MATHEMATICAL METHODS FOR PHYSICISTS

SIXTH EDITION

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CHAPTER 14

FOURIER SERIES

14.1 GENERAL PROPERTIES

Periodic phenomena involving waves, rotating machines (harmonic motion), or other repetitive driving forces are described by periodic functions. Fourier series are a basic tool for solving ordinary differential equations (ODEs) and partial differential equations (PDEs) with periodic boundary conditions. Fourier integrals for nonperiodic phenomena are developed in Chapter 15. The common name for the field is **Fourier analysis**.

A Fourier series is defined as an expansion of a function or representation of a function in a series of sines and cosines, such as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \tag{14.1}$$

The coefficients a_0, a_n, and b_n are related to the periodic function $f(x)$ by definite integrals:

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx, \tag{14.2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx, \qquad n = 0, 1, 2, \ldots, \tag{14.3}$$

which are subject to the requirement that the integrals exist. Notice that a_0 is singled out for special treatment by the inclusion of the factor $\frac{1}{2}$. This is done so that Eq. (14.2) will apply to all a_n, $n = 0$ as well as $n > 0$.

The conditions imposed on $f(x)$ to make Eq. (14.1) valid are that $f(x)$ have only a finite number of finite discontinuities and only a finite number of extreme values, maxima, and minima in the interval $[0, 2\pi]$.[1] Functions satisfying these conditions may be called

[1]These conditions are **sufficient** but not **necessary**.

piecewise regular. The conditions themselves are known as the **Dirichlet conditions.** Although there are some functions that do not obey these Dirichlet conditions, they may well be labeled pathological for purposes of Fourier expansions. In the vast majority of physical problems involving a Fourier series these conditions will be satisfied. In most physical problems we shall be interested in functions that are square integrable (in the Hilbert space L^2 of Section 10.4). In this space the sines and cosines form a complete orthogonal set. And this in turn means that Eq. (14.1) is valid, in the sense of convergence in the mean.

Expressing $\cos nx$ and $\sin nx$ in exponential form, we may rewrite Eq. (14.1) as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{14.4}$$

in which

$$c_n = \frac{1}{2}(a_n - ib_n), \qquad c_{-n} = \frac{1}{2}(a_n + ib_n), \qquad n > 0, \tag{14.5a}$$

and

$$c_0 = \frac{1}{2}a_0. \tag{14.5b}$$

Complex Variables — Abel's Theorem

Consider a function $f(z)$ represented by a convergent power series

$$f(z) = \sum_{n=0}^{\infty} C_n z^n = \sum_{n=0}^{\infty} C_n r^n e^{in\theta}. \tag{14.6}$$

This is our Fourier exponential series, Eq. (14.4). Separating real and imaginary parts we get

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n r^n \cos n\theta, \qquad v(r,\theta) = \sum_{n=1}^{\infty} C_n r^n \sin n\theta, \tag{14.7a}$$

the Fourier cosine and sine series. Abel's theorem asserts that if $u(1,\theta)$ and $v(1,\theta)$ are convergent for a given θ, then

$$u(1,\theta) + iv(1,\theta) = \lim_{r\to 1} f\left(re^{i\theta}\right). \tag{14.7b}$$

An application of this appears as Exercise 14.1.9 and in Example 14.1.1.

Example 14.1.1 SUMMATION OF A FOURIER SERIES

Usually in this chapter we shall be concerned with finding the coefficients of the Fourier expansion of a known function. Occasionally, we may wish to reverse this process and determine the function represented by a given Fourier series.

Consider the series $\sum_{n=1}^{\infty}(1/n)\cos nx, x \in (0, 2\pi)$. Since this series is only conditionally convergent (and diverges at $x = 0$), we take

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = \lim_{r\to 1} \sum_{n=1}^{\infty} \frac{r^n \cos nx}{n}, \tag{14.8}$$

absolutely convergent for $|r| < 1$. Our procedure is to try forming power series by transforming the trigonometric functions into exponential form:

$$\sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{r^n e^{inx}}{n} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{r^n e^{-inx}}{n}. \tag{14.9}$$

Now, these power series may be identified as Maclaurin expansions of $-\ln(1-z), z = re^{ix}, re^{-ix}$ (Eq. (5.95)), and

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} &= -\frac{1}{2}\left[\ln\left(1 - re^{ix}\right) + \ln\left(1 - re^{-ix}\right)\right] \\ &= -\ln\left[\left(1 + r^2\right) - 2r\cos x\right]^{1/2}.\end{aligned} \tag{14.10}$$

Letting $r = 1$ and using Abel's theorem, we see that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\cos nx}{n} &= -\ln(2 - 2\cos x)^{1/2} \\ &= -\ln\left(2\sin\frac{x}{2}\right), \qquad x \in (0, 2\pi).^2\end{aligned} \tag{14.11}$$

Both sides of this expression diverge as $x \to 0$ and 2π. ■

Completeness

The problem of establishing completeness may be approached in a number of different ways. One way is to transform the trigonometric Fourier series into exponential form and to compare it with a Laurent series. If we expand $f(z)$ in a Laurent series[3] (assuming $f(z)$ is analytic),

$$f(z) = \sum_{n=-\infty}^{\infty} d_n z^n. \tag{14.12}$$

On the unit circle $z = e^{i\theta}$ and

$$f(z) = f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} d_n e^{in\theta}. \tag{14.13}$$

[2]The limits may be shifted to $[-\pi, \pi]$ (and $x \neq 0$) using $|x|$ on the right-hand side.
[3]Section 6.5.

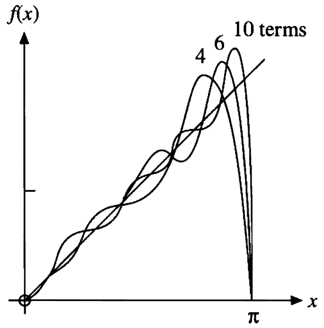

FIGURE 14.1 Fourier representation of sawtooth wave.

The Laurent expansion on the unit circle (Eq. (14.13)) has the same form as the complex Fourier series (Eq. (14.12)), which shows the equivalence between the two expansions. Since the Laurent series as a power series has the property of completeness, we see that the Fourier functions e^{inx} form a complete set. There is a significant limitation here. Laurent series and complex power series cannot handle discontinuities such as a square wave or the sawtooth wave of Fig. 14.1, except on the circle of convergence.

The theory of vector spaces provides a second approach to the completeness of the sines and cosines. Here completeness is established by the Weierstrass theorem for two variables.

The Fourier expansion and the completeness property may be expected, for the functions $\sin nx, \cos nx, e^{inx}$ are all eigenfunctions of a self-adjoint linear ODE,

$$y'' + n^2 y = 0. \tag{14.14}$$

We obtain orthogonal eigenfunctions for different values of the eigenvalue n for the interval $[0, 2\pi]$ that satisfy the boundary conditions in the Sturm–Liouville theory (Chapter 10). Different eigenfunctions for the same eigenvalue n are orthogonal. We have

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} \pi \delta_{mn}, & m \neq 0, \\ 0, & m = 0, \end{cases} \tag{14.15}$$

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} \pi \delta_{mn}, & m \neq 0, \\ 2\pi, & m = n = 0, \end{cases} \tag{14.16}$$

$$\int_0^{2\pi} \sin mx \cos nx \, dx = 0 \qquad \text{for all integral } m \text{ and } n. \tag{14.17}$$

Note that any interval $x_0 \le x \le x_0 + 2\pi$ will be equally satisfactory. Frequently, we shall use $x_0 = -\pi$ to obtain the interval $-\pi \le x \le \pi$. For the complex eigenfunctions $e^{\pm inx}$ orthogonality is usually **defined** in terms of the complex conjugate of one of the two factors,

$$\int_0^{2\pi} \left(e^{imx}\right)^* e^{inx} \, dx = 2\pi \delta_{mn}. \tag{14.18}$$

This agrees with the treatment of the spherical harmonics (Section 12.6).

Sturm–Liouville Theory

The Sturm–Liouville theory guarantees the validity of Eq. (14.1) (for functions satisfying the Dirichlet conditions) and, by use of the orthogonality relations, Eqs. (14.15), (14.16), and (14.17), allows us to compute the expansion coefficients a_n, b_n, as shown in Eqs. (14.2), and (14.3). Substituting Eqs. (14.2) and (14.3) into Eq. (14.1), we write our Fourier expansion as

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt \\ &\quad + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\cos nx \int_0^{2\pi} f(t)\cos nt\,dt + \sin nx \int_0^{2\pi} f(t)\sin nt\,dt\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi} f(t)\cos n(t-x)\,dt, \end{aligned} \tag{14.19}$$

the first (constant) term being the average value of $f(x)$ over the interval $[0, 2\pi]$. Equation (14.19) offers one approach to the development of the Fourier integral and Fourier transforms, Section 15.1.

Another way of describing what we are doing here is to say that $f(x)$ is part of an infinite-dimensional Hilbert space, with the orthogonal $\cos nx$ and $\sin nx$ as the basis. (They can always be renormalized to unity if desired.) The statement that $\cos nx$ and $\sin nx$ $(n = 0, 1, 2, \ldots)$ span this Hilbert space is equivalent to saying that they form a complete set. Finally, the expansion coefficients a_n and b_n correspond to the projections of $f(x)$, with the integral inner products (Eqs. (14.2) and (14.3)) playing the role of the dot product of Section 1.3. These points are outlined in Section 10.4.

Example 14.1.2 SAWTOOTH WAVE

An idea of the convergence of a Fourier series and the error in using only a finite number of terms in the series may be obtained by considering the expansion of

$$f(x) = \begin{cases} x, & 0 \le x < \pi, \\ x - 2\pi, & \pi < x \le 2\pi. \end{cases} \tag{14.20}$$

This is a sawtooth wave, and for convenience we shall shift our interval from $[0, 2\pi]$ to $[-\pi, \pi]$. In this interval we have $f(x) = x$. Using Eqs. (14.2) and (14.3), we show the expansion to be

$$f(x) = x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots + (-1)^{n+1}\frac{\sin nx}{n} + \cdots\right]. \tag{14.21}$$

Figure 14.1 shows $f(x)$ for $0 \le x < \pi$ for the sum of 4, 6, and 10 terms of the series. Three features deserve comment.

1. There is a steady increase in the accuracy of the representation as the number of terms included is increased.

2. All the curves pass through the midpoint, $f(x) = 0$, at $x = \pi$.
3. In the vicinity of $x = \pi$ there is an overshoot that persists and shows no sign of diminishing.

As a matter of incidental interest, setting $x = \pi/2$ in Eq. (14.21) provides an alternate derivation of Leibniz' formula, Exercise 5.7.6. ■

Behavior of Discontinuities

The behavior of the sawtooth wave $f(x)$ at $x = \pi$ is an example of a general rule that at a finite discontinuity the series converges to the arithmetic mean. For a discontinuity at $x = x_0$ the series yields

$$f(x_0) = \tfrac{1}{2}\big[f(x_0 + 0) + f(x_0 - 0)\big], \tag{14.22}$$

the arithmetic mean of the right and left approaches to $x = x_0$. A general proof using partial sums, as in Section 14.5, is given by Jeffreys and Jeffreys and by Carslaw (see the Additional Readings). The proof may be simplified by the use of Dirac delta functions—Exercise 14.5.1.

The overshoot of the sawtooth wave just before $x = \pi$ in Fig. 14.1 is an example of the Gibbs phenomenon, discussed in Section 14.5.

Exercises

14.1.1 A function $f(x)$ (quadratically integrable) is to be represented by a **finite** Fourier series. A convenient measure of the accuracy of the series is given by the integrated square of the deviation,

$$\Delta_p = \int_0^{2\pi} \left[f(x) - \frac{a_0}{2} - \sum_{n=1}^{p} (a_n \cos nx + b_n \sin nx) \right]^2 dx.$$

Show that the requirement that Δ_p be minimized, that is,

$$\frac{\partial \Delta_p}{\partial a_n} = 0, \qquad \frac{\partial \Delta_p}{\partial b_n} = 0,$$

for all n, leads to choosing a_n and b_n as given in Eqs. (14.2) and (14.3).
Note. Your coefficients a_n and b_n are independent of p. This independence is a consequence of orthogonality and would not hold for powers of x, fitting a curve with polynomials.

14.1.2 In the analysis of a complex waveform (ocean tides, earthquakes, musical tones, etc.) it might be more convenient to have the Fourier series written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos(nx - \theta_n).$$

Show that this is equivalent to Eq. (14.1) with

$$a_n = \alpha_n \cos\theta_n, \qquad \alpha_n^2 = a_n^2 + b_n^2,$$
$$b_n = \alpha_n \sin\theta_n, \qquad \tan\theta_n = b_n/a_n.$$

Note. The coefficients α_n^2 as a function of n define what is called the **power spectrum**. The importance of α_n^2 lies in their invariance under a shift in the phase θ_n.

14.1.3 A function $f(x)$ is expanded in an exponential Fourier series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

If $f(x)$ is real, $f(x) = f^*(x)$, what restriction is imposed on the coefficients c_n?

14.1.4 Assuming that $\int_{-\pi}^{\pi} [f(x)]^2 dx$ is finite, show that

$$\lim_{m\to\infty} a_m = 0, \qquad \lim_{m\to\infty} b_m = 0.$$

Hint. Integrate $[f(x) - s_n(x)]^2$, where $s_n(x)$ is the nth partial sum, and use Bessel's inequality, Section 10.4. For our finite interval the assumption that $f(x)$ is square integrable ($\int_{-\pi}^{\pi} |f(x)|^2 dx$ is finite) implies that $\int_{-\pi}^{\pi} |f(x)|\, dx$ is also finite. The converse does not hold.

14.1.5 Apply the summation technique of this section to show that

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} \frac{1}{2}(\pi - x), & 0 < x \le \pi \\ -\frac{1}{2}(\pi + x), & -\pi \le x < 0 \end{cases}$$

(Fig. 14.2).

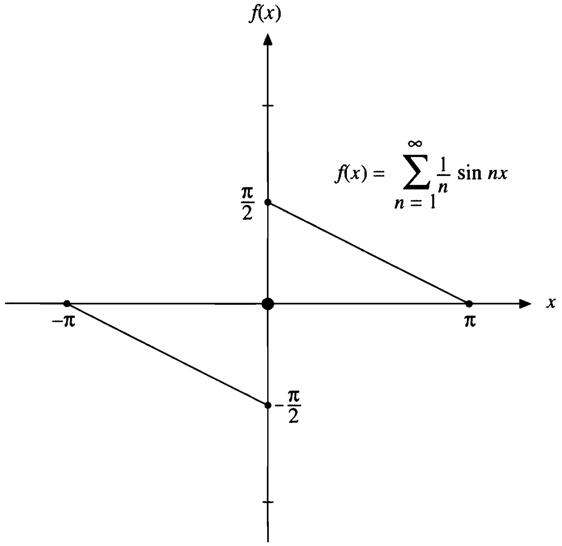

FIGURE 14.2 Reverse sawtooth wave.

14.1.6 Sum the trigonometric series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

and show that it equals $x/2$.

14.1.7 Sum the trigonometric series

$$\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}$$

and show that it equals

$$\begin{cases}\pi/4, & 0<x<\pi \\ -\pi/4, & -\pi<x<0.\end{cases}$$

14.1.8 Calculate the sum of the finite Fourier sine series for the sawtooth wave, $f(x) = x, (-\pi, \pi)$, Eq. (14.21). Use 4-, 6-, 8-, and 10-term series and $x/\pi = 0.00(0.02)1.00$. If a plotting routine is available, plot your results and compare with Fig. 14.1.

14.1.9 Let $f(z) = \ln(1+z) = \sum_{n=1}^{\infty}(-1)^{n+1}z^n/n$. (This series converges to $\ln(1+z)$ for $|z| \le 1$, except at the point $z = -1$.)

(a) From the real parts show that

$$\ln\left(2\cos\frac{\theta}{2}\right) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos n\theta}{n}, \qquad -\pi<\theta<\pi.$$

(b) Using a change of variable, transform part (a) into

$$-\ln\left(2\sin\frac{\theta}{2}\right) = \sum_{n=1}^{\infty}\frac{\cos n\theta}{n}, \qquad 0<\theta<2\pi.$$

14.2 Advantages, Uses of Fourier Series

Discontinuous Functions

One of the advantages of a Fourier representation over some other representation, such as a Taylor series, is that it can represent a discontinuous function. An example is the sawtooth wave in the preceding section. Other examples are considered in Section 14.3 and in the exercises.

Periodic Functions

Related to this advantage is the usefulness of a Fourier series in representing a periodic function. If $f(x)$ has a period of 2π, perhaps it is only natural that we expand it in a series of functions with period $2\pi, 2\pi/2, 2\pi/3, \ldots$. This guarantees that if our periodic $f(x)$ is represented over one interval $[0, 2\pi]$ or $[-\pi, \pi]$, the representation holds for all finite x.

At this point we may conveniently consider the properties of symmetry. Using the interval $[-\pi, \pi]$, $\sin x$ is odd and $\cos x$ is an even function of x. Hence, by Eqs. (14.2) and (14.3),[4] if $f(x)$ is odd, all $a_n = 0$ and if $f(x)$ is even, all $b_n = 0$. In other words,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \qquad f(x) \text{ even}, \tag{14.23}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad f(x) \text{ odd}, \tag{14.24}$$

Frequently these properties are helpful in expanding a given function.

We have noted that the Fourier series is periodic. This is important in considering whether Eq. (14.1) holds outside the initial interval. Suppose we are given only that

$$f(x) = x, \qquad 0 \le x < \pi \tag{14.25}$$

and are asked to represent $f(x)$ by a series expansion. Let us take three of the infinite number of possible expansions.

1. If we assume a Taylor expansion, we have
$$f(x) = x, \tag{14.26}$$
a one-term series. This (one-term) series is defined for all finite x.
2. Using the Fourier cosine series (Eq. (14.23)), thereby assuming the function is represented faithfully in the interval $[0, \pi)$ and extended to neighboring intervals using the known symmetry properties, we predict that
$$\begin{aligned} f(x) &= -x, & -\pi &< x \le 0, \\ f(x) &= 2\pi - x, & \pi &< x < 2\pi. \end{aligned} \tag{14.27}$$
3. Finally, from the Fourier sine series (Eq. (14.24)), we have
$$\begin{aligned} f(x) &= x, & -\pi &< x \le 0, \\ f(x) &= x - 2\pi, & \pi &< x < 2\pi. \end{aligned} \tag{14.28}$$

These three possibilities — Taylor series, Fourier cosine series, and Fourier sine series — are each perfectly valid in the original interval, $[0, \pi]$. Outside, however, their behavior is strikingly different (compare Fig. 14.3). Which of the three, then, is correct? This question has no answer, unless we are given more information about $f(x)$. It may be any of the three or none of them. Our Fourier expansions are valid over the basic interval. Unless the function $f(x)$ is known to be periodic with a period equal to our basic interval or to $(1/n)$th of our basic interval, there is no assurance whatever that the representation (Eq. (14.1)) will have any meaning outside the basic interval.

In addition to the advantages of representing discontinuous and periodic functions, there is a third very real advantage in using a Fourier series. Suppose that we are solving the equation of motion of an oscillating particle subject to a periodic driving force. The Fourier

[4]With the range of integration $-\pi \le x \le \pi$.

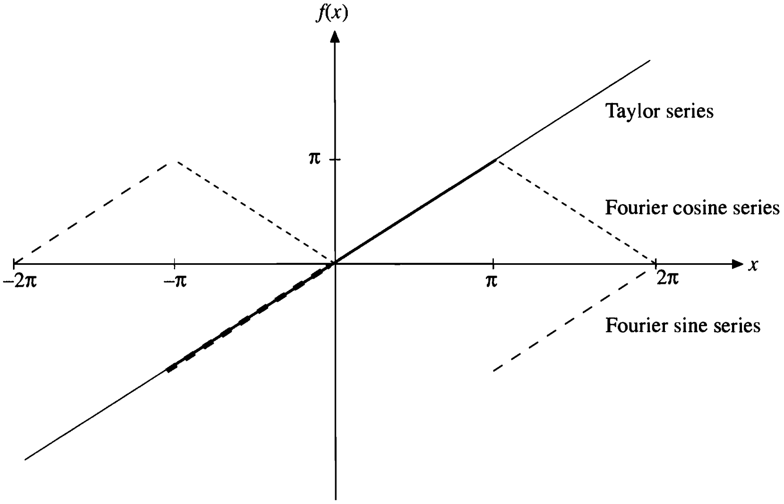

FIGURE 14.3 Comparison of Fourier cosine series, Fourier sine series, and Taylor series.

expansion of the driving force then gives us the fundamental term and a series of harmonics. The (linear) ODE may be solved for each of these harmonics individually, a process that may be much easier than dealing with the original driving force. Then, as long as the ODE is linear, all the solutions may be added together to obtain the final solution.[5] This is more than just a clever mathematical trick.

- It corresponds to finding the response of the system to the fundamental frequency and to each of the harmonic frequencies.

One question that is sometimes raised is: "Were the harmonics there all along, or were they created by our Fourier analysis?" One answer compares the functional resolution into harmonics with the resolution of a vector into rectangular components. The components may have been present, in the sense that they may be isolated and observed, but the resolution is certainly not unique. Hence many authors prefer to say that the harmonics were created by our choice of expansion. Other expansions in other sets of orthogonal functions would give different results. For further discussion we refer to a series of notes and letters in the *American Journal of Physics*.[6]

Change of Interval

So far attention has been restricted to an interval of length 2π. This restriction may easily be relaxed. If $f(x)$ is periodic with a period $2L$, we may write

[5]One of the nastier features of nonlinear differential equations is that this principle of superposition is not valid.

[6]B. L. Robinson, Concerning frequencies resulting from distortion. *Am. J. Phys.* **21**: 391 (1953); F. W. Van Name, Jr., Concerning frequencies resulting from distortion. *ibid.* **22**: 94 (1954).

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right], \tag{14.29}$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt, \qquad n = 0, 1, 2, 3, \ldots, \tag{14.30}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt, \qquad n = 1, 2, 3, \ldots, \tag{14.31}$$

replacing x in Eq. (14.1) with $\pi x/L$ and t in Eqs. (14.2) and (14.3) with $\pi t/L$. (For convenience the interval in Eqs. (14.2) and (14.3) is shifted to $-\pi \le t \le \pi$.) The choice of the symmetric interval $(-L, L)$ is **not** essential. For $f(x)$ periodic with a period of $2L$, any interval $(x_0, x_0 + 2L)$ will do. The choice is a matter of convenience or literally personal preference.

Exercises

14.2.1 The boundary conditions (such as $\psi(0) = \psi(l) = 0$) may suggest solutions of the form $\sin(n\pi x/l)$ and eliminate the corresponding cosines.

(a) Verify that the boundary conditions used in the Sturm–Liouville theory are satisfied for the interval $(0, l)$. Note that this is only half the usual Fourier interval.

(b) Show that the set of functions $\varphi_n(x) = \sin(n\pi x/l),\ n = 1, 2, 3, \ldots,$ satisfies an orthogonality relation

$$\int_0^l \varphi_m(x)\varphi_n(x)\,dx = \frac{l}{2}\delta_{mn}, \qquad n > 0.$$

14.2.2 (a) Expand $f(x) = x$ in the interval $(0, 2L)$. Sketch the series you have found (right-hand side of Ans.) over $(-2L, 2L)$.

$$\text{ANS. } x = L - \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right).$$

(b) Expand $f(x) = x$ as a sine series in the **half** interval $(0, L)$. Sketch the series you have found (right-hand side of Ans.) over $(-2L, 2L)$.

$$\text{ANS. } x = \frac{4L}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\left(\frac{(2n+1)\pi x}{L}\right).$$

14.2.3 In some problems it is convenient to approximate $\sin\pi x$ over the interval $[0, 1]$ by a parabola $ax(1 - x)$, where a is a constant. To get a feeling for the accuracy of this approximation, expand $4x(1 - x)$ in a Fourier sine series $(-1 \le x \le 1)$:

$$f(x) = \begin{Bmatrix} 4x(1-x), & 0 \le x \le 1 \\ 4x(1+x), & -1 \le x \le 0 \end{Bmatrix} = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

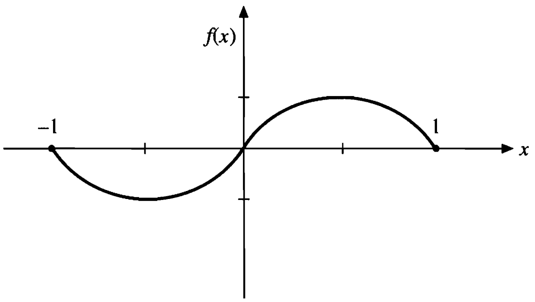

FIGURE 14.4 Parabolic sine wave.

ANS. $b_n = \frac{32}{\pi^3} \cdot \frac{1}{n^3}$, n odd
$b_n = 0$, n even.

(Fig. 14.4.)

14.3 APPLICATIONS OF FOURIER SERIES

Example 14.3.1 SQUARE WAVE — HIGH FREQUENCIES

One application of Fourier series, the analysis of a "square" wave (Fig. 14.5) in terms of its Fourier components, occurs in electronic circuits designed to handle sharply rising pulses. Suppose that our wave is defined by

$$f(x) = 0, \qquad -\pi < x < 0,$$
$$f(x) = h, \qquad 0 < x < \pi. \tag{14.32}$$

From Eqs. (14.2) and (14.3) we find

$$a_0 = \frac{1}{\pi}\int_0^\pi h\,dt = h, \tag{14.33}$$
$$a_n = \frac{1}{\pi}\int_0^\pi h\cos nt\,dt = 0, \qquad n = 1, 2, 3, \dots, \tag{14.34}$$

$$b_n = \frac{1}{\pi}\int_0^\pi h\sin nt\,dt = \frac{h}{n\pi}(1 - \cos n\pi); \tag{14.35}$$
$$b_n = \frac{2h}{n\pi}, \qquad n \text{ odd}, \tag{14.36}$$
$$b_n = 0, \qquad n \text{ even}. \tag{14.37}$$

The resulting series is

$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right). \tag{14.38}$$

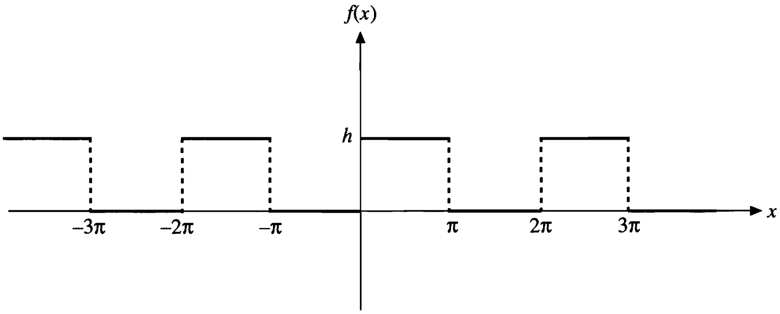

FIGURE 14.5 Square wave.

Except for the first term, which represents an average of $f(x)$ over the interval $[-\pi, \pi]$, all the cosine terms have vanished. Since $f(x) - h/2$ is odd, we have a Fourier sine series. Although only the odd terms in the sine series occur, they fall only as n^{-1}. This **conditional convergence** is like that of the alternating harmonic series. Physically this means that our square wave contains a lot of **high-frequency components**. If the electronic apparatus will not pass these components, our square-wave input will emerge more or less rounded off, perhaps as an amorphous blob. ■

Example 14.3.2 FULL-WAVE RECTIFIER

As a second example, let us ask how well the output of a full-wave rectifier approaches pure direct current (Fig. 14.6). Our rectifier may be thought of as having passed the positive peaks of an incoming sine wave and inverting the negative peaks. This yields

$$\begin{aligned} f(t) &= \sin\omega t, \qquad 0 < \omega t < \pi, \\ f(t) &= -\sin\omega t, \qquad -\pi < \omega t < 0. \end{aligned} \tag{14.39}$$

Since $f(t)$ defined here is even, no terms of the form $\sin n\omega t$ will appear. Again, from Eqs. (14.2) and (14.3), we have

$$\begin{aligned} a_0 &= -\frac{1}{\pi}\int_{-\pi}^{0} \sin\omega t\, d(\omega t) + \frac{1}{\pi}\int_{0}^{\pi} \sin\omega t\, d(\omega t) \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin\omega t\, d(\omega t) = \frac{4}{\pi}, \end{aligned} \tag{14.40}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_{0}^{\pi} \sin\omega t \cos n\omega t\, d(\omega t) \\ &= -\frac{2}{\pi}\frac{2}{n^2-1}, \qquad n \text{ even}, \\ &= 0, \qquad n \text{ odd}. \end{aligned} \tag{14.41}$$

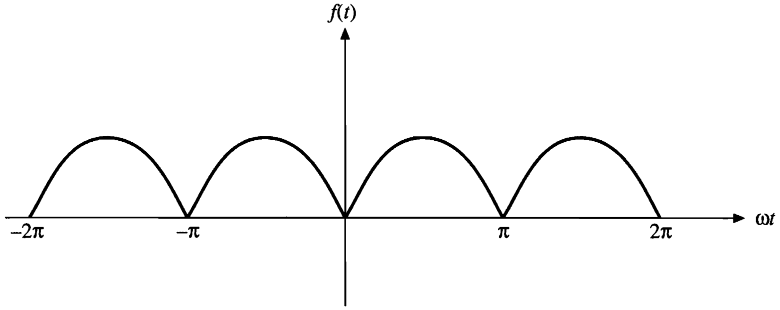

FIGURE 14.6 Full-wave rectifier.

Note that $[0, \pi]$ is not an orthogonality interval for both sines and cosines together and we do not get zero for even n. The resulting series is

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6,\ldots}^{\infty} \frac{\cos n\omega t}{n^2 - 1}. \tag{14.42}$$

The original frequency, ω, has been eliminated. The lowest-frequency oscillation is 2ω. The high-frequency components fall off as n^{-2}, showing that the full-wave rectifier does a fairly good job of approximating direct current. Whether this good approximation is adequate depends on the particular application. If the remaining ac components are objectionable, they may be further suppressed by appropriate filter circuits. These two examples bring out two features characteristic of Fourier expansions.[7]

- If $f(x)$ has discontinuities (as in the square wave in Example 14.3.1), we can expect the nth coefficient to be decreasing as $\mathcal{O}(1/n)$. Convergence is conditional only.
- If $f(x)$ is continuous (although possibly with discontinuous derivatives, as in the full-wave rectifier of Example 14.3.2), we can expect the nth coefficient to be decreasing as $1/n^2$, that is, absolute convergence. ■

Example 14.3.3 INFINITE SERIES, RIEMANN ZETA FUNCTION

As a final example, we consider the problem of expanding x^2. Let

$$f(x) = x^2, \qquad -\pi < x < \pi. \tag{14.43}$$

Since $f(x)$ is even, all $b_n = 0$. For the a_n we have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{2\pi^2}{3}, \tag{14.44}$$

[7] G. Raisbeek, Order of magnitude of Fourier coefficients. *Am. Math. Mon.* **62**: 149–155 (1955).

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx \\ &= \frac{2}{\pi}\cdot(-1)^n \frac{2\pi}{n^2} \\ &= (-1)^n \frac{4}{n^2}. \end{aligned} \tag{14.45}$$

From this we obtain

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}. \tag{14.46}$$

As it stands, Eq. (14.46) is of no particular importance. But if we set $x = \pi$,

$$\cos n\pi = (-1)^n \tag{14.47}$$

and Eq. (14.46) becomes[8]

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}, \tag{14.48}$$

or

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} \equiv \zeta(2), \tag{14.49}$$

thus yielding the Riemann zeta function, $\zeta(2)$, in closed form (in agreement with the Bernoulli number result of Section 5.9). From our expansion of x^2 and expansions of other powers of x, numerous other infinite series can be evaluated. A few are included in this list of exercises:

	Fourier series	Reference
1.	$\sum_{n=1}^{\infty}\frac{1}{n}\sin nx = \begin{cases} -\frac{1}{2}(\pi + x), & -\pi \le x < 0 \\ \frac{1}{2}(\pi - x), & 0 \le x < \pi \end{cases}$	Exercise 14.1.5 Exercise 14.3.3
2.	$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}\sin nx = \frac{1}{2}x, \quad -\pi < x < \pi$	Exercise 14.1.6 Exercise 14.3.2
3.	$\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)x = \begin{cases} -\pi/4, & -\pi < x < 0 \\ +\pi/4, & 0 < x < \pi \end{cases}$	Exercise 14.1.7 Eq. (14.38)
4.	$\sum_{n=1}^{\infty}\frac{\cos nx}{n} = -\ln\left[2\sin\left(\frac{\lvert x\rvert}{2}\right)\right], \quad -\pi < x < \pi$	Eq. (14.11) Exercise 14.1.9(b)
5.	$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}\cos nx = -\ln\left[2\cos\left(\frac{x}{2}\right)\right], \quad -\pi < x < \pi$	Exercise 14.1.9(a)
6.	$\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos(2n+1)x = \frac{1}{2}\ln\left[\cot\frac{\lvert x\rvert}{2}\right], \quad -\pi < x < \pi$	

[8]Note that the point $x = \pi$ is not a point of discontinuity.

The square-wave Fourier series from Eq. (14.38) and item (3) in the table,

$$g(x) = \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1} = (-1)^m \frac{\pi}{4}, \qquad m\pi < x < (m+1)\pi, \tag{14.50}$$

can be used to derive Riemann's **functional equation for the zeta function.** Its defining Dirichlet series can be written in various forms:

$$\begin{aligned}\zeta(s) = \sum_{n=1}^{\infty} n^{-s} &= 1 + \sum_{n=1}^{\infty} (2n)^{-s} + \sum_{n=1}^{\infty} (2n+1)^{-s} \\ &= 2^{-s}\zeta(s) + \sum_{n=0}^{\infty} (2n+1)^{-s}\end{aligned}$$

implying that the function $\lambda(s)$ defined in Section 5.9 (along with $\eta(s)$) satisfies

$$\lambda(s) \equiv \sum_{n=0}^{\infty} (2n+1)^{-s} = \left(1 - 2^{-s}\right)\zeta(s). \tag{14.51}$$

Here s is a complex variable. Both Dirichlet series converge for $\sigma = \Re s > 1$. Alternatively, using Eq. (14.51), we have

$$\eta(s) \equiv \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = \sum_{n=0}^{\infty} (2n+1)^{-s} - \sum_{n=1}^{\infty} (2n)^{-s} = \left(1 - 2^{1-s}\right)\zeta(s), \tag{14.52}$$

which converges already for $\Re s > 0$ using the Leibniz convergence criterion (see Section 5.3).

Another approach to Dirichlet series starts from Euler's integral for the gamma function,

$$\int_0^{\infty} y^{s-1} e^{-ny}\, dy = n^{-s} \int_0^{\infty} e^{-y} y^{s-1}\, dy = n^{-s}\Gamma(s), \tag{14.53}$$

which may be summed using the geometric series

$$\sum_{n=1}^{\infty} e^{-ny} = \frac{e^{-y}}{1 - e^{-y}} = \frac{1}{e^{y} - 1}$$

to yield the integral representation for the zeta function:

$$\int_0^{\infty} \frac{y^{s-1}}{e^{y} - 1}\, dy = \zeta(s)\Gamma(s). \tag{14.54}$$

If we combine the alternative forms of Eq. (14.53),

$$\int_0^{\infty} y^{s-1} e^{-iny}\, dy = n^{-s}\Gamma(s) e^{-i\pi s/2},$$

$$\int_0^{\infty} y^{s-1} e^{iny}\, dy = n^{-s}\Gamma(s) e^{i\pi s/2},$$

we obtain

$$\int_0^\infty y^{s-1}\sin(ny)\,dy = n^{-s}\Gamma(s)\sin\frac{\pi s}{2}. \tag{14.55}$$

Dividing both sides of Eq. (14.55) by n and summing over all odd n yields, for $\sigma = \Re(s) > 0$,

$$\int_0^\infty g(y)y^{s-1}\,dy = \left(1-2^{-s-1}\right)\zeta(s+1)\Gamma(s)\sin\frac{\pi s}{2}, \tag{14.56}$$

using Eqs. (14.50) and (14.51). Here, the interchange of summation and integration is justified by uniform convergence. This relation is at the heart of the functional equation. If we divide the integration range into intervals $m\pi < y < (m+1)\pi$ and substitute Eq. (14.50) into Eq. (14.56) we find

$$\begin{aligned}\int_0^\infty g(y)y^{s-1}\,dy &= \frac{\pi}{4}\sum_{m=0}^{\infty}(-1)^m\int_{m\pi}^{(m+1)\pi} y^{s-1}\,dy\\ &= \frac{\pi^{s+1}}{4s}\left\{\sum_{m=1}^{\infty}(-1)^m\left[(m+1)^s - m^s\right]+1\right\}\\ &= \frac{\pi^{s+1}}{2s}\left(1-2^{s+1}\right)\zeta(-s),\end{aligned} \tag{14.57}$$

using Eq. (14.52). The series in Eq. (14.57) converges for $\Re s < 1$ to an analytic function. Comparing Eqs. (14.56) and (14.57) for the common area of convergence to analytic functions, $0 < \sigma = \Re s < 1$, we get the **functional equation**

$$\frac{\pi^{s+1}}{2s}\left(1-2^{s+1}\right)\zeta(-s) = \left(1-2^{-s-1}\right)\zeta(s+1)\Gamma(s)\sin\frac{\pi s}{2},$$

which can be rewritten as

$$\zeta(1-s) = 2(2\pi)^{-s}\zeta(s)\Gamma(s)\cos\frac{\pi s}{2}. \tag{14.58}$$

This functional equation provides an analytic continuation of $\zeta(s)$ into the negative half-plane of s. For $s \to 1$ the pole of $\zeta(s)$ and the zero of $\cos(\pi s/2)$ cancel in Eq. (14.58), so $\zeta(0) = -1/2$ results. Since $\cos(\pi s/2) = 0$ for $s = 2m+1 =$ odd integer, Eq. (14.58) gives $\zeta(-2m) = 0$, the trivial zeros of the zeta function for $m = 1, 2, \ldots$. All other zeros must lie in the "critical strip" $0 < \sigma = \Re s < 1$. They are closely related to the distribution of prime numbers because the prime number product for $\zeta(s)$ (see Section 5.9) can be converted into a Dirichlet series over prime powers for $\zeta'/\zeta = d\ln\zeta(s)/ds$. From here on we sketch ideas only, without proofs. Using the inverse Mellin transform (see Section 16.2) yields the relation

$$\sum_{\substack{p^m < x,\, p=\text{prime}\\ m=1,2,\ldots}} \ln p = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\zeta'(s)}{\zeta(s)s}x^s\,ds \tag{14.59}$$

for $\sigma > 1$, which is a cornerstone of analytic number theory. Since zeros of $\zeta(s)$ become simple poles of ζ'/ζ, the asymptotic distribution of prime numbers is directly related by

Eq. (14.59) to the zeros of the Riemann zeta function. Riemann conjectured that all zeros lie on the line $\sigma = 1/2$, that is, have the form $1/2 + it$ with real t. If so, one could shift the line of integration to the left to $\sigma = 1/2 + \varepsilon$, the simple pole of $\zeta(s)$ at $s = 1$ giving rise to the residue x, while the integral along the line $\sigma = 1/2 + \varepsilon$ is of order $\mathcal{O}(x^{1/2+\varepsilon})$. Hence, the remarkably small remainder in the asymptotic estimate

$$\sum_{p<x} \ln p \sim x + \mathcal{O}\big(x^{1/2+\varepsilon}\big), \qquad x \to \infty$$

would result for arbitrarily small ε. This is equivalent to the estimate for the number of primes below x,

$$\pi(x) = \sum_{p<x} 1 = \int_2^x (\ln t)^{-1}\, dt + \mathcal{O}\big(x^{1/2+\varepsilon}\big), \qquad x \to \infty.$$

In fact, numerical studies have shown that the first 300×10^9 zeros are simple and lie all on the critical line $\sigma = 1/2$. For more details the reader is referred to the classic text by E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann Zeta Function*, Oxford, UK: Clarendon Press (1986); H. M. Edwards, *Riemann's Zeta Function*, New York: Academic Press (1974) and Dover (2003); J. Van de Lune, H. J. J. Te Riele, and D. T. Winter, On the zeros of the Riemann zeta function in the critical strip. IV. *Math. Comput.* **47**: 667 (1986). Popular accounts can be found in M. du Sautoy, *The Music of the Primes: Searching to Solve the Greatest Mystery in Mathematics*, New York: HarperCollins (2003); J. Derbyshire, *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*. Washington, DC: Joseph Henry Press (2003); K. Sabbagh, *The Riemann Hypothesis: The Greatest Unsolved Problem in Mathematics*, New York: Farrar, Straus and Giroux (2003).

More recently the statistics of the zeros ρ of the Riemann zeta function on the critical line played a prominent role in the development of theories of chaos (see Chapter 18 for an introduction). Assuming that there is a quantum mechanical system whose energies are the imaginary parts of the ρ, then primes determine the primitive periodic orbits of the associated classically chaotic system. For this case Gutzwiller's trace formula, which relates quantum energy levels and classical periodic orbits, plays a central role and can be better understood using properties of the zeta function and primes. For more details see Sections 12.6 and 12.7 by J. Keating, in *The Nature of Chaos* (T. Mullin, ed.), Oxford, UK: Clarendon Press (1993), and references therein. ■

Exercises

14.3.1 Develop the Fourier series representation of

$$f(t) = \begin{cases} 0, & -\pi \le \omega t \le 0, \\ \sin \omega t, & 0 \le \omega t \le \pi. \end{cases}$$

This is the output of a simple half-wave rectifier. It is also an approximation of the solar thermal effect that produces "tides" in the atmosphere.

$$\text{ANS. } f(t) = \frac{1}{\pi} + \frac{1}{2}\sin \omega t - \frac{2}{\pi} \sum_{n=2,4,6,\ldots}^{\infty} \frac{\cos n\omega t}{n^2 - 1}.$$

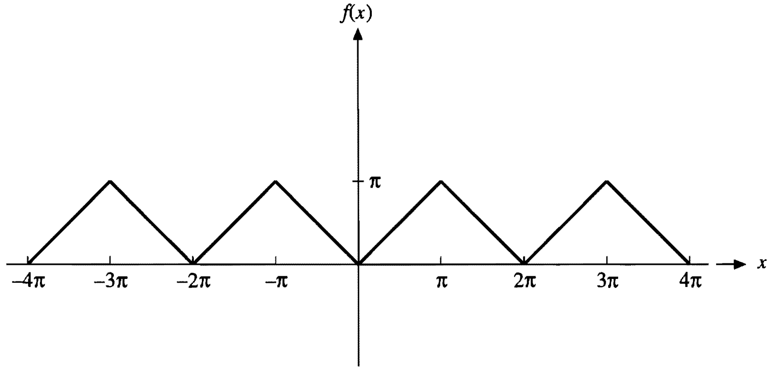

FIGURE 14.7 Triangular wave.

14.3.2 A sawtooth wave is given by

$$f(x) = x, \qquad -\pi < x < \pi.$$

Show that

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

14.3.3 A different sawtooth wave is described by

$$f(x) = \begin{cases} -\frac{1}{2}(\pi + x), & -\pi \le x < 0 \\ +\frac{1}{2}(\pi - x), & 0 < x \le \pi. \end{cases}$$

Show that $f(x) = \sum_{n=1}^{\infty}(\sin nx/n)$.

14.3.4 A triangular wave (Fig. 14.7) is represented by

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -x, & -\pi < x < 0. \end{cases}$$

Represent $f(x)$ by a Fourier series.

ANS. $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi} \displaystyle\sum_{n=1,3,5,\ldots} \frac{\cos nx}{n^2}.$

14.3.5 Expand

$$f(x) = \begin{cases} 1, & x^2 < x_0^2 \\ 0, & x^2 > x_0^2 \end{cases}$$

in the interval $[-\pi, \pi]$.

Note. This variable-width square wave is of some importance in electronic music.

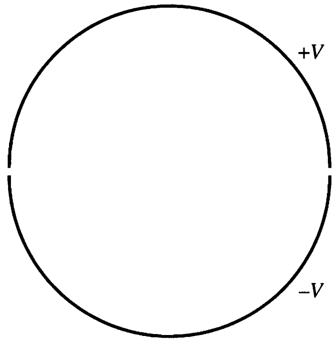

FIGURE 14.8 Cross section of split tube.

14.3.6 A metal cylindrical tube of radius a is split lengthwise into two nontouching halves. The top half is maintained at a potential $+V$, the bottom half at a potential $-V$ (Fig. 14.8). Separate the variables in Laplace's equation and solve for the electrostatic potential for $r \le a$. Observe the resemblance between your solution for $r = a$ and the Fourier series for a square wave.

14.3.7 A metal cylinder is placed in a (previously) uniform electric field, E_0, with the axis of the cylinder perpendicular to that of the original field.

(a) Find the perturbed electrostatic potential.
(b) Find the induced surface charge on the cylinder as a function of angular position.

14.3.8 Transform the Fourier expansion of a square wave, Eq. (14.38), into a power series. Show that the coefficients of x^1 form a **divergent** series. Repeat for the coefficients of x^3.
A power series cannot handle a discontinuity. These infinite coefficients are the result of attempting to beat this basic limitation on power series.

14.3.9 (a) Show that the Fourier expansion of $\cos ax$ is

$$\cos ax = \frac{2a \sin a\pi}{\pi}\left\{\frac{1}{2a^2} - \frac{\cos x}{a^2 - 1^2} + \frac{\cos 2x}{a^2 - 2^2} - \cdots\right\},$$

$$a_n = (-1)^n \frac{2a \sin a\pi}{\pi(a^2 - n^2)}.$$

(b) From the preceding result show that

$$a\pi \cot a\pi = 1 - 2\sum_{p=1}^{\infty} \zeta(2p) a^{2p}.$$

This provides an alternate derivation of the relation between the Riemann zeta function and the Bernoulli numbers, Eq. (5.152).

14.3.10 Derive the Fourier series expansion of the Dirac delta function $\delta(x)$ in the interval $-\pi < x < \pi$.

(a) What significance can be attached to the constant term?
(b) In what region is this representation valid?
(c) With the identity

$$\sum_{n=1}^{N} \cos nx = \frac{\sin(Nx/2)}{\sin(x/2)} \cos\left[\left(N + \frac{1}{2}\right)\frac{x}{2}\right],$$

show that your Fourier representation of $\delta(x)$ is consistent with Eq. (1.190).

14.3.11 Expand $\delta(x - t)$ in a Fourier series. Compare your result with the bilinear form of Eq. (1.190).

$$\text{ANS. } \delta(x-t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}(\cos nx \cos nt + \sin nx \sin nt)$$
$$= \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} \cos n(x-t).$$

14.3.12 Verify that

$$\delta(\varphi_1 - \varphi_2) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi_1 - \varphi_2)}$$

is a Dirac delta function by showing that it satisfies the definition of a Dirac delta function:

$$\int_{-\pi}^{\pi} f(\varphi_1)\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi_1-\varphi_2)}\, d\varphi_1 = f(\varphi_2).$$

Hint. Represent $f(\varphi_1)$ by an exponential Fourier series.
Note. The continuum analog of this expression is developed in Section 15.2. The most important application of this expression is in the determination of Green's functions, Section 9.7.

14.3.13 (a) Using

$$f(x) = x^2, \qquad -\pi < x < \pi,$$

show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} = \eta(2).$$

(b) Using the Fourier series for a triangular wave developed in Exercise 14.3.4, show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} = \lambda(2).$$

(c) Using

$$f(x) = x^4, \qquad -\pi < x < \pi,$$

show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = \zeta(4), \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{720} = \eta(4).$$

(d) Using

$$f(x) = \begin{cases} x(\pi - x), & 0 < x < \pi, \\ x(\pi + x), & \pi < x < 0, \end{cases}$$

derive

$$f(x) = \frac{8}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin nx}{n^3}$$

and show that

$$\sum_{n=1,3,5,\ldots}^{\infty} (-1)^{(n-1)/2} \frac{1}{n^3} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32} = \beta(3).$$

(e) Using the Fourier series for a square wave, show that

$$\sum_{n=1,3,5,\ldots}^{\infty} (-1)^{(n-1)/2} \frac{1}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} = \beta(1).$$

This is Leibniz' formula for π, obtained by a different technique in Exercise 5.7.6. *Note*. The $\eta(2)$, $\eta(4)$, $\lambda(2)$, $\beta(1)$, and $\beta(3)$ functions are defined by the indicated series. General definitions appear in Section 5.9.

14.3.14 (a) Find the Fourier series representation of

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 \le x < \pi. \end{cases}$$

(b) From the Fourier expansion show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

14.3.15 A symmetric triangular pulse of adjustable height and width is described by

$$f(x) = \begin{cases} a(1 - x/b), & 0 \le |x| \le b \\ 0, & b \le |x| \le \pi. \end{cases}$$

(a) Show that the Fourier coefficients are

$$a_0 = \frac{ab}{\pi}, \qquad a_n = \frac{2ab}{\pi(nb)^2}(1 - \cos nb).$$

Sum the finite Fourier series through $n = 10$ and through $n = 100$ for $x/\pi = 0(1/9)1$. Take $a = 1$ and $b = \pi/2$.

(b) Call a Fourier analysis subroutine (if available) to calculate the Fourier coefficients of $f(x)$, a_0 through a_{10}.

14.3.16 (a) Using a Fourier analysis subroutine, calculate the Fourier cosine coefficients a_0 through a_{10} of

$$f(x) = \left[1 - \left(\frac{x}{\pi}\right)^2\right]^{1/2}, \qquad x \in [-\pi, \pi].$$

(b) Spot-check by calculating some of the preceding coefficients by direct numerical quadrature.

Check values. $a_0 = 0.785$, $a_2 = 0.284$.

14.3.17 Using a Fourier analysis subroutine, calculate the Fourier coefficients through a_{10} and b_{10} for

(a) a full-wave rectifier, Example 14.3.2,
(b) a half-wave rectifier, Exercise 14.3.1. Check your results against the analytic forms given (Eq. (14.41) and Exercise 14.3.1).

14.4 PROPERTIES OF FOURIER SERIES

Convergence

It might be noted, first, that our Fourier series should not be expected to be uniformly convergent if it represents a discontinuous function. A uniformly convergent series of continuous functions ($\sin nx$, $\cos nx$) always yields a continuous function (compare Section 5.5). If, however,

(a) $f(x)$ is continuous, $-\pi \le x \le \pi$,
(b) $f(-\pi) = f(+\pi)$, and
(c) $f'(x)$ is sectionally continuous,

the Fourier series for $f(x)$ will converge uniformly. These restrictions do not demand that $f(x)$ be periodic, but they will be satisfied by continuous, differentiable, periodic functions (period of 2π). For a proof of uniform convergence we refer to the literature.[9] With or without a discontinuity in $f(x)$, the Fourier series will yield convergence in the mean, Section 10.4.

[9] See, for instance, R. V. Churchill, *Fourier Series and Boundary Value Problems*, 5th ed., New York: McGraw-Hill (1993), Section 38.

Integration

Term-by-term integration of the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \tag{14.60}$$

yields

$$\int_{x_0}^{x} f(x)\,dx = \frac{a_0 x}{2}\bigg|_{x_0}^{x} + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx\bigg|_{x_0}^{x} - \sum_{n=1}^{\infty} \frac{b_n}{n} \cos nx\bigg|_{x_0}^{x}. \tag{14.61}$$

Clearly, the effect of integration is to place an additional power of n in the denominator of each coefficient. This results in more rapid convergence than before. Consequently, a convergent Fourier series may always be integrated term by term, the resulting series converging uniformly to the integral of the original function. Indeed, term-by-term integration may be valid even if the original series (Eq. (14.60)) is not itself convergent. The function $f(x)$ need only be integrable. A discussion will be found in Jeffreys and Jeffreys, Section 14.06 (see the Additional Readings).

Strictly speaking, Eq. (14.61) may not be a Fourier series; that is, if $a_0 \neq 0$, there will be a term $\frac{1}{2}a_0 x$. However,

$$\int_{x_0}^{x} f(x)\,dx - \frac{1}{2}a_0 x \tag{14.62}$$

will still be a Fourier series.

Differentiation

The situation regarding differentiation is quite different from that of integration. Here the word is caution. Consider the series for

$$f(x) = x, \qquad -\pi < x < \pi. \tag{14.63}$$

We readily find (compare Exercise 14.3.2) that the Fourier series is

$$x = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}, \qquad -\pi < x < \pi. \tag{14.64}$$

Differentiating term by term, we obtain

$$1 = 2\sum_{n=1}^{\infty} (-1)^{n+1} \cos nx, \tag{14.65}$$

which is not convergent. **Warning**: Check your derivative for convergence.

For a triangular wave (Exercise 14.3.4), in which the convergence is more rapid (and uniform),

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,\mathrm{odd}}^{\infty} \frac{\cos nx}{n^2}. \tag{14.66}$$

Differentiating term by term we get

$$f'(x) = \frac{4}{\pi} \sum_{n=1,\text{odd}}^{\infty} \frac{\sin nx}{n}, \tag{14.67}$$

which is the Fourier expansion of a square wave,

$$f'(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0. \end{cases} \tag{14.68}$$

Inspection of Fig. 14.7 verifies that this is indeed the derivative of our triangular wave.

- As the inverse of integration, the operation of differentiation has placed an additional factor n in the numerator of each term. This reduces the rate of convergence and may, as in the first case mentioned, render the differentiated series divergent.
- In general, term-by-term differentiation is permissible under the same conditions listed for uniform convergence.

Exercises

14.4.1 Show that integration of the Fourier expansion of $f(x) = x$, $-\pi < x < \pi$, leads to

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots.$$

14.4.2 Parseval's identity.

(a) Assuming that the Fourier expansion of $f(x)$ is uniformly convergent, show that

$$\frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

This is Parseval's identity. It is actually a special case of the completeness relation, Eq. (10.73).

(b) Given

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}, \qquad -\pi \le x \le \pi,$$

apply Parseval's identity to obtain $\zeta(4)$ in closed form.

(c) The condition of uniform convergence is not necessary. Show this by applying the Parseval identity to the square wave

$$\begin{aligned} f(x) &= \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases} \\ &= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}. \end{aligned}$$

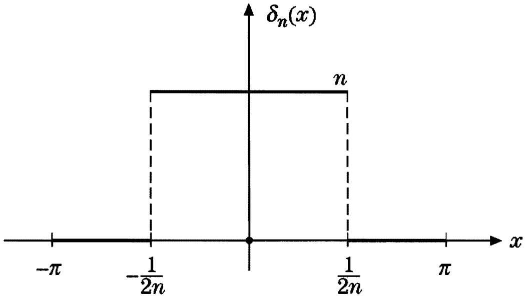

FIGURE 14.9 Rectangular pulse.

14.4.3 Show that integrating the Fourier expansion of the Dirac delta function (Exercise 14.3.10) leads to the Fourier representation of the square wave, Eq. (14.38), with $h = 1$.
Note. Integrating the constant term $(1/2\pi)$ leads to a term $x/2\pi$. What are you going to do with this?

14.4.4 Integrate the Fourier expansion of the unit step function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi. \end{cases}$$

Show that your integrated series agrees with Exercise 14.3.14.

14.4.5 In the interval $(-\pi, \pi)$,

$$\delta_n(x) = \begin{cases} n, & \text{for } |x| < \frac{1}{2n}, \\ 0, & \text{for } |x| > \frac{1}{2n} \end{cases}$$

(Fig. 14.9).

(a) Expand $\delta_n(x)$ as a Fourier cosine series.
(b) Show that your Fourier series agrees with a Fourier expansion of $\delta(x)$ in the limit as $n \to \infty$.

14.4.6 Confirm the delta function nature of your Fourier series of Exercise 14.4.4 by showing that for any $f(x)$ that is finite in the interval $[-\pi, \pi]$ and continuous at $x = 0$,

$$\int_{-\pi}^{\pi} f(x)\big[\text{Fourier expansion of } \delta_\infty(x)\big]\,dx = f(0).$$

14.4.7 (a) Show that the Dirac delta function $\delta(x - a)$, expanded in a Fourier sine series in the half-interval $(0, L)(0 < a < L)$, is given by

$$\delta(x - a) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

Note that this series actually describes

$$-\delta(x + a) + \delta(x - a) \qquad \text{in the interval} \qquad (-L, L).$$

(b) By integrating both sides of the preceding equation from 0 to x, show that the cosine expansion of the square wave

$$f(x) = \begin{cases} 0, & 0 \le x < a \\ 1, & a < x < L, \end{cases}$$

is

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi a}{L}\right) - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi a}{L}\right) \cos\left(\frac{n\pi x}{L}\right),$$

for $0 \le x < L$.

(c) Verify that the term

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi a}{L}\right)$$

is $\langle f(x) \rangle$.

14.4.8 Verify the Fourier cosine expansion of the square wave, Exercise 14.4.7(b), by direct calculation of the Fourier coefficients.

14.4.9 (a) A string is clamped at both ends $x = 0$ and $x = L$. Assuming small-amplitude vibrations, we find that the amplitude $y(x, t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}.$$

Here v is the wave velocity. The string is set in vibration by a sharp blow at $x = a$. Hence we have

$$y(x, 0) = 0, \qquad \frac{\partial y(x, t)}{\partial t} = L v_0 \delta(x - a) \qquad \text{at } t = 0.$$

The constant L is included to compensate for the dimensions (inverse length) of $\delta(x - a)$. With $\delta(x - a)$ given by Exercise 14.4.7(a), solve the wave equation subject to these initial conditions.

$$\text{ANS. } y(x, t) = \frac{2 v_0 L}{\pi v} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi a}{L} \sin \frac{n\pi x}{L} \sin \frac{n\pi v t}{L}.$$

(b) Show that the transverse velocity of the string $\partial y(x, t)/\partial t$ is given by

$$\frac{\partial y(x, t)}{\partial t} = 2 v_0 \sum_{n=1}^{\infty} \sin \frac{n\pi a}{L} \sin \frac{n\pi x}{L} \cos \frac{n\pi v t}{L}.$$

14.4.10 A string, clamped at $x = 0$ and at $x = 1$, is vibrating freely. Its motion is described by the wave equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} = v^2 \frac{\partial^2 u(x, t)}{\partial x^2}.$$

Assume a Fourier expansion of the form

$$u(x,t)=\sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{l}$$

and determine the coefficients $b_n(t)$. The initial conditions are

$$u(x,0)=f(x) \qquad \text{and} \qquad \frac{\partial}{\partial t}u(x,0)=g(x).$$

Note. This is only half the conventional Fourier orthogonality integral interval. However, as long as only the sines are included here, the Sturm–Liouville boundary conditions are still satisfied and the functions are orthogonal.

$$\text{ANS. } b_n(t)=A_n\cos\frac{n\pi vt}{l}+B_n\sin\frac{n\pi vt}{l},$$

$$A_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx,\ B_n=\frac{2}{n\pi v}\int_0^l g(x)\sin\frac{n\pi x}{l}dx.$$

14.4.11 (a) Let us continue the vibrating string problem, Exercise 14.4.10. The presence of a resisting medium will damp the vibrations according to the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}=v^2\frac{\partial^2 u(x,t)}{\partial x^2}-k\frac{\partial u(x,t)}{\partial t}.$$

Assume a Fourier expansion

$$u(x,t)=\sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{l}$$

and again determine the coefficients $b_n(t)$. Take the initial and boundary conditions to be the same as in Exercise 14.4.10. Assume the damping to be small.

(b) Repeat, but assume the damping to be large.

$$\text{ANS. (a) } b_n(t)=e^{-kt/2}\{A_n\cos\omega_n t+B_n\sin\omega_n t\},$$

$$A_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx,$$

$$B_n=\frac{2}{\omega_n l}\int_0^l g(x)\sin\frac{n\pi x}{l}dx+\frac{k}{2\omega_n}A_n,$$

$$\omega_n^2=\left(\frac{n\pi v}{l}\right)-\left(\frac{k}{2}\right)^2>0.$$

(b) $b_n(t) = e^{-kt/2}\{A_n \cosh \sigma_n t + B_n \sinh \sigma_n t\}$,

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx,$$

$$B_n = \frac{2}{\sigma_n l}\int_0^l g(x)\sin\frac{n\pi x}{l}dx + \frac{k}{2\sigma_n}A_n,$$

$$\sigma_n^2 = \left(\frac{k}{2}\right)^2 - \left(\frac{n\pi v}{l}\right)^2 > 0.$$

14.4.12 Find the charge distribution over the interior surfaces of the semicircles of Exercise 14.3.6.

Note. You obtain a divergent series and this Fourier approach fails. Using conformal mapping techniques, we may show the charge density to be proportional to $\csc\theta$. Does $\csc\theta$ have a Fourier expansion?

14.4.13 Given

$$\varphi_1(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n} = \begin{cases} -\frac{1}{2}(\pi + x), & -\pi \le x < 0, \\ \frac{1}{2}(\pi - x), & 0 < x \le \pi, \end{cases}$$

show by integrating that

$$\varphi_2(x) \equiv \sum_{n=1}^{\infty}\frac{\cos nx}{n^2} = \begin{cases} \frac{1}{4}(\pi + x)^2 - \frac{\pi^2}{12}, & -\pi \le x \le 0 \\ \frac{1}{4}(\pi - x)^2 - \frac{\pi^2}{12}, & 0 \le x \le \pi. \end{cases}$$

14.4.14 Given

$$\psi_{2s}(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n^{2s}}, \qquad \psi_{2s+1}(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n^{2s+1}},$$

develop the following recurrence relations:

(a) $\psi_{2s}(x) = \int_0^x \psi_{2s-1}(x)\,dx$

(b) $\psi_{2s+1}(x) = \zeta(2s+1) - \int_0^x \psi_{2s}(x)\,dx.$

Note. The functions $\psi_s(x)$ and the $\varphi_s(x)$ of the preceding two exercises are known as **Clausen functions**. In theory they may be used to improve the rate of convergence of a Fourier series. As with the series of Chapter 5, there is always the question of how much analytical work we do and how much arithmetic work we demand that the computer do. As computers become steadily more powerful, the balance progressively shifts so that we are doing less and demanding that they do more.

14.4.15 Show that

$$f(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n+1}$$

may be written as

$$f(x) = \psi_1(x) - \varphi_2(x) + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2(n+1)}.$$

Note. $\psi_1(x)$ and $\varphi_2(x)$ are defined in the preceding exercises.

14.5 GIBBS PHENOMENON

The Gibbs phenomenon is an overshoot, a peculiarity of the Fourier series and other eigenfunction series at a simple discontinuity. An example is seen in Fig. 14.1.

Summation of Series

In Section 14.1 the sum of the first several terms of the Fourier series for a sawtooth wave was plotted (Fig. 14.10). Now we develop analytic methods of summing the first r terms of our Fourier series.

From Eq. (14.19),

$$a_n \cos nx + b_n \sin nx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos n(t - x)\, dt. \tag{14.69}$$

Then the rth partial sum becomes[10]

$$\begin{aligned} s_r(x) &= \frac{a_0}{2} + \sum_{n=1}^{r} (a_n \cos nx + b_n \sin nx) \\ &= \Re \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{n=1}^{r} e^{-i(t-x)n} \right] dt. \end{aligned} \tag{14.70}$$

Summing the finite series of exponentials (geometric progression),[11] we obtain

$$s_r(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin[(r + \frac{1}{2})(t - x)]}{\sin \frac{1}{2}(t - x)}\, dt. \tag{14.71}$$

This is convergent at all points, including $t = x$. The factor

$$\frac{\sin[(r + \frac{1}{2})(t - x)]}{2\pi \sin \frac{1}{2}(t - x)}$$

is the Dirichlet kernel mentioned in Section 1.15 as a Dirac delta distribution.

[10] It is of some interest to note that this series also occurs in the analysis of the diffraction grating (r slits).

[11] Compare Exercise 6.1.7 with initial value $n = 1$.

Square Wave

For convenience of numerical calculation we consider the behavior of the Fourier series that represents the periodic square wave

$$f(x)=\begin{cases}\frac{h}{2}, & 0<x<\pi,\\ -\frac{h}{2}, & -\pi<x<0.\end{cases} \tag{14.72}$$

This is essentially the square wave used in Section 14.3, and we immediately see that the solution is

$$f(x)=\frac{2h}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right). \tag{14.73}$$

Applying Eq. (14.71) to our square wave (Eq. (14.72)), we have the sum of the first r terms (plus $\frac{1}{2}a_0$, which is zero here):

$$\begin{aligned} s_r(x) &= \frac{h}{4\pi}\int_0^{\pi}\frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt-\frac{h}{4\pi}\int_{-\pi}^{0}\frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt\\ &= \frac{h}{4\pi}\int_0^{\pi}\frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt-\frac{h}{4\pi}\int_0^{\pi}\frac{\sin[(r+\frac{1}{2})(t+x)]}{\sin\frac{1}{2}(t+x)}dt. \end{aligned} \tag{14.74}$$

This last result follows from the transformation

$$\vec{t}-t$$

in the second integral. Replacing $t-x$ in the first term with s and $t+x$ in the second term with s, we obtain

$$s_r(x)=\frac{h}{4\pi}\int_{-x}^{\pi-x}\frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds-\frac{h}{4\pi}\int_{x}^{\pi+x}\frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds. \tag{14.75}$$

The intervals of integration are shown in Fig. 14.10(top). Because the integrands have the same mathematical form, the integrals from x to $\pi-x$ cancel, leaving the integral ranges shown in the bottom portion of Fig. 14.10:

$$s_r(x)=\frac{h}{4\pi}\int_{-x}^{x}\frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds-\frac{h}{4\pi}\int_{\pi-x}^{\pi+x}\frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds. \tag{14.76}$$

Consider the partial sum in the vicinity of the discontinuity at $x=0$. As $x\to 0$, the second integral becomes negligible, and we associate the first integral with the discontinuity at $x=0$. Using $r+\frac{1}{2}=p$ and $ps=\xi$ we obtain

$$s_r(x)=\frac{h}{2\pi}\int_0^{px}\frac{\sin\xi}{\sin(\xi/2p)}\frac{d\xi}{p}. \tag{14.77}$$

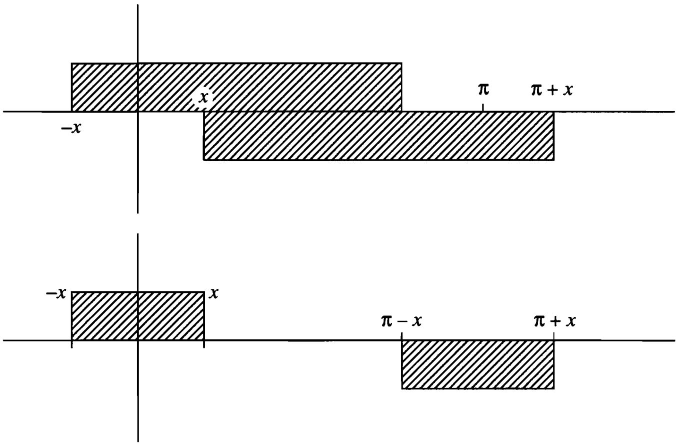

FIGURE 14.10 Intervals of integration—Eq. (14.75).

Calculation of Overshoot

Our partial sum $s_r(x)$ starts at zero when $x = 0$ (in agreement with Eq. (14.22)) and increases until $\xi = ps = \pi$, at which point the numerator, $\sin\xi$, goes negative. For large r, and therefore for large p, our denominator remains positive. We get the maximum value of the partial sum by taking the upper limit $px = \pi$. Right here we see that x, the location of the overshoot maximum, is inversely proportional to the number of terms taken:

$$x = \frac{\pi}{p} \approx \frac{\pi}{r}.$$

The maximum value of the partial sum is then

$$s_r(x)_{\max} = \frac{h}{2} \cdot \frac{1}{\pi} \int_0^{\pi} \frac{\sin\xi \, d\xi}{\sin(\xi/2p)p} \approx \frac{h}{2} \cdot \frac{2}{\pi} \int_0^{\pi} \frac{\sin\xi}{\xi} d\xi. \tag{14.78}$$

In terms of the sine integral, $\mathrm{si}(x)$ of Section 8.5,

$$\int_0^{\pi} \frac{\sin\xi}{\xi} d\xi = \frac{\pi}{2} + \mathrm{si}(\pi). \tag{14.79}$$

The integral is clearly greater than $\pi/2$, since it can be written as

$$\left(\int_0^{\infty} - \int_{\pi}^{3\pi} - \int_{3\pi}^{5\pi} - \cdots \right) \frac{\sin\xi}{\xi} d\xi = \int_0^{\pi} \frac{\sin\xi}{\xi} d\xi. \tag{14.80}$$

We saw in Example 7.1.4 that the integral from 0 to ∞ is $\pi/2$. From this integral we are subtracting a series of negative terms. A Gaussian quadrature or a power-series expansion and term-by-term integration yields

$$\frac{2}{\pi} \int_0^{\pi} \frac{\sin\xi}{\xi} d\xi = 1.1789797\ldots, \tag{14.81}$$

which means that the Fourier series tends to overshoot the positive corner by some 18 percent and to undershoot the negative corner by the same amount, as suggested in Fig. 14.11.

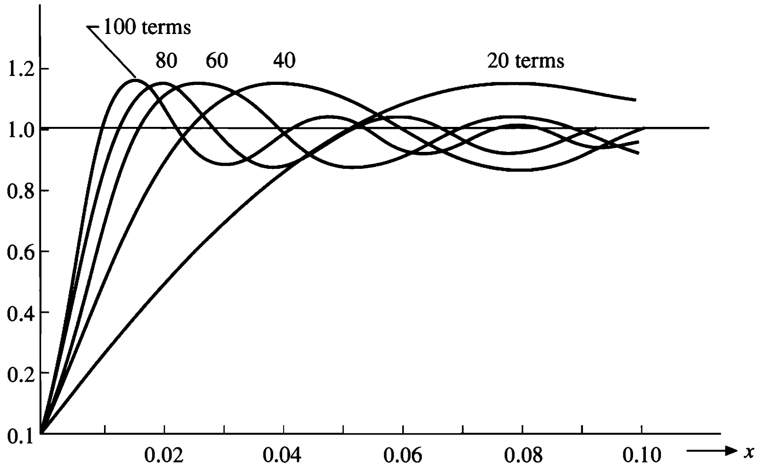

FIGURE 14.11 Square wave — Gibbs phenomenon.

The inclusion of more terms (increasing r) does nothing to remove this overshoot but merely moves it closer to the point of discontinuity. The overshoot is the Gibbs phenomenon, and because of it the Fourier series representation may be highly unreliable for precise numerical work, especially in the vicinity of a discontinuity.

The Gibbs phenomenon is not limited to the Fourier series. It occurs with other eigenfunction expansions. Exercise 12.3.27 is an example of the Gibbs phenomenon for a Legendre series. For more details, see W. J. Thompson, Fourier series and the Gibbs phenomenon, *Am. J. Phys.* **60**: 425 (1992).

Exercises

14.5.1 With the partial sum summation techniques of this section, show that at a discontinuity in $f(x)$ the Fourier series for $f(x)$ takes on the arithmetic mean of the right- and left-hand limits:

$$f(x_0) = \frac{1}{2}\big[f(x_0+0) + f(x_0-0)\big].$$

In evaluating $\lim_{r\to\infty} s_r(x_0)$ you may find it convenient to identify part of the integrand as a Dirac delta function.

14.5.2 Determine the partial sum, s_n, of the series in Eq. (14.73) by using

$$\text{(a)}\ \frac{\sin mx}{m} = \int_0^x \cos my\,dy, \qquad \text{(b)}\ \sum_{p=1}^{n} \cos(2p-1)y = \frac{\sin 2ny}{2\sin y}.$$

Do you agree with the result given in Eq. (14.79)?

14.5.3 Evaluate the finite step function series, Eq. (14.73), $h = 2$, using 100, 200, 300, 400, and 500 terms for $x = 0.0000(0.0005)0.0200$. Sketch your results (five curves) or, if a plotting routine is available, plot your results.

14.5.4 (a) Calculate the value of the Gibbs phenomenon integral

$$I = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\,dt$$

by numerical quadrature accurate to 12 significant figures.

(b) Check your result by (1) expanding the integrand as a series, (2) integrating term by term, and (3) evaluating the integrated series. This calls for double precision calculation.

ANS. $I = 1.178979744472$.

14.6 Discrete Fourier Transform

For many physicists the Fourier transform is automatically the continuous Fourier transform of Chapter 15. The use of digital computers, however, necessarily replaces a continuum of values by a discrete set; an integration is replaced by a summation. The continuous Fourier transform becomes the discrete Fourier transform and an appropriate topic for this chapter.

Orthogonality over Discrete Points

The orthogonality of the trigonometric functions and the imaginary exponentials is expressed in Eqs. (14.15) to (14.18). This is the usual orthogonality for functions: **integration** of a product of functions over the orthogonality interval. The sines, cosines, and imaginary exponentials have the remarkable property that they are also orthogonal over a series of discrete, equally spaced points over the period (the orthogonality interval).

Consider a set of $2N$ time values

$$t_k = 0, \frac{T}{2N}, \frac{2T}{2N}, \ldots, \frac{(2N-1)T}{2N} \tag{14.82}$$

for the time interval $(0, T)$. Then

$$t_k = \frac{kT}{2N}, \qquad k = 0, 1, 2, \ldots, 2N-1. \tag{14.83}$$

We shall prove that the exponential functions $\exp(2\pi i p t_k/T)$ and $\exp(2\pi i q t_k/T)$ satisfy an orthogonality relation over the discrete points t_k:

$$\sum_{k=0}^{2N-1}\left[\exp\left(\frac{2\pi i p t_k}{T}\right)\right]^* \exp\left(\frac{2\pi i q t_k}{T}\right) = 2N\delta_{p,q\pm 2nN}. \tag{14.84}$$

Here n, p, and q are all integers.

Replacing $q-p$ by s, we find that the left-hand side of Eq. (14.84) becomes

$$\sum_{k=0}^{2N-1}\exp\left(\frac{2\pi i s t_k}{T}\right) = \sum_{k=0}^{2N-1}\exp\left(\frac{2\pi i s k}{2N}\right).$$

This right-hand side is obtained by using Eq. (14.83) to replace T. This is a finite geometric series with an initial term 1 and a ratio

$$r = \exp\left(\frac{\pi i s}{N}\right).$$

From Eq. (5.3),

$$\sum_{k=0}^{2N-1} \exp\left(\frac{2\pi i s t_k}{T}\right) = \begin{cases} \dfrac{1-r^{2N}}{1-r} = 0, & r \neq 1 \\ 2N, & r = 1, \end{cases} \tag{14.85}$$

establishing Eq. (14.84), our basic orthogonality relation. The upper value, zero, is a consequence of

$$r^{2N} = \exp(2\pi i s) = 1$$

for s an integer. The lower value, $2N$, for $r = 1$ corresponds to $p = q$. The orthogonality of the corresponding trigonometric functions is left as Exercise 14.6.1.

Discrete Fourier Transform

To simplify the notation and to make more direct contact with physics, we introduce the (reciprocal) ω-space, or angular frequency, with

$$\omega_p = \frac{2\pi p}{T}, \qquad p = 0, 1, 2, \ldots, 2N-1. \tag{14.86}$$

We make p range over the same integers as k. The exponential $\exp(\pm 2\pi i p t_k/T)$ of Eq. (14.84) becomes $\exp(\pm i\omega_p t_k)$. The choice of whether to use the $+$ or the $-$ sign is a matter of convenience or convention. In quantum mechanics the negative sign is selected when expressing the time dependence.

Consider a function of time defined (measured) at the discrete time values t_k. Then we construct

$$F(\omega_p) = \frac{1}{2N}\sum_{k=0}^{2N-1} f(t_k)e^{i\omega_p t_k}. \tag{14.87}$$

Employing the orthogonality relation, we obtain

$$\frac{1}{2N}\sum_{p=0}^{2N-1} \left(e^{i\omega_p t_m}\right)^* e^{i\omega_p t_k} = \delta_{mk}, \tag{14.88}$$

and then replacing the subscript m by k, we find that the amplitudes $f(t_k)$ become

$$f(t_k) = \sum_{p=0}^{2N-1} F(\omega_p)e^{-i\omega_p t_k}. \tag{14.89}$$

The time function $f(t_k)$, $k = 0, 1, 2, \ldots, 2N-1$, and the frequency function $F(\omega_p)$, $p = 0, 1, 2, \ldots, 2N-1$, are discrete Fourier transforms of each other.[12] Compare Eqs. (14.87)

[12] The two transform equations may be symmetrized with a resulting $(2N)^{-1/2}$ in each equation if desired.

and (14.89) with the corresponding continuous Fourier transforms, Eqs. (15.22) and (15.23) of Chapter 15.

Limitations

Taken as a pair of mathematical relations, the discrete Fourier transforms are exact. We can say that the $2N$ $2N$-component vectors $\exp(-i\omega_p t_k), k = 0, 1, 2, \dots, 2N-1$, form a complete set[13] spanning the t_k-space. Then $f(t_k)$ in Eq. (14.89) is simply a particular linear combination of these vectors. Alternatively, we may take the $2N$ measured components $f(t_k)$ as defining a $2N$-component vector in t_k-space. Then, Eq. (14.87) yields the $2N$-component vector $F(\omega_p)$ in the **reciprocal** ω_p-space. Equations (14.87) and (14.89) become matrix equations, with $\exp(i\omega_p t_k)/(2N)^{1/2}$ the elements of a unitary matrix.

The limitations of the discrete Fourier transform arise when we apply Eqs. (14.87) and (14.89) to physical systems and attempt physical interpretation and the limit $F(\omega_p) \to F(\omega)$. Example 14.6.1 illustrates the problems that can occur. The most important precaution to be taken to avoid trouble is to take N sufficiently large so that there is no angular frequency component of a higher angular frequency than $\omega_N = 2\pi N/T$. For details on errors and limitations in the use of the discrete Fourier transform we refer to Hamming in the Additional Readings.

Example 14.6.1 Discrete Fourier Transform — Aliasing

Consider the simple case of $T = 2\pi$, $N = 2$, and $f(t_k) = \cos t_k$. From

$$t_k = \frac{kT}{4} = \frac{k\pi}{2}, \qquad k = 0, 1, 2, 3, \tag{14.90}$$

$f(t_k) = \cos(t_k)$ is represented by the four-component vector

$$f(t_k) = (1, 0, -1, 0). \tag{14.91}$$

The frequencies, ω_p, are given by Eq. (14.86):

$$\omega_p = \frac{2\pi p}{T} = p. \tag{14.92}$$

Clearly, $\cos t_k$ implies a $p = 1$ component and no other frequency components.

The transformation matrix

$$\frac{\exp(i\omega_p t_k)}{2N} = \frac{\exp(ipk\pi/2)}{2N}$$

becomes

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}. \tag{14.93}$$

[13] By Eq. (14.85) these vectors are orthogonal and are therefore linearly independent.

Note that the $2N \times 2N$ matrix has only $2N$ independent components. It is the **repetition of values that makes the fast Fourier transform technique possible.**

Operating on column vector $f(t_k)$, we find that this matrix yields a column vector

$$F(\omega_p) = \left(0, \tfrac{1}{2}, 0, \tfrac{1}{2}\right). \tag{14.94}$$

Apparently, there is a $p = 3$ frequency component present. We reconstruct $f(t_k)$ by Eq. (14.89), obtaining

$$f(t_k) = \tfrac{1}{2}e^{-it_k} + \tfrac{1}{2}e^{-3it_k}. \tag{14.95}$$

Taking real parts, we can rewrite the equation as

$$\Re f(t_k) = \tfrac{1}{2}\cos t_k + \tfrac{1}{2}\cos 3t_k. \tag{14.96}$$

Obviously, this result, Eq. (14.96), is not identical with our original $f(t_k) \cos t_k$. But $\cos t_k = \frac{1}{2}\cos t_k + \frac{1}{2}\cos 3t_k$ at $t_k = 0, \pi/2, \pi$; and $3\pi/2$. The $\cos t_k$ and $\cos 3t_k$ mimic each other because of the limited number of data points (and the particular choice of data points). This error of one frequency mimicking another is known as **aliasing**. The problem can be minimized by taking more data points. ∎

Fast Fourier Transform

The fast Fourier transform is a particular way of factoring and rearranging the terms in the sums of the discrete Fourier transform. Brought to the attention of the scientific community by Cooley and Tukey,[14] its importance lies in the drastic reduction in the number of numerical operations required. Because of the tremendous increase in speed achieved (and reduction in cost), the fast Fourier transform has been hailed as one of the few really significant advances in numerical analysis in the past few decades.

For N time values (measurements), a direct calculation of a discrete Fourier transform would mean about N^2 multiplications. For N a power of 2, the fast Fourier transform technique of Cooley and Tukey cuts the number of multiplications required to $(N/2)\log_2 N$. If $N = 1024$ $(= 2^{10})$, the fast Fourier transform achieves a computational reduction by a factor of over 200. This is why the fast Fourier transform is called fast and why it has revolutionized the digital processing of waveforms. Details on the internal operation will be found in the paper by Cooley and Tukey and in the paper by Bergland.[15]

[14] J. W. Cooley and J. W. Tukey, *Math. Comput.* **19**: 297 (1965).

[15] G. D. Bergland, A guided tour of the fast Fourier transform, *IEEE Spectrum*, July, pp. 41–52 (1969); see also, W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes*, 2nd ed., Cambridge, UK: Cambridge University Press (1996), Section 12.3.

Exercises

14.6.1 Derive the trigonometric forms of discrete orthogonality corresponding to Eq. (14.84):

$$\sum_{k=0}^{2N-1} \cos\left(\frac{2\pi p t_k}{T}\right)\sin\left(\frac{2\pi q t_k}{T}\right) = 0$$

$$\sum_{k=0}^{2N-1} \cos\left(\frac{2\pi p t_k}{T}\right)\cos\left(\frac{2\pi q t_k}{T}\right) = \begin{cases} 0, & p \neq q \\ N, & p = q \neq 0, N \\ 2N, & p = q = 0, N \end{cases}$$

$$\sum_{k=0}^{2N-1} \sin\left(\frac{2\pi p t_k}{T}\right)\sin\left(\frac{2\pi q t_k}{T}\right) = \begin{cases} 0, & p \neq q \\ N, & p = q \neq 0, N \\ 0, & p = q = 0, N. \end{cases}$$

Hint. Trigonometric identities such as

$$\sin A \cos B = \tfrac{1}{2}\big[\sin(A+B) + \sin(A-B)\big]$$

are useful.

14.6.2 Equation (14.84) exhibits orthogonality summing over time points. Show that we have the same orthogonality summing over frequency points

$$\frac{1}{2N}\sum_{p=0}^{2N-1} \left(e^{i\omega_p t_m}\right)^* e^{i\omega_p t_k} = \delta_{mk}.$$

14.6.3 Show in detail how to go from

$$F(\omega_p) = \frac{1}{2N}\sum_{k=0}^{2N-1} f(t_k)e^{i\omega_p t_k}$$

to

$$f(t_k) = \sum_{p=0}^{2N-1} F(\omega_p)e^{-i\omega_p t_k}.$$

14.6.4 The functions $f(t_k)$ and $F(\omega_p)$ are discrete Fourier transforms of each other. Derive the following symmetry relations:

(a) If $f(t_k)$ is real, $F(\omega_p)$ is Hermitian symmetric; that is,

$$F(\omega_p) = F^*\left(\frac{4\pi N}{T} - \omega_p\right).$$

(b) If $f(t_k)$ is pure imaginary,

$$F(\omega_p) = -F^*\left(\frac{4\pi N}{T} - \omega_p\right).$$

Note. The symmetry of part (a) is an illustration of aliasing. The frequency $4\pi N/T - \omega_p$ masquerades as the frequency ω_p.

14.6.5 Given $N = 2$, $T = 2\pi$, and $f(t_k) = \sin t_k$,

(a) find $F(\omega_p)$, $p = 0, 1, 2, 3$, and
(b) reconstruct $f(t_k)$ from $F(\omega_p)$ and exhibit the aliasing of $\omega_1 = 1$ and $\omega_3 = 3$.

ANS. (a) $F(\omega_p) = (0, i/2, 0, -i/2)$

(b) $f(t_k) = \frac{1}{2}\sin t_k - \frac{1}{2}\sin 3t_k$.

14.6.6 Show that the Chebyshev polynomials $T_m(x)$ satisfy a discrete orthogonality relation

$$\frac{1}{2}T_m(-1)T_n(-1) + \sum_{s=1}^{N-1} T_m(x_s)T_n(x_s) + \frac{1}{2}T_m(1)T_n(1) = \begin{cases} 0, & m \neq n \\ N/2, & m = n \neq 0 \\ N, & m = n = 0. \end{cases}$$

Here, $x_s = \cos\theta_s$, where the $(N+1)\theta_s$ are equally spaced along the θ-axis:

$$\theta_s = \frac{s\pi}{N}, \qquad s = 0, 1, 2, \ldots, N.$$

14.7 Fourier Expansions of Mathieu Functions

As a realistic application of Fourier series we now derive first integral equations satisfied by Mathieu functions, from which subsequently their Fourier series are obtained.

Integral Equations and Fourier Series for Mathieu Functions

Our first goal is to establish Whittaker's integral equations that Mathieu functions satisfy, from which we then obtain their Fourier series representations.

We start from an integral representation

$$V(\mathbf{r}) = \int_{-\pi}^{\pi} f(z + ix\cos\theta + iy\sin\theta, \theta)\, d\theta \tag{14.97}$$

of a solution V of Laplace's equation with a twice differentiable function $f(v, \theta)$. Applying ∇^2 to V we verify that it obeys Laplace's PDE. Separating variables in Laplace's PDE suggests choosing the product form $f(v, \theta) = e^{kv}\phi(\theta)$. Substituting the elliptical variables of Eq. (13.163) we rewrite V as

$$R(\xi)\Phi(\eta)e^{kz} = \int_{-\pi}^{\pi} \phi(\theta) e^{k(z + ic\cosh\xi\cos\eta\cos\theta + ic\sinh\xi\sin\eta\sin\theta)}\, d\theta \tag{14.98}$$

with normalization $R(0) = 1$. Since ξ and η are independent variables we may set $\xi = 0$, which leads to Whittaker's integral representation

$$\Phi(\eta) = \int_{-\pi}^{\pi} \phi(\theta)\exp(ick\cos\theta\cos\eta)\, d\theta, \tag{14.99}$$

where $ck = 2\sqrt{q}$ from Eq. (13.180). Clearly, Φ is even in the variable η and periodic with period π. In order to prove that $\phi \sim \Phi$ we check how $\phi(\theta)$ is constrained when $\Phi(\eta)$ is taken to obey the angular Mathieu ODE

$$\begin{aligned}\frac{d^2\Phi}{d\eta^2} &+ (\lambda - 2q\cos 2\eta)\Phi(\eta)\\ &= \int_{-\pi}^{\pi}\phi(\theta)\exp(ick\cos\theta\cos\eta)\\ &\quad\cdot\big[\lambda - 2q\cos 2\eta + (ick\cos\theta\sin\eta)^2 - ick\cos\theta\cos\eta\big]d\theta. \end{aligned}\tag{14.100}$$

Here we integrate the last term on the right-hand side by parts, obtaining

$$\begin{aligned}&\frac{d^2\Phi}{d\eta^2} + (\lambda - 2q\cos 2\eta)\Phi(\eta)\\ &\quad= \phi(\theta)(-ick\cos\eta\sin\theta)\exp(ick\cos\theta\cos\eta)\Big|_{\theta=-\pi}^{\pi}\\ &\qquad+ \int_{-\pi}^{\pi}\phi(\theta)\exp(ick\cos\theta\cos\eta)[\lambda - 2q\cos 2\eta - ick\cos\theta\cos\eta]\,d\theta\\ &\qquad+ \int_{-\pi}^{\pi}\big[-\phi'(\theta)(-ick\cos\eta\sin\theta) + \phi(\theta)ick\cos\eta\cos\theta\big]\exp(ick\cos\theta\cos\eta)\,d\theta\\ &\quad= \int_{-\pi}^{\pi}\exp ick\cos\theta\cos\eta\big[\phi(\theta)(\lambda - 2q\cos 2\eta) + \phi'(\theta)ick\cos\eta\sin\theta\big]\,d\theta, \end{aligned}\tag{14.101}$$

where the integrated term vanishes if $\phi(-\pi) = \phi(\pi)$, which we assume to be the case. Integrating once more by parts yields

$$\begin{aligned}\frac{d^2\Phi}{d\eta^2} + (\lambda - 2q\cos 2\eta)\Phi(\eta) &= -\phi'(\theta)\exp ick\cos\theta\cos\eta\Big|_{\theta=-\pi}^{\pi}\\ &+ \int_{-\pi}^{\pi}\exp(ick\cos\theta\cos\eta)\big[\phi(\theta)(\lambda - 2q\cos 2\eta) + \phi''(\theta)\big]\,d\theta, \end{aligned}\tag{14.102}$$

where the integrated term vanishes if ϕ' is periodic with period π, which we assume is the case. Therefore, if $\phi(\theta)$ obeys the angular Mathieu ODE, so does the integral $\Phi(\eta)$, in Eq. (14.99). As a consequence, $\phi(\theta) \sim \Phi(\theta)$, where the constant may be a function of the parameter q.

Thus, we have the main result that a solution $\Phi(\eta)$ of Mathieu's ODE that is even in the variable η satisfies the integral equation

$$\boxed{\Phi(\eta) = \Lambda_n(q)\int_{-\pi}^{\pi} e^{2i\sqrt{q}\cos\theta\cos\eta}\,\Phi(\theta)\,d\theta.}\tag{14.103}$$

When these Mathieu functions are expanded in a Fourier cosine series and normalized so that the leading term is $\cos n\eta$, they are denoted by $\mathrm{ce}_n(\eta, q)$.

Similarly, solutions of Mathieu's ODE that are odd in η with leading term $\sin n\eta$ in a Fourier series are denoted by $\mathrm{se}_n(\eta, q)$, and they can similarly be shown to obey the integral equation

$$\boxed{\mathrm{se}_n(\eta, q) = s_n(q) \int_{-\pi}^{\pi} \sin\bigl(2i\sqrt{q}\sin\eta\sin\theta\bigr)\mathrm{se}_n(\theta, q)\, d\theta.} \tag{14.104}$$

We now come to the **Fourier expansion for the angular Mathieu functions** and start with

$$\mathrm{se}_1(\eta, q) = \sin\eta + \sum_{\nu=1}^{\infty} \beta_\nu(q)\sin(2\nu+1)\eta, \ \beta_\nu(q) = \sum_{\mu=\nu}^{\infty} \beta_\mu^{(\nu)} q^\mu \tag{14.105}$$

as a paradigm for the systematic construction of Mathieu functions of odd parity. Notice the key point that the coefficient β_ν of $\sin(2\nu+1)\eta$ in the Fourier series depends on the parameter q and is expanded in a power series. Moreover, se_1 is normalized so that the coefficient of the leading term, $\sin\eta$, is unity, that is, independent of q. This feature will become important when se_1 is substituted into the angular Mathieu ODE to determine the eigenvalue $\lambda(q)$.

The fact that the β_ν power series in q starts with exponent ν can be proved by a simpler but similar series for $\mathrm{se}_1(\eta, q)$:

$$\mathrm{se}_1(\eta, q) = \sum_{\nu=0}^{\infty} \gamma_\nu(q)\sin^{2\nu+1}\eta, \qquad \gamma_\nu(q) = \sum_{\mu=0}^{\infty} \gamma_\mu^{(\nu)} q^\mu, \tag{14.106}$$

which is useful for this demonstration alone. However, since we need to expand

$$\sin^{2\nu+1}\eta = \sum_{m=0}^{\nu} B_{\nu m}\sin(2m+1)\eta \tag{14.107}$$

with Fourier coefficients

$$B_{\nu m} = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^{2\nu+1}\eta\sin(2m+1)\eta\, d\eta = \frac{(-1)^m}{2^{2\nu}}\binom{2\nu+1}{\nu-m} \tag{14.108}$$

that we can look up in a table of integrals (see Gradshteyn and Ryzhik in the Additional Readings of Chapter 13), this proof gives us an opportunity to introduce the $B_{\nu m}$ that are nonzero only if $m \le \nu$ and are important ingredients of the recursion relations for the leading terms of se_1 (and all other Mathieu functions of odd parity). Substituting Eq. (14.107) into Eq. (14.106) we obtain

$$\mathrm{se}_1(\eta, q) = \sum_{\nu=0}^{\infty}\sum_{m=0}^{\nu} B_{\nu m}\gamma_\nu(q)\sin(2m+1)\eta. \tag{14.109}$$

Comparing this expression for se_1 with Eq. (14.105) we find

$$\beta_\nu(q) = \sum_{m=\nu}^{\infty} B_{m\nu}\gamma_m(q). \tag{14.110}$$

Here, the sum starts with $m = \nu$ because $B_{m\nu} = 0$ for $m < \nu$.

Next we substitute Eq. (14.106) into the integral Eq. (14.104) for $n = 1$, where we insert the power series for $\sin(2i\sqrt{q}\sin\eta\sin\theta)$. This yields

$$\begin{aligned}\frac{\mathrm{se}_1(\eta,q)}{2\pi s_1(q)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin(2i\sqrt{q}\sin\eta\sin\theta)\mathrm{se}_1(\theta,q)\,d\theta \\ &= \frac{1}{2\pi s_1(q)}\sum_{m=0}^{\infty}\gamma_m(q)\sin^{2m+1}\eta \\ &= i\sqrt{q}\sum_{m,\nu=0}^{\infty}q^m\gamma_\nu(q)\sin^{2m+1}\eta\frac{2^{2m+1}}{(2m+1)!}\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^{2\nu+2m+2}\theta\,d\theta,\end{aligned} \tag{14.111}$$

from which we obtain the recursion relations

$$\gamma_m(q) = \frac{2^{2m+1}}{(2m+1)!}q^m i\sqrt{q}s_1(q)\sum_{\nu=0}^{\infty}\gamma_\nu(q)\int_{-\pi}^{\pi}\sin^{2\nu+2m+2}\theta\,d\theta, \tag{14.112}$$

upon comparing coefficients of $\sin^{2m+1}\eta$. This shows that the power series for $\gamma_m(q)$ starts with q^m. Using Eq. (14.110) proves that the power series for $\beta_m(q)$ also starts with q^m, and this confirms Eq. (14.105). The integral in Eq. (14.112) can be evaluated analytically and expressed via the beta function (Chapter 8) in terms of ratios of factorials, but we do not need this formula here.

Our next goal is to establish a recursion relation for the leading term $\beta_\nu^{(\nu)}$ of se_1, in Eq. (14.105). We substitute Eq. (14.105) into the integral Eq. (14.104) for $n = 1$, where we insert the power series for $\sin(2i\sqrt{q}\sin\eta\sin\theta)$ again, along with the expansion

$$\frac{1}{2\pi s_1(q)} = i\sum_{m=0}^{\infty}\alpha_m q^{m+1/2}. \tag{14.113}$$

Here, the extra factor, $i\sqrt{q}$, cancels the corresponding factor from the sine in the integral equation. This yields

$$\begin{aligned}&\sum_{\nu=0}^{\infty}\beta_\mu^{(\lambda)}q^{\mu+\nu}\sin^{2\nu+1}\eta\frac{2^{2\nu+1}}{(2\nu+1)!}\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin^{2\nu+2\lambda+2}\theta\,d\theta \\ &\quad= \sum_{m,\mu=0}^{\infty}\alpha_m q^{m+\mu}\beta_\mu^{(\nu)}\sin(2\nu+1)\eta.\end{aligned} \tag{14.114}$$

Here, we replace $\sin^{2\nu+1}\eta$ by $\sin(2m+1)\eta$ using Eq. (14.107). Upon comparing the coefficients of $q^N\sin(2\nu+1)\eta$ for $N = \mu + \nu$ we obtain the recursion relation

$$\sum_{\nu=n}^{N}\sum_{\lambda=0}^{N-\nu}\beta_{N-\nu}^{(\lambda)}\frac{2^{2\nu}}{(2\nu+1)!}B_{\nu n}B_{\nu\lambda} = \sum_{m=0}^{N-n}\alpha_m\beta_{N-m}^{(n)}. \tag{14.115}$$

Now we substitute Eq. (14.108) to obtain the **main recursion relation** for the leading coefficients $\beta_\nu^{(\nu)}$ of se_1:

$$\sum_{\nu=n}^{N}\sum_{\lambda=0}^{N-\nu}\frac{\beta_{N-\nu}^{(\lambda)}}{2^{2\nu}(2\nu+1)!}\binom{2\nu+1}{\nu-n}\binom{2\nu+1}{\nu-\lambda}=\sum_{m=0}^{N-n}\alpha_m\beta_{N-m}^{(n)}. \tag{14.116}$$

Example 14.7.1 Leading Coefficients of se_1

We evaluate Eq. (14.116) starting with $N=0, n=0$. For this case we find $\beta_0^{(0)}=\alpha_0\beta_0^{(0)}$, or $\alpha_0=1$ because the coefficient of $\sin\eta$ in se_1, $\beta_0^{(0)}=1$, by normalization. For $N=1$, $n=0$ Eq. (14.116) yields

$$\alpha_0\beta_1^{(0)}+\alpha_1\beta_0^{(0)}=\beta_1^{(0)}+\frac{1}{4\cdot 3!}\binom{3}{1}\left[\beta_0^{(0)}\binom{3}{1}+\beta_0^{(1)}\binom{3}{0}\right], \tag{14.117}$$

where $\beta_0^{(1)}=0$ and $\beta_1^{(0)}$ drops out, a general feature. Of course, $\beta_1^{(0)}=0$ because $\sin\eta$ in se_1 has coefficient unity. This yields $\alpha_1=3/8$.

The case $N=1, n=1$ yields

$$\frac{-1}{4\cdot 3!}\binom{3}{0}\beta_0^{(0)}\binom{3}{1}=\alpha_0\beta_1^{(1)}, \tag{14.118}$$

or $\beta_1^{(1)}=-1/8$. The leading term is obtained from the general case $n=N$,

$$\frac{(-1)^N}{2^{2N}(2N+1)!}\binom{2N+1}{0}\beta_0^{(0)}\binom{2N+1}{N}=\alpha_0\beta_N^{(N)}, \tag{14.119}$$

as

$$\boxed{\beta_N^{(N)}=\frac{(-1)^N}{2^{2N}(2N+1)!}\binom{2N+1}{N},} \tag{14.120}$$

which was first derived by Mathieu. For $N=1$ this formula reproduces our earlier result, $\beta_1^{(1)}=-1/8$. ■

In order to determine the first nonleading term $\beta_{N+1}^{(N)}$ of se_1, Eq. (14.105), and the eigenvalue $\lambda_1(q)$ we substitute se_1 into the angular Mathieu ODE, Eq. (13.181), using the trigonometric identities

$$2\cos 2\eta\sin(2\nu+1)\eta=\sin(2\nu+3)\eta+\sin(2\nu-1)\eta$$

and

$$\frac{d^2\sin(2\nu+1)\eta}{d\eta^2}=-(2\nu+1)^2\sin(2\nu+1)\eta.$$

This yields

$$
\begin{aligned}
0 &= \frac{d^2\mathrm{se}_1}{d\eta^2} + (\lambda_1 - 2q\cos 2\eta)\mathrm{se}_1 = q(\sin\eta - \sin 3\eta) + \lambda_1 \sin\eta - \sin\eta \\
&\quad + \sum_{\nu=1}^{\infty} [\lambda_1 - (2\nu+1)^2] \left[\frac{(-1)^\nu q^\nu}{2^{2\nu}\nu!(\nu+1)!} + \beta_{\nu+1}^{(\nu)} q^{\nu+1} + \cdots \right] \sin(2\nu+1)\eta \\
&\quad - q \sum_{\nu=1}^{\infty} \left[\frac{(-q)^\nu}{2^{2\nu}\nu!(\nu+1)!} + \beta_{\nu+1}^{(\nu)} q^{\nu+1} + \cdots \right] \big(\sin(2\nu+3)\eta + \sin(2\nu-1)\eta\big) \\
&= \left(\lambda_1 - 1 + q - q\left[-\frac{q}{2^2 2!} + \beta_2^{(1)} q^2 + \cdots \right] \right) \sin\eta \\
&\quad + \sin 3\eta \left[-q - q\left(\frac{q^2}{2^4 2! 3!} + \beta_3^{(2)} q^3 \right) + (\lambda_1 - 3^2)\left(-\frac{q}{2^2 2!} + \beta_2^{(1)} q^2 \right) \right] \\
&\quad + \sin(2\nu+1)\eta [\lambda_1 - (2\nu+1)^2] \left(\frac{(-q)^\nu}{2^{2\nu}\nu!(\nu+1)!} + \beta_{\nu+1}^{(\nu)} q^{\nu+1} \right) \\
&\quad - q\sin(2\nu+1)\eta \left(\frac{(-q)^{\nu+1}}{2^{2(\nu+1)}(\nu+1)!(\nu+2)!} + \beta_{\nu+2}^{(\nu+1)} q^{\nu+2} \right) \\
&\quad - q\sin(2\nu+1)\eta \left(\frac{(-q)^{\nu-1}}{2^{2(\nu-1)}(\nu-1)!\nu!} + \beta_{\nu}^{(\nu-1)} q^{\nu} \right) + \cdots .
\end{aligned} \tag{14.121}
$$

In this series the coefficient of each power of q within different sine terms must vanish; that of $\sin\eta$ being zero yields the eigenvalue

$$
\lambda_1(q) = 1 - q - \frac{1}{8} q^2 + \beta_2^{(1)} q^3 + \cdots, \tag{14.122}
$$

with $\beta_2^{(1)} = 1/2^6$ coming from the vanishing coefficient of q^2 in $\sin 3\eta$. Setting the coefficient of $(-q)^\nu$ in $\sin(2\nu+1)\eta$ equal to zero yields the identity

$$
[1 - (2\nu+1)^2] \frac{1}{2^{2\nu}\nu!(\nu+1)!} + \frac{1}{2^{2(\nu-1)}(\nu-1)!\nu!} = 0, \tag{14.123}
$$

which verifies the correct determination of the leading terms $\beta_\nu^{(\nu)}$ in Eq. (14.120). The vanishing coefficient of $q^{\nu+1}$ in $\sin(2\nu+1)\eta$ yields

$$
\frac{(-1)^{\nu+1}}{2^{2\nu}\nu!(\nu+1)!} + [1 - (2\nu+1)^2]\beta_{\nu+1}^{(\nu)} - \beta_\nu^{(\nu-1)} = 0, \tag{14.124}
$$

which implies the main **recursion relation for nonleading coefficients**,

$$
4\nu(\nu+1)\beta_{\nu+1}^{(\nu)} = -\beta_\nu^{(\nu-1)} + \frac{(-1)^{\nu+1}}{2^{2\nu}\nu!(\nu+1)!}, \tag{14.125}
$$

for the first nonleading terms. We verify that

$$
\boxed{\beta_{\nu+1}^{(\nu)} = \frac{(-1)^{\nu+1}\nu}{2^{2\nu+2}(\nu+1)!^2}} \tag{14.126}
$$

satisfies this recursion relation. Higher nonleading terms may be obtained by setting to zero the coefficient of $q^{\nu+2}$, etc. Altogether we have derived the Fourier series for

$$\boxed{\mathrm{se}_1(\eta,q)=\sin\eta+\sum_{\nu=1}^{\infty}\left[\frac{(-q)^{\nu}}{2^{2\nu}\nu!(\nu+1)!}+\frac{(-q)^{\nu+1}\nu}{2^{2\nu+2}(\nu+1)!^2}+\cdots\right]\cdot\sin(2\nu+1)\eta.} \tag{14.127}$$

A similar treatment yields the Fourier series for $\mathrm{se}_{2n+1}(\eta,q)$ and $\mathrm{se}_{2n}(\eta,q)$. An invariance of Mathieu's ODE leads to the symmetry relation

$$\mathrm{ce}_{2n+1}(\eta,q)=(-1)^n\mathrm{se}_{2n+1}(\eta+\pi/2,-q), \tag{14.128}$$

which allows us to determine the ce_{2n+1} of period 2π from se_{2n+1}. Similarly, $\mathrm{ce}_{2n}(\eta+\pi/2,-q)=\mathrm{se}_{2n}(\eta,q)$ relates these Mathieu functions of period π to each other.

Finally, we briefly outline a derivation of the Fourier series for

$$\mathrm{ce}_0(\eta,q)=1+\sum_{n=1}^{\infty}\beta_n(q)\cos 2n\eta, \qquad \beta_n(q)=\sum_{m=n}^{\infty}\beta_m^{(n)}q^m, \tag{14.129}$$

as a paradigm for the Mathieu functions of period π. Note that this normalization agrees with Whittaker and Watson and with Hochstadt in the Additional Readings of Chapter 13, whereas in AMS-55 (for the full reference see footnote 4 in Chapter 5) ce_0 differs by a factor of $1/\sqrt{2}$. The symmetry relation from the Mathieu ODE,

$$\mathrm{ce}_0\left(\frac{\pi}{2}-\eta,-q\right)=\mathrm{ce}_0(\eta,q), \tag{14.130}$$

implies

$$\beta_n(-q)=(-1)^n\beta_n(q); \tag{14.131}$$

that is, β_{2n} contains only even powers of q and β_{2n+1} only odd powers.

The fact that the power series for $\beta_n(q)$ in Eq. (14.129) starts with q^n can be proved by the similar expansion

$$\mathrm{ce}_0(\eta,q)=\sum_{n=0}^{\infty}\gamma_n(q)\cos^{2n}\eta, \qquad \gamma_n(q)=\sum_{\mu=0}^{\infty}\gamma_\mu^{(n)}q^\mu, \tag{14.132}$$

as for se_1 in Eqs. (14.105) to (14.112). Substituting Eq. (14.132) into the integral equation

$$\mathrm{ce}_0(\eta,q)=c_0(q)\int_{-\pi}^{\pi}e^{2i\sqrt{q}\cos\theta\cos\eta}\mathrm{ce}(\theta,q)\,d\theta, \tag{14.133}$$

inserting the power series for the exponential function (odd powers $\cos^{2m+1}\theta$ drop out) and equating the coefficients of $\cos^{2m}\eta$ yields

$$\gamma_m(q)=c_1(q)(-q)^m\frac{2^{2m}}{(2m)!}\sum_{\mu=0}^{\infty}\gamma_\mu(q)\int_{-\pi}^{\pi}\cos^{2m+2\mu}\theta\,d\theta. \tag{14.134}$$

This recursion relation shows that the power series for $\gamma_m(q)$ starts with q^m. We expand

$$\cos^{2n}\eta = \sum_{m=0}^{n} A_{nm}\cos 2m\eta \tag{14.135}$$

with Fourier coefficients

$$A_{nm} = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\cos^{2n}\eta\cos 2m\eta\, d\eta = \frac{1}{2^{2n-1}}\binom{2n}{n-m}, \tag{14.136}$$

which are nonzero only when $m \le n$. Using this result to replace the cosine powers in Eq. (14.132) by $\cos 2m\eta$ we obtain

$$\beta_n(q) = \sum_{m=n}^{\infty} A_{mn}\gamma_m(q), \tag{14.137}$$

confirming Eq. (14.129).

Proceeding as for se_1 in Eqs. (14.113) to (14.120) we substitute Eq. (14.129) into the integral Eq. (14.133) and obtain

$$\begin{aligned}\sum_{m,\mu,\nu,\lambda=0}^{\infty} &(-1)^m\frac{2^{2m}}{(2m)!}q^{\mu+m}\sum_{\nu=0}^{m} A_{m\nu}\cos(2\nu\eta)A_{m\lambda}\\ &= \sum_{m,\mu,\nu=0}^{\infty}\alpha_m\beta_\mu^{(\nu)}q^{m+\mu}\cos(2\nu\eta),\end{aligned} \tag{14.138}$$

with

$$\frac{1}{2\pi c_1(q)} = \sum_{m=0}^{\infty}\alpha_m q^m. \tag{14.139}$$

Upon comparing the coefficient of $q^N\cos(2\nu\eta)$ with $N = m+\mu$, we extract the **recursion relation for leading coefficients** $\beta_n^{(n)}$ **of** ce_0

$$\sum_{m=\nu}^{N}(-1)^m\frac{2^{2m}}{(2m)!}A_{m\nu}\sum_{\lambda=0}^{m}\beta_{N-m}^{(\lambda)}A_{m\lambda} = \sum_{m=0}^{N}\alpha_m\beta_{N-m}^{(\nu)}, \tag{14.140}$$

with A_{nm} in Eq. (14.136).

Example 14.7.2 LEADING COEFFICIENTS FOR ce_0

The case $N=0$, $\nu=0$ of Eq. (14.140) yields

$$A_{00}^2\beta_0^{(0)} = \alpha_0\beta_0^{(0)}, \tag{14.141}$$

with $A_{00}=1$ and $\beta_0^{(0)}=1$ from normalizing the leading term of ce_0 to unity so that $\alpha_0 = 1$ results.

The case $N=1$, $\nu=0$ yields

$$A_{00}\beta_1^{(0)}A_{00} - 2A_{10}\left[\beta_0^{(0)}A_{10} + \beta_0^{(1)}A_{11}\right] = \alpha_0\beta_1^{(0)} + \alpha_1\beta_0^{(0)}, \tag{14.142}$$

with $\beta_0^{(1)} = 0$ by Eq. (14.129). This simplifies to

$$\beta_1^{(0)} - \frac{1}{2}\beta_0^{(0)} = \alpha_1\beta_0^{(0)} + \beta_1^{(0)}, \tag{14.143}$$

where $\beta_1^{(0)}$ drops out. We know already that $\beta_1^{(0)} = 0$ from the leading term unity of ce_0. Therefore, $\alpha_1 = -1/2$.

For the case $N = 1$, $\nu = 1$ we obtain

$$-2A_{11}\beta_0^{(0)}A_{10} = \alpha_0\beta_1^{(1)}, \tag{14.144}$$

with $A_{10} = 1/2 = A_{11}$, from which $\beta_1^{(1)} = -1/2$ follows. For the case $N = 2$, $\nu = 2$ we find

$$\frac{2^4}{4!2^3}\beta_0^{(0)}A_{20} = \alpha_0\beta_2^{(2)}, \tag{14.145}$$

with $A_{20} = 3/8$, from which $\beta_2^{(2)} = 2^{-4}$ follows. The general case N, $\nu = N$ yields

$$(-1)^N\frac{2^{2N}}{(2N)!}A_{NN}\beta_0^{(0)}A_{N0} = \alpha_0\beta_N^{(N)}, \tag{14.146}$$

with $A_{NN} = 2^{-2N+1}$, $A_{N0} = \frac{1}{2^{2N-1}}\binom{2N}{N}$, from which the leading term

$$\boxed{\beta_N^{(N)} = \frac{(-1)^N}{2^{2N-1}(2N)!}\binom{2N}{N} = \frac{(-1)^N}{2^{2N-1}N!^2}} \tag{14.147}$$

follows. ■

The nonleading terms $\beta_{N+1}^{(N)}$ of ce_0 are best determined from the angular Mathieu ODE by substitution of Eq. (14.129), in analogy with se_1, Eqs. (14.121) to (14.127). Using the identities

$$\begin{aligned} 2\cos(2n\eta)\cos 2\eta &= \cos(2n+2)\eta + \cos(2n-2)\eta, \\ \frac{d^2}{d\eta^2}\cos(2n\eta) &= -(2n)^2\cos(2n\eta), \end{aligned} \tag{14.148}$$

we obtain

$$\begin{aligned} &\frac{d^2\mathrm{ce}_0}{d\eta^2} + \big(\lambda_0(q) - 2q\cos 2\eta\big)\mathrm{ce}_0 = 0 \\ &\qquad = \lambda_0(q) - q\left(-\frac{q}{2} + \frac{7}{2^7}q^3\right) + \cdots \\ &\qquad + \sum_{n=1}^{\infty}\big(\lambda_0 - 4n^2\big)\cos(2n\eta)\left[\frac{(-q)^n}{2^{2n-1}n!^2} + \beta_{n+2}^{(n)}q^{n+2}\right] \\ &\qquad - q\sum_{n=1}^{\infty}\left[\frac{(-q)^n}{2^{2n-1}n!^2} + \beta_{n+2}^{(n)}q^{n+2}\right] \\ &\qquad \times\big[\cos(2n+2)\eta + \cos(2n-2)\eta\big]. \end{aligned} \tag{14.149}$$

Setting the coefficient of $\cos(2n\eta)$ for $n = 0$ to zero yields the eigenvalue

$$\lambda_0 = -\frac{1}{2}q^2 + \frac{7}{2^7}q^4 + \cdots . \tag{14.150}$$

The coefficient of $\cos(2n\eta)q^n$ yields an identity,

$$(-1)^{n+1}\frac{4n^2}{2^{2n-1}n!^2} + \frac{(-1)^n}{2^{2n-3}(n-1)!^2} = 0, \tag{14.151}$$

which shows that the leading term in Eq. (14.147) was correctly determined. The coefficient of $q^{n+2}\cos(2n\eta)$ yields the recursion relation

$$-4n^2\beta_{n+2}^{(n)} - \beta_{n+1}^{(n-1)} + \frac{(-1)^{n+1}}{2^{2n}n!^2} + \frac{(-1)^n}{2^{2n+1}(n+1)!^2} = 0. \tag{14.152}$$

It is straightforward to check that

$$\beta_{n+2}^{(n)} = (-1)^{n+1}\frac{n(3n+4)}{2^{2n+3}(n+1)!^2} \tag{14.153}$$

satisfies this recursion relation. Altogether we have derived the formula

$$\boxed{\begin{aligned} ce_0(\eta, q) &= 1 + \cos 2\eta\left[-\frac{1}{2}q^2 + \frac{7}{2^7}q^3 + \cdots\right] \\ &\quad + \cos 4\eta\left[\frac{q^2}{2^5} + \cdots\right] + \cos 6\eta\left[-\frac{q^3}{2^7 3^2} + \cdots\right] \\ &= 1 + \sum_{n=1}^{\infty}\cos(2n\eta)\left[\frac{(-q)^n}{2^{2n-1}n!^2} + \frac{(-1)^{n+1}n(3n+4)q^{n+2}}{2^{2n+3}(n+1)!^2} + \cdots\right]. \end{aligned}} \tag{14.154}$$

Similarly one can derive

$$\boxed{ce_1(\eta, q) = \cos\eta + \sum_{n=1}^{\infty}\cos(2n+1)\eta\left[\frac{(-q)^n}{2^{2n}n!(n+1)!} - \frac{(-q)^{n+1}n}{2^{2n+2}(n+1)1^2} + \cdots\right],} \tag{14.155}$$

whose eigenvalue is given by the power series

$$\boxed{\lambda_1(q) = 1 + q - \frac{1}{8}q^2 - \frac{1}{2^6}q^3 + \cdots .} \tag{14.156}$$

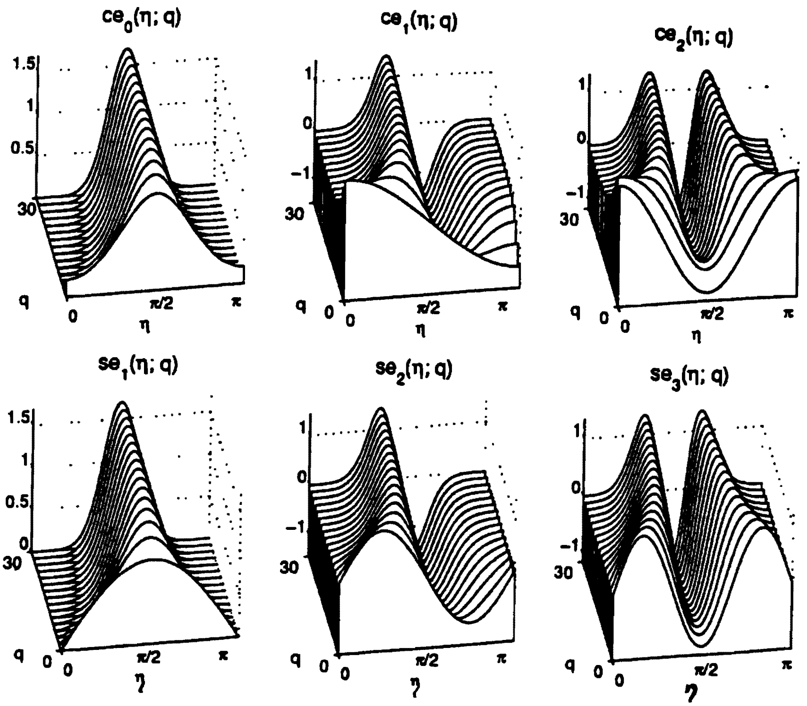

FIGURE 14.12 Angular Mathieu functions. (From Gutiérrez-Vega *et al.*, *Am. J. Phys.* **71**: 233 (2003).)

Exercises

14.7.1 Determine the nonleading coefficients $\beta^{(n)}_{n+2}$ for se_1. Derive a suitable recursion relation.

14.7.2 Determine the nonleading coefficients $\beta^{(n)}_{n+4}$ for ce_0. Derive the corresponding recursion relation.

14.7.3 Derive the formula for ce_1, Eq. (14.155), and its eigenvalue, Eq. (14.156).

Additional Readings

Carslaw, H. S., *Introduction to the Theory of Fourier's Series and Integrals*, 2nd ed. London: Macmillan (1921); 3rd ed., paperback, New York: Dover (1952). This is a detailed and classic work; includes a considerable discussion of Gibbs phenomenon in Chapter IX.

Hamming, R. W., *Numerical Methods for Scientists and Engineers*, 2nd ed. New York: McGraw-Hill (1973), reprinted Dover (1987). Chapter 33 provides an excellent description of the fast Fourier transform.

Jeffreys, H., and B. S. Jeffreys, *Methods of Mathematical Physics*, 3rd ed. Cambridge, UK: Cambridge University Press (1972).

Kufner, A., and J. Kadlec, *Fourier Series*. London: Iliffe (1971). This book is a clear account of Fourier series in the context of Hilbert space.

Lanczos, C., *Applied Analysis*, Englewood Cliffs, NJ: Prentice-Hall (1956), reprinted Dover (1988). The book gives a well-written presentation of the Lanczos convergence technique (which suppresses the Gibbs phenomenon oscillations). This and several other topics are presented from the point of view of a mathematician who wants useful numerical results and not just abstract existence theorems.

Oberhettinger, F., *Fourier Expansions, A Collection of Formulas*. New York, Academic Press (1973).

Zygmund, A., *Trigonometric Series*. Cambridge, UK: Cambridge University Press (1988). The volume contains an extremely complete exposition, including relatively recent results in the realm of pure mathematics.