CHAPTER 7

FUNCTIONS OF A COMPLEX VARIABLE II

In this chapter we return to the analysis that started with the Cauchy–Riemann conditions in Chapter 6 and develop the residue theorem, with major applications to the evaluation of definite and principal part integrals of interest to scientists and asymptotic expansion of integrals by the method of steepest descent. We also develop further specific analytic functions, such as pole expansions of meromorphic functions and product expansions of entire functions. Dispersion relations are included because they represent an important application of complex variable methods for physicists.

7.1 CALCULUS OF RESIDUES

Residue Theorem

If the Laurent expansion of a function $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ is integrated term by term by using a closed contour that encircles one isolated singular point z_0 once in a counterclockwise sense, we obtain (Exercise 6.4.1)

$$a_n \oint (z - z_0)^n \, dz = a_n \frac{(z - z_0)^{n+1}}{n+1}\bigg|_{z_1}^{z_1} = 0, \qquad n \neq -1. \tag{7.1}$$

However, if $n = -1$,

$$a_{-1} \oint (z - z_0)^{-1} \, dz = a_{-1} \oint \frac{ire^{i\theta} \, d\theta}{re^{i\theta}} = 2\pi i a_{-1}. \tag{7.2}$$

Summarizing Eqs. (7.1) and (7.2), we have

$$\frac{1}{2\pi i} \oint f(z) \, dz = a_{-1}. \tag{7.3}$$

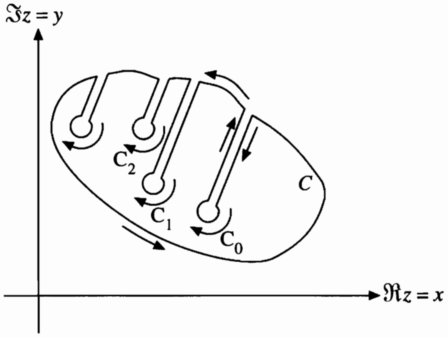

FIGURE 7.1 Excluding isolated singularities.

The constant a_{-1}, the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion, is called the residue of $f(z)$ at $z = z_0$.

A set of isolated singularities can be handled by deforming our contour as shown in Fig. 7.1. Cauchy's integral theorem (Section 6.3) leads to

$$\oint_C f(z)\,dz + \oint_{C_0} f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots = 0. \tag{7.4}$$

The circular integral around any given singular point is given by Eq. (7.3),

$$\oint_{C_i} f(z)\,dz = -2\pi i a_{-1,z_i}, \tag{7.5}$$

assuming a Laurent expansion about the singular point $z = z_i$. The negative sign comes from the clockwise integration, as shown in Fig. 7.1. Combining Eqs. (7.4) and (7.5), we have

$$\boxed{\oint_C f(z)\,dz = 2\pi i(a_{-1z_0} + a_{-1z_1} + a_{-1z_2} + \cdots)}$$
$$= 2\pi i \times (\text{sum of enclosed residues}). \tag{7.6}$$

This is the **residue theorem**. The problem of evaluating one or more contour integrals is replaced by the algebraic problem of computing residues at the enclosed singular points.

We first use this residue theorem to develop the concept of the Cauchy principal value. Then in the remainder of this section we apply the residue theorem to a wide variety of definite integrals of mathematical and physical interest.

Using the transformation $z = 1/w$ for w approaching 0, we can find the nature of a singularity at z going to ∞ and the residue of a function $f(z)$ with just isolated singularities and no branch points. In such cases we know that

$$\sum\{\text{residues in the finite } z\text{-plane}\} + \{\text{residue at } z \to \infty\} = 0.$$

Cauchy Principal Value

Occasionally an isolated pole will be directly on the contour of integration, causing the integral to diverge. Let us illustrate a physical case.

Example 7.1.1 FORCED CLASSICAL OSCILLATOR

The inhomogeneous differential equation for a classical, undamped, driven harmonic oscillator,

$$\ddot{x}(t) + \omega_0^2 x(t) = f(t), \tag{7.7}$$

may be solved by representing the driving force $f(t) = \int \delta(t' - t) f(t')\, dt'$ as a superposition of impulses by analogy with an extended charge distribution in electrostatics.[1] If we solve first the simpler differential equation

$$\ddot{G} + \omega_0^2 G = \delta(t - t') \tag{7.8}$$

for $G(t, t')$, which is independent of the driving term f (model dependent), then $x(t) = \int G(t, t') f(t')\, dt'$ solves the original problem. First, we verify this by substituting the integrals for $x(t)$ and its time derivatives into the differential equation for $x(t)$ using the differential equation for G. Then we look for $G(t, t') = \int \tilde{G}(\omega) e^{i\omega t} \frac{d\omega}{2\pi}$ in terms of an integral weighted by $\tilde{G}$, which is suggested by a similar integral form for $\delta(t - t') = \int e^{i\omega(t-t')} \frac{d\omega}{2\pi}$ (see Eq. (1.193c) in Section 1.15).

Upon substituting G and $\ddot{G}$ into the differential equation for G, we obtain

$$\int \left[(\omega_0^2 - \omega^2)\tilde{G} - e^{-i\omega t'}\right] e^{i\omega t}\, d\omega = 0. \tag{7.9}$$

Because this integral is zero for all t, the expression in brackets must vanish for all ω. This relation is no longer a differential equation but an algebraic relation that we can solve for $\tilde{G}$:

$$\tilde{G}(\omega) = \frac{e^{-i\omega t'}}{\omega_0^2 - \omega^2} = \frac{e^{-i\omega t'}}{2\omega_0(\omega + \omega_0)} - \frac{e^{-i\omega t'}}{2\omega_0(\omega - \omega_0)}. \tag{7.10}$$

Substituting $\tilde{G}$ into the integral for G yields

$$G(t, t') = \frac{1}{4\pi\omega_0} \int_{-\infty}^{\infty} \left[\frac{e^{i\omega(t-t')}}{\omega + \omega_0} - \frac{e^{i\omega(t-t')}}{\omega - \omega_0}\right] d\omega. \tag{7.11}$$

Here, the dependence of G on $t - t'$ in the exponential is consistent with the same dependence of $\delta(t - t')$, its driving term. Now, the problem is that this integral diverges because the integrand blows up at $\omega = \pm\omega_0$, since the integration goes right through the first-order poles. To explain why this happens, we note that the δ-function driving term for G includes all frequencies with the same amplitude. Next, we see that the equation for $\tilde{G}$ at $t' = 0$ has its driving term equal to unity for all frequencies ω, including the resonant ω_0.

[1] Adapted from A. Yu. Grosberg, priv. comm.

We know from physics that forcing an oscillator at resonance leads to an indefinitely growing amplitude when there is no friction. With friction, the amplitude remains finite, even at resonance. This suggests including a small friction term in the differential equations for $x(t)$ and G.

With a small friction term $\eta\dot{G}$, $\eta > 0$, in the differential equation for $G(t, t')$ (and $\eta\dot{x}$ for $x(t)$), we can still solve the algebraic equation

$$\left(\omega_0^2 - \omega^2 + i\eta\omega\right)\tilde{G} = e^{-i\omega t'} \tag{7.12}$$

for $\tilde{G}$ with friction. The solution is

$$\tilde{G} = \frac{e^{-i\omega t'}}{\omega_0^2 - \omega^2 + i\eta\omega} = \frac{e^{-i\omega t'}}{2\Omega}\left(\frac{1}{\omega - \omega_-} - \frac{1}{\omega - \omega_+}\right), \tag{7.13}$$

$$\omega_\pm = \pm\Omega + \frac{i\eta}{2}, \qquad \Omega = \omega_0\sqrt{1 - \left(\frac{\eta}{2\omega_0}\right)^2}. \tag{7.14}$$

For small friction, $0 < \eta \ll \omega_0$, Ω is nearly equal to ω_0 and real, whereas $\omega_\pm$ each pick up a small imaginary part. This means that the integration of the integral for G,

$$G(t, t') = \frac{1}{4\pi\Omega}\int_{-\infty}^{\infty}\left[\frac{e^{i\omega(t-t')}}{\omega - \omega_-} - \frac{e^{i\omega(t-t')}}{\omega - \omega_+}\right]d\omega, \tag{7.15}$$

no longer encounters a pole and remains finite. ■

This treatment of an integral with a pole moves the pole off the contour and then considers the limiting behavior as it is brought back, as in Example 7.1.1 for $\eta \to 0$. This example also suggests treating ω as a complex variable in case the singularity is a first-order pole, deforming the integration path to avoid the singularity, which is equivalent to adding a small imaginary part to the pole position, and evaluating the integral by means of the residue theorem.

Therefore, if the integration path of an integral $\int \frac{dz}{z - x_0}$ for real x_0 goes right through the pole x_0, we may deform the contour to include or exclude the residue, as desired, by including a semicircular detour of **infinitesimal radius**. This is shown in Fig. 7.2. The integration over the semicircle then gives, with $z - x_0 = \delta e^{i\varphi}$, $dz = i\,\delta e^{i\varphi}\,d\varphi$ (see Eq. (6.27a)),

$$\int \frac{dz}{z - x_0} = i\int_{\pi}^{2\pi} d\varphi = i\pi, \text{ i.e., } \pi i a_{-1} \qquad \text{if counterclockwise,}$$

$$\int \frac{dz}{z - x_0} = i\int_{\pi}^{0} d\varphi = -i\pi, \text{ i.e., } -\pi i a_{-1} \qquad \text{if clockwise.}$$

This contribution, $+$ or $-$, appears on the left-hand side of Eq. (7.6). If our detour were clockwise, the residue would not be enclosed and there would be no corresponding term on the right-hand side of Eq. (7.6).

However, if our detour were counterclockwise, this residue would be enclosed by the contour C and a term $2\pi i a_{-1}$ would appear on the right-hand side of Eq. (7.6).

The net result for either clockwise or counterclockwise detour is that a simple pole on the contour is counted as one-half of what it would be if it were within the contour. This corresponds to taking the Cauchy principal value.

FIGURE 7.2 Bypassing singular points.

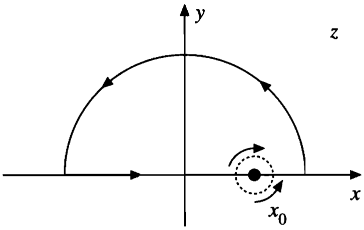

FIGURE 7.3 Closing the contour with an infinite-radius semicircle.

For instance, let us suppose that $f(z)$ with a simple pole at $z = x_0$ is integrated over the entire real axis. The contour is closed with an infinite semicircle in the upper half-plane (Fig. 7.3). Then

$$\begin{aligned}\oint f(z)\,dz &= \int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{C_{x_0}} f(z)\,dz \\ &\quad + \int_{x_0+\delta}^{\infty} f(x)\,dx + \int_C \text{infinite semicircle} \\ &= 2\pi i \sum \textbf{enclosed} \text{ residues.}\end{aligned} \tag{7.16}$$

If the small semicircle C_{x_0}, includes x_0 (by going below the x-axis, counterclockwise), x_0 is enclosed, and its contribution appears **twice**—as $\pi i a_{-1}$ in $\int_{C_{x_0}}$ and as $2\pi i a_{-1}$ in the term $2\pi i \sum$ enclosed residues—for a net contribution of $\pi i a_{-1}$. If the upper small semicircle is selected, x_0 is excluded. The only contribution is from the **clockwise** integration over C_{x_0}, which yields $-\pi i a_{-1}$. Moving this to the extreme right of Eq. (7.16), we have $+\pi i a_{-1}$, as before.

The integrals along the x-axis may be combined and the semicircle radius permitted to approach zero. We therefore define

$$\lim_{\delta\to 0}\left\{\int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{x_0+\delta}^{\infty} f(x)\,dx\right\} = P\int_{-\infty}^{\infty} f(x)\,dx. \tag{7.17}$$

P indicates the Cauchy **principal value** and represents the preceding limiting process. Note that the Cauchy principal value is a balancing (or canceling) process. In the vicinity of our singularity at $z = x_0$,

$$f(x) \approx \frac{a_{-1}}{x - x_0}. \tag{7.18}$$

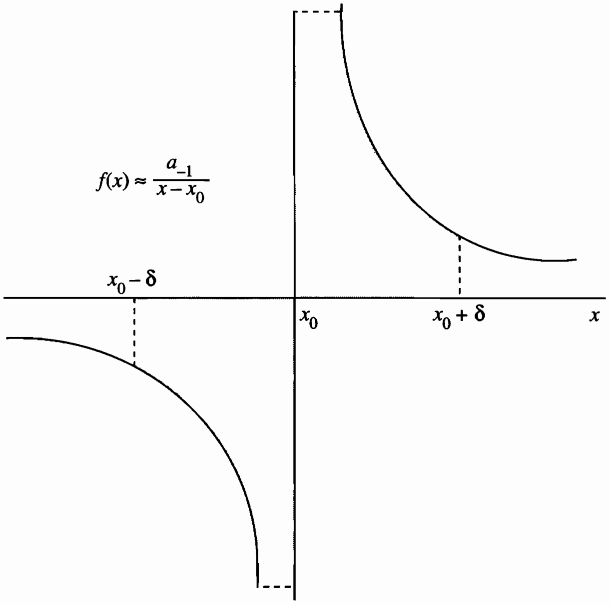

FIGURE 7.4 Cancellation at a simple pole.

This is odd, relative to x_0. The symmetric or even interval (relative to x_0) provides cancellation of the shaded areas, Fig. 7.4. The contribution of the singularity is in the integration about the semicircle.

In general, if a function $f(x)$ has a singularity x_0 somewhere inside the interval $a \le x_0 \le b$ and is integrable over every portion of this interval that does not contain the point x_0, then we **define**

$$\int_a^b f(x)\,dx = \lim_{\delta_1 \to 0} \int_a^{x_0-\delta_1} f(x)\,dx + \lim_{\delta_2 \to 0} \int_{x_0+\delta_2}^b f(x)\,dx,$$

when the limit exists as $\delta_j \to 0$ **independently**, else the integral is said to diverge. If this limit does not exist but the limit $\delta_1 = \delta_2 = \delta \to 0$ exists, it is defined to be the principal value of the integral.

This same limiting technique is applicable to the integration limits $\pm\infty$. We define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a \to -\infty, b \to \infty} \int_a^b f(x)\,dx, \tag{7.19a}$$

if the integral exists with a, b approaching their limits independently, else the integral diverges. In case the integral diverges but

$$\lim_{a \to \infty} \int_{-a}^{a} f(x)\,dx = P \int_{-\infty}^{\infty} f(x)\,dx \tag{7.19b}$$

exist, it is defined as its principal value.

Pole Expansion of Meromorphic Functions

Analytic functions $f(z)$ that have only isolated poles as singularities are called **meromorphic**. Examples are $\cot z$ [from $\frac{d}{dz}\ln\sin z$ in Eq. (5.210)] and ratios of polynomials. For simplicity we assume that these poles at finite $z = a_n$ with $0 < |a_1| < |a_2| < \cdots$ are all simple with residues b_n. Then an expansion of $f(z)$ in terms of $b_n(z-a_n)^{-1}$ depends in a systematic way on all singularities of $f(z)$, in contrast to the Taylor expansion about an arbitrarily chosen analytic point z_0 of $f(z)$ or the Laurent expansion about one of the singular points of $f(z)$.

Let us consider a series of concentric circles C_n about the origin so that C_n includes $a_1, a_2, \ldots, a_n$ but no other poles, its radius $R_n \to \infty$ as $n \to \infty$. To guarantee convergence we assume that $|f(z)| < \varepsilon R_n$ for any small positive constant ε and all z on C_n. Then the series

$$\boxed{f(z) = f(0) + \sum_{n=1}^{\infty} b_n\left\{(z-a_n)^{-1} + a_n^{-1}\right\}} \tag{7.20}$$

converges to $f(z)$. To prove this **theorem** (due to Mittag–Leffler) we use the residue theorem to evaluate the contour integral for z inside C_n:

$$\begin{aligned} I_n &= \frac{1}{2\pi i}\int_{C_n} \frac{f(w)}{w(w-z)}\,dw \\ &= \sum_{m=1}^{n} \frac{b_m}{a_m(a_m - z)} + \frac{f(z) - f(0)}{z}. \end{aligned} \tag{7.21}$$

On C_n we have, for $n \to \infty$,

$$|I_n| \le 2\pi R_n \frac{\max_{w \text{ on } C_n} |f(w)|}{2\pi R_n(R_n - |z|)} < \frac{\varepsilon R_n}{R_n - |z|} \to \varepsilon$$

for $R_n \gg |z|$. Using $I_n \to 0$ in Eq. (7.21) proves Eq. (7.20).

If $|f(z)| < \varepsilon R_n^{p+1}$, then we evaluate similarly the integral

$$I_n = \frac{1}{2\pi i}\int \frac{f(w)}{w^{p+1}(w-z)}\,dw \to 0 \qquad \text{as } n \to \infty$$

and obtain the analogous pole expansion

$$f(z) = f(0) + zf'(0) + \cdots + \frac{z^p f^{(p)}(0)}{p!} + \sum_{n=1}^{\infty} \frac{b_n z^{p+1}/a_n^{p+1}}{z - a_n}. \tag{7.22}$$

Note that the convergence of the series in Eqs. (7.20) and (7.22) is implied by the bound of $|f(z)|$ for $|z| \to \infty$.

Product Expansion of Entire Functions

A function $f(z)$ that is analytic for all finite z is called an **entire** function. The logarithmic derivative f'/f is a meromorphic function with a pole expansion.

If $f(z)$ has a simple zero at $z = a_n$, then $f(z) = (z - a_n)g(z)$ with analytic $g(z)$ and $g(a_n) \neq 0$. Hence the logarithmic derivative

$$\frac{f'(z)}{f(z)} = (z - a_n)^{-1} + \frac{g'(z)}{g(z)} \tag{7.23}$$

has a simple pole at $z = a_n$ with residue 1, and g'/g is analytic there. If f'/f satisfies the conditions that lead to the pole expansion in Eq. (7.20), then

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\frac{1}{a_n} + \frac{1}{z - a_n}\right] \tag{7.24}$$

holds. Integrating Eq. (7.24) yields

$$\begin{aligned}\int_0^z \frac{f'(z)}{f(z)}dz &= \ln f(z) - \ln f(0)\\ &= \frac{zf'(0)}{f(0)} + \sum_{n=1}^{\infty}\left\{\ln(z - a_n) - \ln(-a_n) + \frac{z}{a_n}\right\},\end{aligned}$$

and exponentiating we obtain the product expansion

$$\boxed{f(z) = f(0)\exp\left(\frac{zf'(0)}{f(0)}\right)\prod_1^{\infty}\left(1 - \frac{z}{a_n}\right)e^{z/a_n}.} \tag{7.25}$$

Examples are the product expansions (see Chapter 5) for

$$\boxed{\begin{aligned}\sin z &= z\prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(1 - \frac{z}{n\pi}\right)e^{z/n\pi} = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right),\\ \cos z &= \prod_{n=1}^{\infty}\left\{1 - \frac{z^2}{(n - 1/2)^2\pi^2}\right\}.\end{aligned}} \tag{7.26}$$

Another example is the product expansion of the gamma function, which will be discussed in Chapter 8.

As a consequence of Eq. (7.23) the contour integral of the logarithmic derivative may be used to count the number N_f of zeros (including their multiplicities) of the function $f(z)$ inside the contour C:

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N_f. \tag{7.27}$$

Moreover, using

$$\int \frac{f'(z)}{f(z)}dz = \ln f(z) = \ln\big|f(z)\big| + i \arg f(z), \tag{7.28}$$

we see that the real part in Eq. (7.28) does not change as z moves once around the contour, while the corresponding change in $\arg f$ must be

$$\Delta_C \arg(f) = 2\pi N_f. \tag{7.29}$$

This leads to **Rouché's theorem**: *If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C and $|g(z)| < |f(z)|$ on C then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.*

To show this we use

$$2\pi N_{f+g} = \Delta_C \arg(f+g) = \Delta_C \arg(f) + \Delta_C \arg\left(1 + \frac{g}{f}\right).$$

Since $|g| < |f|$ on C, the point $w = 1 + g(z)/f(z)$ is always an interior point of the circle in the w-plane with center at 1 and radius 1. Hence $\arg(1 + g/f)$ must return to its original value when z moves around C (it does not circle the origin); it cannot decrease or increase by a multiple of 2π so that $\Delta_C \arg(1 + g/f) = 0$.

Rouché's theorem may be used for an alternative proof of the fundamental theorem of algebra: A polynomial $\sum_{m=0}^{n} a_m z^m$ with $a_n \neq 0$ has n zeros. We define $f(z) = a_n z^n$. Then f has an n-fold zero at the origin and no other zeros. Let $g(z) = \sum_{m=0}^{n-1} a_m z^m$. We apply Rouché's theorem to a circle C with center at the origin and radius $R > 1$. On C, $|f(z)| = |a_n| R^n$ and

$$\big|g(z)\big| \le |a_0| + |a_1|R + \cdots + |a_{n-1}|R^{n-1} \le \left(\sum_{m=0}^{n-1} |a_m|\right) R^{n-1}.$$

Hence $|g(z)| < |f(z)|$ for z on C, provided $R > (\sum_{m=0}^{n-1} |a_m|)/|a_n|$. For all sufficiently large circles C therefore, $f + g = \sum_{m=0}^{n} a_m z^m$ has n zeros inside C according to Rouché's theorem.

Evaluation of Definite Integrals

Definite integrals appear repeatedly in problems of mathematical physics as well as in pure mathematics. Three moderately general techniques are useful in evaluating definite integrals: (1) contour integration, (2) conversion to gamma or beta functions (Chapter 8), and (3) numerical quadrature. Other approaches include series expansion with term-by-term integration and integral transforms. As will be seen subsequently, the method of contour integration is perhaps the most versatile of these methods, since it is applicable to a wide variety of integrals.

Definite Integrals: $\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta$

The calculus of residues is useful in evaluating a wide variety of definite integrals in both physical and purely mathematical problems. We consider, first, integrals of the form

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta, \tag{7.30}$$

where f is finite for all values of θ. We also require f to be a rational function of $\sin\theta$ and $\cos\theta$ so that it will be single-valued. Let

$$z = e^{i\theta}, \qquad dz = ie^{i\theta}\, d\theta.$$

From this,

$$d\theta = -i\frac{dz}{z}, \qquad \sin\theta = \frac{z - z^{-1}}{2i}, \qquad \cos\theta = \frac{z + z^{-1}}{2}. \tag{7.31}$$

Our integral becomes

$$I = -i \oint f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{z}, \tag{7.32}$$

with the path of integration the unit circle. By the residue theorem, Eq. (7.16),

$$I = (-i)2\pi i \sum \text{residues within the unit circle.} \tag{7.33}$$

Note that we are after the residues of f/z. Illustrations of integrals of this type are provided by Exercises 7.1.7–7.1.10.

Example 7.1.2 INTEGRAL OF COS IN DENOMINATOR

Our problem is to evaluate the definite integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon\cos\theta}, \qquad |\varepsilon| < 1.$$

By Eq. (7.32) this becomes

$$I = -i \oint_{\text{unit circle}} \frac{dz}{z[1 + (\varepsilon/2)(z + z^{-1})]}$$
$$= -i\frac{2}{\varepsilon} \oint \frac{dz}{z^2 + (2/\varepsilon)z + 1}.$$

The denominator has roots

$$z_- = -\frac{1}{\varepsilon} - \frac{1}{\varepsilon}\sqrt{1 - \varepsilon^2} \qquad \text{and} \qquad z_+ = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1 - \varepsilon^2},$$

where z_+ is within the unit circle and z_- is outside. Then by Eq. (7.33) and Exercise 6.6.1,

$$I = -i\frac{2}{\varepsilon} \cdot 2\pi i \frac{1}{2z + 2/\varepsilon}\bigg|_{z=-1/\varepsilon+(1/\varepsilon)\sqrt{1-\varepsilon^2}}.$$

We obtain

$$\int_0^{2\pi} \frac{d\theta}{1+\varepsilon\cos\theta} = \frac{2\pi}{\sqrt{1-\varepsilon^2}}, \qquad |\varepsilon| < 1.$$

■

Evaluation of Definite Integrals: $\int_{-\infty}^{\infty} f(x)\,dx$

Suppose that our definite integral has the form

$$I = \int_{-\infty}^{\infty} f(x)\,dx \tag{7.34}$$

and satisfies the two conditions:

- $f(z)$ is analytic in the upper half-plane except for a finite number of poles. (It will be assumed that there are no poles on the real axis. If poles are present on the real axis, they may be included or excluded as discussed earlier in this section.)
- $f(z)$ vanishes as strongly[2] as $1/z^2$ for $|z| \to \infty$, $0 \le \arg z \le \pi$.

With these conditions, we may take as a contour of integration the real axis and a semicircle in the upper half-plane, as shown in Fig. 7.5. We let the radius R of the semicircle become infinitely large. Then

$$\begin{aligned}\oint f(z)\,dz &= \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty}\int_0^{\pi} f\big(Re^{i\theta}\big)iRe^{i\theta}\,d\theta \\ &= 2\pi i \sum \text{residues (upper half-plane).}\end{aligned} \tag{7.35}$$

From the second condition the second integral (over the semicircle) vanishes and

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{residues (upper half-plane).} \tag{7.36}$$

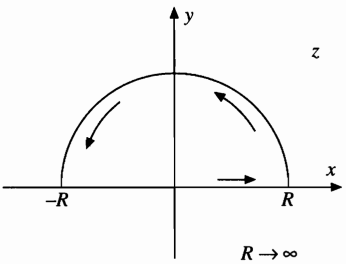

FIGURE 7.5 Half-circle contour.

[2]We could use $f(z)$ vanishes faster than $1/z$, and we wish to have $f(z)$ single-valued.

Example 7.1.3 INTEGRAL OF MEROMORPHIC FUNCTION

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}. \tag{7.37}$$

From Eq. (7.36),

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \sum \text{residues (upper half-plane)}.$$

Here and in every other similar problem we have the question: Where are the poles? Rewriting the integrand as

$$\frac{1}{z^2+1} = \frac{1}{z+i} \cdot \frac{1}{z-i}, \tag{7.38}$$

we see that there are simple poles (order 1) at $z = i$ and $z = -i$.

A simple pole at $z = z_0$ indicates (and is indicated by) a Laurent expansion of the form

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + \sum_{n=1}^{\infty} a_n (z-z_0)^n. \tag{7.39}$$

The residue a_{-1} is easily isolated as (Exercise 6.6.1)

$$a_{-1} = (z-z_0) f(z)|_{z=z_0}. \tag{7.40}$$

Using Eq. (7.40), we find that the residue at $z = i$ is $1/2i$, whereas that at $z = -i$ is $-1/2i$.

Then

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \cdot \frac{1}{2i} = \pi. \tag{7.41}$$

Here we have used $a_{-1} = 1/2i$ for the residue of the one included pole at $z = i$. Note that it is possible to use the lower semicircle and that this choice will lead to the same result, $I = \pi$. A somewhat more delicate problem is provided by the next example. ■

Evaluation of Definite Integrals: $\int_{-\infty}^{\infty} f(x)e^{iax}\,dx$

Consider the definite integral

$$I = \int_{-\infty}^{\infty} f(x) e^{iax}\, dx, \tag{7.42}$$

with a real and positive. (This is a Fourier transform, Chapter 15.) We assume the two conditions:

- $f(z)$ is analytic in the upper half-plane except for a finite number of poles.

- $\lim_{|z|\to\infty} f(z) = 0, \qquad 0 \le \arg z \le \pi.$ (7.43)

Note that this is a less restrictive condition than the second condition imposed on $f(z)$ for integrating $\int_{-\infty}^{\infty} f(x)\,dx$ previously.

We employ the contour shown in Fig. 7.5. The application of the calculus of residues is the same as the one just considered, but here we have to work harder to show that the integral over the (infinite) semicircle goes to zero. This integral becomes

$$I_R = \int_0^{\pi} f(Re^{i\theta}) e^{iaR\cos\theta - aR\sin\theta} iRe^{i\theta}\,d\theta. \tag{7.44}$$

Let R be so large that $|f(z)| = |f(Re^{i\theta})| < \varepsilon$. Then

$$|I_R| \le \varepsilon R \int_0^{\pi} e^{-aR\sin\theta}\,d\theta = 2\varepsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\,d\theta. \tag{7.45}$$

In the range $[0, \pi/2]$,

$$\frac{2}{\pi}\theta \le \sin\theta.$$

Therefore (Fig. 7.6)

$$|I_R| \le 2\varepsilon R \int_0^{\pi/2} e^{-2aR\theta/\pi}\,d\theta. \tag{7.46}$$

Now, integrating by inspection, we obtain

$$|I_R| \le 2\varepsilon R \frac{1 - e^{-aR}}{2aR/\pi}.$$

Finally,

$$\lim_{R\to\infty} |I_R| \le \frac{\pi}{a}\varepsilon. \tag{7.47}$$

From Eq. (7.43), $\varepsilon \to 0$ as $R \to \infty$ and

$$\lim_{R\to\infty} |I_R| = 0. \tag{7.48}$$

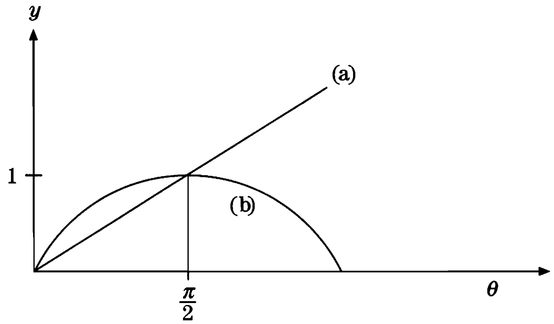

FIGURE 7.6 (a) $y = (2/\pi)\theta$, (b) $y = \sin\theta$.

This useful result is sometimes called **Jordan's lemma**. With it, we are prepared to tackle Fourier integrals of the form shown in Eq. (7.42).

Using the contour shown in Fig. 7.5, we have

$$\int_{-\infty}^{\infty} f(x)e^{iax}\,dx + \lim_{R\to\infty} I_R = 2\pi i \sum \text{residues (upper half-plane)}.$$

Since the integral over the upper semicircle I_R vanishes as $R \to \infty$ (Jordan's lemma),

$$\int_{-\infty}^{\infty} f(x)e^{iax}\,dx = 2\pi i \sum \text{residues (upper half-plane)} \qquad (a > 0). \tag{7.49}$$

Example 7.1.4 SIMPLE POLE ON CONTOUR OF INTEGRATION

The problem is to evaluate

$$I = \int_0^{\infty} \frac{\sin x}{x} dx. \tag{7.50}$$

This may be taken as the imaginary part[3] of

$$I_2 = P\int_{-\infty}^{\infty} \frac{e^{iz}dz}{z}. \tag{7.51}$$

Now the only pole is a simple pole at $z = 0$ and the residue there by Eq. (7.40) is $a_{-1} = 1$. We choose the contour shown in Fig. 7.7 (1) to avoid the pole, (2) to include the real axis, and (3) to yield a vanishingly small integrand for $z = iy$, $y \to \infty$. Note that in this case a large (infinite) semicircle in the lower half-plane would be disastrous. We have

$$\oint \frac{e^{iz}\,dz}{z} = \int_{-R}^{-r} e^{ix}\frac{dx}{x} + \int_{C_1} \frac{e^{iz}\,dz}{z} + \int_r^R \frac{e^{ix}\,dx}{x} + \int_{C_2} \frac{e^{iz}\,dz}{z} = 0, \tag{7.52}$$

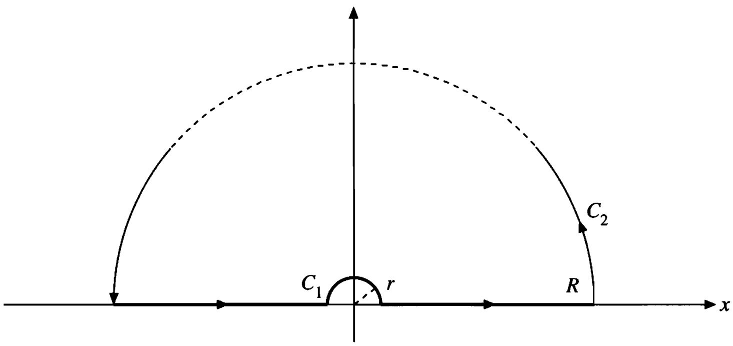

FIGURE 7.7 Singularity on contour.

[3]One can use $\int[(e^{iz} - e^{-iz})/2iz]\,dz$, but then two different contours will be needed for the two exponentials (compare Example 7.1.5)

the final zero coming from the residue theorem (Eq. (7.6)). By Jordan's lemma

$$\int_{C_2} \frac{e^{iz}\,dz}{z} = 0, \tag{7.53}$$

and

$$\oint \frac{e^{iz}\,dz}{z} = \int_{C_1} \frac{e^{iz}\,dz}{z} + P\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x} = 0. \tag{7.54}$$

The integral over the small semicircle yields $(-)\pi i$ times the residue of 1, and minus, as a result of going clockwise. Taking the imaginary part,[4] we have

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi \tag{7.55}$$

or

$$\int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \tag{7.56}$$

The contour of Fig. 7.7, although convenient, is not at all unique. Another choice of contour for evaluating Eq. (7.50) is presented as Exercise 7.1.15. ■

Example 7.1.5 QUANTUM MECHANICAL SCATTERING

The quantum mechanical analysis of scattering leads to the function

$$I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x\,dx}{x^2 - \sigma^2}, \tag{7.57}$$

where σ is real and positive. This integral is divergent and therefore ambiguous. From the physical conditions of the problem there is a further requirement: $I(\sigma)$ is to have the form $e^{i\sigma}$ so that it will represent an outgoing scattered wave.

Using

$$\sin z = \frac{1}{i}\sinh iz = \frac{1}{2i}e^{iz} - \frac{1}{2i}e^{-iz}, \tag{7.58}$$

we write Eq. (7.57) in the complex plane as

$$I(\sigma) = I_1 + I_2, \tag{7.59}$$

with

$$I_1 = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 - \sigma^2}\,dz,$$

$$I_2 = -\frac{1}{2i}\int_{-\infty}^{\infty} \frac{ze^{-iz}}{z^2 - \sigma^2}\,dz. \tag{7.60}$$

[4]Alternatively, we may combine the integrals of Eq. (7.52) as

$$\int_{-R}^{-r} e^{ix}\frac{dx}{x} + \int_{r}^{R} e^{ix}\frac{dx}{x} = \int_{r}^{R} \left(e^{ix} - e^{-ix}\right)\frac{dx}{x} = 2i\int_{r}^{R} \frac{\sin x}{x}\,dx.$$

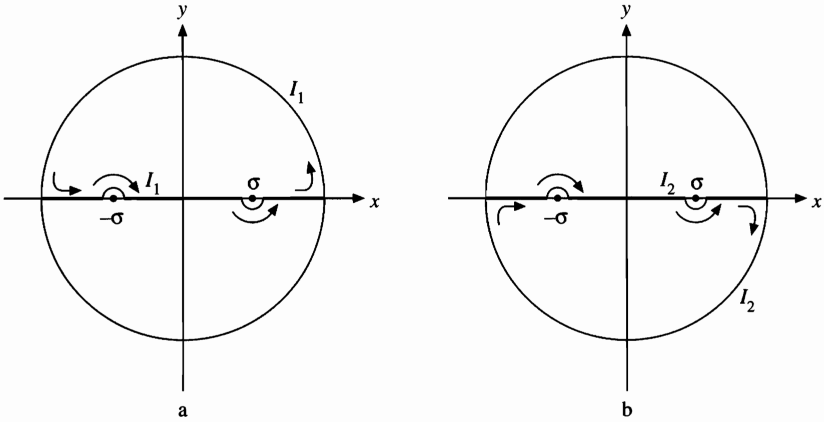

FIGURE 7.8 Contours.

Integral I_1 is similar to Example 7.1.4 and, as in that case, we may complete the contour by an infinite semicircle in the upper half-plane, as shown in Fig. 7.8a. For I_2 the exponential is negative and we complete the contour by an infinite semicircle in the lower half-plane, as shown in Fig. 7.8b. As in Example 7.1.4, neither semicircle contributes anything to the integral — Jordan's lemma.

There is still the problem of locating the poles and evaluating the residues. We find poles at $z = +\sigma$ and $z = -\sigma$ **on the contour of integration**. The residues are (Exercises 6.6.1 and 7.1.1)

	$z = \sigma$	$z = -\sigma$
I_1	$\frac{e^{i\sigma}}{2}$	$\frac{e^{-i\sigma}}{2}$
I_2	$\frac{e^{-i\sigma}}{2}$	$\frac{e^{i\sigma}}{2}$

Detouring around the poles, as shown in Fig. 7.8 (it matters little whether we go above or below), we find that the residue theorem leads to

$$PI_1 - \pi i\left(\frac{1}{2i}\right)\frac{e^{-i\sigma}}{2} + \pi i\left(\frac{1}{2i}\right)\frac{e^{i\sigma}}{2} = 2\pi i\left(\frac{1}{2i}\right)\frac{e^{i\sigma}}{2}, \tag{7.61}$$

for we have enclosed the singularity at $z = \sigma$ but excluded the one at $z = -\sigma$. In similar fashion, but noting that the contour for I_2 is clockwise,

$$PI_2 - \pi i\left(\frac{-1}{2i}\right)\frac{e^{i\sigma}}{2} + \pi i\left(\frac{-1}{2i}\right)\frac{e^{-i\sigma}}{2} = -2\pi i\left(\frac{-1}{2i}\right)\frac{e^{i\sigma}}{2}. \tag{7.62}$$

Adding Eqs. (7.61) and (7.62), we have

$$PI(\sigma) = PI_1 + PI_2 = \frac{\pi}{2}\left(e^{i\sigma} + e^{-i\sigma}\right) = \pi \cosh i\sigma = \pi \cos\sigma. \tag{7.63}$$

This is a perfectly good evaluation of Eq. (7.57), but unfortunately the cosine dependence is appropriate for a standing wave and not for the outgoing scattered wave as specified.

To obtain the desired form, we try a different technique (compare Example 7.1.1). Instead of dodging around the singular points, let us move them off the real axis. Specifically, let $\sigma \to \sigma + i\gamma$, $-\sigma \to -\sigma - i\gamma$, where γ is positive but small and will eventually be made to approach zero; that is, for I_1 we include one pole and for I_2 the other one,

$$I_+(\sigma) = \lim_{\gamma \to 0} I(\sigma + i\gamma). \tag{7.64}$$

With this simple substitution, the first integral I_1 becomes

$$I_1(\sigma + i\gamma) = 2\pi i \left(\frac{1}{2i}\right) \frac{e^{i(\sigma + i\gamma)}}{2} \tag{7.65}$$

by direct application of the residue theorem. Also,

$$I_2(\sigma + i\gamma) = -2\pi i \left(\frac{-1}{2i}\right) \frac{e^{i(\sigma + i\gamma)}}{2}. \tag{7.66}$$

Adding Eqs. (7.65) and (7.66) and then letting $\gamma \to 0$, we obtain

$$\begin{aligned} I_+(\sigma) &= \lim_{\gamma \to 0} \left[I_1(\sigma + i\gamma) + I_2(\sigma + i\gamma) \right] \\ &= \lim_{\gamma \to 0} \pi e^{i(\sigma + i\gamma)} = \pi e^{i\sigma}, \end{aligned} \tag{7.67}$$

a result that does fit the boundary conditions of our scattering problem.

It is interesting to note that the substitution $\sigma \to \sigma - i\gamma$ would have led to

$$I_-(\sigma) = \pi e^{-i\sigma}, \tag{7.68}$$

which could represent an incoming wave. Our earlier result (Eq. (7.63)) is seen to be the arithmetic average of Eqs. (7.67) and (7.68). This average is the Cauchy principal value of the integral. Note that we have these possibilities (Eqs. (7.63), (7.67), and (7.68)) because our integral is not uniquely defined until we specify the particular limiting process (or average) to be used. ■

Evaluation of Definite Integrals: Exponential Forms

With exponential or hyperbolic functions present in the integrand, life gets somewhat more complicated than before. Instead of a general overall prescription, the contour must be chosen to fit the specific integral. These cases are also opportunities to illustrate the versatility and power of contour integration.

As an example, we consider an integral that will be quite useful in developing a relation between $\Gamma(1 + z)$ and $\Gamma(1 - z)$. Notice how the periodicity along the imaginary axis is exploited.

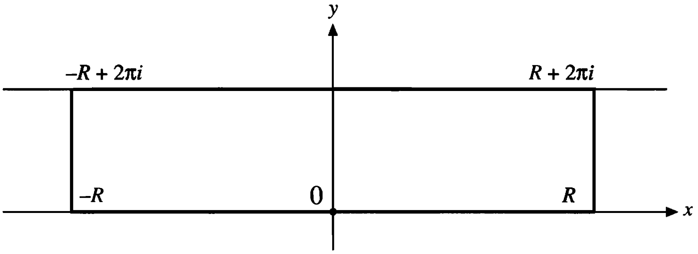

FIGURE 7.9 Rectangular contour.

Example 7.1.6 FACTORIAL FUNCTION

We wish to evaluate

$$I=\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}dx, \qquad 0<a<1. \tag{7.69}$$

The limits on a are sufficient (but not necessary) to prevent the integral from diverging as $x \to \pm\infty$. This integral (Eq. (7.69)) may be handled by replacing the real variable x by the complex variable z and integrating around the contour shown in Fig. 7.9. If we take the limit as $R \to \infty$, the real axis, of course, leads to the integral we want. The return path along $y = 2\pi$ is chosen to leave the denominator of the integral invariant, at the same time introducing a constant factor $e^{i2\pi a}$ in the numerator. We have, in the complex plane,

$$\oint \frac{e^{az}}{1+e^{z}}dz = \lim_{R\to\infty}\left(\int_{-R}^{R}\frac{e^{ax}}{1+e^{x}}dx - e^{i2\pi a}\int_{-R}^{R}\frac{e^{ax}}{1+e^{x}}dx\right)$$
$$=\left(1-e^{i2\pi a}\right)\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}dx. \tag{7.70}$$

In addition there are two vertical sections $(0 \le y \le 2\pi)$, which vanish (exponentially) as $R \to \infty$.

Now where are the poles and what are the residues? We have a pole when

$$e^{z}=e^{x}e^{iy}=-1. \tag{7.71}$$

Equation (7.71) is satisfied at $z = 0 + i\pi$. By a Laurent expansion[5] in powers of $(z - i\pi)$ the pole is seen to be a simple pole with a residue of $-e^{i\pi a}$. Then, applying the residue theorem,

$$\left(1-e^{i2\pi a}\right)\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}dx = 2\pi i\left(-e^{i\pi a}\right). \tag{7.72}$$

This quickly reduces to

$$\int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^{x}}dx=\frac{\pi}{\sin a\pi}, \qquad 0<a<1. \tag{7.73}$$

[5] $1+e^{z}=1+e^{z-i\pi}e^{i\pi}=1-e^{z-i\pi}=-(z-i\pi)(1+\frac{z-i\pi}{2!}+\frac{(z-i\pi)^2}{3!}+\cdots)$.

Using the beta function (Section 8.4), we can show the integral to be equal to the product $\Gamma(a)\Gamma(1-a)$. This results in the interesting and useful factorial function relation

$$\Gamma(a+1)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}. \tag{7.74}$$

Although Eq. (7.73) holds for real $a, 0 < a < 1$, Eq. (7.74) may be extended by analytic continuation to all values of a, real and complex, excluding only real integral values. ■

As a final example of contour integrals of exponential functions, we consider Bernoulli numbers again.

Example 7.1.7 BERNOULLI NUMBERS

In Section 5.9 the Bernoulli numbers were defined by the expansion

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n. \tag{7.75}$$

Replacing x with z (analytic continuation), we have a Taylor series (compare Eq. (6.47)) with

$$B_n = \frac{n!}{2\pi i} \oint_{C_0} \frac{z}{e^z - 1} \frac{dz}{z^{n+1}}, \tag{7.76}$$

where the contour C_0 is around the origin counterclockwise with $|z| < 2\pi$ to avoid the poles at $2\pi i n$.

For $n = 0$ we have a simple pole at $z = 0$ with a residue of $+1$. Hence by Eq. (7.25),

$$B_0 = \frac{0!}{2\pi i} \cdot 2\pi i(1) = 1. \tag{7.77}$$

For $n = 1$ the singularity at $z = 0$ becomes a second-order pole. The residue may be shown to be $-\frac{1}{2}$ by series expansion of the exponential, followed by a binomial expansion. This results in

$$B_1 = \frac{1!}{2\pi i} \cdot 2\pi i\left(-\frac{1}{2}\right) = -\frac{1}{2}. \tag{7.78}$$

For $n \geq 2$ this procedure becomes rather tedious, and we resort to a different means of evaluating Eq. (7.76). The contour is deformed, as shown in Fig. 7.10.

The new contour C still encircles the origin, as required, but now it also encircles (in a negative direction) an infinite series of singular points along the imaginary axis at $z = \pm p2\pi i$, $p = 1, 2, 3, \ldots$. The integration back and forth along the x-axis cancels out, and for $R \to \infty$ the integration over the infinite circle yields zero. Remember that $n \geq 2$. Therefore

$$\oint_{C_0} \frac{z}{e^z - 1} \frac{dz}{z^{n+1}} = -2\pi i \sum_{p=1}^{\infty} \text{residues} \qquad (z = \pm p2\pi i). \tag{7.79}$$

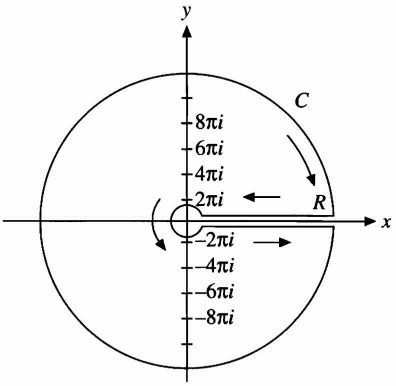

FIGURE 7.10 Contour of integration for Bernoulli numbers.

At $z = p2\pi i$ we have a simple pole with a residue $(p2\pi i)^{-n}$. When n is odd, the residue from $z = p2\pi i$ exactly cancels that from $z = -p2\pi i$ and $B_n = 0, n = 3, 5, 7$, and so on. For n even the residues add, giving

$$B_n = \frac{n!}{2\pi i}(-2\pi i)2\sum_{p=1}^{\infty}\frac{1}{p^n(2\pi i)^n}$$

$$= -\frac{(-1)^{n/2}2n!}{(2\pi)^n}\sum_{p=1}^{\infty}p^{-n} = -\frac{(-1)^{n/2}2n!}{(2\pi)^n}\zeta(n) \qquad (n \text{ even}), \qquad (7.80)$$

where $\zeta(n)$ is the Riemann zeta function introduced in Section 5.9. Equation (7.80) corresponds to Eq. (5.152) of Section 5.9. ■

Exercises

7.1.1 Determine the nature of the singularities of each of the following functions and evaluate the residues $(a > 0)$.

(a) $\dfrac{1}{z^2 + a^2}$. (b) $\dfrac{1}{(z^2 + a^2)^2}$.

(c) $\dfrac{z^2}{(z^2 + a^2)^2}$. (d) $\dfrac{\sin 1/z}{z^2 + a^2}$.

(e) $\dfrac{ze^{+iz}}{z^2 + a^2}$. (f) $\dfrac{ze^{+iz}}{z^2 - a^2}$.

(g) $\dfrac{e^{+iz}}{z^2 - a^2}$. (h) $\dfrac{z^{-k}}{z + 1}$, $0 < k < 1$.

Hint. For the point at infinity, use the transformation $w = 1/z$ for $|z| \to 0$. For the residue, transform $f(z)\,dz$ into $g(w)\,dw$ and look at the behavior of $g(w)$.

7.1.2 Locate the singularities and evaluate the residues of each of the following functions.

(a) $z^{-n}(e^z - 1)^{-1}, \qquad z \neq 0,$

(b) $\dfrac{z^2 e^z}{1 + e^{2z}}.$

(c) Find a closed-form expression (that is, not a sum) for the sum of the finite-plane singularities.

(d) Using the result in part (c), what is the residue at $|z| \to \infty$?

Hint. See Section 5.9 for expressions involving Bernoulli numbers. Note that Eq. (5.144) cannot be used to investigate the singularity at $z \to \infty$, since this series is only valid for $|z| < 2\pi$.

7.1.3 The statement that the integral halfway around a singular point is equal to one-half the integral all the way around was limited to simple poles. Show, by a specific example, that

$$\int_{\text{Semicircle}} f(z)\,dz = \frac{1}{2}\oint_{\text{Circle}} f(z)\,dz$$

does not necessarily hold if the integral encircles a pole of higher order.
Hint. Try $f(z) = z^{-2}$.

7.1.4 A function $f(z)$ is analytic along the real axis except for a third-order pole at $z = x_0$. The Laurent expansion about $z = x_0$ has the form

$$f(z) = \frac{a_{-3}}{(z - x_0)^3} + \frac{a_{-1}}{z - x_0} + g(z),$$

with $g(z)$ analytic at $z = x_0$. Show that the Cauchy principal value technique is applicable, in the sense that

(a) $\lim_{\delta \to 0}\left\{\int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{x_0+\delta}^{\infty} f(x)\,dx\right\}$ is finite.

(b) $\int_{C_{x_0}} f(z)\,dz = \pm i\pi a_{-1},$
where C_{x_0} denotes a **small semicircle** about $z = x_0$.

7.1.5 The unit step function is defined as (compare Exercise 1.15.13)

$$u(s - a) = \begin{cases} 0, & s < a \\ 1, & s > a. \end{cases}$$

Show that $u(s)$ has the integral representations

(a) $u(s) = \lim_{\varepsilon \to 0^+} \dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ixs}}{x - i\varepsilon}\,dx,$

(b) $u(s) = \frac{1}{2} + \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \frac{e^{ixs}}{x} dx.$

Note. The parameter s is real.

7.1.6 Most of the special functions of mathematical physics may be generated (defined) by a generating function of the form

$$g(t,x) = \sum_n f_n(x) t^n.$$

Given the following integral representations, derive the corresponding generating function:

(a) Bessel:

$$J_n(x) = \frac{1}{2\pi i} \oint e^{(x/2)(t-1/t)} t^{-n-1}\, dt.$$

(b) Modified Bessel:

$$I_n(x) = \frac{1}{2\pi i} \oint e^{(x/2)(t+1/t)} t^{-n-1}\, dt.$$

(c) Legendre:

$$P_n(x) = \frac{1}{2\pi i} \oint \left(1 - 2tx + t^2\right)^{-1/2} t^{-n-1}\, dt.$$

(d) Hermite:

$$H_n(x) = \frac{n!}{2\pi i} \oint e^{-t^2+2tx} t^{-n-1}\, dt.$$

(e) Laguerre:

$$L_n(x) = \frac{1}{2\pi i} \oint \frac{e^{-xt/(1-t)}}{(1-t)t^{n+1}} dt.$$

(f) Chebyshev:

$$T_n(x) = \frac{1}{4\pi i} \oint \frac{(1-t^2)t^{-n-1}}{(1-2tx+t^2)} dt.$$

Each of the contours encircles the origin and no other singular points.

7.1.7 Generalizing Example 7.1.2, show that

$$\int_0^{2\pi} \frac{d\theta}{a \pm b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a \pm b\sin\theta} = \frac{2\pi}{(a^2-b^2)^{1/2}}, \qquad \text{for } a > |b|.$$

What happens if $|b| > |a|$?

7.1.8 Show that

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{3/2}}, \qquad a > 1.$$

7.1.9 Show that

$$\int_0^{2\pi} \frac{d\theta}{1 - 2t\cos\theta + t^2} = \frac{2\pi}{1 - t^2}, \qquad \text{for } |t| < 1.$$

What happens if $|t| > 1$? What happens if $|t| = 1$?

7.1.10 With the calculus of residues show that

$$\int_0^{\pi} \cos^{2n}\theta\, d\theta = \pi \frac{(2n)!}{2^{2n}(n!)^2} = \pi \frac{(2n-1)!!}{(2n)!!}, \qquad n = 0, 1, 2, \ldots.$$

(The double factorial notation is defined in Section 8.1.)
Hint. $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1}), \qquad |z| = 1.$

7.1.11 Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos bx - \cos ax}{x^2} dx, \qquad a > b > 0.$$

ANS. $\pi(a - b)$.

7.1.12 Prove that

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}.$$

Hint. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.

7.1.13 A quantum mechanical calculation of a transition probability leads to the function $f(t, \omega) = 2(1 - \cos\omega t)/\omega^2$. Show that

$$\int_{-\infty}^{\infty} f(t, \omega)d\omega = 2\pi t.$$

7.1.14 Show that $(a > 0)$

(a) $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx = \frac{\pi}{a} e^{-a}.$$

How is the right side modified if $\cos x$ is replaced by $\cos kx$?

(b) $$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2 + a^2} dx = \pi e^{-a}.$$

How is the right side modified if $\sin x$ is replaced by $\sin kx$?

These integrals may also be interpreted as Fourier cosine and sine transforms—Chapter 15.

7.1.15 Use the contour shown (Fig. 7.11) with $R \to \infty$ to prove that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$

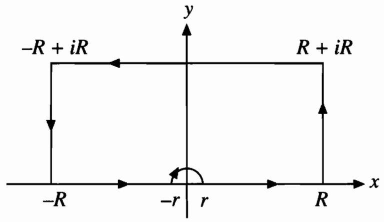

FIGURE 7.11 Large square contour.

7.1.16 In the quantum theory of atomic collisions we encounter the integral

$$I=\int_{-\infty}^{\infty}\frac{\sin t}{t}e^{ipt}\,dt,$$

in which p is real. Show that

$$\begin{aligned} I&=0, \qquad |p|>1\\ I&=\pi, \qquad |p|<1. \end{aligned}$$

What happens if $p=\pm 1$?

7.1.17 Evaluate

$$\int_0^{\infty}\frac{(\ln x)^2}{1+x^2}dx$$

(a) by appropriate series expansion of the integrand to obtain

$$4\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-3},$$

(b) and by contour integration to obtain

$$\frac{\pi^3}{8}.$$

Hint. $x\to z=e^t$. Try the contour shown in Fig. 7.12, letting $R\to\infty$.

7.1.18 Show that

$$\int_0^{\infty}\frac{x^a}{(x+1)^2}dx=\frac{\pi a}{\sin\pi a},$$

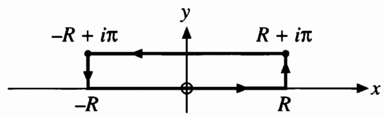

FIGURE 7.12 Small square contour.

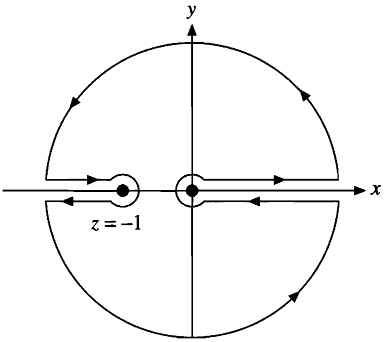

FIGURE 7.13 Contour avoiding branch point and pole.

where $-1 < a < 1$. Here is still another way of deriving Eq. (7.74).
Hint. Use the contour shown in Fig. 7.13, noting that $z = 0$ is a branch point and the positive x-axis is a cut line. Note also the comments on phases following Example 6.6.1.

7.1.19 Show that

$$\int_0^\infty \frac{x^{-a}}{x+1} dx = \frac{\pi}{\sin a\pi},$$

where $0 < a < 1$. This opens up another way of deriving the factorial function relation given by Eq. (7.74).
Hint. You have a branch point and you will need a cut line. Recall that $z^{-a} = w$ in polar form is

$$\left[re^{i(\theta+2\pi n)}\right]^{-a} = \rho e^{i\varphi},$$

which leads to $-a\theta - 2an\pi = \varphi$. You must restrict n to zero (or any other single integer) in order that φ may be uniquely specified. Try the contour shown in Fig. 7.14.

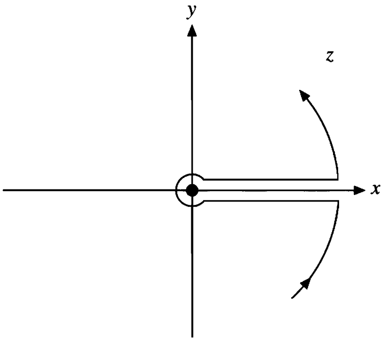

FIGURE 7.14 Alternative contour avoiding branch point.

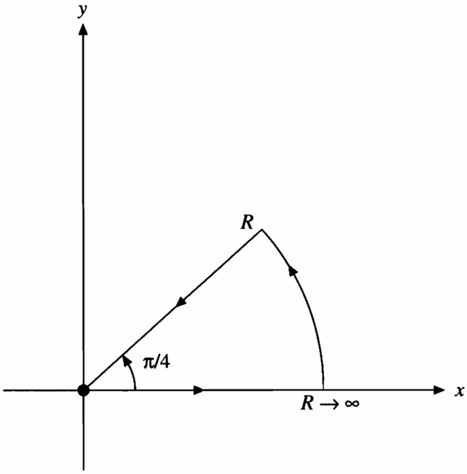

FIGURE 7.15 Angle contour.

7.1.20 Show that

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, \qquad a > 0.$$

7.1.21 Evaluate

$$\int_{-\infty}^\infty \frac{x^2}{1+x^4}dx.$$

ANS. $\pi/\sqrt{2}$.

7.1.22 Show that

$$\int_0^\infty \cos(t^2)\,dt = \int_0^\infty \sin(t^2)\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Hint. Try the contour shown in Fig. 7.15.
Note. These are the Fresnel integrals for the special case of infinity as the upper limit. For the general case of a varying upper limit, asymptotic expansions of the Fresnel integrals are the topic of Exercise 5.10.2. Spherical Bessel expansions are the subject of Exercise 11.7.13.

7.1.23 Several of the Bromwich integrals, Section 15.12, involve a portion that may be approximated by

$$I(y) = \int_{a-iy}^{a+iy} \frac{e^{zt}}{z^{1/2}}dz.$$

Here a and t are positive and finite. Show that

$$\lim_{y\to\infty} I(y) = 0.$$

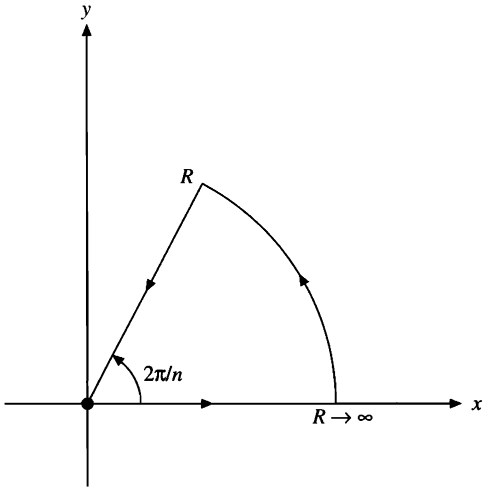

FIGURE 7.16 Sector contour.

7.1.24 Show that

$$\int_0^\infty \frac{1}{1+x^n}dx = \frac{\pi/n}{\sin(\pi/n)}.$$

Hint. Try the contour shown in Fig. 7.16.

7.1.25 (a) Show that

$$f(z) = z^4 - 2\cos 2\theta z^2 + 1$$

has zeros at $e^{i\theta}, e^{-i\theta}, -e^{i\theta}$, and $-e^{-i\theta}$.

(b) Show that

$$\int_{-\infty}^\infty \frac{dx}{x^4 - 2\cos 2\theta x^2 + 1} = \frac{\pi}{2\sin\theta} = \frac{\pi}{2^{1/2}(1-\cos 2\theta)^{1/2}}.$$

Exercise 7.1.24 ($n = 4$) is a special case of this result.

7.1.26 Show that

$$\int_{-\infty}^\infty \frac{x^2\,dx}{x^4 - 2\cos 2\theta x^2 + 1} = \frac{\pi}{2\sin\theta} = \frac{\pi}{2^{1/2}(1-\cos 2\theta)^{1/2}}.$$

Exercise 7.1.21 is a special case of this result.

7.1.27 Apply the techniques of Example 7.1.5 to the evaluation of the improper integral

$$I = \int_{-\infty}^\infty \frac{dx}{x^2 - \sigma^2}.$$

(a) Let $\sigma \to \sigma + i\gamma$.
(b) Let $\sigma \to \sigma - i\gamma$.
(c) Take the Cauchy principal value.

7.1.28 The integral in Exercise 7.1.17 may be transformed into

$$\int_0^\infty e^{-y}\frac{y^2}{1+e^{-2y}}dy = \frac{\pi^3}{16}.$$

Evaluate this integral by the Gauss–Laguerre quadrature and compare your result with $\pi^3/16$.

ANS. Integral = 1.93775 (10 points).

7.2 DISPERSION RELATIONS

The concept of dispersion relations entered physics with the work of Kronig and Kramers in optics. The name **dispersion** comes from optical dispersion, a result of the dependence of the index of refraction on wavelength, or angular frequency. The index of refraction n may have a real part determined by the phase velocity and a (negative) imaginary part determined by the absorption—see Eq. (7.94). Kronig and Kramers showed in 1926–1927 that the real part of $(n^2 - 1)$ could be expressed as an integral of the imaginary part. Generalizing this, we shall apply the label **dispersion relations** to any pair of equations giving the real part of a function as an integral of its imaginary part and the imaginary part as an integral of its real part—Eqs. (7.86a) and (7.86b), which follow. The existence of such integral relations might be suspected as an integral analog of the Cauchy–Riemann differential relations, Section 6.2.

The applications in modern physics are widespread. For instance, the real part of the function might describe the forward scattering of a gamma ray in a nuclear Coulomb field (a dispersive process). Then the imaginary part would describe the electron–positron pair production in that same Coulomb field (the absorptive process). As will be seen later, the dispersion relations may be taken as a consequence of causality and therefore are independent of the details of the particular interaction.

We consider a complex function $f(z)$ that is analytic in the upper half-plane and on the real axis. We also require that

$$\lim_{|z|\to\infty} |f(z)| = 0, \qquad 0 \le \arg z \le \pi, \tag{7.81}$$

in order that the integral over an infinite semicircle will vanish. The point of these conditions is that we may express $f(z)$ by the Cauchy integral formula, Eq. (6.43),

$$f(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{z - z_0}dz. \tag{7.82}$$

The integral over the upper semicircle[6] vanishes and we have

$$f(z_0) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(x)}{x - z_0}dx. \tag{7.83}$$

The integral over the contour shown in Fig. 7.17 has become an integral along the x-axis.

Equation (7.83) assumes that z_0 is in the upper half-plane—interior to the closed contour. If z_0 were in the lower half-plane, the integral would yield zero by the Cauchy integral

[6]The use of a semicircle to close the path of integration is convenient, not mandatory. Other paths are possible.

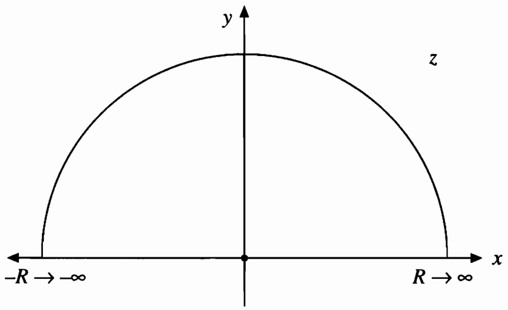

FIGURE 7.17 Semicircle contour.

theorem, Section 6.3. Now, either letting z_0 approach the real axis from above $(z_0 - x_0)$ or placing it on the real axis and taking an average of Eq. (7.83) and zero, we find that Eq. (7.83) becomes

$$f(x_0) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x - x_0} dx, \tag{7.84}$$

where P indicates the Cauchy principal value. Splitting Eq. (7.84) into real and imaginary parts[7] yields

$$\begin{aligned} f(x_0) &= u(x_0) + i v(x_0) \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)}{x - x_0} dx - \frac{i}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)}{x - x_0} dx. \end{aligned} \tag{7.85}$$

Finally, equating real part to real part and imaginary part to imaginary part, we obtain

$$u(x_0) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{v(x)}{x - x_0} dx \tag{7.86a}$$

$$v(x_0) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)}{x - x_0} dx. \tag{7.86b}$$

These are the dispersion relations. The real part of our complex function is expressed as an integral over the imaginary part. The imaginary part is expressed as an integral over the real part. The real and imaginary parts are **Hilbert transforms** of each other. Note that these relations are meaningful only when $f(x)$ is a complex function of the real variable x. Compare Exercise 7.2.1.

From a physical point of view $u(x)$ and/or $v(x)$ represent some physical measurements. Then $f(z) = u(z) + iv(z)$ is an analytic continuation over the upper half-plane, with the value on the real axis serving as a boundary condition.

[7]The second argument, $y = 0$, is dropped: $u(x_0, 0) \to u(x_0)$.

Symmetry Relations

On occasion $f(x)$ will satisfy a symmetry relation and the integral from $-\infty$ to $+\infty$ may be replaced by an integral over positive values only. This is of considerable physical importance because the variable x might represent a frequency and only zero and positive frequencies are available for physical measurements. Suppose[8]

$$f(-x) = f^*(x). \tag{7.87}$$

Then

$$u(-x) + iv(-x) = u(x) - iv(x). \tag{7.88}$$

The real part of $f(x)$ is even and the imaginary part is odd.[9] In quantum mechanical scattering problems these relations (Eq. (7.88)) are called crossing conditions. To exploit these **crossing conditions**, we rewrite Eq. (7.86a) as

$$u(x_0) = \frac{1}{\pi} P \int_{-\infty}^{0} \frac{v(x)}{x - x_0} dx + \frac{1}{\pi} P \int_{0}^{\infty} \frac{v(x)}{x - x_0} dx. \tag{7.89}$$

Letting $x \to -x$ in the first integral on the right-hand side of Eq. (7.89) and substituting $v(-x) = -v(x)$ from Eq. (7.88), we obtain

$$\begin{aligned} u(x_0) &= \frac{1}{\pi} P \int_{0}^{\infty} v(x) \left\{ \frac{1}{x + x_0} + \frac{1}{x - x_0} \right\} dx \\ &= \frac{2}{\pi} P \int_{0}^{\infty} \frac{x v(x)}{x^2 - x_0^2} dx. \end{aligned} \tag{7.90}$$

Similarly,

$$v(x_0) = -\frac{2}{\pi} P \int_{0}^{\infty} \frac{x_0 u(x)}{x^2 - x_0^2} dx. \tag{7.91}$$

The original Kronig–Kramers optical dispersion relations were in this form. The asymptotic behavior ($x_0 \to \infty$) of Eqs. (7.90) and (7.91) lead to quantum mechanical **sum rules**, Exercise 7.2.4.

Optical Dispersion

The function $\exp[i(kx - \omega t)]$ describes an electromagnetic wave moving along the x-axis in the positive direction with velocity $v = \omega/k$; ω is the angular frequency, k the wave number or propagation vector, and $n = ck/\omega$ the index of refraction. From Maxwell's

[8] This is not just a happy coincidence. It ensures that the Fourier transform of $f(x)$ will be real. In turn, Eq. (7.87) is a consequence of obtaining $f(x)$ as the Fourier transform of a real function.

[9] $u(x, 0) = u(-x, 0)$, $v(x, 0) = -v(-x, 0)$. Compare these symmetry conditions with those that follow from the Schwarz reflection principle, Section 6.5.

equations, electric permittivity ε, and Ohm's law with conductivity σ, the propagation vector k for a dielectric becomes[10]

$$k^2 = \varepsilon \frac{\omega^2}{c^2}\left(1 + i\frac{4\pi\sigma}{\omega\varepsilon}\right) \tag{7.92}$$

(with μ, the magnetic permeability, taken to be unity). The presence of the conductivity (which means absorption) gives rise to an imaginary part. The propagation vector k (and therefore the index of refraction n) have become complex.

Conversely, the (positive) imaginary part implies absorption. For poor conductivity ($4\pi\sigma/\omega\varepsilon \ll 1$) a binomial expansion yields

$$k = \sqrt{\varepsilon}\frac{\omega}{c} + i\frac{2\pi\sigma}{c\sqrt{\varepsilon}}$$

and

$$e^{i(kx-\omega t)} = e^{i\omega(x\sqrt{\varepsilon}/c-t)}e^{-2\pi\sigma x/c\sqrt{\varepsilon}},$$

an attenuated wave.

Returning to the general expression for k^2, Eq. (7.92), we find that the index of refraction becomes

$$n^2 = \frac{c^2k^2}{\omega^2} = \varepsilon + i\frac{4\pi\sigma}{\omega}. \tag{7.93}$$

We take n^2 to be a function of the **complex** variable ω (with ε and σ depending on ω). However, n^2 does not vanish as $\omega \to \infty$ but instead approaches unity. So to satisfy the condition, Eq. (7.81), one works with $f(\omega) = n^2(\omega) - 1$. The original Kronig–Kramers optical dispersion relations were in the form of

$$\begin{aligned}\Re\left[n^2(\omega_0) - 1\right] &= \frac{2}{\pi}P\int_0^\infty \frac{\omega\Im[n^2(\omega) - 1]}{\omega^2 - \omega_0^2}d\omega,\\ \Im\left[n^2(\omega_0) - 1\right] &= -\frac{2}{\pi}P\int_0^\infty \frac{\omega_0\Re[n^2(\omega) - 1]}{\omega^2 - \omega_0^2}d\omega.\end{aligned} \tag{7.94}$$

Knowledge of the absorption coefficient at all frequencies specifies the real part of the index of refraction, and vice versa.

The Parseval Relation

When the functions $u(x)$ and $v(x)$ are Hilbert transforms of each other (given by Eqs. (7.86)) and each is square integrable,[11] the two functions are related by

$$\int_{-\infty}^{\infty} |u(x)|^2\,dx = \int_{-\infty}^{\infty} |v(x)|^2\,dx. \tag{7.95}$$

[10] See J. D. Jackson, *Classical Electrodynamics*, 3rd ed. New York: Wiley (1999), Sections 7.7 and 7.10. Equation (7.92) is in Gaussian units.

[11] This means that $\int_{-\infty}^{\infty} |u(x)|^2\,dx$ and $\int_{-\infty}^{\infty} |v(x)|^2\,dx$ are finite.

This is the Parseval relation.

To derive Eq. (7.95), we start with

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx = \int_{-\infty}^{\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(s)\, ds}{s-x} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(t)\, dt}{t-x} dx,$$

using Eq. (7.86a) twice. Integrating first with respect to x, we have

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx = \int_{-\infty}^{\infty} v(s)\, ds \int_{-\infty}^{\infty} \frac{v(t)\, dt}{\pi^2} \int_{-\infty}^{\infty} \frac{dx}{(s-x)(t-x)}. \tag{7.96}$$

From Exercise 7.2.8, the x integration yields a delta function:

$$\frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dx}{(s-x)(t-x)} = \delta(s-t).$$

We have

$$\int_{-\infty}^{\infty} |u(x)|^2 \, dx = \int_{-\infty}^{\infty} v(t)\, dt \int_{-\infty}^{\infty} v(s)\delta(s-t)\, ds. \tag{7.97}$$

Then the s integration is carried out by inspection, using the defining property of the delta function:

$$\int_{-\infty}^{\infty} v(s)\delta(s-t)\, ds = v(t). \tag{7.98}$$

Substituting Eq. (7.98) into Eq. (7.97), we have Eq. (7.95), the Parseval relation. Again, in terms of optics, the presence of refraction over some frequency range ($n \neq 1$) implies the existence of absorption, and vice versa.

Causality

The real significance of dispersion relations in physics is that they are a direct consequence of assuming that the particular physical system obeys causality. Causality is awkward to define precisely, but the general meaning is that the effect cannot precede the cause. A scattered wave cannot be emitted by the scattering center before the incident wave has arrived. For linear systems the most general relation between an input function G (the cause) and an output function H (the effect) may be written as

$$H(t) = \int_{-\infty}^{\infty} F(t-t')G(t')\, dt'. \tag{7.99}$$

Causality is imposed by requiring that

$$F(t-t') = 0 \qquad \text{for } t-t' < 0.$$

Equation (7.99) gives the time dependence. The frequency dependence is obtained by taking Fourier transforms. By the Fourier convolution theorem, Section 15.5,

$$h(\omega) = f(\omega)g(\omega),$$

where $f(\omega)$ is the Fourier transform of $F(t)$, and so on. Conversely, $F(t)$ is the Fourier transform of $f(\omega)$.

The connection with the dispersion relations is provided by the Titchmarsh **theorem.**[12] This states that if $f(\omega)$ is square integrable over the real ω-axis, then any one of the following three statements implies the other two.

1. The Fourier transform of $f(\omega)$ is zero for $t < 0$: Eq. (7.99).
2. Replacing ω by z, the function $f(z)$ is analytic in the complex z-plane for $y > 0$ and approaches $f(x)$ almost everywhere as $y \to 0$. Further,
$$\int_{-\infty}^{\infty} \left| f(x+iy) \right|^2 dx < K \qquad \text{for } y > 0;$$
that is, the integral is bounded.
3. The real and imaginary parts of $f(z)$ are Hilbert transforms of each other: Eqs. (7.86a) and (7.86b).

The assumption that the relationship between the input and the output of our linear system is causal (Eq. (7.99)) means that the first statement is satisfied. If $f(\omega)$ is square integrable, then the Titchmarsh theorem has the third statement as a consequence and we have dispersion relations.

Exercises

7.2.1 The function $f(z)$ satisfies the conditions for the dispersion relations. In addition, $f(z) = f^*(z^*)$, the Schwarz reflection principle, Section 6.5. Show that $f(z)$ is identically zero.

7.2.2 For $f(z)$ such that we may replace the closed contour of the Cauchy integral formula by an integral over the real axis we have
$$f(x_0) = \frac{1}{2\pi i}\left\{\int_{-\infty}^{x_0-\delta} \frac{f(x)}{x-x_0}dx + \int_{x_0+\delta}^{\infty} \frac{f(x)}{x-x_0}dx\right\} + \frac{1}{2\pi i}\int_{C_{x_0}} \frac{f(x)}{x-x_0}dx.$$
Here C_{x_0} designates a small semicircle about x_0 in the lower half-plane. Show that this reduces to
$$f(x_0) = \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{f(x)}{x-x_0}dx,$$
which is Eq. (7.84).

7.2.3 (a) For $f(z) = e^{iz}$, Eq. (7.81) does not hold at the endpoints, $\arg z = 0, \pi$. Show, with the help of Jordan's lemma, Section 7.1, that Eq. (7.82) still holds.
(b) For $f(z) = e^{iz}$ verify the dispersion relations, Eq. (7.89) or Eqs. (7.90) and (7.91), by direct integration.

7.2.4 With $f(x) = u(x) + iv(x)$ and $f(x) = f^*(-x)$, show that as $x_0 \to \infty$,

[12]Refer to E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd ed. New York: Oxford University Press (1937). For a more informal discussion of the Titchmarsh theorem and further details on causality see J. Hilgevoord, *Dispersion Relations and Causal Description*. Amsterdam: North-Holland (1962).

(a) $u(x_0) \sim -\frac{2}{\pi x_0^2} \int_0^\infty x v(x)\, dx,$

(b) $v(x_0) \sim \frac{2}{\pi x_0} \int_0^\infty u(x)\, dx.$

In quantum mechanics relations of this form are often called **sum rules**.

7.2.5 (a) Given the integral equation

$$\frac{1}{1+x_0^2} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(x)}{x - x_0} dx,$$

use Hilbert transforms to determine $u(x_0)$.

(b) Verify that the integral equation of part (a) is satisfied.

(c) From $f(z)|_{y=0} = u(x) + i v(x)$, replace x by z and determine $f(z)$. Verify that the conditions for the Hilbert transforms are satisfied.

(d) Are the crossing conditions satisfied?

ANS. (a) $u(x_0) = \frac{x_0}{1+x_0^2}$, (c) $f(z) = (z+i)^{-1}$.

7.2.6 (a) If the real part of the complex index of refraction (squared) is constant (no optical dispersion), show that the imaginary part is zero (no absorption).

(b) Conversely, if there is absorption, show that there must be dispersion. In other words, if the imaginary part of $n^2 - 1$ is not zero, show that the real part of $n^2 - 1$ is not constant.

7.2.7 Given $u(x) = x/(x^2+1)$ and $v(x) = -1/(x^2+1)$, show by direct evaluation of each integral that

$$\int_{-\infty}^{\infty} |u(x)|^2 dx = \int_{-\infty}^{\infty} |v(x)|^2 dx.$$

ANS. $\int_{-\infty}^{\infty} |u(x)|^2 dx = \int_{-\infty}^{\infty} |v(x)|^2 dx = \frac{\pi}{2}$.

7.2.8 Take $u(x) = \delta(x)$, a delta function, and **assume** that the Hilbert transform equations hold.

(a) Show that

$$\delta(w) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dy}{y(y-w)}.$$

(b) With changes of variables $w = s - t$ and $x = s - y$, transform the δ representation of part (a) into

$$\delta(s-t) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dx}{(x-s)(s-t)}.$$

Note. The δ function is discussed in Section 1.15.

7.2.9 Show that

$$\delta(x) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dt}{t(t-x)}$$

is a valid representation of the delta function in the sense that

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0).$$

Assume that $f(x)$ satisfies the condition for the existence of a Hilbert transform.
Hint. Apply Eq. (7.84) twice.

7.3 Method of Steepest Descents

Analytic Landscape

In analyzing problems in mathematical physics, one often finds it desirable to know the behavior of a function for large values of the variable or some parameter s, that is, the asymptotic behavior of the function. Specific examples are furnished by the gamma function (Chapter 8) and various Bessel functions (Chapter 11). All these analytic functions are defined by integrals

$$I(s) = \int_C F(z, s)\,dz, \tag{7.100}$$

where F is analytic in z and depends on a real parameter s. We write $F(z)$ whenever possible.

So far we have evaluated such definite integrals of analytic functions along the real axis by deforming the path C to C' in the complex plane, so $|F|$ becomes small for all z on C'. This method succeeds as long as only isolated poles occur in the area between C and C'. The poles are taken into account by applying the residue theorem of Section 7.1. The residues give a measure of the simple poles, where $|F| \to \infty$, which usually dominate and determine the value of the integral.

The behavior of the integral in Eq. (7.100) clearly depends on the absolute value $|F|$ of the integrand. Moreover, the contours of $|F|$ often become more pronounced as s becomes large. Let us focus on a plot of $|F(x+iy)|^2 = U^2(x,y) + V^2(x,y)$, rather than the real part $\Re F = U$ and the imaginary part $\Im F = V$ separately. Such a plot of $|F|^2$ over the complex plane is called the **analytic landscape**, after Jensen, who, in 1912, proved that it has **only saddle points and troughs but no peaks**. Moreover, the troughs reach down all the way to the complex plane. In the absence of (simple) poles, **saddle points** are next in line to **dominate the integral** in Eq. (7.100). Hence the name **saddle point method**. At a saddle point the real (or imaginary) part U of F has a local maximum, which implies that

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial y} = 0,$$

and therefore by the use of the Cauchy–Riemann conditions of Section 6.2,

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0,$$

so V has a minimum, or vice versa, and $F'(z) = 0$. Jensen's theorem prevents U and V from having either a maximum or a minimum. See Fig. 7.18 for a typical shape (and Exercises 6.2.3 and 6.2.4). **Our strategy will be to choose the path C so that it runs over the saddle point, which gives the dominant contribution, and in the valleys elsewhere.** If there are several saddle points, we treat each alike, and their contributions will add to $I(s \to \infty)$.

To prove that there are no peaks, assume there is one at z_0. That is, $|F(z_0)|^2 > |F(z)|^2$ for all z of a neighborhood $|z - z_0| \le r$. If

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

is the Taylor expansion at z_0, the mean value $m(F)$ on the circle $z = z_0 + r\exp(i\varphi)$ becomes

$$\begin{aligned} m(F) &\equiv \frac{1}{2\pi}\int_0^{2\pi} \left|F\left(z_0 + re^{i\varphi}\right)\right|^2 d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{m,n=0}^{\infty} a_m^* a_n r^{m+n} e^{i(n-m)\varphi}\, d\varphi \\ &= \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \ge |a_0|^2 = \left|F(z_0)\right|^2, \end{aligned} \tag{7.101}$$

using orthogonality, $\frac{1}{2\pi}\int_0^{2\pi} \exp i(n-m)\varphi\, d\varphi = \delta_{nm}$. Since $m(F)$ is the mean value of $|F|^2$ on the circle of radius r, there must be a point z_1 on it so that $|F(z_1)|^2 \ge m(F) \ge |F(z_0)|^2$, which contradicts our assumption. Hence there can be no such peak.

Next, let us assume there is a minimum at z_0 so that $0 < |F(z_0)|^2 < |F(z)|^2$ for all z of a neighborhood of z_0. In other words, the dip in the valley does not go down to the complex plane. Then $|F(z)|^2 > 0$ and, since $1/F(z)$ is analytic there, it has a Taylor expansion and z_0 would be a peak of $1/|F(z)|^2$, which is impossible. This proves Jensen's theorem. We now turn our attention back to the integral in Eq. (7.100).

Saddle Point Method

Since each saddle point z_0 necessarily lies above the complex plane, that is, $|F(z_0)|^2 > 0$, we write F in exponential form, $e^{f(z,s)}$, in its vicinity without loss of generality. Note that having no zero in the complex plane is a characteristic property of the exponential function. Moreover, any saddle point with $F(z) = 0$ becomes a trough of $|F(z)|^2$ because $|F(z)|^2 \ge 0$. A case in point is the function z^2 at $z = 0$, where $d(z^2)/dz = 2z = 0$. Here $z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$, and $2xy$ has a saddle point at $z = 0$, and so has $x^2 - y^2$, but $|z|^4$ has a trough there.

At z_0 the tangential plane is horizontal; that is, $\frac{\partial F}{\partial z}|_{z=z_0} = 0$, or equivalently $\frac{\partial f}{\partial z}|_{z=z_0} = 0$. **This condition locates the saddle point.** Our next goal is to determine the **direction of steepest descent.** At z_0, f has a power series

$$f(z) = f(z_0) + \frac{1}{2} f''(z_0)(z - z_0)^2 + \cdots, \tag{7.102}$$

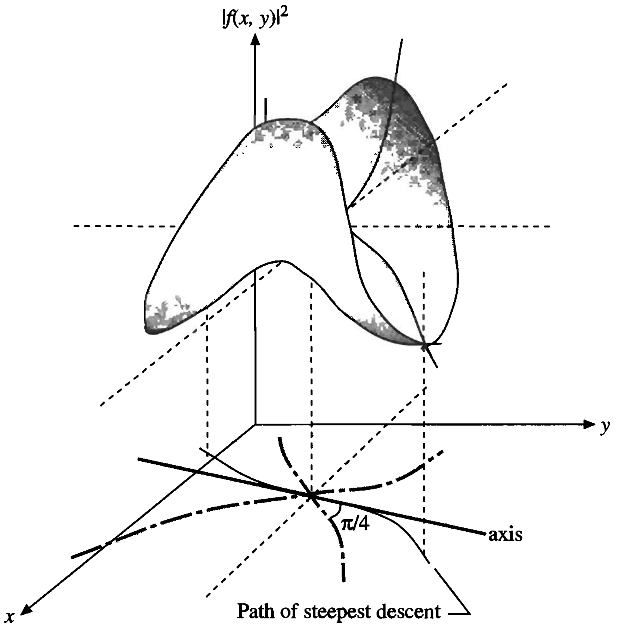

FIGURE 7.18 A saddle point.

or

$$f(z) = f(z_0) + \frac{1}{2}\big(f''(z_0) + \varepsilon\big)(z - z_0)^2, \tag{7.103}$$

upon collecting all higher powers in the (small) ε. Let us take $f''(z_0) \neq 0$ for simplicity. Then

$$f''(z_0)(z - z_0)^2 = -t^2, \qquad t \text{ real}, \tag{7.104}$$

defines a line through z_0 (saddle point **axis** in Fig. 7.18). At z_0, $t = 0$. Along the axis $\Im f''(z_0)(z - z_0)^2$ is zero and $v = \Im f(z) \approx \Im f(z_0)$ is constant if ε in Eq. (7.103) is neglected. Equation (7.104) can also be expressed in terms of angles,

$$\arg(z - z_0) = \frac{\pi}{2} - \frac{1}{2}\arg f''(z_0) = \text{constant}. \tag{7.105}$$

Since $|F(z)|^2 = \exp(2\Re f)$ varies monotonically with $\Re f$, $|F(z)|^2 \approx \exp(-t^2)$ falls off exponentially from its maximum at $t = 0$ along this axis. Hence the name **steepest descent**. The line through z_0 defined by

$$f''(z_0)(z - z_0)^2 = +t^2 \tag{7.106}$$

is orthogonal to this axis (**dashed** in Fig. 7.18), which is evident from its angle,

$$\arg(z - z_0) = -\frac{1}{2}\arg f''(z_0) = \text{constant}, \tag{7.107}$$

when compared with Eq. (7.105). Here $|F(z)|^2$ grows exponentially.

The curves $\Re f(z) = \Re f(z_0)$ go through z_0, so $\Re[(f''(z_0) + \varepsilon)(z - z_0)^2] = 0$, or $(f''(z_0) + \varepsilon)(z - z_0)^2 = it$ for real t. Expressing this in angles as

$$\arg(z - z_0) = \frac{\pi}{4} - \frac{1}{2}\arg\bigl(f''(z_0) + \varepsilon\bigr), \qquad t > 0, \tag{7.108a}$$

$$\arg(z - z_0) = -\frac{\pi}{4} - \frac{1}{2}\arg\bigl(f''(z_0) + \varepsilon\bigr), \qquad t < 0, \tag{7.108b}$$

and comparing with Eqs. (7.105) and (7.107) we note that these curves (**dot-dashed** in Fig. 7.18) divide the saddle point region into four sectors, two with $\Re f(z) > \Re f(z_0)$ (hence $|F(z)| > |F(z_0)|$), shown shaded in Fig. 7.18, and two with $\Re f(z) < \Re f(z_0)$ (hence $|F(z)| < |F(z_0)|$). They are at $\pm\frac{\pi}{4}$ angles from the axis. Thus, the integration path has to avoid the shaded areas, where $|F|$ rises. If a path is chosen to run up the slopes above the saddle point, the large imaginary part of $f(z)$ leads to rapid oscillations of $F(z) = e^{f(z)}$ and cancelling contributions to the integral.

So far, our **treatment has been general**, except for $f''(z_0) \neq 0$, which can be relaxed. Now we are ready to **specialize the integrand** F further in order to tie up the path selection with the asymptotic behavior as $s \to \infty$.

We assume that s appears linearly in the exponent, that is, we replace $\exp f(z, s) \to \exp(sf(z))$. This dependence on s ensures that the saddle point contribution at z_0 grows with $s \to \infty$ providing steep slopes, as is the case in most applications in physics. In order to account for the region far away from the saddle point that is not influenced by s, we include another analytic function, $g(z)$, which varies slowly near the saddle point and is independent of s.

Altogether, then, **our integral has the more appropriate and specific form**

$$I(s) = \int_C g(z)e^{sf(z)}\,dz. \tag{7.109}$$

The path of steepest descent is the saddle point axis when we neglect the higher-order terms, ε, in Eq. (7.103). With ε, the path of steepest descent is the curve close to the axis within the unshaded sectors, where $v = \Im f(z)$ is strictly constant, while $\Im f(z)$ is only approximately constant on the axis. We approximate $I(s)$ by the integral along the piece of the axis inside the patch in Fig. 7.18, where (compare with Eq. (7.104))

$$z = z_0 + xe^{i\alpha}, \qquad \alpha = \frac{\pi}{2} - \frac{1}{2}\arg f''(z_0), \qquad a \le x \le b. \tag{7.110}$$

We find

$$I(s) \approx e^{i\alpha}\int_a^b g\bigl(z_0 + xe^{i\alpha}\bigr)\exp\bigl[sf\bigl(z_0 + xe^{i\alpha}\bigr)\bigr]\,dx, \tag{7.111a}$$

and the omitted part is small and can be estimated because $\Re(f(z) - f(z_0))$ has an upper negative bound, $-R$ say, that depends on the size of the saddle point patch in Fig. 7.18 (that is, the values of a, b in Eq. (7.110)) that we choose. In Eq. (7.111) we use the power expansions

$$\begin{aligned} f\bigl(z_0 + xe^{i\alpha}\bigr) &= f(z_0) + \frac{1}{2}f''(z_0)e^{2i\alpha}x^2 + \cdots, \\ g\bigl(z_0 + xe^{i\alpha}\bigr) &= g(z_0) + g'(z_0)e^{i\alpha}x + \cdots, \end{aligned} \tag{7.111b}$$

and recall from Eq. (7.110) that

$$\frac{1}{2}f''(z_0)e^{2i\alpha} = -\frac{1}{2}\left|f''(z_0)\right| < 0.$$

We find for the leading term for $s \to \infty$:

$$I(s) = g(z_0)e^{sf(z_0)+i\alpha}\int_a^b e^{-\frac{1}{2}s|f''(z_0)|x^2}\,dx. \tag{7.112}$$

Since the integrand in Eq. (7.112) is essentially zero when x departs appreciably from the origin, we let $b \to \infty$ and $a \to -\infty$. The small error involved is straightforward to estimate. Noting that the remaining integral is just a Gauss error integral,

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}a^2x^2}\,dx = \frac{1}{a}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = \frac{\sqrt{2\pi}}{a},$$

we finally obtain

$$\boxed{I(s) = \frac{\sqrt{2\pi}\,g(z_0)e^{sf(z_0)}e^{i\alpha}}{|sf''(z_0)|^{1/2}},} \tag{7.113}$$

where the phase α was introduced in Eqs. (7.110) and (7.105).

A note of warning: We assumed that the only significant contribution to the integral came from the immediate vicinity of the saddle point(s) $z = z_0$. This condition must be checked for each new problem (Exercise 7.3.5).

Example 7.3.1 ASYMPTOTIC FORM OF THE HANKEL FUNCTION $H_\nu^{(1)}(s)$

In Section 11.4 it is shown that the Hankel functions, which satisfy Bessel's equation, may be defined by

$$H_\nu^{(1)}(s) = \frac{1}{\pi i}\int_{C_1,0}^{\infty e^{i\pi}} e^{(s/2)(z-1/z)}\frac{dz}{z^{\nu+1}}, \tag{7.114}$$

$$H_\nu^{(2)}(s) = \frac{1}{\pi i}\int_{C_2,\infty e^{-i\pi}}^{0} e^{(s/2)(z-1/z)}\frac{dz}{z^{\nu+1}}. \tag{7.115}$$

The contour C_1 is the curve in the upper half-plane of Fig. 7.19. The contour C_2 is in the lower half-plane. We apply the method of steepest descents to the first Hankel function, $H_\nu^{(1)}(s)$, which is conveniently in the form specified by Eq. (7.109), with $f(z)$ given by

$$f(z) = \frac{1}{2}\left(z - \frac{1}{z}\right). \tag{7.116}$$

By differentiating, we obtain

$$f'(z) = \frac{1}{2} + \frac{1}{2z^2}. \tag{7.117}$$

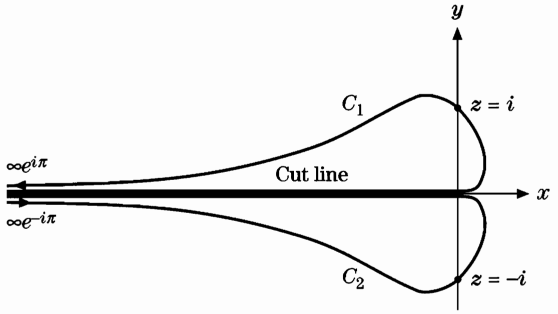

FIGURE 7.19 Hankel function contours.

Setting $f'(z) = 0$, we obtain

$$z = i, -i. \tag{7.118}$$

Hence there are saddle points at $z = +i$ and $z = -i$. At $z = i$, $f''(i) = -i$, or $\arg f''(i) = -\pi/2$, so the saddle point direction is given by Eq. (7.110) as $\alpha = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3}{4}\pi$. For the integral for $H_\nu^{(1)}(s)$ we must choose the contour through the point $z = +i$ so that it starts at the origin, moves out tangentially to the positive real axis, and then moves around through the saddle point at $z = +i$ in the direction given by the angle $\alpha = 3\pi/4$ and then on out to minus infinity, asymptotic with the negative real axis. The path of steepest ascent, which we must avoid, has the phase $-\frac{1}{2}\arg f''(i) = \frac{\pi}{4}$, according to Eq. (7.107), and is orthogonal to the axis, our path of steepest descent.

Direct substitution into Eq. (7.113) with $\alpha = 3\pi/4$ now yields

$$\begin{aligned} H_\nu^{(1)}(s) &= \frac{1}{\pi i}\frac{\sqrt{2\pi}i^{-\nu-1}e^{(s/2)(i-1/i)}e^{3\pi i/4}}{|(s/2)(-2/i^3)|^{1/2}} \\ &= \sqrt{\frac{2}{\pi s}}e^{(i\pi/2)(-\nu-2)}e^{is}e^{i(3\pi/4)}. \end{aligned} \tag{7.119}$$

By combining terms, we obtain

$$H_\nu^{(1)}(s) \approx \sqrt{\frac{2}{\pi s}}e^{i(s-\nu(\pi/2)-\pi/4)} \tag{7.120}$$

as the leading term of the asymptotic expansion of the Hankel function $H_\nu^{(1)}(s)$. Additional terms, if desired, may be picked up from the power series of f and g in Eq. (7.111b). The other Hankel function can be treated similarly using the saddle point at $z = -i$. ■

Example 7.3.2 ASYMPTOTIC FORM OF THE FACTORIAL FUNCTION $\Gamma(1+s)$

In many physical problems, particularly in the field of statistical mechanics, it is desirable to have an accurate approximation of the gamma or factorial function of very large numbers. As developed in Section 8.1, the factorial function may be defined by the Euler integral

$$\Gamma(1+s) = \int_0^\infty \rho^s e^{-\rho}\, d\rho = s^{s+1} \int_0^\infty e^{s(\ln z - z)}\, dz. \tag{7.121}$$

Here we have made the substitution $\rho = zs$ in order to convert the integral to the form required by Eq. (7.109). As before, we assume that s is real and positive, from which it follows that the integrand vanishes at the limits 0 and ∞. By differentiating the z-dependence appearing in the exponent, we obtain

$$\frac{df(z)}{dz} = \frac{d}{dz}(\ln z - z) = \frac{1}{z} - 1, \qquad f''(z) = -\frac{1}{z^2}, \tag{7.122}$$

which shows that the point $z = 1$ is a saddle point and $\arg f''(1) = \arg(-1) = \pi$. According to Eq. (7.109) we let

$$z - 1 = xe^{i\alpha}, \qquad \alpha = \frac{\pi}{2} - \frac{1}{2}\arg f''(1) = \frac{\pi}{2} - \frac{\pi}{2} = 0, \tag{7.123}$$

with x small, to describe the contour in the vicinity of the saddle point. From this we see that the direction of steepest descent is along the real axis, a conclusion that we could have reached more or less intuitively.

Direct substitution into Eq. (7.113) with $\alpha = 0$ now gives

$$\Gamma(1+s) \approx \frac{\sqrt{2\pi}\, s^{s+1} e^{-s}}{|s(-1^{-2})|^{1/2}}. \tag{7.124}$$

Thus the first term in the asymptotic expansion of the factorial function is

$$\Gamma(1+s) \approx \sqrt{2\pi s}\, s^s e^{-s}. \tag{7.125}$$

This result is the first term in Stirling's expansion of the factorial function. The method of steepest descent is probably the easiest way of obtaining this first term. If more terms in the expansion are desired, then the method of Section 8.3 is preferable. ■

In the foregoing example the calculation was carried out by assuming s to be real. This assumption is not necessary. We may show (Exercise 7.3.6) that Eq. (7.125) also holds when s is replaced by the complex variable w, provided only that the real part of w be required to be large and positive.

Asymptotic limits of integral representations of functions are extremely important in many approximations and applications in physics:

$$\int_C g(z) e^{sf(z)}\, dz \sim \frac{\sqrt{2\pi}\, g(z_0) e^{sf(z_0)} e^{i\alpha}}{\sqrt{|sf''(z_0)|}}, \qquad f'(z_0) = 0.$$

The saddle point method is one method of choice for deriving them and belongs in the toolkit of every physicist and engineer.

Exercises

7.3.1 Using the method of steepest descents, evaluate the second Hankel function, given by

$$H_\nu^{(2)}(s) = \frac{1}{\pi i}\int_{-\infty C_2}^{0} e^{(s/2)(z-1/z)}\frac{dz}{z^{\nu+1}},$$

with contour C_2 as shown in Fig. 7.19.

ANS. $H_\nu^{(2)}(s) \approx \sqrt{\frac{2}{\pi s}}e^{-i(s-\pi/4-\nu\pi/2)}$.

7.3.2 Find the steepest path and leading asymptotic expansion for the Fresnel integrals $\int_0^s \cos x^2\,dx$, $\int_0^s \sin x^2\,dx$.
Hint. Use $\int_0^1 e^{isz^2}\,dz$.

7.3.3 (a) In applying the method of steepest descent to the Hankel function $H_\nu^{(1)}(s)$, show that

$$\Re\big[f(z)\big] < \Re\big[f(z_0)\big] = 0$$

for z on the contour C_1 but away from the point $z = z_0 = i$.

(b) Show that

$$\Re\big[f(z)\big] > 0 \qquad \text{for} \quad 0 < r < 1, \qquad \begin{cases} \frac{\pi}{2} < \theta \le \pi \\ -\pi \le \theta < \frac{\pi}{2} \end{cases}$$

and

$$\Re\big[f(z)\big] < 0 \qquad \text{for} \quad r > 1, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

(Fig. 7.20). This is why C_1 may not be deformed to pass through the second saddle point, $z = -i$. Compare with and verify the dot-dashed lines in Fig. 7.18 for this case.

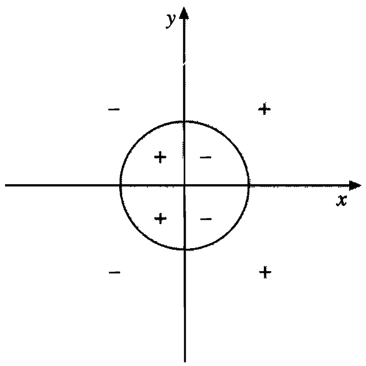

FIGURE 7.20

7.3.4 Determine the asymptotic dependence of the modified Bessel functions $I_\nu(x)$, given

$$I_\nu(x) = \frac{1}{2\pi i}\int_C e^{(x/2)(t+1/t)}\frac{dt}{t^{\nu+1}}.$$

The contour starts and ends at $t = -\infty$, encircling the origin in a positive sense. There are two saddle points. Only the one at $z = +1$ contributes significantly to the asymptotic form.

7.3.5 Determine the asymptotic dependence of the modified Bessel function of the second kind, $K_\nu(x)$, by using

$$K_\nu(x) = \frac{1}{2}\int_0^\infty e^{(-x/2)(s+1/s)}\frac{ds}{s^{1-\nu}}.$$

7.3.6 Show that Stirling's formula,

$$\Gamma(1+s) \approx \sqrt{2\pi s}\, s^s e^{-s},$$

holds for complex values of s (with $\Re(s)$ large and positive).
Hint. This involves assigning a phase to s and then demanding that $\Im[sf(z)] = \text{constant}$ in the vicinity of the saddle point.

7.3.7 Assume $H_\nu^{(1)}(s)$ to have a negative power-series expansion of the form

$$H_\nu^{(1)}(s) = \sqrt{\frac{2}{\pi s}}\, e^{i(s-\nu(\pi/2)-\pi/4)}\sum_{n=0}^{\infty} a_{-n}s^{-n},$$

with the coefficient of the summation obtained by the method of steepest descent. Substitute into Bessel's equation and show that you reproduce the asymptotic series for $H_\nu^{(1)}(s)$ given in Section 11.6.

Additional Readings

Nussenzveig, H. M., *Causality and Dispersion Relations*, Mathematics in Science and Engineering Series, Vol. 95. New York: Academic Press (1972). This is an advanced text covering causality and dispersion relations in the first chapter and then moving on to develop the implications in a variety of areas of theoretical physics.

Wyld, H. W., *Mathematical Methods for Physics*. Reading, MA: Benjamin/Cummings (1976), Perseus Books (1999). This is a relatively advanced text that contains an extensive discussion of the dispersion relations.