MATHEMATICAL METHODS FOR PHYSICISTS

SIXTH EDITION

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CHAPTER 6

FUNCTIONS OF A COMPLEX VARIABLE I

ANALYTIC PROPERTIES, MAPPING

The imaginary numbers are a wonderful flight of God's spirit; they are almost an amphibian between being and not being.

GOTTFRIED WILHELM VON LEIBNIZ, 1702

We turn now to a study of functions of a complex variable. In this area we develop some of the most powerful and widely useful tools in all of analysis. To indicate, at least partly, why complex variables are important, we mention briefly several areas of application.

1. For many pairs of functions u and v, both u and v satisfy Laplace's equation,

$$\nabla^2\psi = \frac{\partial^2\psi(x,y)}{\partial x^2} + \frac{\partial^2\psi(x,y)}{\partial y^2} = 0.$$

Hence either u or v may be used to describe a two-dimensional electrostatic potential. The other function, which gives a family of curves orthogonal to those of the first function, may then be used to describe the electric field $\mathbf{E}$. A similar situation holds for the hydrodynamics of an ideal fluid in irrotational motion. The function u might describe the velocity potential, whereas the function v would then be the stream function.

In many cases in which the functions u and v are unknown, mapping or transforming in the complex plane permits us to create a coordinate system tailored to the particular problem.

2. In Chapter 9 we shall see that the second-order differential equations of interest in physics may be solved by power series. The same power series may be used in the complex plane to replace x by the complex variable z. The dependence of the solution $f(z)$ at a given z_0 on the behavior of $f(z)$ elsewhere gives us greater insight into the behavior of our

solution and a powerful tool (analytic continuation) for extending the region in which the solution is valid.

3. The change of a parameter k from real to imaginary, $k \to ik$, transforms the Helmholtz equation into the diffusion equation. The same change transforms the Helmholtz equation solutions (Bessel and spherical Bessel functions) into the diffusion equation solutions (modified Bessel and modified spherical Bessel functions).

4. Integrals in the complex plane have a wide variety of useful applications:

- Evaluating definite integrals;
- Inverting power series;
- Forming infinite products;
- Obtaining solutions of differential equations for large values of the variable (asymptotic solutions);
- Investigating the stability of potentially oscillatory systems;
- Inverting integral transforms.

5. Many physical quantities that were originally real become complex as a simple physical theory is made more general. The real index of refraction of light becomes a complex quantity when absorption is included. The real energy associated with an energy level becomes complex when the finite lifetime of the level is considered.

6.1 COMPLEX ALGEBRA

A complex number is nothing more than an ordered pair of two real numbers, (a, b). Similarly, a complex variable is an ordered pair of two real variables,[1]

$$z \equiv (x, y). \tag{6.1}$$

The ordering is significant. In general (a, b) is not equal to (b, a) and (x, y) is not equal to (y, x). As usual, we continue writing a real number $(x, 0)$ simply as x, and we call $i \equiv (0, 1)$ the imaginary unit.

All our complex variable analysis can be developed in terms of ordered pairs of numbers (a, b), variables (x, y), and functions $(u(x, y), v(x, y))$.

We now define **addition** of complex numbers in terms of their Cartesian components as

$$\boxed{z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),} \tag{6.2a}$$

that is, two-dimensional vector addition. In Chapter 1 the points in the xy-plane are identified with the two-dimensional displacement vector $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. As a result, two-dimensional vector analogs can be developed for much of our complex analysis. Exercise 6.1.2 is one simple example; Cauchy's theorem, Section 6.3, is another.

Multiplication of complex numbers is defined as

$$\boxed{z_1 z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).} \tag{6.2b}$$

[1]This is precisely how a computer does complex arithmetic.

Using Eq. (6.2b) we verify that $i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1$, so we can also identify $i = \sqrt{-1}$, as usual and further rewrite Eq. (6.1) as

$$\boxed{z = (x, y) = (x, 0) + (0, y) = x + (0, 1) \cdot (y, 0) = x + iy.} \tag{6.2c}$$

Clearly, the i is not necessary here but it is convenient. It serves to keep pairs in order—somewhat like the unit vectors of Chapter 1.[2]

Permanence of Algebraic Form

All our elementary functions, e^z, $\sin z$, and so on, can be extended into the complex plane (compare Exercise 6.1.9). For instance, they can be defined by power-series expansions, such as

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{6.3}$$

for the exponential. Such definitions agree with the real variable definitions along the real x-axis and extend the corresponding real functions into the complex plane. This result is often called **permanence of the algebraic form**.

It is convenient to employ a graphical representation of the complex variable. By plotting x—the real part of z—as the abscissa and y—the imaginary part of z—as the ordinate, we have the complex plane, or Argand plane, shown in Fig. 6.1. If we assign specific values to x and y, then z corresponds to a point (x, y) in the plane. In terms of the ordering mentioned before, it is obvious that the point (x, y) does not coincide with the point (y, x) except for the special case of $x = y$. Further, from Fig. 6.1 we may write

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{6.4a}$$

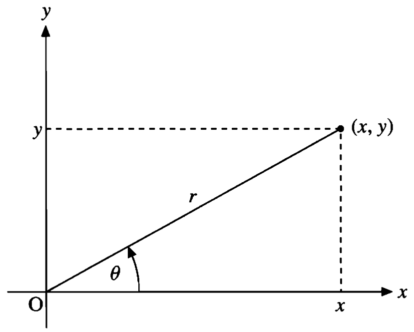

FIGURE 6.1 Complex plane—Argand diagram.

[2]The algebra of complex numbers, (a, b), is isomorphic with that of matrices of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

(compare Exercise 3.2.4).

and

$$z = r(\cos\theta + i\sin\theta). \tag{6.4b}$$

Using a result that is suggested (but not rigorously proved)[3] by Section 5.6 and Exercise 5.6.1, we have the useful polar representation

$$\boxed{z = r(\cos\theta + i\sin\theta) = re^{i\theta}.} \tag{6.4c}$$

In order to prove this identity, we use $i^3 = -i$, $i^4 = 1, \ldots$ in the Taylor expansion of the exponential and trigonometric functions and separate even and odd powers in

$$\begin{aligned} e^{i\theta} &= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{\nu=0}^{\infty} \frac{(i\theta)^{2\nu}}{(2\nu)!} + \sum_{\nu=0}^{\infty} \frac{(i\theta)^{2\nu+1}}{(2\nu+1)!} \\ &= \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{\theta^{2\nu}}{(2\nu)!} + i \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{\theta^{2\nu+1}}{(2\nu+1)!} = \cos\theta + i\sin\theta. \end{aligned}$$

For the special values $\theta = \pi/2$ and $\theta = \pi$, we obtain

$$e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i, \qquad e^{i\pi} = \cos(\pi) = -1,$$

intriguing connections between e, i, and π. Moreover, the exponential function $e^{i\theta}$ is periodic with period 2π, just like $\sin\theta$ and $\cos\theta$.

In this representation r is called the **modulus** or **magnitude** of z ($r = |z| = (x^2 + y^2)^{1/2}$) and the angle θ ($= \tan^{-1}(y/x)$) is labeled the argument or **phase** of z. (Note that the arctan function $\tan^{-1}(y/x)$ has infinitely many branches.)

The choice of polar representation, Eq. (6.4c), or Cartesian representation, Eqs. (6.1) and (6.2c), is a matter of convenience. Addition and subtraction of complex variables are easier in the Cartesian representation, Eq. (6.2a). Multiplication, division, powers, and roots are easier to handle in polar form, Eq. (6.4c).

Analytically or graphically, using the vector analogy, we may show that the modulus of the sum of two complex numbers is no greater than the sum of the moduli and no less than the difference, Exercise 6.1.3,

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|. \tag{6.5}$$

Because of the vector analogy, these are called the **triangle** inequalities.

Using the polar form, Eq. (6.4c), we find that the magnitude of a product is the product of the magnitudes:

$$|z_1 \cdot z_2| = |z_1| \cdot |z_2|. \tag{6.6}$$

Also,

$$\arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2. \tag{6.7}$$

[3] Strictly speaking, Chapter 5 was limited to real variables. The development of power-series expansions for complex functions is taken up in Section 6.5 (Laurent expansion).

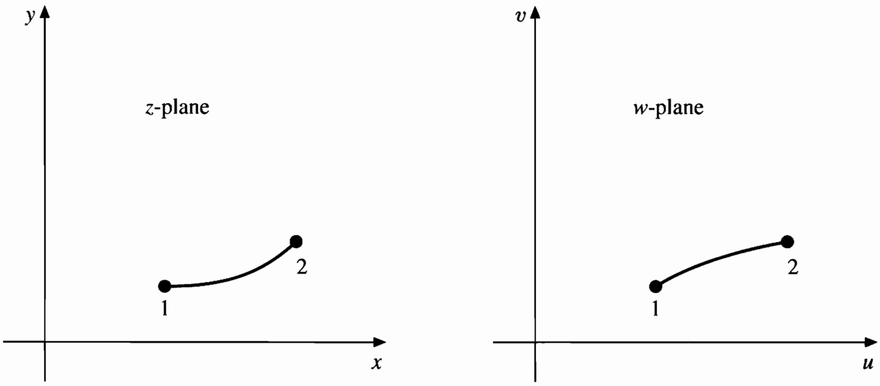

FIGURE 6.2 The function $w(z) = u(x, y) + iv(x, y)$ maps points in the xy-plane into points in the uv-plane.

From our complex variable z complex functions $f(z)$ or $w(z)$ may be constructed. These complex functions may then be resolved into real and imaginary parts,

$$w(z) = u(x, y) + iv(x, y), \tag{6.8}$$

in which the separate functions $u(x, y)$ and $v(x, y)$ are pure real. For example, if $f(z) = z^2$, we have

$$f(z) = (x + iy)^2 = \left(x^2 - y^2\right) + i2xy.$$

The **real part** of a function $f(z)$ will be labeled $\Re f(z)$, whereas the **imaginary part** will be labeled $\Im f(z)$. In Eq. (6.8)

$$\Re w(z) = \mathrm{Re}(w) = u(x, y), \qquad \Im w(z) = \mathrm{Im}(w) = v(x, y).$$

The relationship between the independent variable z and the dependent variable w is perhaps best pictured as a mapping operation. A given $z = x + iy$ means a given point in the z-plane. The complex value of $w(z)$ is then a point in the w-plane. Points in the z-plane map into points in the w-plane and curves in the z-plane map into curves in the w-plane, as indicated in Fig. 6.2.

Complex Conjugation

In all these steps, complex number, variable, and function, the operation of replacing i by $-i$ is called "taking the complex conjugate." The complex conjugate of z is denoted by z^*, where[4]

$$\boxed{z^* = x - iy.} \tag{6.9}$$

[4]The complex conjugate is often denoted by $\bar{z}$ in the mathematical literature.

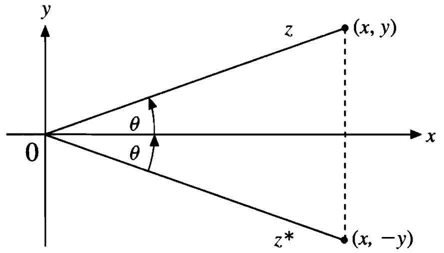

FIGURE 6.3 Complex conjugate points.

The complex variable z and its complex conjugate z^* are mirror images of each other reflected in the x-axis, that is, inversion of the y-axis (compare Fig. 6.3). The product zz^* leads to

$$\boxed{zz^* = (x+iy)(x-iy) = x^2+y^2 = r^2.} \tag{6.10}$$

Hence

$$(zz^*)^{1/2} = |z|,$$

the **magnitude** of z.

Functions of a Complex Variable

All the elementary functions of real variables may be extended into the complex plane—replacing the real variable x by the complex variable z. This is an example of the analytic continuation mentioned in Section 6.5. The extremely important relation of Eq. (6.4c) is an illustration. Moving into the complex plane opens up new opportunities for analysis.

Example 6.1.1 De Moivre's Formula

If Eq. (6.4c) (setting $r = 1$) is raised to the nth power, we have

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n. \tag{6.11}$$

Expanding the exponential now with argument $n\theta$, we obtain

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n. \tag{6.12}$$

De Moivre's formula is generated if the right-hand side of Eq. (6.12) is expanded by the binomial theorem; we obtain $\cos n\theta$ as a series of powers of $\cos\theta$ and $\sin\theta$, Exercise 6.1.6. ■

Numerous other examples of relations among the exponential, hyperbolic, and trigonometric functions in the complex plane appear in the exercises.

Occasionally there are complications. The logarithm of a complex variable may be expanded using the polar representation

$$\ln z = \ln re^{i\theta} = \ln r + i\theta. \tag{6.13a}$$

This is not complete. To the phase angle, θ, we may add any integral multiple of 2π without changing z. Hence Eq. (6.13a) should read

$$\ln z = \ln r e^{i(\theta+2n\pi)} = \ln r + i(\theta + 2n\pi). \qquad (6.13b)$$

The parameter n may be any integer. This means that $\ln z$ is a **multivalued** function having an infinite number of values for a single pair of real values r and θ. To avoid ambiguity, the simplest choice is $n = 0$ and limitation of the phase to an interval of length 2π, such as $(-\pi, \pi)$.[5] The line in the z-plane that is not crossed, the negative real axis in this case, is labeled a **cut line** or **branch cut**. The value of $\ln z$ with $n = 0$ is called the **principal value** of $\ln z$. Further discussion of these functions, including the logarithm, appears in Section 6.7.

Exercises

6.1.1 (a) Find the reciprocal of $x + iy$, working entirely in the Cartesian representation.
(b) Repeat part (a), working in polar form but expressing the final result in Cartesian form.

6.1.2 The complex quantities $a = u + iv$ and $b = x + iy$ may also be represented as two-dimensional vectors $\mathbf{a} = \hat{\mathbf{x}}u + \hat{\mathbf{y}}v$, $\mathbf{b} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$. Show that

$$a^* b = \mathbf{a} \cdot \mathbf{b} + i\hat{\mathbf{z}} \cdot \mathbf{a} \times \mathbf{b}.$$

6.1.3 Prove algebraically that for complex numbers,

$$|z_1| - |z_2| \le |z_1 + z_2| \le |z_1| + |z_2|.$$

Interpret this result in terms of two-dimensional vectors. Prove that

$$|z - 1| < \left|\sqrt{z^2 - 1}\right| < |z + 1|, \qquad \text{for} \quad \Re(z) > 0.$$

6.1.4 We may define a complex conjugation operator K such that $Kz = z^*$. Show that K is not a linear operator.

6.1.5 Show that complex numbers have square roots and that the square roots are contained in the complex plane. What are the square roots of i?

6.1.6 Show that

(a) $\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \cdots$.
(b) $\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \cdots$.

Note. The quantities $\binom{n}{m}$ are binomial coefficients: $\binom{n}{m} = n!/[(n-m)!m!]$.

6.1.7 Prove that

(a) $\displaystyle\sum_{n=0}^{N-1} \cos nx = \frac{\sin(Nx/2)}{\sin x/2}\cos(N-1)\frac{x}{2}$,

[5] There is no standard choice of phase; the appropriate phase depends on each problem.

(b) $$\sum_{n=0}^{N-1} \sin nx = \frac{\sin(Nx/2)}{\sin x/2} \sin(N-1)\frac{x}{2}.$$

These series occur in the analysis of the multiple-slit diffraction pattern. Another application is the analysis of the Gibbs phenomenon, Section 14.5.
Hint. Parts (a) and (b) may be combined to form a geometric series (compare Section 5.1).

6.1.8 For $-1 < p < 1$ prove that

(a) $$\sum_{n=0}^{\infty} p^n \cos nx = \frac{1 - p\cos x}{1 - 2p\cos x + p^2},$$

(b) $$\sum_{n=0}^{\infty} p^n \sin nx = \frac{p \sin x}{1 - 2p\cos x + p^2}.$$

These series occur in the theory of the Fabry–Perot interferometer.

6.1.9 Assume that the trigonometric functions and the hyperbolic functions are defined for complex argument by the appropriate power series

$$\sin z = \sum_{n=1,\text{odd}}^{\infty} (-1)^{(n-1)/2} \frac{z^n}{n!} = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s+1}}{(2s+1)!},$$

$$\cos z = \sum_{n=0,\text{even}}^{\infty} (-1)^{n/2} \frac{z^n}{n!} = \sum_{s=0}^{\infty} (-1)^s \frac{z^{2s}}{(2s)!},$$

$$\sinh z = \sum_{n=1,\text{odd}}^{\infty} \frac{z^n}{n!} = \sum_{s=0}^{\infty} \frac{z^{2s+1}}{(2s+1)!},$$

$$\cosh z = \sum_{n=0,\text{even}}^{\infty} \frac{z^n}{n!} = \sum_{s=0}^{\infty} \frac{z^{2s}}{(2s)!}.$$

(a) Show that

$$i \sin z = \sinh iz, \qquad \sin iz = i \sinh z,$$
$$\cos z = \cosh iz, \qquad \cos iz = \cosh z.$$

(b) Verify that familiar functional relations such as

$$\cosh z = \tfrac{e^z + e^{-z}}{2},$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1,$$

still hold in the complex plane.

6.1.10 Using the identities

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

established from comparison of power series, show that

(a) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$

$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y,$

(b) $|\sin z|^2 = \sin^2 x + \sinh^2 y, \qquad |\cos z|^2 = \cos^2 x + \sinh^2 y.$

This demonstrates that we may have $|\sin z|, |\cos z| > 1$ in the complex plane.

6.1.11 From the identities in Exercises 6.1.9 and 6.1.10 show that

(a) $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y,$

$\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y,$

(b) $|\sinh z|^2 = \sinh^2 x + \sin^2 y, \qquad |\cosh z|^2 = \cosh^2 x + \sin^2 y.$

6.1.12 Prove that

(a) $|\sin z| \geq |\sin x|$ (b) $|\cos z| \geq |\cos x|$.

6.1.13 Show that the exponential function e^z is periodic with a pure imaginary period of $2\pi i$.

6.1.14 Show that

(a) $\tanh \frac{z}{2} = \frac{\sinh x + i \sin y}{\cosh x + \cos y}$, (b) $\coth \frac{z}{2} = \frac{\sinh x - i \sin y}{\cosh x - \cos y}$.

6.1.15 Find all the zeros of

(a) $\sin z$, (b) $\cos z$, (c) $\sinh z$, (d) $\cosh z$.

6.1.16 Show that

(a) $\sin^{-1} z = -i \ln\left(iz \pm \sqrt{1 - z^2}\right)$, (d) $\sinh^{-1} z = \ln\left(z + \sqrt{z^2 + 1}\right)$,

(b) $\cos^{-1} z = -i \ln\left(z \pm \sqrt{z^2 - 1}\right)$, (e) $\cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$,

(c) $\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i + z}{i - z}\right)$, (f) $\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1 + z}{1 - z}\right)$.

Hint. 1. Express the trigonometric and hyperbolic functions in terms of exponentials. 2. Solve for the exponential and then for the exponent.

6.1.17 In the quantum theory of the photoionization we encounter the identity

$$\left(\frac{ia - 1}{ia + 1}\right)^{ib} = \exp(-2b \cot^{-1} a),$$

in which a and b are real. Verify this identity.

6.1.18 A plane wave of light of angular frequency ω is represented by

$$e^{i\omega(t-nx/c)}.$$

In a certain substance the simple real index of refraction n is replaced by the complex quantity $n - ik$. What is the effect of k on the wave? What does k correspond to physically? The generalization of a quantity from real to complex form occurs frequently in physics. Examples range from the complex Young's modulus of viscoelastic materials to the complex (optical) potential of the "cloudy crystal ball" model of the atomic nucleus.

6.1.19 We see that for the angular momentum components defined in Exercise 2.5.14,

$$L_x - iL_y \neq (L_x + iL_y)^*.$$

Explain why this occurs.

6.1.20 Show that the **phase** of $f(z) = u + iv$ is equal to the imaginary part of the logarithm of $f(z)$. Exercise 8.2.13 depends on this result.

6.1.21 (a) Show that $e^{\ln z}$ always equals z.
(b) Show that $\ln e^z$ does not always equal z.

6.1.22 The infinite product representations of Section 5.11 hold when the real variable x is replaced by the complex variable z. From this, develop infinite product representations for
(a) $\sinh z$, (b) $\cosh z$.

6.1.23 The equation of motion of a mass m **relative to a rotating coordinate system** is

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\left(\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt}\right) - m\left(\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}\right).$$

Consider the case $\mathbf{F} = 0$, $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$, and $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, with ω constant. Show that the replacement of $\mathbf{r} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$ by $z = x + iy$ leads to

$$\frac{d^2z}{dt^2} + i2\omega\frac{dz}{dt} - \omega^2 z = 0.$$

Note. This ODE may be solved by the substitution $z = fe^{-i\omega t}$.

6.1.24 Using the complex arithmetic available in FORTRAN, write a program that will calculate the complex exponential e^z from its series expansion (definition). Calculate e^z for $z = e^{in\pi/6}$, $n = 0, 1, 2, \ldots, 12$. Tabulate the phase angle ($\theta = n\pi/6$), $\Re z$, $\Im z$, $\Re(e^z)$, $\Im(e^z)$, $|e^z|$, and the phase of e^z.

Check value. $n = 5$, $\theta = 2.61799$, $\Re(z) = -0.86602$,
$\Im z = 0.50000$, $\Re(e^z) = 0.36913$, $\Im(e^z) = 0.20166$,
$|e^z| = 0.42062$, phase$(e^z) = 0.50000$.

6.1.25 Using the complex arithmetic available in FORTRAN, calculate and tabulate $\Re(\sinh z)$, $\Im(\sinh z)$, $|\sinh z|$, and phase $(\sinh z)$ for $x = 0.0(0.1)1.0$ and $y = 0.0(0.1)1.0$.

Hint. Beware of dividing by zero when calculating an angle as an arc tangent.

Check value. $z = 0.2 + 0.1i$, $\Re(\sinh z) = 0.20033$, $\Im(\sinh z) = 0.10184$, $|\sinh z| = 0.22473$, $\text{phase}(\sinh z) = 0.47030$.

6.1.26 Repeat Exercise 6.1.25 for $\cosh z$.

6.2 Cauchy–Riemann Conditions

Having established complex functions of a complex variable, we now proceed to differentiate them. The derivative of $f(z)$, like that of a real function, is defined by

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{z + \delta z - z} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} = f'(z), \tag{6.14}$$

provided that the limit is **independent** of the particular approach to the point z. For real variables we require that the right-hand limit ($x \to x_0$ from above) and the left-hand limit ($x \to x_0$ from below) be equal for the derivative $df(x)/dx$ to exist at $x = x_0$. Now, with z (or z_0) some point in a plane, our requirement that the limit be independent of the direction of approach is very restrictive.

Consider increments δx and δy of the variables x and y, respectively. Then

$$\delta z = \delta x + i\delta y. \tag{6.15}$$

Also,

$$\delta f = \delta u + i\delta v, \tag{6.16}$$

so that

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}. \tag{6.17}$$

Let us take the limit indicated by Eq. (6.14) by two different approaches, as shown in Fig. 6.4. First, with $\delta y = 0$, we let $\delta x \to 0$. Equation (6.14) yields

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{6.18}$$

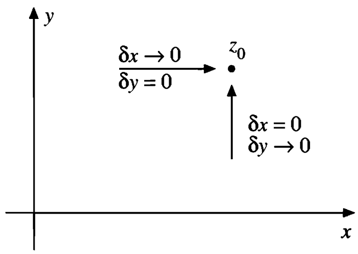

FIGURE 6.4 Alternate approaches to z_0.

assuming the partial derivatives exist. For a second approach, we set $\delta x = 0$ and then let $\delta y \to 0$. This leads to

$$\lim_{\delta z\to 0}\frac{\delta f}{\delta z} = \lim_{\delta y\to 0}\left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}\right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{6.19}$$

If we are to have a derivative df/dz, Eqs. (6.18) and (6.19) must be identical. Equating real parts to real parts and imaginary parts to imaginary parts (like components of vectors), we obtain

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.} \tag{6.20}$$

These are the famous **Cauchy–Riemann** conditions. They were discovered by Cauchy and used extensively by Riemann in his theory of analytic functions. These Cauchy–Riemann conditions are necessary for the existence of a derivative of $f(z)$; that is, if df/dz exists, the Cauchy–Riemann conditions must hold.

Conversely, if the Cauchy–Riemann conditions are satisfied and the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous, the derivative df/dz exists. This may be shown by writing

$$\delta f = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y. \tag{6.21}$$

The justification for this expression depends on the continuity of the partial derivatives of u and v. Dividing by δz, we have

$$\begin{aligned}\frac{\delta f}{\delta z} &= \frac{(\partial u/\partial x + i(\partial v/\partial x))\delta x + (\partial u/\partial y + i(\partial v/\partial y))\delta y}{\delta x + i\delta y}\\ &= \frac{(\partial u/\partial x + i(\partial v/\partial x)) + (\partial u/\partial y + i(\partial v/\partial y))\delta y/\delta x}{1 + i(\delta y/\delta x)}.\end{aligned} \tag{6.22}$$

If $\delta f/\delta z$ is to have a unique value, the dependence on $\delta y/\delta x$ must be eliminated. Applying the Cauchy–Riemann conditions to the y derivatives, we obtain

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}. \tag{6.23}$$

Substituting Eq. (6.23) into Eq. (6.22), we may cancel out the $\delta y/\delta x$ dependence and

$$\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{6.24}$$

which shows that $\lim \delta f/\delta z$ is independent of the direction of approach in the complex plane as long as the partial derivatives are continuous. Thus, $\frac{df}{dz}$ exists and f is analytic at z.

It is worthwhile noting that the Cauchy–Riemann conditions guarantee that the curves $u = c_1$ will be orthogonal to the curves $v = c_2$ (compare Section 2.1). This is fundamental in application to potential problems in a variety of areas of physics. If $u = c_1$ is a line of electric force, then $v = c_2$ is an equipotential line (surface), and vice versa. To see this, let us write the Cauchy–Riemann conditions as a product of ratios of partial derivatives,

$$\frac{u_x}{u_y} \cdot \frac{v_x}{v_y} = -1, \tag{6.25}$$

with the abbreviations

$$\frac{\partial u}{\partial x} \equiv u_x, \qquad \frac{\partial u}{\partial y} \equiv u_y, \qquad \frac{\partial v}{\partial x} \equiv v_x, \qquad \frac{\partial v}{\partial y} \equiv v_y.$$

Now recall the geometric meaning of $-u_x/u_y$ as the slope of the tangent of each curve $u(x, y) = \text{const.}$ and similarly for $v(x, y) = \text{const.}$ This means that the $u = \text{const.}$ and $v = \text{const.}$ curves are mutually orthogonal at each intersection. Alternatively,

$$u_x\, dx + u_y\, dy = 0 = v_y\, dx - v_x\, dy$$

says that, if (dx, dy) is tangent to the u-curve, then the orthogonal $(-dy, dx)$ is tangent to the v-curve at the intersection point, $z = (x, y)$. Or equivalently, $u_x v_x + u_y v_y = 0$ implies that the **gradient vectors** (u_x, u_y) and (v_x, v_y) **are perpendicular**. A further implication for potential theory is developed in Exercise 6.2.1.

Analytic Functions

Finally, if $f(z)$ is differentiable at $z = z_0$ and in some small region around z_0, we say that $f(z)$ is **analytic**[6] at $z = z_0$. If $f(z)$ is analytic everywhere in the (finite) complex plane, we call it an **entire** function. Our theory of complex variables here is one of analytic functions of a complex variable, which points up the crucial importance of the Cauchy–Riemann conditions. The concept of analyticity carried on in advanced theories of modern physics plays a crucial role in dispersion theory (of elementary particles). If $f'(z)$ does not exist at $z = z_0$, then z_0 is labeled a singular point and consideration of it is postponed until Section 6.6.

To illustrate the Cauchy–Riemann conditions, consider two very simple examples.

Example 6.2.1 z^2 Is Analytic

Let $f(z) = z^2$. Then the real part $u(x, y) = x^2 - y^2$ and the imaginary part $v(x, y) = 2xy$. Following Eq. (6.20),

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

We see that $f(z) = z^2$ satisfies the Cauchy–Riemann conditions throughout the complex plane. Since the partial derivatives are clearly continuous, we conclude that $f(z) = z^2$ is analytic. ■

[6]Some writers use the term **holomorphic** or **regular**.

Example 6.2.2 z^* IS NOT ANALYTIC

Let $f(z) = z^*$. Now $u = x$ and $v = -y$. Applying the Cauchy–Riemann conditions, we obtain

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1.$$

The Cauchy–Riemann conditions are not satisfied and $f(z) = z^*$ is not an analytic function of z. It is interesting to note that $f(z) = z^*$ is continuous, thus providing an example of a function that is everywhere continuous but nowhere differentiable in the complex plane.

The derivative of a real function of a real variable is essentially a local characteristic, in that it provides information about the function only in a local neighborhood—for instance, as a truncated Taylor expansion. The existence of a derivative of a function of a complex variable has much more far-reaching implications. The real and imaginary parts of our analytic function must separately satisfy Laplace's equation. This is Exercise 6.2.1. Further, our analytic function is guaranteed derivatives of all orders, Section 6.4. In this sense the derivative not only governs the local behavior of the complex function, but controls the distant behavior as well. ■

Exercises

6.2.1 The functions $u(x, y)$ and $v(x, y)$ are the real and imaginary parts, respectively, of an analytic function $w(z)$.

(a) Assuming that the required derivatives exist, show that

$$\nabla^2 u = \nabla^2 v = 0.$$

Solutions of Laplace's equation such as $u(x, y)$ and $v(x, y)$ are called **harmonic** functions.

(b) Show that

$$\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0,$$

and give a geometric interpretation.

Hint. The technique of Section 1.6 allows you to construct vectors normal to the curves $u(x, y) = c_i$ and $v(x, y) = c_j$.

6.2.2 Show whether or not the function $f(z) = \Re(z) = x$ is analytic.

6.2.3 Having shown that the real part $u(x, y)$ and the imaginary part $v(x, y)$ of an analytic function $w(z)$ each satisfy Laplace's equation, show that $u(x, y)$ and $v(x, y)$ **cannot both have either a maximum or a minimum** in the interior of any region in which $w(z)$ is analytic. (They can have saddle points only.)

6.2.4 Let $A = \partial^2 w/\partial x^2$, $B = \partial^2 w/\partial x \partial y$, $C = \partial^2 w/\partial y^2$. From the calculus of functions of two variables, $w(x, y)$, we have a **saddle point** if

$$B^2 - AC > 0.$$

With $f(z) = u(x, y) + iv(x, y)$, apply the Cauchy–Riemann conditions and show that **neither** $u(x, y)$ nor $v(x, y)$ **has a maximum or a minimum** in a finite region of the complex plane. (See also Section 7.3.)

6.2.5 Find the analytic function

$$w(z) = u(x, y) + iv(x, y)$$

if (a) $u(x, y) = x^3 - 3xy^2$, (b) $v(x, y) = e^{-y} \sin x$.

6.2.6 If there is some common region in which $w_1 = u(x, y) + iv(x, y)$ and $w_2 = w_1^* = u(x, y) - iv(x, y)$ are both analytic, prove that $u(x, y)$ and $v(x, y)$ are constants.

6.2.7 The function $f(z) = u(x, y) + iv(x, y)$ is analytic. Show that $f^*(z^*)$ is also analytic.

6.2.8 Using $f(re^{i\theta}) = R(r, \theta)e^{i\Phi(r,\theta)}$, in which $R(r, \theta)$ and $\Phi(r, \theta)$ are differentiable real functions of r and θ, show that the Cauchy–Riemann conditions in polar coordinates become

$$\text{(a)}\ \frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta}, \qquad \text{(b)}\ \frac{1}{r}\frac{\partial R}{\partial \theta} = -R\frac{\partial \Theta}{\partial r}.$$

Hint. Set up the derivative first with δz radial and then with δz tangential.

6.2.9 As an extension of Exercise 6.2.8 show that $\Theta(r, \theta)$ satisfies Laplace's equation in polar coordinates. Equation (2.35) (without the final term and set to zero) is the Laplacian in polar coordinates.

6.2.10 Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x, v) + iv(x, y)$. We label the real part, $u(x, y)$, the velocity potential and the imaginary part, $v(x, y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic,

(a) Show that $df/dz = V_x - iV_y$;
(b) Show that $\nabla \cdot \mathbf{V} = 0$ (no sources or sinks);
(c) Show that $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow).

6.2.11 A proof of the Schwarz inequality (Section 10.4) involves minimizing an expression,

$$f = \psi_{aa} + \lambda\psi_{ab} + \lambda^*\psi_{ab}^* + \lambda\lambda^*\psi_{bb} \geq 0.$$

The ψ are integrals of products of functions; ψ_{aa} and ψ_{bb} are real, ψ_{ab} is complex and λ is a complex parameter.

(a) Differentiate the preceding expression with respect to λ^*, treating λ as an independent parameter, independent of λ^*. Show that setting the derivative $\partial f/\partial\lambda^*$ equal to zero yields

$$\lambda = -\frac{\psi_{ab}^*}{\psi_{bb}}.$$

(b) Show that $\partial f/\partial\lambda = 0$ leads to the same result.

(c) Let $\lambda = x + iy$, $\lambda^* = x - iy$. Set the x and y derivatives equal to zero and show that again

$$\lambda = -\frac{\psi_{ab}^*}{\psi_{bb}}.$$

This independence of λ and λ^* appears again in Section 17.7.

6.2.12 The function $f(z)$ is analytic. Show that the derivative of $f(z)$ with respect to z^* does not exist unless $f(z)$ is a constant.
Hint. Use the chain rule and take $x = (z + z^*)/2$, $y = (z - z^*)/2i$.
Note. This result emphasizes that our analytic function $f(z)$ is not just a complex function of two real variables x and y. It is a function of the complex variable $x + iy$.

6.3 Cauchy's Integral Theorem

Contour Integrals

With differentiation under control, we turn to integration. The integral of a complex variable over a contour in the complex plane may be defined in close analogy to the (Riemann) integral of a real function integrated along the real x-axis.

We divide the contour from z_0 to z_0' into n intervals by picking $n - 1$ intermediate points $z_1, z_2, \ldots$ on the contour (Fig. 6.5). Consider the sum

$$S_n = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}), \tag{6.26}$$

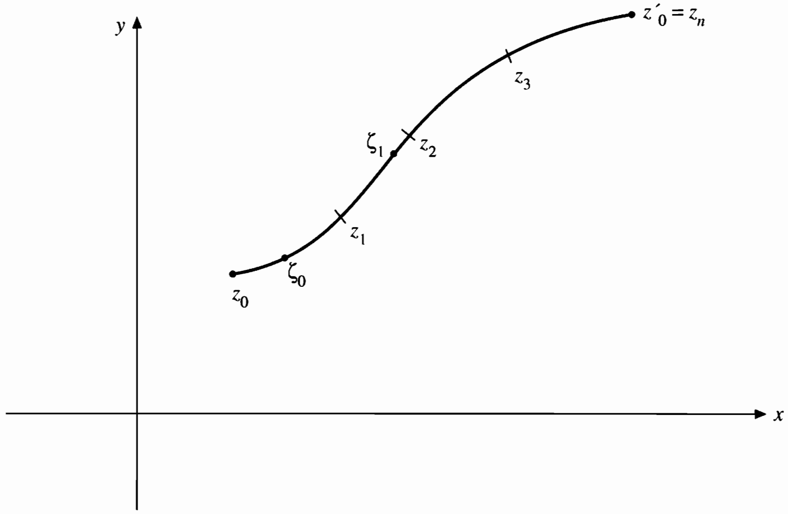

FIGURE 6.5 Integration path.

where ζ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with

$$|z_j - z_{j-1}| \to 0$$

for all j. If the $\lim_{n\to\infty} S_n$ exists and is independent of the details of choosing the points z_j and ζ_j, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)\, dz. \tag{6.27}$$

The right-hand side of Eq. (6.27) is called the contour integral of $f(z)$ (along the specified contour C from $z = z_0$ to $z = z_0'$).

The preceding development of the contour integral is closely analogous to the Riemann integral of a real function of a real variable. As an alternative, the contour integral may be defined by

$$\begin{aligned}\int_{z_1}^{z_2} f(z)dz &= \int_{x_1,y_1}^{x_2,y_2} \big[u(x,y) + iv(x,y)\big][dx + idy] \\ &= \int_{x_1,y_1}^{x_2,y_2} \big[u(x,y)\,dx - v(x,y)\,dy\big] + i\int_{x_1,y_1}^{x_2,y_2} \big[v(x,y)\,dx + u(x,y)\,dy\big]\end{aligned}$$

with the path joining (x_1, y_1) and (x_2, y_2) specified. This reduces the complex integral to the complex sum of real integrals. It is somewhat analogous to the replacement of a vector integral by the vector sum of scalar integrals, Section 1.10.

An important example is the contour integral $\int_C z^n\, dz$, where C is a circle of radius $r > 0$ around the origin $z = 0$ in the positive mathematical sense (counterclockwise). In polar coordinates of Eq. (6.4c) we parameterize the circle as $z = re^{i\theta}$ and $dz = ire^{i\theta}d\theta$. For $n \neq -1$, n an integer, we then obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_C z^n\, dz &= \frac{r^{n+1}}{2\pi}\int_0^{2\pi} \exp\big[i(n+1)\theta\big]\,d\theta \\ &= \big[2\pi i(n+1)\big]^{-1} r^{n+1}\big[e^{i(n+1)\theta}\big]\Big|_0^{2\pi} = 0\end{aligned} \tag{6.27a}$$

because 2π is a period of $e^{i(n+1)\theta}$, while for $n = -1$

$$\frac{1}{2\pi i}\int_C \frac{dz}{z} = \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1, \tag{6.27b}$$

again independent of r.

Alternatively, we can **integrate around a rectangle** with the corners z_1, z_2, z_3, z_4 to obtain for $n \neq -1$

$$\int z^n\, dz = \frac{z^{n+1}}{n+1}\Big|_{z_1}^{z_2} + \frac{z^{n+1}}{n+1}\Big|_{z_2}^{z_3} + \frac{z^{n+1}}{n+1}\Big|_{z_3}^{z_4} + \frac{z^{n+1}}{n+1}\Big|_{z_4}^{z_1} = 0,$$

because each corner point appears once as an upper and a lower limit that cancel. For $n = -1$ the corresponding real parts of the logarithms cancel similarly, but their imaginary parts involve the increasing arguments of the points from z_1 to z_4 and, when we come back to the first corner z_1, its argument has increased by 2π due to the multivaluedness of the logarithm, so $2\pi i$ is left over as the value of the integral. Thus, **the value of the integral involving a multivalued function must be that which is reached in a continuous fashion on the path being taken**. These integrals are examples of Cauchy's integral theorem, which we consider in the next section.

Stokes' Theorem Proof

Cauchy's integral theorem is the first of two basic theorems in the theory of the behavior of functions of a complex variable. First, we offer a proof under relatively restrictive conditions — conditions that are intolerable to the mathematician developing a beautiful abstract theory but that are usually satisfied in physical problems.

If a function $f(z)$ is analytic, that is, if its partial derivatives are continuous throughout some **simply connected region** R,[7] for every closed path C (Fig. 6.6) in R, and if it is single-valued (assumed for simplicity here), the line integral of $f(z)$ around C is zero, or

$$\int_C f(z)\,dz = \oint_C f(z)\,dz = 0. \tag{6.27c}$$

Recall that in Section 1.13 such a function $f(z)$, identified as a force, was labeled conservative. The symbol $\oint$ is used to emphasize that the path is closed. Note that the interior of the simply connected region bounded by a contour is that region lying to the left when moving in the direction implied by the contour; as a rule, a simply connected region is bounded by a single closed curve.

In this form the Cauchy integral theorem may be proved by direct application of Stokes' theorem (Section 1.12). With $f(z) = u(x, y) + iv(x, y)$ and $dz = dx + i dy$,

$$\begin{aligned}\oint_C f(z)\,dz &= \oint_C (u + iv)(dx + i dy)\\ &= \oint_C (u\,dx - v\,dy) + i\oint (v\,dx + u\,dy).\end{aligned} \tag{6.28}$$

These two line integrals may be converted to surface integrals by Stokes' theorem, a procedure that is justified if the partial derivatives are continuous within C. In applying Stokes' theorem, note that the final two integrals of Eq. (6.28) are real. Using

$$\mathbf{V} = \hat{\mathbf{x}}V_x + \hat{\mathbf{y}}V_y,$$

Stokes' theorem says that

$$\oint_C (V_x\,dx + V_y\,dy) = \int\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dx\,dy. \tag{6.29}$$

For the first integral in the last part of Eq. (6.28) let $u = V_x$ and $v = -V_y$.[8] Then

[7] Any closed simple curve (one that does not intersect itself) inside a simply connected region or domain may be contracted to a single point that still belongs to the region. If a region is not simply connected, it is called multiply connected. As an example of a multiply connected region, consider the z-plane with the interior of the unit circle **excluded**.

[8] In the proof of Stokes' theorem, Section 1.12, V_x and V_y are any two functions (with continuous partial derivatives).

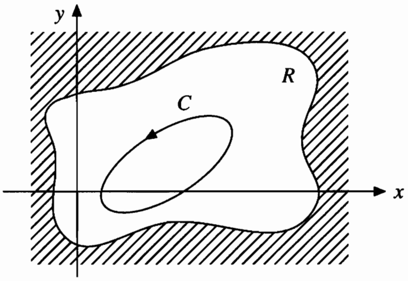

FIGURE 6.6 A closed contour C within a simply connected region R.

$$\oint_C (u\,dx - v\,dy) = \oint_C (V_x\,dx + V_y\,dy)$$
$$= \int\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)dx\,dy = -\int\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx\,dy. \quad (6.30)$$

For the second integral on the right side of Eq. (6.28) we let $u = V_y$ and $v = V_x$. Using Stokes' theorem again, we obtain

$$\oint (v\,dx + u\,dy) = \int\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy. \quad (6.31)$$

On application of the Cauchy–Riemann conditions, which must hold, since $f(z)$ is assumed analytic, each integrand vanishes and

$$\oint f(z)\,dz = -\int\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dx\,dy + i\int\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy = 0. \quad (6.32)$$

Cauchy–Goursat Proof

This completes the proof of Cauchy's integral theorem. However, the proof is marred from a theoretical point of view by the need for continuity of the first partial derivatives. Actually, as shown by Goursat, this condition is not necessary. An outline of the Goursat proof is as follows. We subdivide the region inside the contour C into a network of small squares, as indicated in Fig. 6.7. Then

$$\oint_C f(z)\,dz = \sum_j \oint_{C_j} f(z)\,dz, \quad (6.33)$$

all integrals along interior lines canceling out. To estimate the $\oint_{C_j} f(z)\,dz$, we construct the function

$$\delta_j(z, z_j) = \frac{f(z) - f(z_j)}{z - z_j} - \left.\frac{df(z)}{dz}\right|_{z=z_j}, \quad (6.34)$$

with z_j an interior point of the jth subregion. Note that $[f(z) - f(z_j)]/(z - z_j)$ is an approximation to the derivative at $z = z_j$. Equivalently, we may note that if $f(z)$ had

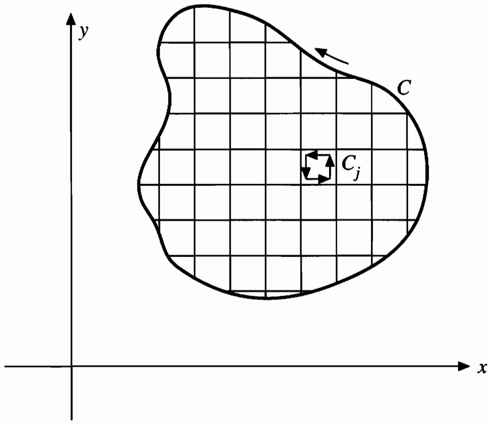

FIGURE 6.7 Cauchy–Goursat contours.

a Taylor expansion (which we have not yet proved), then $\delta_j(z, z_j)$ would be of order $z - z_j$, approaching zero as the network was made finer. But since $f'(z_j)$ exists, that is, is finite, we may make

$$\left|\delta_j(z, z_j)\right| < \varepsilon, \tag{6.35}$$

where ε is an arbitrarily chosen small positive quantity. Solving Eq. (6.34) for $f(z)$ and integrating around C_j, we obtain

$$\oint_{C_j} f(z)\,dz = \oint_{C_j} (z - z_j)\delta_j(z, z_j)\,dz, \tag{6.36}$$

the integrals of the other terms vanishing.[9] When Eqs. (6.35) and (6.36) are combined, one shows that

$$\left|\sum_j \oint_{C_j} f(z)\,dz\right| < A\varepsilon, \tag{6.37}$$

where A is a term of the order of the area of the enclosed region. Since ε is arbitrary, we let $\varepsilon \to 0$ and conclude that if a function $f(z)$ is analytic on and within a closed path C,

$$\oint_C f(z)\,dz = 0. \tag{6.38}$$

Details of the proof of this significantly more general and more powerful form can be found in Churchill in the Additional Readings. Actually we can still prove the theorem for $f(z)$ analytic within the interior of C and only continuous on C.

The consequence of the Cauchy integral theorem is that for analytic functions the line integral is a function only of its endpoints, independent of the path of integration,

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1) = -\int_{z_2}^{z_1} f(z)\,dz, \tag{6.39}$$

again exactly like the case of a conservative force, Section 1.13.

[9] $\oint dz$ and $\oint z\,dz = 0$ by Eq. (6.27a).

Multiply Connected Regions

The original statement of Cauchy's integral theorem demanded a simply connected region. This restriction may be relaxed by the creation of a barrier, a contour line. The purpose of the following contour-line construction is to permit, within a multiply connected region, the identification of curves that can be shrunk to a point within the region, that is, the construction of a subregion that is simply connected.

Consider the multiply connected region of Fig. 6.8, in which $f(z)$ is not defined for the interior, R'. Cauchy's integral theorem is not valid for the contour C, as shown, but we can construct a contour C' for which the theorem holds. We draw a line from the interior forbidden region, R', to the forbidden region exterior to R and then run a new contour, C', as shown in Fig. 6.9.

The new contour, C', through $ABDEFGA$ never crosses the contour line that literally converts R into a simply connected region. The three-dimensional analog of this technique was used in Section 1.14 to prove Gauss' law. By Eq. (6.39),

$$\int_G^A f(z)\,dz = -\int_E^D f(z)\,dz, \tag{6.40}$$

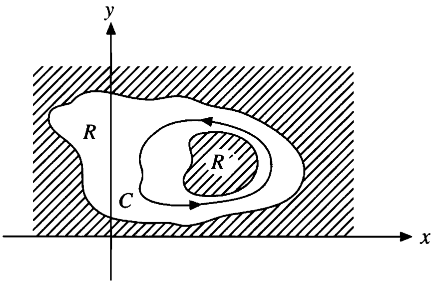

FIGURE 6.8 A closed contour C in a multiply connected region.

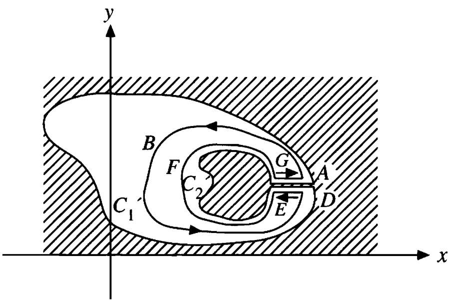

FIGURE 6.9 Conversion of a multiply connected region into a simply connected region.

with $f(z)$ having been continuous across the contour line and line segments DE and GA arbitrarily close together. Then

$$\oint_{C'} f(z)\,dz = \int_{ABD} f(z)\,dz + \int_{EFG} f(z)\,dz = 0 \tag{6.41}$$

by Cauchy's integral theorem, with region R now simply connected. Applying Eq. (6.39) once again with $ABD \to C_1'$ and $EFG \to -C_2'$, we obtain

$$\oint_{C_1'} f(z)\,dz = \oint_{C_2'} f(z)\,dz, \tag{6.42}$$

in which C_1' and C_2' are both traversed in the same (counterclockwise, that is, positive) direction.

Let us emphasize that the contour line here is a matter of mathematical convenience, to permit the application of Cauchy's integral theorem. Since $f(z)$ is analytic in the annular region, it is necessarily single-valued and continuous across any such contour line.

Exercises

6.3.1 Show that $\int_{z_1}^{z_2} f(z)\,dz = -\int_{z_2}^{z_1} f(z)\,dz$.

6.3.2 Prove that

$$\left|\int_C f(z)\,dz\right| \le |f|_{\max} \cdot L,$$

where $|f|_{\max}$ is the maximum value of $|f(z)|$ along the contour C and L is the length of the contour.

6.3.3 Verify that

$$\int_{0,0}^{1,1} z^*\,dz$$

depends on the path by evaluating the integral for the two paths shown in Fig. 6.10. Recall that $f(z) = z^*$ is not an analytic function of z and that Cauchy's integral theorem therefore does not apply.

6.3.4 Show that

$$\oint_C \frac{dz}{z^2 + z} = 0,$$

in which the contour C is a circle defined by $|z| = R > 1$.

Hint. Direct use of the Cauchy integral theorem is illegal. Why? The integral may be evaluated by transforming to polar coordinates and using tables. This yields 0 for $R > 1$ and $2\pi i$ for $R < 1$.

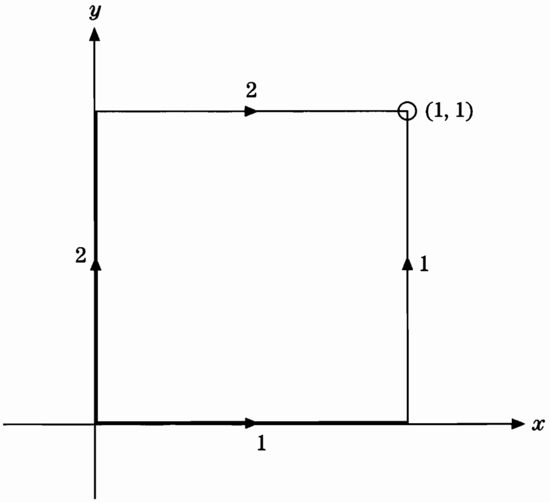

FIGURE 6.10 Contour.

6.4 CAUCHY'S INTEGRAL FORMULA

As in the preceding section, we consider a function $f(z)$ that is analytic on a closed contour C and within the interior region bounded by C. We seek to prove that

$$\boxed{\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz = f(z_0),} \tag{6.43}$$

in which z_0 is any point in the interior region bounded by C. This is the second of the two basic theorems mentioned in Section 6.3. Note that since z is on the contour C while z_0 is in the interior, $z - z_0 \neq 0$ and the integral Eq. (6.43) is well defined. Although $f(z)$ is assumed analytic, the integrand is $f(z)/(z-z_0)$ and is not analytic at $z = z_0$ unless $f(z_0) = 0$. If the contour is deformed as shown in Fig. 6.11 (or Fig. 6.9, Section 6.3), Cauchy's integral theorem applies. By Eq. (6.42),

$$\oint_C \frac{f(z)}{z-z_0}dz - \oint_{C_2} \frac{f(z)}{z-z_0}dz = 0, \tag{6.44}$$

where C is the original outer contour and C_2 is the circle surrounding the point z_0 traversed in a **counterclockwise** direction. Let $z = z_0 + re^{i\theta}$, using the polar representation because of the circular shape of the path around z_0. Here r is small and will eventually be made to approach zero. We have (with $dz = ire^{ir\theta}d\theta$ from Eq. (6.27a))

$$\oint_{C_2} \frac{f(z)}{z-z_0}dz = \oint_{C_2} \frac{f(z_0+re^{i\theta})}{re^{i\theta}} rie^{i\theta}\,d\theta.$$

Taking the limit as $r \to 0$, we obtain

$$\oint_{C_2} \frac{f(z)}{z-z_0}dz = if(z_0)\int_{C_2} d\theta = 2\pi i f(z_0), \tag{6.45}$$

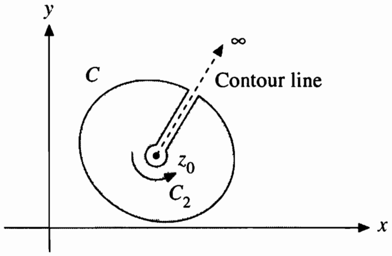

FIGURE 6.11 Exclusion of a singular point.

since $f(z)$ is analytic and therefore continuous at $z = z_0$. This proves the Cauchy integral formula.

Here is a remarkable result. The value of an analytic function $f(z)$ is given at an interior point $z = z_0$ once the values on the boundary C are specified. This is closely analogous to a two-dimensional form of Gauss' law (Section 1.14) in which the magnitude of an interior line charge would be given in terms of the cylindrical surface integral of the electric field **E**.

A further analogy is the determination of a function in real space by an integral of the function and the corresponding Green's function (and their derivatives) over the bounding surface. Kirchhoff diffraction theory is an example of this.

It has been emphasized that z_0 is an interior point. What happens if z_0 is exterior to C? In this case the entire integrand is analytic on and within C. Cauchy's integral theorem, Section 6.3, applies and the integral vanishes. We have

$$\frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z - z_0} = \begin{cases} f(z_0), & z_0 \quad \text{interior} \\ 0, & z_0 \quad \text{exterior.} \end{cases}$$

Derivatives

Cauchy's integral formula may be used to obtain an expression for the derivative of $f(z)$. From Eq. (6.43), with $f(z)$ analytic,

$$\frac{f(z_0 + \delta z_0) - f(z_0)}{\delta z_0} = \frac{1}{2\pi i\,\delta z_0}\left(\oint \frac{f(z)}{z - z_0 - \delta z_0}dz - \oint \frac{f(z)}{z - z_0}dz\right).$$

Then, by definition of derivative (Eq. (6.14)),

$$\begin{aligned} f'(z_0) &= \lim_{\delta z_0 \to 0} \frac{1}{2\pi i\,\delta z_0}\oint \frac{\delta z_0 f(z)}{(z - z_0 - \delta z_0)(z - z_0)}dz \\ &= \frac{1}{2\pi i}\oint \frac{f(z)}{(z - z_0)^2}dz. \end{aligned} \tag{6.46}$$

This result could have been obtained by differentiating Eq. (6.43) under the integral sign with respect to z_0. This formal, or turning-the-crank, approach is valid, but the justification for it is contained in the preceding analysis.

This technique for constructing derivatives may be repeated. We write $f'(z_0 + \delta z_0)$ and $f'(z_0)$, using Eq. (6.46). Subtracting, dividing by δz_0, and finally taking the limit as $\delta z_0 \to 0$, we have

$$f^{(2)}(z_0) = \frac{2}{2\pi i} \oint \frac{f(z)\,dz}{(z - z_0)^3}.$$

Note that $f^{(2)}(z_0)$ is independent of the direction of δz_0, as it must be. Continuing, we get[10]

$$\boxed{f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)\,dz}{(z - z_0)^{n+1}};} \tag{6.47}$$

that is, the requirement that $f(z)$ be analytic guarantees not only a first derivative but derivatives of **all** orders as well! The derivatives of $f(z)$ are automatically analytic. Notice that this statement assumes the Goursat version of the Cauchy integral theorem. This is also why Goursat's contribution is so significant in the development of the theory of complex variables.

Morera's Theorem

A further application of Cauchy's integral formula is in the proof of Morera's **theorem**, which is the converse of Cauchy's integral theorem. The theorem states the following:

> *If a function $f(z)$ is continuous in a simply connected region R and $\oint_C f(z)\,dz = 0$ for every closed contour C within R, then $f(z)$ is analytic throughout R.*

Let us integrate $f(z)$ from z_1 to z_2. Since every closed-path integral of $f(z)$ vanishes, the integral is independent of path and depends only on its endpoints. We label the result of the integration $F(z)$, with

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(z)\,dz. \tag{6.48}$$

As an identity,

$$\frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1) = \frac{\int_{z_1}^{z_2} [f(t) - f(z_1)]\,dt}{z_2 - z_1}, \tag{6.49}$$

using t as another complex variable. Now we take the limit as $z_2 \to z_1$:

$$\lim_{z_2 \to z_1} \frac{\int_{z_1}^{z_2} [f(t) - f(z_1)]\,dt}{z_2 - z_1} = 0, \tag{6.50}$$

[10]This expression is the starting point for defining derivatives of **fractional order**. See A. Erdelyi (ed.), *Tables of Integral Transforms*, Vol. 2. New York: McGraw-Hill (1954). For recent applications to mathematical analysis, see T. J. Osler, An integral analogue of Taylor's series and its use in computing Fourier transforms. *Math. Comput.* **26**: 449 (1972), and references therein.

since $f(t)$ is continuous.[11] Therefore

$$\lim_{z_2 \to z_1} \frac{F(z_2) - F(z_1)}{z_2 - z_1} = F'(z)\Big|_{z=z_1} = f(z_1) \tag{6.51}$$

by definition of derivative (Eq. (6.14)). We have proved that $F'(z)$ at $z = z_1$ exists and equals $f(z_1)$. Since z_1 is any point in R, we see that $F(z)$ is analytic. Then by Cauchy's integral formula (compare Eq. (6.47)), $F'(z) = f(z)$ is also analytic, proving Morera's theorem.

Drawing once more on our electrostatic analog, we might use $f(z)$ to represent the electrostatic field **E**. If the net charge within every closed region in R is zero (Gauss' law), the charge density is everywhere zero in R. Alternatively, in terms of the analysis of Section 1.13, $f(z)$ represents a conservative force (by definition of conservative), and then we find that it is always possible to express it as the derivative of a potential function $F(z)$.

An important application of Cauchy's integral formula is the following **Cauchy inequality**. If $f(z) = \sum a_n z^n$ is analytic and bounded, $|f(z)| \le M$ on a circle of radius r about the origin, then

$$|a_n| r^n \le M \qquad \text{(Cauchy's inequality)} \tag{6.52}$$

gives upper bounds for the coefficients of its Taylor expansion. To prove Eq. (6.52) let us define $M(r) = \max_{|z|=r} |f(z)|$ and use the Cauchy integral for a_n:

$$|a_n| = \frac{1}{2\pi} \left| \int_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right| \le M(r) \frac{2\pi r}{2\pi r^{n+1}}.$$

An immediate consequence of the inequality (6.52) is **Liouville's theorem**: If $f(z)$ is analytic and bounded in the entire complex plane it is a constant. In fact, if $|f(z)| \le M$ for all z, then Cauchy's inequality (6.52) gives $|a_n| \le M r^{-n} \to 0$ as $r \to \infty$ for $n > 0$. Hence $f(z) = a_0$.

Conversely, the slightest deviation of an analytic function from a constant value implies that there must be at least one singularity somewhere in the infinite complex plane. Apart from the trivial constant functions, then, singularities are a fact of life, and we must learn to live with them. But we shall do more than that. We shall next expand a function in a Laurent series at a singularity, and we shall use singularities to develop the powerful and useful calculus of residues in Chapter 7.

A famous application of Liouville's theorem yields the **fundamental theorem of algebra** (due to C. F. Gauss), which says that any polynomial $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ with $n > 0$ and $a_n \ne 0$ has n roots. To prove this, suppose $P(z)$ has no zero. Then $1/P(z)$ is analytic and bounded as $|z| \to \infty$. Hence $P(z)$ is a constant by Liouville's theorem, q.e.a. Thus, $P(z)$ has at least one root that we can divide out. Then we repeat the process for the resulting polynomial of degree $n - 1$. This leads to the conclusion that $P(z)$ has exactly n roots.

[11] We quote the mean value theorem of calculus here.

Exercises

6.4.1 Show that

$$\oint_C (z-z_0)^n\,dz = \begin{cases} 2\pi i, & n=-1, \\ 0, & n\neq -1, \end{cases}$$

where the contour C encircles the point $z=z_0$ in a positive (counterclockwise) sense. The exponent n is an integer. See also Eq. (6.27a). The calculus of residues, Chapter 7, is based on this result.

6.4.2 Show that

$$\frac{1}{2\pi i}\oint z^{m-n-1}\,dz, \qquad m \text{ and } n \text{ integers}$$

(with the contour encircling the origin once counterclockwise) is a representation of the Kronecker δ_{mn}.

6.4.3 Solve Exercise 6.3.4 by separating the integrand into partial fractions and then applying Cauchy's integral theorem for multiply connected regions.
Note. Partial fractions are explained in Section 15.8 in connection with Laplace transforms.

6.4.4 Evaluate

$$\oint_C \frac{dz}{z^2-1},$$

where C is the circle $|z|=2$.

6.4.5 Assuming that $f(z)$ is analytic on and within a closed contour C and that the point z_0 is within C, show that

$$\oint_C \frac{f'(z)}{z-z_0}\,dz = \oint_C \frac{f(z)}{(z-z_0)^2}\,dz.$$

6.4.6 You know that $f(z)$ is analytic on and within a closed contour C. You suspect that the nth derivative $f^{(n)}(z_0)$ is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz.$$

Using mathematical induction, prove that this expression is correct.

6.4.7 (a) A function $f(z)$ is analytic within a closed contour C (and continuous on C). If $f(z)\neq 0$ within C and $|f(z)|\leq M$ on C, show that

$$|f(z)|\leq M$$

for all points within C.
Hint. Consider $w(z)=1/f(z)$.

(b) If $f(z)=0$ within the contour C, show that the foregoing result does not hold and that it is possible to have $|f(z)|=0$ at one or more points in the interior with $|f(z)|>0$ over the entire bounding contour. Cite a specific example of an analytic function that behaves this way.

6.4.8 Using the Cauchy integral formula for the nth derivative, convert the following Rodrigues formulas into the corresponding so-called Schlaefli integrals.

(a) Legendre:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n.$$

ANS. $\dfrac{(-1)^n}{2^n}\cdot\dfrac{1}{2\pi i}\oint\dfrac{(1-z^2)^n}{(z-x)^{n+1}}dz.$

(b) Hermite:

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}.$$

(c) Laguerre:

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right).$$

Note. From the Schlaefli integral representations one can develop generating functions for these special functions. Compare Sections 12.4, 13.1, and 13.2.

6.5 LAURENT EXPANSION

Taylor Expansion

The Cauchy integral formula of the preceding section opens up the way for another derivation of Taylor's series (Section 5.6), but this time for functions of a complex variable. Suppose we are trying to expand $f(z)$ about $z = z_0$ and we have $z = z_1$ as the nearest point on the Argand diagram for which $f(z)$ is not analytic. We construct a circle C centered at $z = z_0$ with radius less than $|z_1 - z_0|$ (Fig. 6.12). Since z_1 was assumed to be the nearest point at which $f(z)$ was not analytic, $f(z)$ is necessarily analytic on and within C.

From Eq. (6.43), the Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)[1-(z-z_0)/(z'-z_0)]}. \end{aligned} \tag{6.53}$$

Here z' is a point on the contour C and z is any point interior to C. It is not legal yet to expand the denominator of the integrand in Eq. (6.53) by the binomial theorem, for we have not yet proved the binomial theorem for complex variables. Instead, we note the identity

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n, \tag{6.54}$$

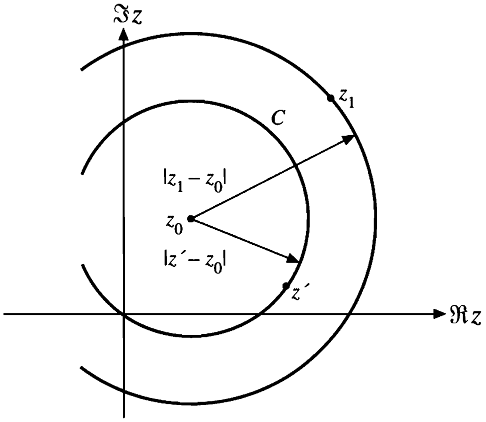

FIGURE 6.12 Circular domain for Taylor expansion.

which may easily be verified by multiplying both sides by $1 - t$. The infinite series, following the methods of Section 5.2, is convergent for $|t| < 1$.

Now, for a point z interior to C, $|z - z_0| < |z' - z_0|$, and, using Eq. (6.54), Eq. (6.53) becomes

$$f(z) = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{(z - z_0)^n f(z')\, dz'}{(z' - z_0)^{n+1}}. \tag{6.55}$$

Interchanging the order of integration and summation (valid because Eq. (6.54) is uniformly convergent for $|t| < 1$), we obtain

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')\, dz'}{(z' - z_0)^{n+1}}. \tag{6.56}$$

Referring to Eq. (6.47), we get

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \frac{f^{(n)}(z_0)}{n!}, \tag{6.57}$$

which is our desired Taylor expansion. Note that it is based only on the assumption that $f(z)$ is analytic for $|z - z_0| < |z_1 - z_0|$. Just as for real variable power series (Section 5.7), this expansion is unique for a given z_0.

From the Taylor expansion for $f(z)$ a binomial theorem may be derived (Exercise 6.5.2).

Schwarz Reflection Principle

From the binomial expansion of $g(z) = (z - x_0)^n$ for integral n it is easy to see that the complex conjugate of the function g is the function of the complex conjugate for real x_0:

$$g^*(z) = \left[(z - x_0)^n\right]^* = (z^* - x_0)^n = g(z^*). \tag{6.58}$$

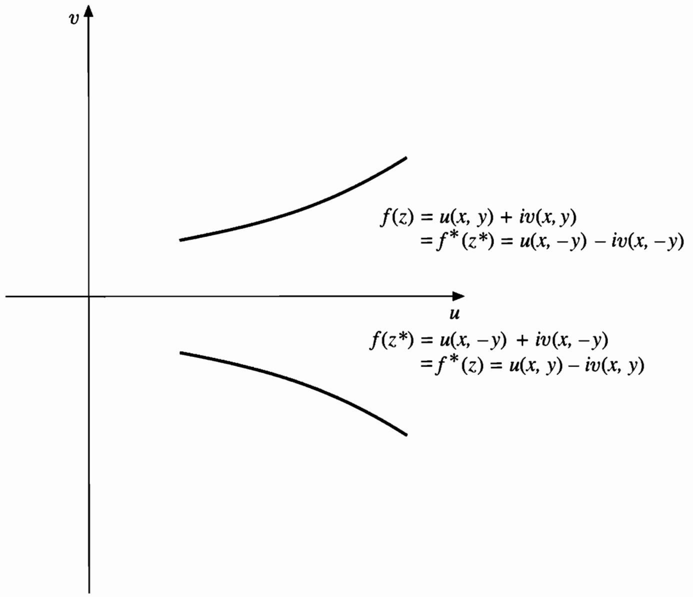

FIGURE 6.13 Schwarz reflection.

This leads us to the Schwarz reflection principle:

> *If a function $f(z)$ is* (1) *analytic over some region including the real axis and* (2) *real when z is real, then*
>
> $$f^*(z) = f(z^*). \tag{6.59}$$

(See Fig. 6.13.)

Expanding $f(z)$ about some (nonsingular) point x_0 on the real axis,

$$f(z) = \sum_{n=0}^{\infty} (z - x_0)^n \frac{f^{(n)}(x_0)}{n!} \tag{6.60}$$

by Eq. (6.56). Since $f(z)$ is analytic at $z = x_0$, this Taylor expansion exists. Since $f(z)$ is real when z is real, $f^{(n)}(x_0)$ must be real for all n. Then when we use Eq. (6.58), Eq. (6.59), the Schwarz reflection principle, follows immediately. Exercise 6.5.6 is another form of this principle. This completes the proof within a circle of convergence. Analytic continuation then permits extending this result to the entire region of analyticity.

Analytic Continuation

It is natural to think of the values $f(z)$ of an analytic function f as a single entity, which is usually defined in some restricted region S_1 of the complex plane, for example, by a Taylor series (see Fig. 6.14). Then f is analytic inside the **circle of convergence** C_1, whose radius is given by the distance r_1 from the center of C_1 to the **nearest singularity** of f at z_1 (in Fig. 6.14). A singularity is any point where f is not analytic. If we choose a point inside C_1

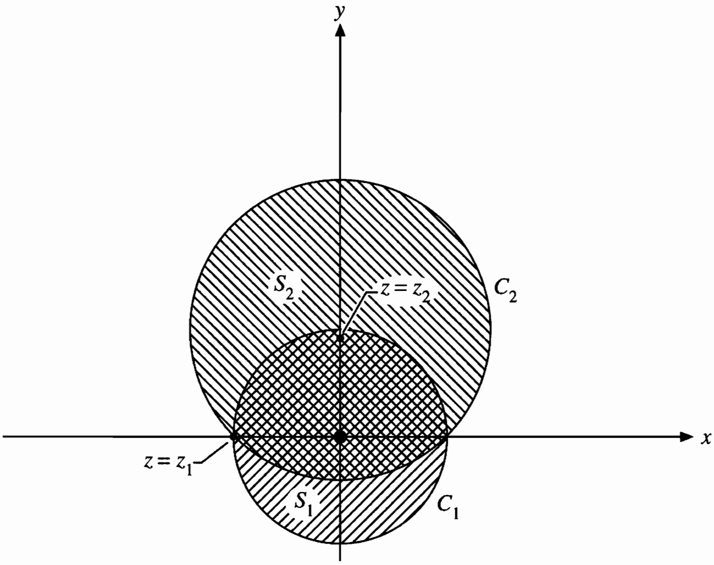

FIGURE 6.14 Analytic continuation.

that is farther than r_1 from the singularity z_1 and make a Taylor expansion of f about it (z_2 in Fig. 6.14), then the circle of convergence, C_2 will usually extend beyond the first circle, C_1. In the overlap region of both circles, C_1, C_2, the function f is uniquely defined. In the region of the circle C_2 that extends beyond C_1, $f(z)$ is uniquely defined by the Taylor series about the center of C_2 and is analytic there, although the Taylor series about the center of C_1 is no longer convergent there. After Weierstrass this process is called **analytic continuation**. It defines the analytic functions in terms of its original definition (in C_1, say) and all its continuations.

A specific example is the function

$$f(z) = \frac{1}{1+z}, \tag{6.61}$$

which has a (simple) pole at $z = -1$ and is analytic elsewhere. The geometric series expansion

$$\frac{1}{1+z} = 1 - z + z^2 + \cdots = \sum_{n=0}^{\infty}(-z)^n \tag{6.62}$$

converges for $|z| < 1$, that is, inside the circle C_1 in Fig. 6.14.

Suppose we expand $f(z)$ about $z = i$, so

$$f(z) = \frac{1}{1+z} = \frac{1}{1+i+(z-i)} = \frac{1}{(1+i)(1+(z-i)/(1+i))}$$
$$= \left[1 - \frac{z-i}{1+i} + \frac{(z-i)^2}{(1+i)^2} - \cdots\right]\frac{1}{1+i} \tag{6.63}$$

converges for $|z - i| < |1 + i| = \sqrt{2}$. Our circle of convergence is C_2 in Fig. 6.14. Now

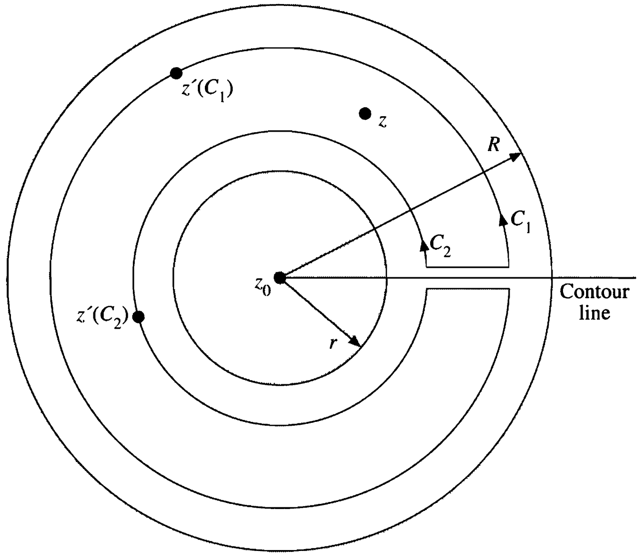

FIGURE 6.15 $|z' - z_0|_{C_1} > |z - z_0|$; $|z' - z_0|_{C_2} < |z - z_0|$.

$f(z)$ is defined by the expansion (6.63) in S_2, which overlaps S_1 and extends further out in the complex plane.[12] This extension is an analytic continuation, and when we have only isolated singular points to contend with, the function can be extended indefinitely. Equations (6.61), (6.62), and (6.63) are three different representations of the same function. Each representation has its own domain of convergence. Equation (6.62) is a Maclaurin series. Equation (6.63) is a Taylor expansion about $z = i$ and from the following paragraphs Eq. (6.61) is seen to be a one-term Laurent series.

Analytic continuation may take many forms, and the series expansion just considered is not necessarily the most convenient technique. As an alternate technique we shall use a functional relation in Section 8.1 to extend the factorial function around the isolated singular points $z = -n$, $n = 1, 2, 3, \ldots$. As another example, the hypergeometric equation is satisfied by the hypergeometric function defined by the series, Eq. (13.115), for $|z| < 1$. The integral representation given in Exercise 13.4.7 permits a continuation into the complex plane.

[12]One of the most powerful and beautiful results of the more abstract theory of functions of a complex variable is that if two analytic functions coincide in any region, such as the overlap of S_1 and S_2, or coincide on any line segment, they are the same function, in the sense that they will coincide everywhere as long as they are both well defined. In this case the agreement of the expansions (Eqs. (6.62) and (6.63)) over the region common to S_1 and S_2 would establish the identity of the functions these expansions represent. Then Eq. (6.63) would represent an analytic continuation or extension of $f(z)$ into regions not covered by Eq. (6.62). We could equally well say that $f(z) = 1/(1 + z)$ is itself an analytic continuation of either of the series given by Eqs. (6.62) and (6.63).

Laurent Series

We frequently encounter functions that are analytic and single-valued in an annular region, say, of inner radius r and outer radius R, as shown in Fig. 6.15. Drawing an imaginary contour line to convert our region into a simply connected region, we apply Cauchy's integral formula, and for two circles C_2 and C_1 centered at $z = z_0$ and with radii r_2 and r_1, respectively, where $r < r_2 < r_1 < R$, we have[13]

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\,dz'}{z' - z} - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')\,dz'}{z' - z}. \tag{6.64}$$

Note that in Eq. (6.64) an explicit minus sign has been introduced so that the contour C_2 (like C_1) is to be traversed in the positive (counterclockwise) sense. The treatment of Eq. (6.64) now proceeds exactly like that of Eq. (6.53) in the development of the Taylor series. Each denominator is written as $(z' - z_0) - (z - z_0)$ and expanded by the binomial theorem, which now follows from the Taylor series (Eq. (6.57)).

Noting that for C_1, $|z' - z_0| > |z - z_0|$ while for C_2, $|z' - z_0| < |z - z_0|$, we find

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_{C_1} \frac{f(z')\,dz'}{(z' - z_0)^{n+1}} + \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z')\,dz'. \tag{6.65}$$

The minus sign of Eq. (6.64) has been absorbed by the binomial expansion. Labeling the first series S_1 and the second S_2 we have

$$S_1 = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \oint_{C_1} \frac{f(z')\,dz'}{(z' - z_0)^{n+1}}, \tag{6.66}$$

which is the regular Taylor expansion, convergent for $|z - z_0| < |z' - z_0| = r_1$, that is, for all z **interior** to the larger circle, C_1. For the second series in Eq. (6.65) we have

$$S_2 = \frac{1}{2\pi i}\sum_{n=1}^{\infty}(z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z')\,dz', \tag{6.67}$$

convergent for $|z - z_0| > |z' - z_0| = r_2$, that is, for all z **exterior** to the smaller circle, C_2. Remember, C_2 now goes counterclockwise.

These two series are combined into one series[14] (a Laurent series) by

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,} \tag{6.68}$$

[13]We may take r_2 arbitrarily close to r and r_1 arbitrarily close to R, maximizing the area enclosed between C_1 and C_2.

[14]Replace n by $-n$ in S_2 and add.

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}. \tag{6.69}$$

Since, in Eq. (6.69), convergence of a binomial expansion is no longer a problem, C may be any contour within the annular region $r < |z - z_0| < R$ encircling z_0 once in a counterclockwise sense. If we assume that such an annular region of convergence does exist, then Eq. (6.68) is the Laurent series, or Laurent expansion, of $f(z)$.

The use of the contour line (Fig. 6.15) is convenient in converting the annular region into a simply connected region. Since our function is analytic in this annular region (and single-valued), the contour line is not essential and, indeed, does not appear in the final result, Eq. (6.69).

Laurent series coefficients need not come from evaluation of contour integrals (which may be very intractable). Other techniques, such as ordinary series expansions, may provide the coefficients.

Numerous examples of Laurent series appear in Chapter 7. We limit ourselves here to one simple example to illustrate the application of Eq. (6.68).

Example 6.5.1 Laurent Expansion

Let $f(z) = [z(z-1)]^{-1}$. If we choose $z_0 = 0$, then $r = 0$ and $R = 1$, $f(z)$ diverging at $z = 1$. A partial fraction expansion yields the Laurent series

$$\frac{1}{z(z-1)} = -\frac{1}{1-z} - \frac{1}{z} = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots = -\sum_{n=-1}^{\infty} z^n. \tag{6.70}$$

From Eqs. (6.70), (6.68), and (6.69) we then have

$$a_n = \frac{1}{2\pi i}\oint \frac{dz'}{(z')^{n+2}(z'-1)} = \begin{cases} -1 & \text{for } n \ge -1, \\ 0 & \text{for } n < -1. \end{cases} \tag{6.71}$$

The integrals in Eq. (6.71) can also be directly evaluated by substituting the geometric-series expansion of $(1-z')^{-1}$ used already in Eq. (6.70) for $(1-z)^{-1}$:

$$a_n = \frac{-1}{2\pi i}\oint \sum_{m=0}^{\infty} (z')^m \frac{dz'}{(z')^{n+2}}. \tag{6.72}$$

Upon interchanging the order of summation and integration (uniformly convergent series), we have

$$a_n = -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint \frac{dz'}{(z')^{n+2-m}}. \tag{6.73}$$

If we employ the polar form, as in Eq. (6.47) (or compare Exercise 6.4.1),

$$a_n = -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint \frac{rie^{i\theta}\,d\theta}{r^{n+2-m}e^{i(n+2-m)\theta}}$$

$$= -\frac{1}{2\pi i}\cdot 2\pi i\sum_{m=0}^{\infty}\delta_{n+2-m,1}, \tag{6.74}$$

which agrees with Eq. (6.71). ■

The Laurent series differs from the Taylor series by the obvious feature of negative powers of $(z - z_0)$. For this reason the Laurent series will always diverge at least at $z = z_0$ and perhaps as far out as some distance r (Fig. 6.15).

Exercises

6.5.1 Develop the Taylor expansion of $\ln(1 + z)$.

ANS. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$.

6.5.2 Derive the binomial expansion

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1\cdot 2}z^2 + \cdots = \sum_{n=0}^{\infty}\binom{m}{n}z^n$$

for m any real number. The expansion is convergent for $|z| < 1$. Why?

6.5.3 A function $f(z)$ is analytic on and within the unit circle. Also, $|f(z)| < 1$ for $|z| \le 1$ and $f(0) = 0$. Show that $|f(z)| < |z|$ for $|z| \le 1$.
Hint. One approach is to show that $f(z)/z$ is analytic and then to express $[f(z_0)/z_0]^n$ by the Cauchy integral formula. Finally, consider absolute magnitudes and take the nth root. This exercise is sometimes called Schwarz's theorem.

6.5.4 If $f(z)$ is a real function of the complex variable $z = x + iy$, that is, if $f(x) = f^*(x)$, and the Laurent expansion about the origin, $f(z) = \sum a_n z^n$, has $a_n = 0$ for $n < -N$, show that all of the coefficients a_n are real.
Hint. Show that $z^N f(z)$ is analytic (via Morera's theorem, Section 6.4).

6.5.5 A function $f(z) = u(x, y) + iv(x, y)$ satisfies the conditions for the Schwarz reflection principle. Show that
(a) u is an even function of y. (b) v is an odd function of y.

6.5.6 A function $f(z)$ can be expanded in a Laurent series about the origin with the coefficients a_n real. Show that the complex conjugate of this function of z is the same function of the complex conjugate of z; that is,

$$f^*(z) = f(z^*).$$

Verify this explicitly for
(a) $f(z) = z^n$, n an integer, (b) $f(z) = \sin z$.
If $f(z) = iz$ $(a_1 = i)$, show that the foregoing statement does not hold.

6.5.7 The function $f(z)$ is analytic in a domain that includes the real axis. When z is real ($z = x$), $f(x)$ is pure imaginary.

(a) Show that

$$f(z^*) = -[f(z)]^*.$$

(b) For the specific case $f(z) = iz$, develop the Cartesian forms of $f(z)$, $f(z^*)$, and $f^*(z)$. Do not quote the general result of part (a).

6.5.8 Develop the first three nonzero terms of the Laurent expansion of

$$f(z) = (e^z - 1)^{-1}$$

about the origin. Notice the resemblance to the Bernoulli number–generating function, Eq. (5.144) of Section 5.9.

6.5.9 Prove that the Laurent expansion of a given function about a given point is unique; that is, if

$$f(z) = \sum_{n=-N}^{\infty} a_n(z - z_0)^n = \sum_{n=-N}^{\infty} b_n(z - z_0)^n,$$

show that $a_n = b_n$ for all n.
Hint. Use the Cauchy integral formula.

6.5.10 (a) Develop a Laurent expansion of $f(z) = [z(z-1)]^{-1}$ about the point $z = 1$ valid for small values of $|z - 1|$. Specify the exact range over which your expansion holds. This is an analytic continuation of Eq. (6.70).

(b) Determine the Laurent expansion of $f(z)$ about $z = 1$ but for $|z - 1|$ large.

Hint. Partial fraction this function and use the geometric series.

6.5.11 (a) Given $f_1(z) = \int_0^{\infty} e^{-zt}\,dt$ (with t real), show that the domain in which $f_1(z)$ exists (and is analytic) is $\Re(z) > 0$.

(b) Show that $f_2(z) = 1/z$ equals $f_1(z)$ over $\Re(z) > 0$ and is therefore an analytic continuation of $f_1(z)$ over the entire z-plane except for $z = 0$.

(c) Expand $1/z$ about the point $z = i$. You will have $f_3(z) = \sum_{n=0}^{\infty} a_n(z - i)^n$. What is the domain of $f_3(z)$?

$$\text{ANS. } \frac{1}{z} = -i\sum_{n=0}^{\infty} i^n (z - i)^n, \qquad |z - i| < 1.$$

6.6 SINGULARITIES

The Laurent expansion represents a generalization of the Taylor series in the presence of singularities. We define the point z_0 as an **isolated singular point** of the function $f(z)$ if $f(z)$ is not analytic at $z = z_0$ but is analytic at all neighboring points.

Poles

In the Laurent expansion of $f(z)$ about z_0,

$$f(z) = \sum_{m=-\infty}^{\infty} a_m (z - z_0)^m, \tag{6.75}$$

if $a_m = 0$ for $m < -n < 0$ and $a_{-n} \neq 0$, we say that z_0 is a pole of order n. For instance, if $n = 1$, that is, if $a_{-1}/(z - z_0)$ is the first nonvanishing term in the Laurent series, we have a pole of order 1, often called a **simple** pole.

If, on the other hand, the summation continues to $m = -\infty$, then z_0 is a pole of infinite order and is called an **essential singularity**. These essential singularities have many pathological features. For instance, we can show that in any small neighborhood of an essential singularity of $f(z)$ the function $f(z)$ comes arbitrarily close to any (and therefore every) preselected complex quantity w_0.[15] Here, the entire w-plane is mapped by f into the neighborhood of the point z_0. One point of fundamental difference between a pole of finite order n and an essential singularity is that by multiplying $f(z)$ by $(z - z_0)^n$, $f(z)(z - z_0)^n$ is no longer singular at z_0. This obviously cannot be done for an essential singularity.

The behavior of $f(z)$ as $z \to \infty$ is defined in terms of the behavior of $f(1/t)$ as $t \to 0$. Consider the function

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}. \tag{6.76}$$

As $z \to \infty$, we replace the z by $1/t$ to obtain

$$\sin\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,t^{2n+1}}. \tag{6.77}$$

From the definition, $\sin z$ has an essential singularity at infinity. This result could be anticipated from Exercise 6.1.9 since

$$\sin z = \sin iy = i \sinh y, \qquad \text{when } x = 0,$$

which approaches infinity exponentially as $y \to \infty$. Thus, although the absolute value of $\sin x$ for real x is equal to or less than unity, the absolute value of $\sin z$ is not bounded.

A function that is analytic throughout the finite complex plane **except** for isolated poles is called **meromorphic**, such as ratios of two polynomials or $\tan z$, $\cot z$. Examples are also **entire** functions that have no singularities in the finite complex plane, such as $\exp(z)$, $\sin z$, $\cos z$ (see Sections 5.9, 5.11).

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[15]This theorem is due to Picard. A proof is given by E. C. Titchmarsh, *The Theory of Functions*, 2nd ed. New York: Oxford University Press (1939).

Branch Points

There is another sort of singularity that will be important in Chapter 7. Consider

$$f(z) = z^a,$$

in which a is not an integer.[16] As z moves around the unit circle from e^0 to $e^{2\pi i}$,

$$f(z) \to e^{2\pi a i} \neq e^{0 \cdot a} = 1,$$

for nonintegral a. We have a branch point at the origin and another at infinity. If we set $z = 1/t$, a similar analysis of $f(z)$ for $t \to 0$ shows that $t = 0$; that is, $z = \infty$ is also a branch point. The points e^{0i} and $e^{2\pi i}$ in the z-plane coincide, but these **coincident points lead to different values** of $f(z)$; that is, $f(z)$ is a **multivalued function**. The problem is resolved by constructing a **cut line joining both branch points** so that $f(z)$ will be uniquely specified for a given point in the z-plane. For z^a, the cut line can go out at any angle. Note that the point at infinity must be included here; that is, the cut line may join finite branch points via the point at infinity. The next example is a case in point. If $a = p/q$ is a rational number, then q is called the order of the branch point, because one needs to go around the branch point q times before coming back to the starting point. If a is irrational, then the order of the branch point is infinite, just as for the logarithm.

Note that a function with a branch point and a required cut line will not be continuous across the cut line. Often there will be a phase difference on opposite sides of this cut line. Hence line integrals on opposite sides of this branch point cut line will not generally cancel each other. Numerous examples of this case appear in the exercises.

The contour line used to convert a multiply connected region into a simply connected region (Section 6.3) is completely different. Our function is continuous across that contour line, and no phase difference exists.

Example 6.6.1 BRANCH POINTS OF ORDER 2

Consider the function

$$f(z) = \left(z^2 - 1\right)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}. \tag{6.78}$$

The first factor on the right-hand side, $(z+1)^{1/2}$, has a branch point at $z = -1$. The second factor has a branch point at $z = +1$. At infinity $f(z)$ has a simple pole. This is best seen by substituting $z = 1/t$ and making a binomial expansion at $t = 0$:

$$\left(z^2 - 1\right)^{1/2} = \frac{1}{t}\left(1 - t^2\right)^{1/2} = \frac{1}{t}\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n t^{2n} = \frac{1}{t} - \frac{1}{2}t - \frac{1}{8}t^3 + \cdots.$$

The cut line has to connect both branch points, so it is not possible to encircle either branch point completely. To check on the possibility of taking the line segment joining $z = +1$ and

[16] $z = 0$ is a singular point, for z^a has only a finite number of derivatives, whereas an analytic function is guaranteed an infinite number of derivatives (Section 6.4). The problem is that $f(z)$ is not single-valued as we encircle the origin. The Cauchy integral formula may not be applied.

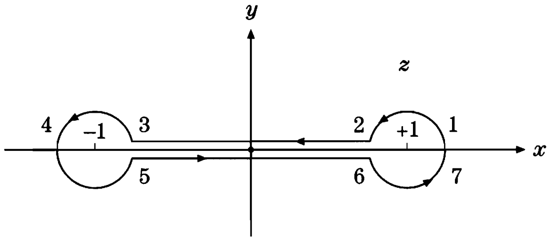

FIGURE 6.16 Branch cut and phases of Table 6.1.

Table 6.1 Phase Angle

Point	θ	φ	$\frac{\theta+\varphi}{2}$
1	0	0	0
2	0	π	$\frac{\pi}{2}$
3	0	π	$\frac{\pi}{2}$
4	π	π	π
5	2π	π	$\frac{3\pi}{2}$
6	2π	π	$\frac{3\pi}{2}$
7	2π	2π	2π

$z = -1$ as a cut line, let us follow the phases of these two factors as we move along the contour shown in Fig. 6.16.

For convenience in following the changes of phase let $z + 1 = re^{i\theta}$ and $z - 1 = \rho e^{i\varphi}$. Then the phase of $f(z)$ is $(\theta + \varphi)/2$. We start at point 1, where both $z + 1$ and $z - 1$ have a phase of zero. Moving from point 1 to point 2, φ, the phase of $z - 1 = \rho e^{i\varphi}$, increases by π. ($z - 1$ becomes negative.) φ then stays constant until the circle is completed, moving from 6 to 7. θ, the phase of $z + 1 = re^{i\theta}$, shows a similar behavior, increasing by 2π as we move from 3 to 5. The phase of the function $f(z) = (z + 1)^{1/2}(z - 1)^{1/2} = r^{1/2}\rho^{1/2}e^{i(\theta+\varphi)/2}$ is $(\theta + \varphi)/2$. This is tabulated in the final column of Table 6.1.

Two features emerge:

1. The phase at points 5 and 6 is not the same as the phase at points 2 and 3. This behavior can be expected at a branch cut.

2. The phase at point 7 exceeds that at point 1 by 2π, and the function $f(z) = (z^2 - 1)^{1/2}$ is therefore **single-valued** for the contour shown, encircling **both** branch points.

If we take the x-axis, $-1 \le x \le 1$, as a cut line, $f(z)$ is uniquely specified. Alternatively, the positive x-axis for $x > 1$ and the negative x-axis for $x < -1$ may be taken as cut lines. The branch points cannot be encircled, and the function remains single-valued. These two cut lines are, in fact, one branch cut from -1 to $+1$ via the point at infinity. ■

Generalizing from this example, we have that the phase of a function

$$f(z) = f_1(z) \cdot f_2(z) \cdot f_3(z) \cdots$$

is the algebraic sum of the phase of its individual factors:

$$\arg f(z) = \arg f_1(z) + \arg f_2(z) + \arg f_3(z) + \cdots.$$

The phase of an individual factor may be taken as the arctangent of the ratio of its imaginary part to its real part (choosing the appropriate branch of the arctan function $\tan^{-1} y/x$, which has infinitely many branches),

$$\arg f_i(z) = \tan^{-1}\left(\frac{v_i}{u_i}\right).$$

For the case of a factor of the form

$$f_i(z) = (z - z_0),$$

the phase corresponds to the phase angle of a two-dimensional vector from $+z_0$ to z, the phase increasing by 2π as the point $+z_0$ is encircled. Conversely, the traversal of any closed loop not encircling z_0 does not change the phase of $z - z_0$.

Exercises

6.6.1 The function $f(z)$ expanded in a Laurent series exhibits a pole of order m at $z = z_0$. Show that the coefficient of $(z - z_0)^{-1}$, a_{-1}, is given by

$$a_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z - z_0)^m f(z)\right]_{z=z_0},$$

with

$$a_{-1} = \left[(z - z_0) f(z)\right]_{z=z_0},$$

when the pole is a simple pole ($m = 1$). These equations for a_{-1} are extremely useful in determining the residue to be used in the residue theorem of Section 7.1.
Hint. The technique that was so successful in proving the uniqueness of power series, Section 5.7, will work here also.

6.6.2 A function $f(z)$ can be represented by

$$f(z) = \frac{f_1(z)}{f_2(z)},$$

in which $f_1(z)$ and $f_2(z)$ are analytic. The denominator, $f_2(z)$, vanishes at $z = z_0$, showing that $f(z)$ has a pole at $z = z_0$. However, $f_1(z_0) \neq 0$, $f_2'(z_0) \neq 0$. Show that a_{-1}, the coefficient of $(z - z_0)^{-1}$ in a Laurent expansion of $f(z)$ at $z = z_0$, is given by

$$a_{-1} = \frac{f_1(z_0)}{f_2'(z_0)}.$$

(This result leads to the Heaviside expansion theorem, Exercise 15.12.11.)

6.6.3 In analogy with Example 6.6.1, consider in detail the phase of each factor and the resultant overall phase of $f(z) = (z^2 + 1)^{1/2}$ following a contour similar to that of Fig. 6.16 but encircling the new branch points.

6.6.4 The Legendre function of the second kind, $Q_\nu(z)$, has branch points at $z = \pm 1$. The branch points are joined by a cut line along the real (x) axis.

(a) Show that $Q_0(z) = \frac{1}{2}\ln((z+1)/(z-1))$ is single-valued (with the real axis $-1 \le x \le 1$ taken as a cut line).

(b) For real argument x and $|x| < 1$ it is convenient to take

$$Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

Show that

$$Q_0(x) = \frac{1}{2}\bigl[Q_0(x+i0) + Q_0(x-i0)\bigr].$$

Here $x + i0$ indicates that z approaches the real axis from above, and $x - i0$ indicates an approach from below.

6.6.5 As an example of an essential singularity, consider $e^{1/z}$ as z approaches zero. For any complex number z_0, $z_0 \neq 0$, show that

$$e^{1/z} = z_0$$

has an infinite number of solutions.

6.7 MAPPING

In the preceding sections we have defined analytic functions and developed some of their main features. Here we introduce some of the more geometric aspects of functions of complex variables, aspects that will be useful in visualizing the integral operations in Chapter 7 and that are valuable in their own right in solving Laplace's equation in two-dimensional systems.

In ordinary analytic geometry we may take $y = f(x)$ and then plot y versus x. Our problem here is more complicated, for z is a function of two variables, x and y. We use the notation

$$w = f(z) = u(x, y) + iv(x, y). \tag{6.79}$$

Then for a point in the z-plane (specific values for x and y) there may correspond specific values for $u(x, y)$ and $v(x, y)$ that then yield a point in the w-plane. As points in the z-plane transform, or are mapped into points in the w-plane, lines or areas in the z-plane will be mapped into lines or areas in the w-plane. Our immediate purpose is to see how lines and areas map from the z-plane to the w-plane for a number of simple functions.

Translation

$$w = z + z_0. \tag{6.80}$$

The function w is equal to the variable z plus a constant, $z_0 = x_0 + iy_0$. By Eqs. (6.1) and (6.79),

$$u = x + x_0, \qquad v = y + y_0, \tag{6.81}$$

representing a pure translation of the coordinate axes, as shown in Fig. 6.17.

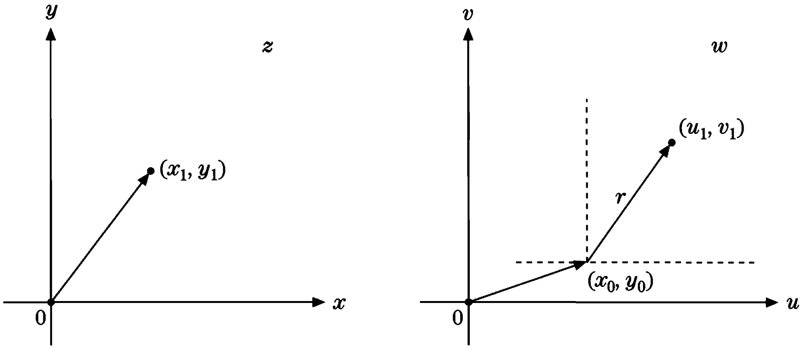

FIGURE 6.17 Translation.

Rotation

$$w = zz_0. \tag{6.82}$$

Here it is convenient to return to the polar representation, using

$$w = \rho e^{i\varphi}, \qquad z = re^{i\theta}, \qquad \text{and} \qquad z_0 = r_0 e^{i\theta_0}, \tag{6.83}$$

then

$$\rho e^{i\varphi} = rr_0 e^{i(\theta+\theta_0)}, \tag{6.84}$$

or

$$\rho = rr_0, \qquad \varphi = \theta + \theta_0. \tag{6.85}$$

Two things have occurred. First, the modulus r has been modified, either expanded or contracted, by the factor r_0. Second, the argument θ has been increased by the additive constant θ_0 (Fig. 6.18). This represents a rotation of the complex variable through an angle θ_0. For the special case of $z_0 = i$, we have a pure rotation through $\pi/2$ radians.

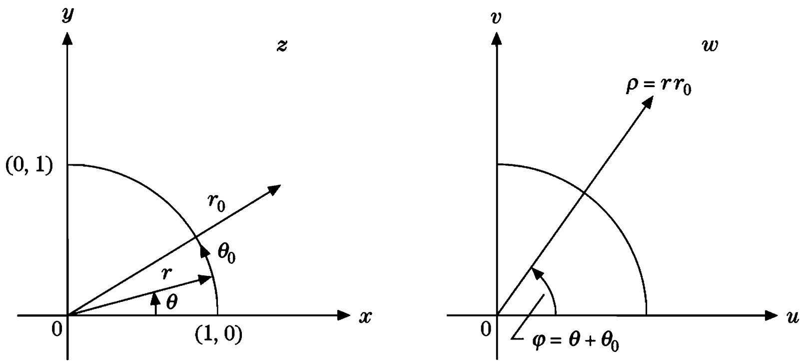

FIGURE 6.18 Rotation.

Inversion

$$w = \frac{1}{z}. \tag{6.86}$$

Again, using the polar form, we have

$$\rho e^{i\varphi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}, \tag{6.87}$$

which shows that

$$\rho = \frac{1}{r}, \qquad \varphi = -\theta. \tag{6.88}$$

The first part of Eq. (6.87) shows that inversion clearly. The interior of the unit circle is mapped onto the exterior and vice versa (Fig. 6.19). In addition, the second part of Eq. (6.87) shows that the polar angle is reversed in sign. Equation (6.88) therefore also involves a reflection of the y-axis, exactly like the complex conjugate equation.

To see how curves in the z-plane transform into the w-plane, we return to the Cartesian form:

$$u + iv = \frac{1}{x + iy}. \tag{6.89}$$

Rationalizing the right-hand side by multiplying numerator and denominator by z^* and then equating the real parts and the imaginary parts, we have

$$\begin{aligned} u &= \frac{x}{x^2+y^2}, & x &= \frac{u}{u^2+v^2}, \\ v &= -\frac{y}{x^2+y^2}, & y &= -\frac{v}{u^2+v^2}. \end{aligned} \tag{6.90}$$

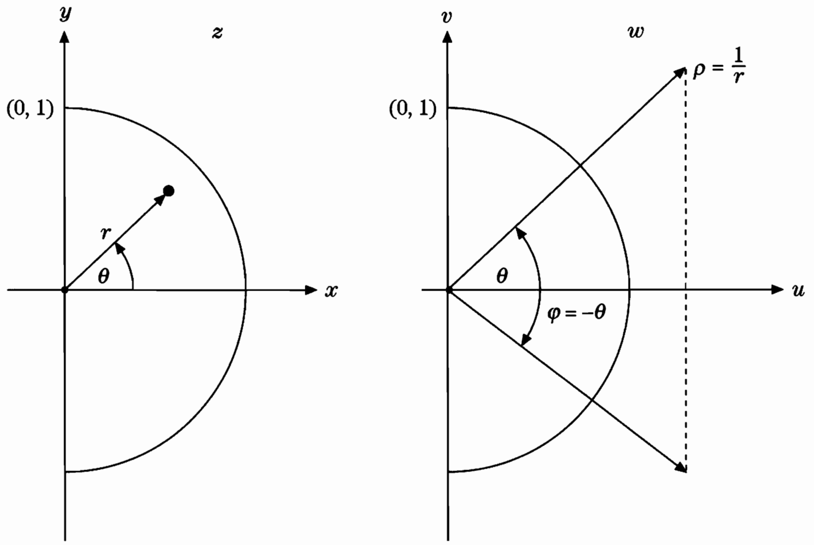

FIGURE 6.19 Inversion.

A circle centered at the origin in the z-plane has the form

$$x^2 + y^2 = r^2 \tag{6.91}$$

and by Eqs. (6.90) transforms into

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = r^2. \tag{6.92}$$

Simplifying Eq. (6.92), we obtain

$$u^2 + v^2 = \frac{1}{r^2} = \rho^2, \tag{6.93}$$

which describes a circle in the w-plane also centered at the origin.

The horizontal line $y = c_1$ transforms into

$$\frac{-v}{u^2+v^2} = c_1, \tag{6.94}$$

or

$$u^2 + \left(v + \frac{1}{2c_1}\right)^2 = \frac{1}{(2c_1)^2}, \tag{6.95}$$

which describes a circle in the w-plane of radius $(1/2c_1)$ and centered at $u = 0, v = -\frac{1}{2c_1}$ (Fig. 6.20).

We pick up the other three possibilities, $x = \pm c_1, y = -c_1$, by rotating the xy-axes. In general, any straight line or circle in the z-plane will transform into a straight line or a circle in the w-plane (compare Exercise 6.7.1).

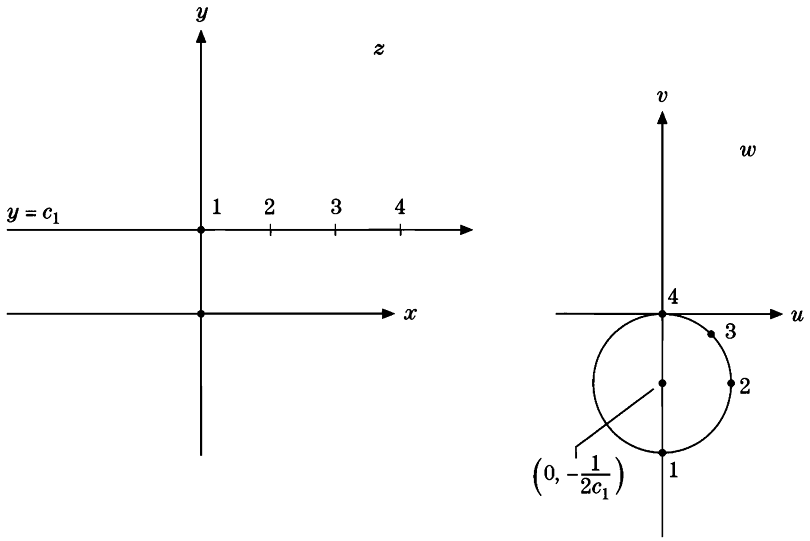

FIGURE 6.20 Inversion, line ↔ circle.

Branch Points and Multivalent Functions

The three transformations just discussed have all involved one-to-one correspondence of points in the z-plane to points in the w-plane. Now to illustrate the variety of transformations that are possible and the problems that can arise, we introduce first a two-to-one correspondence and then a many-to-one correspondence. Finally, we take up the inverses of these two transformations.

Consider first the transformation

$$w = z^2, \tag{6.96}$$

which leads to

$$\rho = r^2, \qquad \varphi = 2\theta. \tag{6.97}$$

Clearly, our transformation is nonlinear, for the modulus is squared, but the significant feature of Eq. (6.96) is that the phase angle or argument is doubled. This means that the

- first quadrant of z, $0 \le \theta < \frac{\pi}{2}$, $\rightarrow$ upper half-plane of w, $0 \le \varphi < \pi$,
- upper half-plane of z, $0 \le \theta < \pi$, $\rightarrow$ whole plane of w, $0 \le \varphi < 2\pi$.

The lower half-plane of z maps into the already covered entire plane of w, thus covering the w-plane a second time. This is our two-to-one correspondence, that is, two distinct points in the z-plane, z_0 and $z_0 e^{i\pi} = -z_0$, corresponding to the single point $w = z_0^2$.

In Cartesian representation,

$$u + iv = (x + iy)^2 = x^2 - y^2 + i2xy, \tag{6.98}$$

leading to

$$u = x^2 - y^2, \qquad v = 2xy. \tag{6.99}$$

Hence the lines $u = c_1$, $v = c_2$ in the w-plane correspond to $x^2 - y^2 = c_1$, $2xy = c_2$, rectangular (and orthogonal) hyperbolas in the z-plane (Fig. 6.21). To every point on the hyperbola $x^2 - y^2 = c_1$ in the right half-plane, $x > 0$, one point on the line $u = c_1$ corresponds, and vice versa. However, every point on the line $u = c_1$ also corresponds to a point on the hyperbola $x^2 - y^2 = c_1$ in the left half-plane, $x < 0$, as already explained.

It will be shown in Section 6.8 that if lines in the w-plane are orthogonal, the corresponding lines in the z-plane are also orthogonal, as long as the transformation is analytic. Since $u = c_1$ and $v = c_2$ are constructed perpendicular to each other, the corresponding hyperbolas in the z-plane are orthogonal. We have constructed a new orthogonal system of hyperbolic lines (or surfaces if we add an axis perpendicular to x and y). Exercise 2.1.3 was an analysis of this system. It might be noted that if the hyperbolic lines are electric or magnetic lines of force, then we have a quadrupole lens useful in focusing beams of high-energy particles.

The inverse of the fourth transformation (Eq. (6.96)) is

$$w = z^{1/2}. \tag{6.100}$$

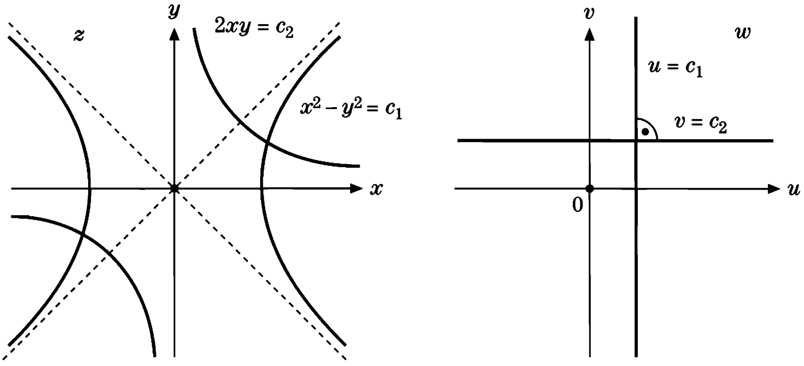

FIGURE 6.21 Mapping — hyperbolic coordinates.

From the relation

$$\rho e^{i\varphi} = r^{1/2} e^{i\theta/2} \tag{6.101}$$

and

$$2\varphi = \theta, \tag{6.102}$$

we now have two points in the w-plane (arguments φ and $\varphi + \pi$) corresponding to one point in the z-plane (except for the point $z = 0$). Or, to put it another way, θ and $\theta + 2\pi$ correspond to φ and $\varphi + \pi$, two distinct points in the w-plane. This is the complex variable analog of the simple real variable equation $y^2 = x$, in which two values of y, plus and minus, correspond to each value of x.

The important point here is that we can make the function w of Eq. (6.100) a single-valued function instead of a double-valued function if we agree to restrict θ to a range such as $0 \le \theta < 2\pi$. This may be done by agreeing never to cross the line $\theta = 0$ in the z-plane (Fig. 6.22). Such a line of demarcation is called a **cut line** or **branch cut**. Note that branch points occur in pairs.

The **cut line joins the two branch point singularities**, here at 0 and ∞ (for the latter, transform $z = 1/t$ for $t \to 0$). Any line from $z = 0$ to infinity would serve equally well. The purpose of the cut line is to restrict the argument of z. The points z and $z\exp(2\pi i)$ coincide in the z-plane but yield different points w and $-w = w\exp(\pi i)$ in the w-plane. Hence in the absence of a cut line, the function $w = z^{1/2}$ is ambiguous. Alternatively, since the function $w = z^{1/2}$ is double-valued, we can also glue two sheets of the complex z-plane together along the branch cut so that $\arg(z)$ increases beyond 2π along the branch cut and continues from 4π on the second sheet to reach the same function values for z as for $ze^{-4\pi i}$, that is, the start on the first sheet again. This construction is called the **Riemann surface** of $w = z^{1/2}$. We shall encounter branch points and cut lines (branch cuts) frequently in Chapter 7.

The transformation

$$w = e^{z} \tag{6.103}$$

leads to

$$\rho e^{i\varphi} = e^{x+iy}, \tag{6.104}$$

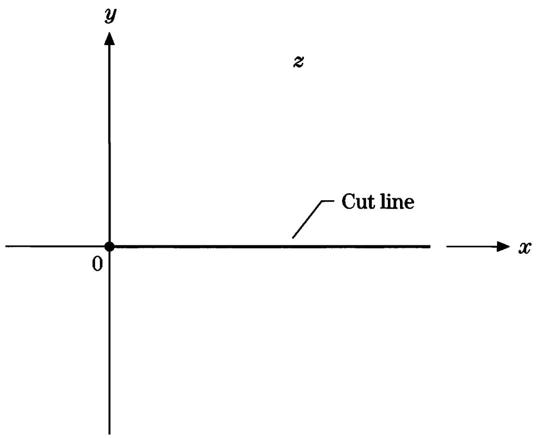

FIGURE 6.22 A cut line.

or

$$\rho = e^x, \qquad \varphi = y. \tag{6.105}$$

If y ranges from $0 \leq y < 2\pi$ (or $-\pi < y \leq \pi$), then φ covers the same range. But this is the whole w-plane. In other words, a horizontal strip in the z-plane of width 2π maps into the entire w-plane. Further, any point $x + i(y + 2n\pi)$, in which n is any integer, maps into the same point (by Eq. (6.104)) in the w-plane. We have a many-(infinitely many)-to-one correspondence.

Finally, as the inverse of the fifth transformation (Eq. (6.103)), we have

$$w = \ln z. \tag{6.106}$$

By expanding it, we obtain

$$u + iv = \ln re^{i\theta} = \ln r + i\theta. \tag{6.107}$$

For a given point z_0 in the z-plane the argument θ is unspecified within an integral multiple of 2π. This means that

$$v = \theta + 2n\pi, \tag{6.108}$$

and, as in the exponential transformation, we have an infinitely many-to-one correspondence.

Equation (6.108) has a nice physical representation. If we go around the unit circle in the z-plane, $r = 1$, and by Eq. (6.107), $u = \ln r = 0$; but $v = \theta$, and θ is steadily increasing and continues to increase as θ continues past 2π.

The cut line joins the branch point at the origin with infinity. As θ increases past 2π we glue a new sheet of the complex z-plane along the cut line, etc. Going around the unit circle in the z-plane is like the advance of a screw as it is rotated or the ascent of a person walking up a spiral staircase (Fig. 6.23), which is the **Riemann surface** of $w = \ln z$.

As in the preceding example, we can also make the correspondence unique (and Eq. (6.106) unambiguous) by restricting θ to a range such as $0 \leq \theta < 2\pi$ by taking the

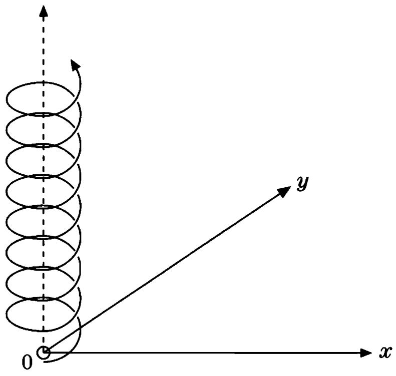

FIGURE 6.23 This is the Riemann surface for $\ln z$, a multivalued function.

line $\theta = 0$ (positive real axis) as a cut line. This is equivalent to taking one and only one complete turn of the spiral staircase.

The concept of mapping is a very broad and useful one in mathematics. Our mapping from a complex z-plane to a complex w-plane is a simple generalization of one definition of function: a mapping of x (from one set) into y in a second set. A more sophisticated form of mapping appears in Section 1.15 where we use the Dirac delta function $\delta(x-a)$ to map a function $f(x)$ into its value at the point a. Then in Chapter 15 integral transforms are used to map one function $f(x)$ in x-space into a second (related) function $F(t)$ in t-space.

Exercises

6.7.1 How do circles centered on the origin in the z-plane transform for

(a) $w_1(z) = z + \dfrac{1}{z}$, (b) $w_2(z) = z - \dfrac{1}{z}$, for $z \neq 0$?

What happens when $|z| \to 1$?

6.7.2 What part of the z-plane corresponds to the interior of the unit circle in the w-plane if

(a) $w = \dfrac{z-1}{z+1}$, (b) $w = \dfrac{z-i}{z+i}$?

6.7.3 Discuss the transformations

(a) $w(z) = \sin z$, (c) $w(z) = \sinh z$,

(b) $w(z) = \cos z$, (d) $w(z) = \cosh z$.

Show how the lines $x = c_1$, $y = c_2$ map into the w-plane. Note that the last three transformations can be obtained from the first one by appropriate translation and/or rotation.

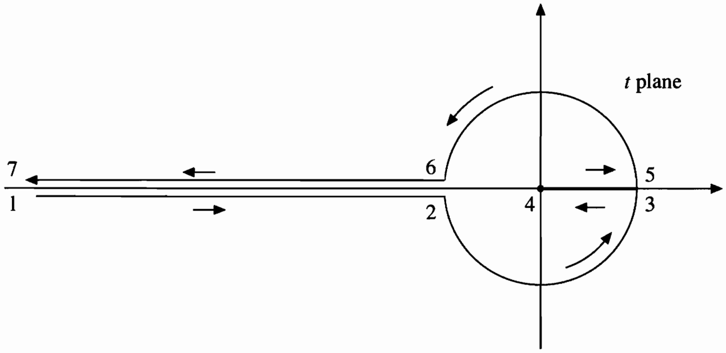

FIGURE 6.24 Bessel function integration contour.

6.7.4 Show that the function

$$w(z) = \left(z^2 - 1\right)^{1/2}$$

is single-valued if we take $-1 \le x \le 1$, $y = 0$ as a cut line.

6.7.5 Show that negative numbers have logarithms in the complex plane. In particular, find $\ln(-1)$.

ANS. $\ln(-1) = i\pi$.

6.7.6 An integral representation of the Bessel function follows the contour in the t-plane shown in Fig. 6.24. Map this contour into the θ-plane with $t = e^{\theta}$. Many additional examples of mapping are given in Chapters 11, 12, and 13.

6.7.7 For noninteger m, show that the binomial expansion of Exercise 6.5.2 holds only for a suitably defined branch of the function $(1 + z)^m$. Show how the z-plane is cut. Explain why $|z| < 1$ may be taken as the circle of convergence for the expansion of this branch, in light of the cut you have chosen.

6.7.8 The Taylor expansion of Exercises 6.5.2 and 6.7.7 is **not** suitable for branches other than the one suitably defined branch of the function $(1 + z)^m$ for noninteger m. [Note that other branches cannot have the same Taylor expansion since they must be distinguishable.] Using the same branch cut of the earlier exercises for all other branches, find the corresponding Taylor expansions, detailing the phase assignments and Taylor coefficients.

6.8 CONFORMAL MAPPING

In Section 6.7 hyperbolas were mapped into straight lines and straight lines were mapped into circles. Yet in all these transformations one feature stayed constant. This constancy was a result of the fact that all the transformations of Section 6.7 were analytic.

As long as $w = f(z)$ is an analytic function, we have

$$\frac{df}{dz} = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}. \tag{6.109}$$

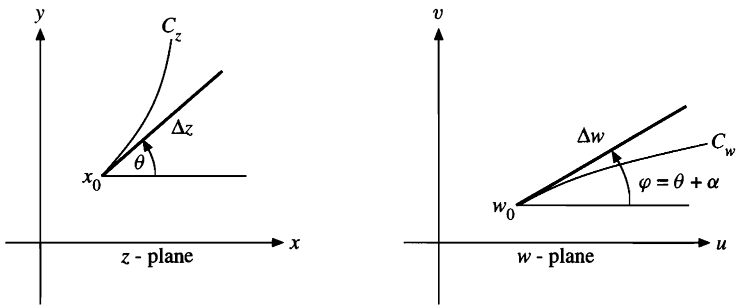

FIGURE 6.25 Conformal mapping—preservation of angles.

Assuming that this equation is in polar form, we may equate modulus to modulus and argument to argument. For the latter (assuming that $df/dz \neq 0$),

$$\arg \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \arg \frac{\Delta w}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \arg \Delta w - \lim_{\Delta z \to 0} \arg \Delta z = \arg \frac{df}{dz} = \alpha, \tag{6.110}$$

where α, the argument of the derivative, may depend on z but is a constant for a fixed z, independent of the direction of approach. To see the significance of this, consider two curves C_z in the z-plane and the corresponding curve C_w in the w-plane (Fig. 6.25). The increment Δz is shown at an angle of θ relative to the real (x) axis, whereas the corresponding increment Δw forms an angle of φ with the real (u) axis. From Eq. (6.110),

$$\varphi = \theta + \alpha, \tag{6.111}$$

or any line in the z-plane is rotated through an angle α in the w-plane as long as w is an analytic transformation and the derivative is not zero.[17]

Since this result holds for any line through z_0, it will hold for a pair of lines. Then for the angle between these two lines,

$$\varphi_2 - \varphi_1 = (\theta_2 + \alpha) - (\theta_1 + \alpha) = \theta_2 - \theta_1, \tag{6.112}$$

which shows that the included angle is preserved under an analytic transformation. Such angle-preserving transformations are called **conformal**. The rotation angle α will, in general, depend on z. In addition $|f'(z)|$ will usually be a function of z.

Historically, these conformal transformations have been of great importance to scientists and engineers in solving Laplace's equation for problems of electrostatics, hydrodynamics, heat flow, and so on. Unfortunately, the conformal transformation approach, however elegant, is limited to problems that can be reduced to two dimensions. The method is often beautiful if there is a high degree of symmetry present but often impossible if the symmetry is broken or absent. Because of these limitations and primarily because electronic computers offer a useful alternative (iterative solution of the partial differential equation), the details and applications of conformal mappings are omitted.

[17]If $df/dz = 0$, its argument or phase is undefined and the (analytic) transformation will not necessarily preserve angles.

Exercises

6.8.1 Expand $w(x)$ in a Taylor series about the point $z = z_0$, where $f'(z_0) = 0$. (Angles are not preserved.) Show that if the first $n - 1$ derivatives vanish but $f^{(n)}(z_0) \neq 0$, then angles in the z-plane with vertices at $z = z_0$ appear in the w-plane multiplied by n.

6.8.2 Develop the transformations that create each of the four cylindrical coordinate systems:

(a) Circular cylindrical: $x = \rho \cos\varphi$,
$y = \rho \sin\varphi$.

(b) Elliptic cylindrical: $x = a \cosh u \cos v$,
$y = a \sinh u \sin v$.

(c) Parabolic cylindrical: $x = \xi\eta$,
$y = \frac{1}{2}(\eta^2 - \xi^2)$.

(d) Bipolar: $x = \dfrac{a \sinh \eta}{\cosh \eta - \cos \xi}$,
$y = \dfrac{a \sin \xi}{\cosh \eta - \cos \xi}$.

Note. These transformations are not necessarily analytic.

6.8.3 In the transformation

$$e^z = \frac{a - w}{a + w},$$

how do the coordinate lines in the z-plane transform? What coordinate system have you constructed?

Additional Readings

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Other references are given at the end of Chapter 15.