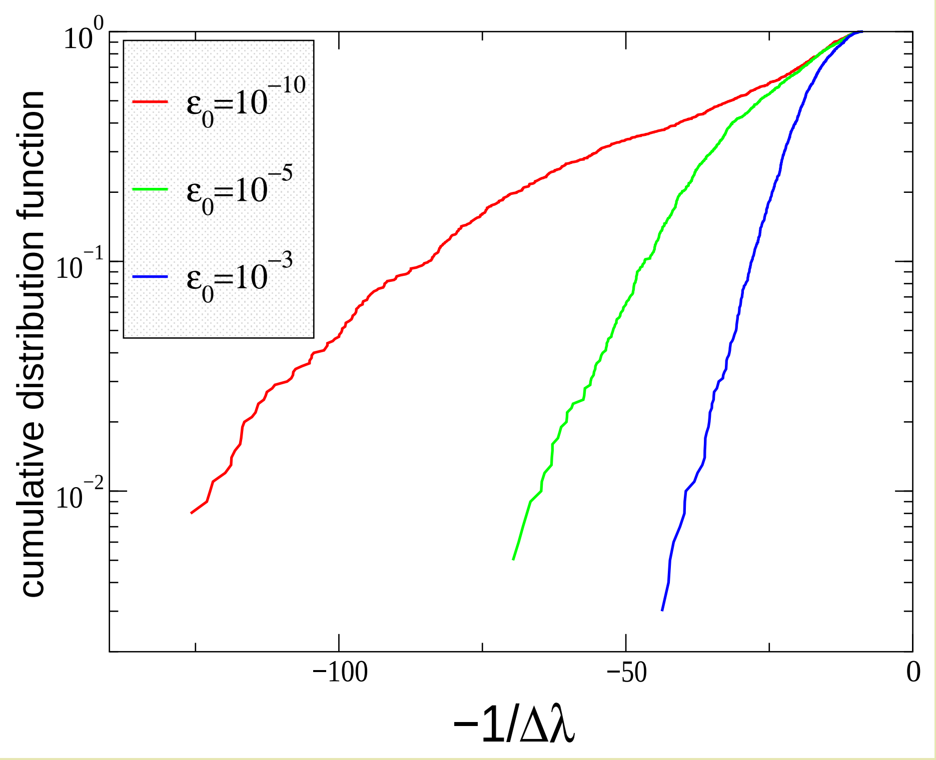
Repulsion of LEs in random systems





Distribution of the "Lyapunov exponent spacing" has a strong depletion for small $\Delta\lambda$





Conditional/transversal LEs and synchronization conditions

Stability of symmetric sets with respect to particular perturbations that break the symmetry is described by linear systems of the type $\dot{\mathbf{v}} = M(t)\mathbf{v}$ with a chaotically time-varying matrix $M(t)$

Example: an ensemble of identical systems $\mathbf{x}_k$, $k = 1, \ldots, N$ coupled via mean fields $\mathbf{g}(\mathbf{x})$ and global variables $\mathbf{y}$:

$$\dot{\mathbf{x}}_k = \mathbf{F}(\mathbf{x}_k, \mathbf{y}, \mathbf{g}; \varepsilon) \ , \qquad \dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}, \mathbf{g}; \varepsilon) \ ,$$

Full synchrony $\mathbf{x}_k = \mathbf{x}$ is described by a low-order system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{g}; \varepsilon) \ , \qquad \dot{\mathbf{y}} = \mathbf{G}(\mathbf{y}, \mathbf{g}; \varepsilon) \ ,$$

“Split” or “evaporation” LEs describing the stability of the synchronous cluster are determined in thermodynamic limit $N \to \infty$

$$\frac{d\delta\mathbf{x}}{dt} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \delta\mathbf{x}$$

LEs in noisy systems and sycnhronization by common noise

Lyapunov exponents in noisy systems characterize sensitivity to initial conditions for the same realization of noise

Synchronization by common noise:

largest LE negative: synchronization to a common identical state

largest LE positive: desynchronization

Example: reliability of neurons under repititions of the same noise

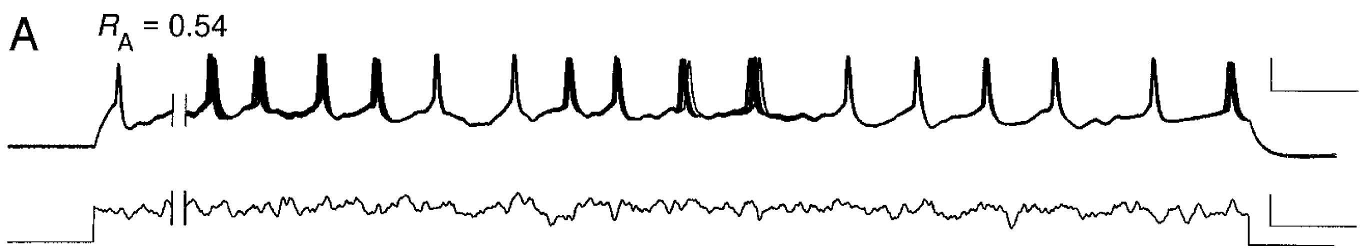


FIG. 4. Spike time reliability in *Aplysia* motoneuron with aperiodic inputs. Superposed voltage traces from 10 different trials recorded from a buccal motoneuron for 4 different input signals. *A*: broadband aperiodic input

Nonlinear exponents

Fix the level of the perturbation $\varepsilon = \|\mathbf{v}\|$ and calculate the time to reach the level 2ε:

This gives the level-dependent exponent $\lambda(\varepsilon)$

Typically $\lambda(\varepsilon)$ decreases for large ε and $\lambda(0) = \lambda$

Lyapunov Exponents in extended systems

- Take a large finite system of length L, calculate the LEs and look how they change with increase of L:
  A spectrum f of LEs $\lambda_k = f(k/L)$ which defines the density of the KS-entropy and of the Lyapunov dimension
- Take an infinite system: Different metrics are not equivalent
  Physically: a perturbation may not simply grow but come from remote parts of the system

- Velocity-dependent exponent: Take a local perturbation $\mathbf{v}(x_0, 0)$ and follow it along the constant-velocity rays: $\|\mathbf{v}(x_0 + Vt, t)\| \propto e^{t\lambda(V)}$
- Chronotopic Lyapunov analysis: Take a perturbation that is exponential in space $\mathbf{v} \sim e^{\mu x}$ and calculate its LE $\lambda(\mu)$. Velocity-dependent exponent is the Legendre transform of the chronotopic one
- Lyapunov-Bloch exponent: Take a system of size L and calculate the Lyapunov exponent $\lambda(\kappa)$ of the perturbation $\mathbf{v} \sim e^{i\kappa x}$. It determines stability of space-periodic states
- Statistics of Lyapunov vectors in extended systems may be nontrivial, in many cases it seemingly belongs to a Kardar-Parisi-Zhang universality class