Intermittency, escape and periodic orbits

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October 15, 1999

Abstract

We prove that the survival probabilitiy of a class of open, nonuniformly hyperbolic maps can be bounded close to sums over periodic orbit stabilities. The survival probabilitiy is the probality of of not escaping an open map during n steps, starting from an initially uniform distribution. A crucial ingredient is an averaging procedure along the periodic orbits, as bounds cannot be established for individual intervals of the partitions, in contrast to expanding maps.

1 Introduction

The periodic orbit theory offers an intriguing possibility to compute properties like

1. Escape from repellers.
2. Chaotic averages, such as Lyapunov exponents, diffusion constants etc.
3. Decay of correlations, Ruelle resonances.

A firm link between the periodic orbits and their invariants on one side and the desired quantities on the other has so far been established for quite a restricted class of systems. The ideal situations is the existence of a finite

Markov partitions and uniform hyperbolicity. Heuristic arguments and numerical experience suggest that periodic orbit theory is applicable to a much wider class of systems, so, what is the limit of periodic orbit theories.

Above we grouped the chaotic properties according to increasing conceptual complexity in establishing a form link to periodic orbits.

Group one, that is the problem of escape, require only knowledge of the measure (size) of dynamically generated partitions of phase space, and a relation to certain periodic orbit sums.

Group two involve the concept of an invariant density, and the possibility of expressing it in terms of periodic orbits.

Group three, correlation decay, is related to the rate at which the invariant density is approached.

In this paper, we will restrict ourselves to escape, and show that, escape from open system, not being uniformly hyperbolic, still can be related to periodic orbits and their stabilities. More precisely, we will show this for a class of 1-d maps, having a very simple Markov partition.

2 Escape and periodic orbits

Consider a 1-d map, defined on some interval I. The map consist of a (possibly infinite) number of monotone branches $f_q(x)$, where $q \in \mathcal{A}$ is a symbol taken from an Alphabet $\mathcal{A}$. Each branch $f_q(x)$ is defined on an interval I_q. A generating partition is then given by $\mathcal{C}^{(1)} = \{I_q; q \in \mathcal{A}\}$. We want the map to admit an unrestricted symbolic dynamics. We therefore require all branches to map their domain $f_q(I_q) = I$ onto some interval $I \supset \mathcal{C}^{(1)}$ covering $\mathcal{C}^{(1)}$. A trajectory escapes whenever some iterate of the map $x \notin \mathcal{C}^{(1)}$.

The n'th level partition $\mathcal{C}^{(n)} = \{I_Q; |Q| = n\}$ can be constructed iteratively. Here Q are words of length $|Q| = n$. An interval is thus defined recursively according to

$$I_{qR} = f_q^{-1}(I_R) \quad , \tag{1}$$

where qQ is the concatenation of letter q with word Q. An initial point surviving n iterations must be contained in $\mathcal{C}^{(n)}$.

Starting from an initial (normalized) distribution we can express the fraction that survives n iterations as

$$\Gamma_n = \left(\sum_{|Q|=n} |I_Q| \right) / |I| \quad , \tag{2}$$

a quantity we will refer to as the *survival probability* or *partition sum*.

In I_Q there is a point x_Q along the *periodic orbit* $\overline{Q}$, that is

$$f^{\circ|Q|}(x_Q) = x_Q \quad x_Q \in I_Q \quad . \tag{3}$$

Its stability is

$$\Lambda_Q = \frac{df^{\circ|Q|}(x)}{dx} \,|_{x=x_Q} \tag{4}$$

Here $\mathcal{S}$ denotes the cyclic shift operator: $\mathcal{S}(Q = q_1 q_2 \dots q_n) = q_2 \dots q_n q_1$.

If all branches are expanding, i.e. there is some number Λ_{min} such that $|f'_q(x)| > \Lambda_{min}$, $\forall q$, $\forall x \in I_q$, then one can bind the size of I_q to the stability of $\overline{Q}$

$$\mathcal{C}_1 \frac{|I|}{\Lambda_Q} < |I_Q| < \mathcal{C}_2 \frac{|I|}{\Lambda_Q} \tag{5}$$

(Constants denoted by a calligraphic $\mathcal{C}$ are arbitrary, in the sense, that they may vary from one place to another.) This implies that the survival fraction can be bounded by a sum over periodic orbit according to

$$\mathcal{C}_1 \sum_{|Q|=n} \frac{1}{|\Lambda_Q|} < \Gamma_n < \mathcal{C}_2 \sum_{|Q|=n} \frac{1}{|\Lambda_Q|} \quad . \tag{6}$$

The periodic orbit sum in (6) will be denoted Z_n

$$\sum_{|Q|=n} \frac{1}{|\Lambda_Q|} \equiv Z_n \quad . \tag{7}$$

One can hardly expect to find a bound like (5), for non-uniformly hyperbolic systems, as we will see, any such attempt is doomed to fail. However, this is no reason to distrust. The following bound is also sufficient to establish (6)

$$\mathcal{C}_1 \frac{|I|}{\Lambda_Q} < \frac{1}{|Q|} \sum_{k=1}^{|Q|} |I_{\mathcal{S}^k Q}| < \mathcal{C}_2 \frac{|I|}{\Lambda_Q} \tag{8}$$

Here $\mathcal{S}$ denotes the cyclic shift operator: $\mathcal{S}(p = q_1 q_2 \dots q_n) = s_2 \dots q_n q_1$.

3 Warmup, expanding map

For comparison, and for later use, we will begin with some considerations on expanding maps. We thus consider maps obeying the following assumptions (call assumption E as in Expanding).

1. The map is defined on an enumerable set of (non overlapping intervals) I_q. The branch of the map f restricted to I_q is called f_q.
2. Each branch is *complete*, $f_q(I_q) = I$.
3. Each branch is *expanding*, i.e. there is a (positive) number $\Lambda_{min} > 0$ such that $|\frac{df_q(x)}{dx}| > \Lambda_{min}$.
4. The *regularity* of the mapping function is $f_q \in C^{1+\text{Lipshitz}}$.

Condition 3 and 4 implies

$$|\frac{df_q(x_1)}{dx_1} - \frac{df_q(x_2)}{dx_2}| < \mathcal{C}|\frac{df_q(x_1)}{dx_1}| \cdot |x_2 - x_1| \tag{9}$$

And equivalently for the inverse

$$|\frac{df_q^{-1}(x_1)}{dx_1} - \frac{df_q^{-1}(x_2)}{dx_2}| < \mathcal{N}_q|\frac{df_q^{-1}(x_1)}{dx_1}| \cdot |x_2 - x_1| \tag{10}$$

where we refer to $\mathcal{N}$ as the nonlinearity. We add to assumption E:

5. The nonlinearity $\mathcal{N}_q$ can be chosen independent on q: $\mathcal{N}_q = \mathcal{N}$.

The existence of a minimal expansion rate immediately implies.

$$|I_R| < \frac{|I|}{\Lambda_{min}^{|R|}} \tag{11}$$

According to the mean value theorem one gets

$$|I_{qR}| = |I_R| \cdot |\frac{df_q{}^{-1}(x)}{dx}| \tag{12}$$

for some $x \in I_R$. We do not know x, but if we choose arbitrarily a reference point $\tilde{x}$ in I_R, we know that $|x - \tilde{x}| \leq |I_R|$, and we can estimate the error induced by shifting x to $\tilde{x}$

$$|I_{qR}| < |I_R| \cdot |\frac{d{f_q}^{-1}(\tilde{x})}{d\tilde{x}}| \cdot |1 + \mathcal{N}|I_R|| < |I_R| \cdot |\frac{d{f_q}^{-1}(\tilde{x})}{d\tilde{x}}| \cdot |1 + \mathcal{N}\frac{|I|}{\Lambda_{min}^{|R|}}| \quad (13)$$

and similarly

$$|I_{qR}| > |I_R| \cdot |\frac{d{f_q}^{-1}(\tilde{x})}{d\tilde{x}}| \cdot |1 - \mathcal{N}\frac{|I|}{\Lambda_{min}^{|R|}}| \quad (14)$$

We get

$$|I_R| \cdot |\tilde{\Lambda}_Q^{-1}| \prod_{i=0}^{|Q|-1} \left(1 - \mathcal{N}\frac{|I_R|}{\Lambda_{min}^i}\right) < |I_{QR}| < |I| \cdot |\tilde{\Lambda}_R^{-1}| \prod_{i=0}^{|Q|-1} \left(1 + \mathcal{N}\frac{|I_R|}{\Lambda_{min}^i}\right) \quad (15)$$

where

$$\tilde{\Lambda}_Q^{-1} = \prod_{i=0}^{|Q|-1} |\frac{d{f_q}^{-1}(\tilde{x}_i)}{d\tilde{x}_i}| \quad (16)$$

Note that $\tilde{\Lambda}_Q$ is computed along an arbitrary sequence of reference points $\tilde{x}_i \in I_{q_i \ldots q_1 R}$ (if $Q = q_{|Q|} \ldots q_1$), and should not be confused with the stability of of periodic orbit $\overline{Q}$, here denoted Λ_Q.

The very useful inequality $1 + x < \exp(x)$ enables us to write

$$\prod_{i=0}^{|Q|-1} \left(1 + \mathcal{N}\frac{|I_R|}{\Lambda_{min}^i}\right) < e^{\mathcal{N}|I_R| \sum_{i=0}^{|Q|-1} \frac{1}{\Lambda_{min}^i}} < e^{\mathcal{N}|I_R| \sum_{i=0}^{\infty} \frac{1}{\Lambda_{min}^i}} = e^{\mathcal{N}|I_R| \frac{\Lambda_{min}}{\Lambda_{min}-1}} \quad (17)$$

and we can state

Lemma 3.1

$$|I_R| \cdot |\tilde{\Lambda}_Q^{-1}| e^{-\mathcal{N}|I_R| \frac{\Lambda_{min}}{\Lambda_{min}-1}} < |I_{QR}| < |I| \cdot |\tilde{\Lambda}_Q^{-1}| e^{\mathcal{N}|I_R| \frac{\Lambda_{min}}{\Lambda_{min}-1}} \quad (18)$$

As a special case we let R be the null string, and $\tilde{\Lambda}_Q = \Lambda_Q$, and get

Corollary 3.1

$$|I| \cdot |\Lambda_Q|^{-1} e^{-\mathcal{N}|I|\frac{\Lambda_{min}}{\Lambda_{min}-1}} < |I_Q| < |I| \cdot |\Lambda_Q|^{-1} e^{\mathcal{N}|I_Q|\frac{\Lambda_{min}}{\Lambda_{min}-1}} \tag{19}$$

which we claimed before in eq. (5)

We make two immediate remarks

Remarks:

1. The bound closes up if $\mathcal{N} \to 0$; the periodic orbit representation is exact if f_q $(\forall q)$ is linear.

2. The bound collapses if $\Lambda_{min} \to 1$, the requirement that the map is expanding seems to be essential.

$\diamond$

4 The intermittent map

4.1 Specification of the map

It is time to specify the particular class of maps we will consider in detail. The map f obey assumption I (as in Intermittency) if

1. $I = [0, 1]$.

2. The map has two branches, f_0 defined on $I_0 = [0, q_0[$, and f_1 on $I_1 = [q_1, 1]$, where $q_0 \leq q_1$. Both branches are monotone (positive slope) and complete, that is $f_0(0) = 0$, $f_0(q_0) = 1$, $f_1(q_1) = 0$ and $f_1(1) = 1$.

3. The first branch can be written as $f_0(x) = x + rx^{1+s} + u(x)$ where $u(x)$ where $u(x)$ is subject to the Lipshitz-like condition $|u'(x) - u'(y)| < \mathcal{C}\,(\max(x, y))^{s+t-1}\,|x - y|$, where $t > 0$. The map has an neutral fixpoint at $x = 0$: $f_0'(0) = 1$, but $f_0'(x) > 1$ elsewhere $(x > 0)$.

4. The second branch f_1 is expanding, $|\frac{df_1(x)}{dx}| > \Lambda_{min} > 1$, and $f_1 \in C^{1+\text{Lipshitz}}$.

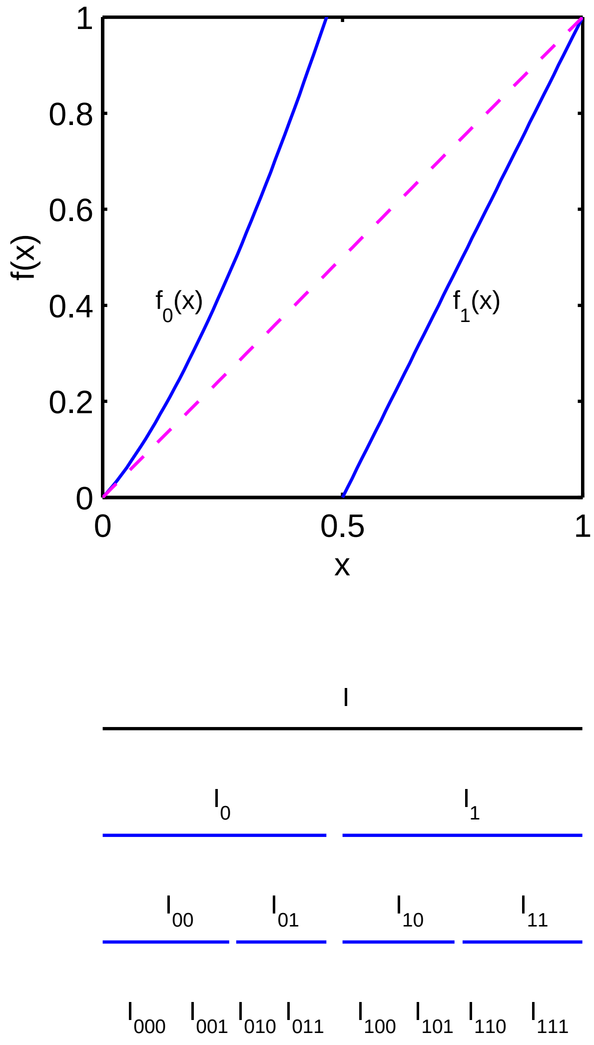

Figure 1: Example of a map abeying assumption I. Below the map is also shown the partitions $\mathcal{C}^{(1)} = \{I_0, I_1\}$, $\mathcal{C}^{(2)}$ and $\mathcal{C}^{(3)}$.

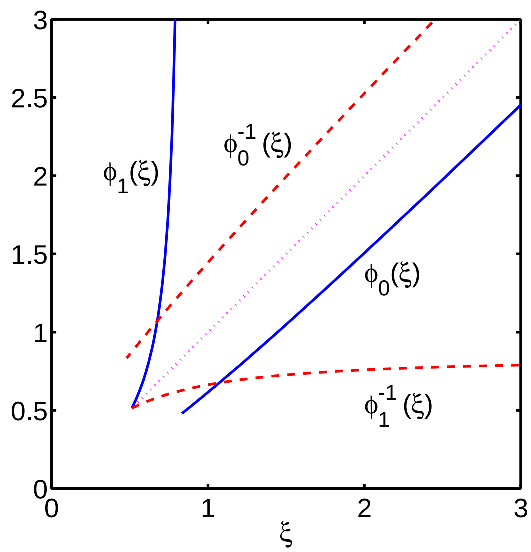

Figure 2: The conjugate of the map in fig 1

Remark:

From the above requirement we immediately deduce that $u'(x) = O(x^{s+t})$ and $u(x) = O(x^{s+t+1})$.

$\diamond$

4.2 Conjugation of the map

It will show very convenient to consider a conjugation of the map $\xi \mapsto \phi(\xi)$

$$\phi(\xi) = h^{-1} \circ f \circ h \; (\xi) \tag{20}$$

given by the change of variables.

$$x = h(\xi) = (r\xi)^{-1/s} \tag{21}$$

Mapping functions and coordinates in the conjugate space will typically be denoted by greek letters. Intervals in ξ- space will be denoted

$$i_Q = h^{-1}(I_Q) \tag{22}$$

We will need the following elementary results later

Lemma 4.2.1 *(a)* $(h(\xi) - h(\eta))/h(\xi) < \frac{1}{s}(\eta - \xi)/\xi$ *if* $\eta > \xi$ *we have.*
(b) $(h(\xi) - h(\eta))/h(\xi) = \frac{1}{s}(\eta - \xi)/\xi + O([(\eta - \xi)/\xi]^2)$

An example of a conjugated maps can be seen in fig. 2.

4.2.1 The intermittent branch

The main virtue of the conjugation is that the action of the intermittent branch ϕ_0

$$({\phi_0}^{-1})^{\circ n}(\xi) = h^{-1} \circ ({f_0}^{-1})^{\circ n} \circ h\ (\xi) \tag{23}$$

will be very simple, as demonstrated by the following

Lemma 4.2.2 *Under assumptions I on map* f*, we have the following result for the intermittent branch* f*:*
(a)

$$({\phi_0}^{-1})^{\circ n}(\xi) = \xi + sn + B_n(\xi) \tag{24}$$

where

$$B_n(\xi) = O((\xi + sn)^{1-\nu}) \quad \nu = min(1, t/s) \tag{25}$$

[1]

(b) There exist constants $\mathcal{C}_1$ $\mathcal{C}_2$*,* $-1 < \mathcal{C}_1 < 1 < \mathcal{C}_2 < \infty$ *such that*

$$\mathcal{C}_1 < \frac{d({\phi_0}^{-1})^{\circ n}(\xi)}{d\xi} = 1 + B_n'(\xi) < \mathcal{C}_2 \tag{26}$$

(c) The derivative of $({\phi_0}^{-1})^{\circ n}$ *is subject to the Lipshitz condition*

$$\mid \frac{1 + B_n'(\eta)}{1 + B_n'(\xi)} - 1 \mid < \mathcal{C}|\eta - \xi| \tag{27}$$

Remark: The bound on $B_n(\xi)$ is actually a bit pessimistic for small n, since $B_0(\xi) = 0$. However, a convenient form to express a) is

$$\mathcal{C}_1(\xi + sn) < \xi + sn + B_n(\xi) < \mathcal{C}_2(\xi + sn). \tag{28}$$

◇

[1] Don't forget $\nu = 1$ case

First we need to express the action of the map ${\phi_0}^{-1}(\xi)$ in a convenient form

$$ {\phi_0}^{-1}(\xi) = \xi + s + b(\xi) \tag{29} $$

where (i)

$$ \begin{cases} b(\xi) = O(1/\xi^\nu) \\ b(\xi) > -s \end{cases} \quad \xi \in i(I_0) \tag{30} $$

The derivative of $b(\xi)$ has a bound (ii)

$$ \begin{cases} b'(\xi) = O(1/\xi^{\nu+1}) \\ b'(\xi) > -1 \end{cases} \quad \xi \in i(I_0) \tag{31} $$

and a Lipzhitz condition (iii)

$$ \mid \frac{1+b'(\eta)}{1+b'(\xi)} - 1 \mid < \mathcal{C} \frac{1}{[\min(\xi,\eta)]^{\nu+2}} |\xi - \eta| \quad \xi, \eta \in i_0 \tag{32} $$

The proof is a straightforward calculation and is omitted.

Proof of (a): The function $B_n(\xi)$ fulfills the following recurrence relation

$$ B_{n+1}(\xi) = B_n(\xi) + b(\xi + sn + B_n(\xi)) \tag{33} $$

We now want to show that

$$ |B_n(\xi)| < D(\xi + sn)^{1-\nu} \quad \xi \in h^{-1}(I_0) \tag{34} $$

by assuming

$$ |b(\xi)| < \frac{C}{\xi^\nu} \quad \xi \in h^{-1}(I_0) \tag{35} $$

We do this by induction, via eq (33). The statement is obviously true for $n = 0$ since $B_0(\xi) = 0$. Assuming (34) is true for n, we get

$$ |B_{n+1}(\xi)| \le |B_n(\xi)| + |b(\xi + sn + B_n(\xi))| \le D(\xi+sn)^{1-\nu} + \frac{C}{[\xi + sn - D(\xi + sn)^{1-\nu}]^\nu} \tag{36} $$

So, given the constant, C, if we can find a constant D such that

$$ D(\xi + sn)^{1-\nu} + \frac{C}{[\xi + sn - D(\xi + sn)^{1-\nu}]^\nu} < D(\xi + sn + s)^{1-\nu} \tag{37} $$

holds for all $\xi \in h^{-1}(I_0)$, we are done. Refurnishing the above expression, using the substitution $z = 1/(\xi + sn)$ (z is then subject to the condition $z < r\, q_0^s$) we get

$$\frac{C}{D}\frac{z}{1 - Dz^{\nu}} < (1 + sz)^{1-\nu} - 1 \tag{38}$$

This can actually be achieved, as follows from elementary analysis.

Proof of (b): Our task is to show that $B'_n(\xi)$ is limited as $n \to \infty$: $-1 < \mathcal{C}_1 < B'_n(\xi) < \mathcal{C}_1 < \infty$.

Next we differentiate (33) to get

$$B'_{n+1}(\xi) = B'_n(\xi) + (1 + B'_n(\xi))b'(\xi + sn + B_n(\xi)) \tag{39}$$

By introducing the function $\Delta_n(\xi) \equiv \ln(1 + B'_n(\xi))$ we recast it into

$$\Delta_{n+1}(\xi) = \Delta_n(\xi) + \ln(1 + b'(\xi + sn + B_n(\xi))) \tag{40}$$

Since $B_0(\xi) = 0$, it is supplemented by the initial condition $\Delta_0(\xi) = 0$ and we get

$$\Delta_n(\xi) = \sum_{m=0}^{n-1} \ln(1 + b'(\xi + sm + B_m(\xi))) \tag{41}$$

It is straightforward to show that there is an upper and lower bound of the sum above, which proves our assertion.

Now remains (c). We begin by writing

$$\frac{1 + B'_n(\eta)}{1 + B'_n(\xi)} = e^{\Delta_n(\xi) - \Delta_n(\eta)} \tag{42}$$

where according to (41)

$$\Delta_n(\xi) - \Delta_n(\eta) = \sum_{m=0}^{n-1} \ln \frac{1 + b'(\xi + sm + B_m(\xi))}{1 + b'(\xi + sm + B_m(\xi))} \tag{43}$$

and by assuming $\eta > \xi$

$$|\Delta_n(\xi) - \Delta_n(\eta)| \leq= \sum_{m=0}^{n-1} |\ln \frac{1 + b'(\xi + sm + B_m(\xi))}{1 + b'(\eta + sm + B_m(\eta))}| \tag{44}$$

$$< \sum_{m=0}^{n-1} |\ln(1 + \mathcal{C}\frac{1}{(\xi + sm + B_m(\xi))^{\nu+2}}|\eta + B_m(\eta)) - \xi - B_m(\xi)|)|$$

$$< \mathcal{C} \sum_{m=0}^{n-1} \frac{1}{(\xi + sm + B_m(\xi))^{\nu+2}} |\eta + B_m(\eta)) - \xi - B_m(\xi)|$$

According to (b)

$$|B_m(\eta) - B_m(\xi)| \leq \int_\xi^\eta |B'_m|(\xi')|d\xi' < \mathcal{C}|\eta - \xi| \tag{45}$$

yielding

$$|\Delta_n(\xi) - \Delta_n(\eta)| < \mathcal{C}|\eta - \xi| \sum_{m=0}^{n-1} \frac{1}{[\xi + sm + B_m(\xi)]^{\nu+2}} \tag{46}$$

$$< \mathcal{C}|\eta - \xi|$$

since the sum converges. The last equation, together with (b) (saying that $1 + B'_n(\xi)$ is bounded) implies the announced result (c).

4.2.2 The expanding branch

We will need one asymptotic result, concerning the conjugation of expanding branch

$$\phi_1^{-1}(\xi) = h^{-1} \circ f_1^{-1} \circ h\ (\xi) \tag{47}$$

This function maps the interval $i = [\frac{1}{r}, \infty[$ to $i_1 = [\frac{1}{r}, \frac{1}{rq_0^s}] \equiv [\frac{1}{r}, \xi_0]$.

Lemma 4.2.3 *There are constants $\mathcal{C}_1$, $\mathcal{C}_2$ ($0 < \mathcal{C}_1 < \mathcal{C}_2 < 1$) such that*

$$\mathcal{C}_1 \left(\frac{\xi_1}{\xi}\right)^{1/s} < \xi_1 - \phi_1^{-1}(\xi) < \mathcal{C}_2 \left(\frac{\xi_1}{\xi}\right)^{1/s} \tag{48}$$

where $\xi_1 = h^{-1}(q_1)$

[2]

[2]rewrite as O(...)

4.3 The induced map

The induced version $F : I_1 \mapsto I_1$ of the map $f : I \mapsto I$ is a restriction of the map, to the interval I_1, achieved in the following way

$$f(x) = F_n(x) \equiv (f_0)^{\circ n} \circ f_1\ (x) \quad x \in I_{10^n 1} \tag{49}$$

for $n \geq 0$.

Next we want to show that assumptions E are fulfilled by the map

$$\Phi(\xi) = h^{-1} \circ F \circ h\ (\xi) \tag{50}$$

an in particular the inverse of branch n

$$\Phi_n^{-1}(\xi) = h^{-1} \circ F_n^{-1} \circ h(\xi) = \phi_1^{-1} \circ (\phi_0^{-1})^{\circ n} \tag{51}$$

Lemma 4.3.1 *(a) There are constants $\mathcal{C}_1$, $\mathcal{C}_2$ $(0 < \mathcal{C}_1 < \mathcal{C}_2 < \infty)$ such that*

$$\mathcal{C}_1 \frac{1}{(1+rsn)^{1+1/s}} < \frac{d\Phi_n^{-1}(\xi)}{d\xi} < \mathcal{C}_2 \frac{1}{(1+rsn)^{1+1/s}} \tag{52}$$

(b) There is a number $\mathcal{N}$ (independent of n) such that

$$|\frac{d\Phi_n^{-1}(\eta)}{d\eta} / \frac{d\Phi_n^{-1}(\xi)}{d\xi} - 1| < \mathcal{N}|\eta - \xi| \tag{53}$$

This means that assumptions E apply on the induced map ϕ and the results of sec 3

Proof:

Schematically

$$\begin{array}{ccccc} & (\phi_0^{-1})^{\circ n} & & \phi_1^{-1} & \\ \xi & \longmapsto & \xi' & \longmapsto & \xi'' \\ \in i_1 & & \in i_{0^n 1} & & \in i_{10^n 1} \end{array} \tag{54}$$

$$\frac{d\Phi_n^{-1}(\xi)}{d\xi} = (1 + B_n'(\xi)) \frac{1}{f_1'(h(\xi''))} \left(\frac{\xi''}{\xi'}\right)^{1+1/s} \tag{55}$$

$$= (1 + B_n'(\xi)) \frac{1}{f_1'(h(\xi''))} \left(\frac{\xi''}{\xi + sn + B_n(\xi)}\right)^{1+1/s}$$

Statement (a) now follows easily, from lemma **??**b, properties of the expanding branch f_1, and the fact the ξ and ξ'' are bounded.

To show (b) we consider a neighboring orbit $\eta \mapsto \eta' \mapsto \eta''$

$$\frac{d\Phi_n^{-1}(\eta)}{d\eta} \Big/ \frac{d\Phi_n^{-1}(\xi)}{d\xi} = \frac{1+B_n'(\eta)}{1+B_n'(\xi)} \cdot \left(\frac{\xi + sn + B_n(\xi)}{\eta + sn + B_n(\eta)} \right)^{1+1/s} \cdot \left(\frac{\eta''}{\xi''} \right)^{1+1/s} \cdot \frac{f_1'(h(\eta''))}{f_1'(h(\xi''))} \tag{56}$$

We consider each of the four factors separately.

Lemma 4.2.2c applies directly to the first factor:

$$\left| \frac{1+B_n'(\eta)}{1+B_n'(\xi)} - 1 \right| < \mathcal{C} |\eta - \xi| \tag{57}$$

To treat the second factor we apply lemma 4.2.2a and get

$$\left| \left(\frac{\xi + sn + B_n(\xi)}{\eta + sn + B_n(\eta)} \right)^{1+1/s} - 1 \right| < O\left(\frac{1}{1+rsn} \right) |\eta - \xi| \tag{58}$$

To obtain a bound on the third factor we observe

$$|\eta'' - \xi''| < \sup_{\xi \in i_1} \left| \frac{d\Phi_n^{-1}(\xi)}{d\xi} \right| \cdot |\eta - \xi| \tag{59}$$

$$= O\left(\frac{1}{(1+rsn)^{1+1/s}} \right) \cdot |\eta - \xi|$$

according to (a). From the fact that ξ'' is limited follows

$$\left| \left(\frac{\eta''}{\xi''} \right)^{1+1/s} - 1 \right| < O\left(\frac{1}{(1+rsn)^{1+1/s}} \right) \cdot |\eta - \xi| \tag{60}$$

Finally for the fourth factor

$$\left| \frac{f_1'(h(\eta''))}{f_1'(h(\xi''))} - 1 \right| < |h(\eta'') - h(\xi'')| < |\eta'' - \xi''| < O\left(\frac{1}{(1+rsn)^{1+1/s}} \right) \cdot |\eta - \xi| \tag{61}$$

where we have used assumption I4, lemma 4.2.3, eq. (59) and the fact that ξ'' and η'' are bounded. Statement (b) can now be verified without difficulties.

5 The main result

All periodic orbits (except $\overline{0}$) can be written on the form $\overline{10^{m_N}\dots 10^{m_2}10^{m_1}}$. Will be interested in few intervals I_{Q_m} for a few cyclic permutations $Q_m = 0^{n_1-m}R10^m$, where $R = 10^{m_N}\dots 10^{m_2}$.

We will approach the result in several steps:

$$i \overset{(\phi_0^{-1})^{\circ m}}{\longmapsto} i_{R^m} \overset{\phi_1^{-1}}{\longmapsto} i_{10^m} \overset{\Phi_{n_N}^{-1}\circ\dots\circ\Phi_{n_2}^{-1}}{\longmapsto} i_{R10^m} \tag{62}$$

$$\overset{(\phi_0^{-1})^{\circ(n_1-m)}}{\longmapsto} i_{0^{n_1-m}R10^m} \overset{h}{\longmapsto} I_{0^{n_1-m}R10^m}$$

We begin with.

$$i = [\frac{1}{r}, \infty[\tag{63}$$

Step 1: $(\phi_0^{-1})^{\circ m}$

After m applications of ϕ_0^{-1} we get

$$i_{0^m} = [\frac{1}{r} + sm + B_m(\frac{1}{r}), \infty] \tag{64}$$

Step 2: ϕ_1^{-1}

By applying ϕ_1^{-1} we are thrown back to i_1

$$i_{10^m} = [\xi_1 - \Delta, \xi_1] \tag{65}$$

where

$$\xi_1 = h^{-1}(q_1) \tag{66}$$

and

$$\mathcal{C}_1 \frac{1}{(1+srm)^{1/s}} < \Delta < \mathcal{C}_2 \frac{1}{(1+srm)^{1/s}} \tag{67}$$

Step 3: $\Phi_{n_N}^{-1} \circ \ldots \circ \Phi_{n_2}^{-1}$

Being back to i_1 we can apply the induced map

$$i_{R10^m} = [\xi_R - \Delta_R, \xi_R] \tag{68}$$

$$\xi_R \leq \xi_1 \tag{69}$$

(equality is R is the null string). Since assumption E is fulfilled we will be able to use Lemma 3.1 in a slightly weakened version.

$$|i_{10^m}| \cdot |\tilde{\Lambda}_R^{-1}| e^{-\mathcal{N}|i_1| \frac{\Lambda_{min}}{\Lambda_{min}-1}} < |i_{R10^m}| < |i_{10^m}| \cdot |\tilde{\Lambda}_R^{-1}| e^{\mathcal{N}|i_1| \frac{\Lambda_{min}}{\Lambda_{min}-1}} \tag{70}$$

(weakened because in principle we could have had $|i_{10^m}|$ instead of $|i_1|$ in the exponent), or equivalently

$$\mathcal{C}_1 \, \Delta \frac{1}{|\tilde{\Lambda_R}|} < \Delta_R < \mathcal{C}_2 \, \Delta \frac{1}{|\tilde{\Lambda_R}|} \tag{71}$$

Combining the result so far we get

$$\mathcal{C}_1 \, \frac{1}{|\tilde{\Lambda_R}|} \frac{1}{(1+srm)^{1/s}} < \Delta_R < \mathcal{C}_2 \, \frac{1}{|\tilde{\Lambda_R}|} \frac{1}{(1+srm)^{1/s}} \tag{72}$$

Step 4; $(\phi_0^{-1})^{\circ(n_1-m)}$

It only remains to apply ϕ_0^{-1} $n_1 - m$ times

$$\begin{aligned} & i_{0^{n_1-m}R10^m} = i_{Q_m} \\ & = [\xi_R - \Delta_R + s(n_1 - m) + B_{n_1-m}(\xi_R - \Delta_R), \xi_R + s(n_1 - m) + B_{n_1-m}(\xi_R)] \\ & \equiv [\hat{\eta}, \hat{\xi}] \end{aligned} \tag{73}$$

yielding

$$\mathcal{C}_1 \frac{1}{|\tilde{\Lambda_R}|} \frac{1}{(1+srm)^{1/s}} < \mathcal{C}_2 \Delta_R < |i_{Q_m}| = \hat{\xi} - \hat{\eta} < \mathcal{C}_3 \Delta_R < \mathcal{C}_4 \frac{1}{|\tilde{\Lambda_R}|} \frac{1}{(1+srm)^{1/s}} \tag{74}$$

Step 5: Finally back to x-space

$$I_{Q_m} = [\hat{x}, \hat{y}] = [h(\hat{\xi}), h(\hat{\eta})] \tag{75}$$

According to Lemma 4.2.1b $|I_{Q_m}| = \hat{x}[(\hat{\xi} - \hat{\eta})/\hat{\xi} + O([(\hat{\xi} - \hat{\eta})/\hat{\xi}]^2)]$ and since $(\hat{\xi} - \hat{\eta})/\hat{\xi}$ is limited we get

$$\mathcal{C}_1 \frac{\hat{x}}{\hat{\xi}}(\hat{\xi} - \hat{\eta}) < |I_{Q_m}| < \mathcal{C}_2 \frac{\hat{x}}{\hat{\xi}}(\hat{\xi} - \hat{\eta}) \tag{76}$$

and finally

$$\mathcal{C}_1 \frac{1}{|\tilde{\Lambda_R}|} \frac{1}{(1+srm)^{1/s}} \frac{1}{(1+sr(n_1-m))^{1+1/s}} < |I_{Q_m}| < \tag{77}$$

$$\mathcal{C}_2 \frac{1}{|\tilde{\Lambda_R}|} \frac{1}{(1+srm)^{1/s}} \frac{1}{(1+sr(n_1-m))^{1+1/s}}$$

$$\frac{1}{|\Lambda_Q|} = \frac{1}{|\tilde{\Lambda_R}|} \frac{d\Phi_{n_1}^{-1}(\xi)}{d\xi} \tag{78}$$

for some $\xi \in i_1$, and due to Lemma 4.3.1a

$$\mathcal{C}_1 \frac{1}{|\Lambda_Q|} \frac{(1+srn_1)^{1+1/s}}{(1+srm)^{1/s}\ (1+sr(n_1-m))^{1+1/s}} < |I_{Q_m}| < \tag{79}$$

$$\mathcal{C}_2 \frac{1}{|\Lambda_Q|} \frac{(1+srn_1)^{1+1/s}}{(1+srm)^{1/s}\ (1+sr(n_1-m))^{1+1/s}}$$

The key idea is to average along a fraction of the periodic orbit

$$\frac{1}{n_1+1} \sum_{m=0}^{n_1} |I_{Q_m}| \tag{80}$$

The relevant sum is

$$\sum_{m=0}^{n_1} \frac{1}{(1+srm)^{1/s}\ (1+sr(n_1-m))^{1+1/s}} \tag{81}$$

$$= \begin{cases} \frac{1}{(1+srn_1)^{1/s}}\mathcal{C} + O(\frac{1}{(1+srn_1)^{1/s+1}}) & s<1 \\ \frac{1}{(1+srn_1)^{1/s}}\mathcal{C} + O(\frac{1}{(1+srn_1)^{2/s}}) & s>1 \end{cases}$$

and is computed in the appendix. The result enables us to write

$$\mathcal{C}_1 \frac{1}{|\Lambda_Q|} < \frac{1}{n_1+1} \sum_{m=0}^{n_1} |I_{Q_m}| < \mathcal{C}_2 \frac{1}{|\Lambda_Q|} \tag{82}$$

which imply (8) and can state our main result

Theorem 5.1 *For a map obeying assumption I, the partition sum*

$$\Gamma_n = \sum_{\substack{|Q| = n \\ Q \neq 0^n}} \frac{|I_Q|}{|I|} \quad . \tag{83}$$

and the periodic orbit sum

$$Z_n = \sum_{\substack{|Q| = n \\ Q \neq 0^n}} \frac{1}{|\Lambda_Q|} \quad , \tag{84}$$

obey the following bound

$$\mathcal{C}_1 Z_n < \Gamma_n < \mathcal{C}_2 Z_n \tag{85}$$

for some constants $0 < \mathcal{C}_1 < \mathcal{C}_2 < \infty$*.*

6 Discussion

6.1 Escape probabilities

We have chosen to focus on the survival probability, and its representation in terms of periodic orbits, from which the escape probability can be deduced.

$$\Psi_n = \Gamma_n - \Gamma_{n+1} \tag{86}$$

An (more dangerous) alternative is to consider a periodic orbit representation of the escape probability directly. One would hen start from the complement of the generating partition $J = [q_1, q_2]$ and write

$$\Psi_n = \sum_{|Q|=n} \frac{|J_Q|}{I} \tag{87}$$

and the iterative rule

$$J_{qR} = f_q^{-1}(J_R) \tag{88}$$

In terms of periodic orbits one would get

$$\Psi_n \approx \frac{|J|}{|I|} Z_n \tag{89}$$

This formula should make us very suspicious, because if Γ_n, and thus Z_n asymptotically follows a power law $\Gamma_n \sim Z_n \sim n^\alpha$, then we know from eq. (86) that $\Psi \sim n^{\alpha-1}$, whereas eq. (89) would predict $\Psi \sim n^\alpha$. How can this paradox be resolved.

An analysis similar to the one in section 5 givew

$$\mathcal{C}_1 \frac{1}{|\tilde{\Lambda_R}|} \frac{1}{(1+sr(n_1-m))^{1+1/s}} < |J_{Q_m}| < \mathcal{C}_2 \frac{1}{|\tilde{\Lambda_R}|} \frac{1}{(1+sr(n_1-m))^{1+1/s}} \tag{90}$$

or

$$\mathcal{C}_1 \frac{1}{|\Lambda_Q|} \frac{(1+srn_1))^{1+1/s}}{(1+sr(n_1-m))^{1+1/s}} < |J_{Q_m}| < \mathcal{C}_2 \frac{1}{|\Lambda_Q|} \frac{(1+srn_1))^{1+1/s}}{(1+sr(n_1-m))^{1+1/s}} \tag{91}$$

and there is no way of rescuing bounds by any averaging procedure.

Non uniform initial distribution

The contibutions to the sum (81) are very unevenly distributed. Therefore, by starting from a nonuniform distribution, the bounds easily detoriate, and may collaps alltogether. En example of such a collaps is the following. Instead of starting with a uniform distribution over the entire I, we start with a uniform disytribution only over I_1. Call the corresponding survival probability Γ'_n. If, for simplicity, the expanding branch f_1 is strictly linear, then $\Gamma'_{n+1} = \Gamma_n$. But, it is not possible to bind the corresponding periodic orbit sum

$$Z'_n = \sum_{Q:|Q|=n-1} \frac{1}{\Lambda_{1Q}} \tag{92}$$

close to Γ'_n.

The moral is that periodic orbit expression for escape from nonuniformly hyperbolic systems is extremely fragile.

Appendix

In this appendix we evaluate the sum

$$S = \sum_{n=0}^{N} \frac{1}{(1+srn)^{1/s}} \frac{1}{(1+sr(N-n))^{1/s+1}} \equiv S_1 + S_2 \tag{93}$$

asymptotically for large values of N. If N is odd we can write (the generalization to even N is trivial and omitted)

$$S_1 = \sum_{n=0}^{(N-1)/2} \frac{1}{(1+srn)^{1/s}} \frac{1}{(1+sr(N-n))^{1/s+1}} \tag{94}$$

and

$$S_2 = \sum_{n=0}^{(N-1)/2} \frac{1}{(1+srn)^{1/s+1}} \frac{1}{(1+sr(N-n))^{1/s}} \tag{95}$$

To treat S_1 we use

$$\tfrac{1}{(1+sr(N-n))^{1/s+1}} = O(\tfrac{1}{(1+srN))^{1/s+1}}), \quad N \geq 0, \quad 0 \leq n \leq (N-1)/2 \tag{96}$$

and get

$$S_1 = \begin{cases} O(\frac{1}{(1+srN)^{1/s+1}}) & s < 1 \\ O(\frac{1}{(1+srN)^{2/s}}) & s > 1 \end{cases} \tag{97}$$

To treat S_2 we use

$$\tfrac{1}{(1+sr(N-n))^{1/s}} = \tfrac{1}{(1+srN))^{1/s}} + O(\tfrac{1+srn}{(1+srN))^{1/s+1}}), \quad N \geq 0, \quad 0 \leq n \leq (N-1)/2 \tag{98}$$

and get

$$S_2 = \begin{cases} \frac{1}{(1+srN)^{1/s}} \sum_{n=0}^{\infty} \frac{1}{(1+srn)^{1/s+1}} + O(\frac{1}{(1+srN)^{1/s+1}}) & s < 1 \\ \frac{1}{(1+srN)^{1/s}} \sum_{n=0}^{\infty} \frac{1}{(1+srn)^{1/s+1}} + O(\frac{1}{(1+srN)^{2/s}}) & s > 1 \end{cases} \tag{99}$$

Summing S_1 and S_2 we arrive at

$$S = \begin{cases} \frac{1}{(1+srN)^{1/s}} \sum_{n=0}^{\infty} \frac{1}{(1+srn)^{1/s+1}} + O(\frac{1}{(1+srN)^{1/s+1}}) & s < 1 \\ \frac{1}{(1+srN)^{1/s}} \sum_{n=0}^{\infty} \frac{1}{(1+srn)^{1/s+1}} + O(\frac{1}{(1+srN)^{2/s}}) & s > 1 \end{cases} \tag{100}$$