mathematical methods - week 8

Discrete Fourier representation

Georgia Tech PHYS-6124

Homework HW #8 due Thursday, October 15, 2020

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 8.1 *Laplacian is a non-local operator*	4 points
Exercise 8.2 *Lattice Laplacian diagonalized*	8 points

Total of 12 points = 100 % score.

edited October 13, 2020

Week 8 syllabus

Tuesday, October 5, 2020

Discretization of continuum, lattices, discrete derivatives, discrete Fourier representations.

- *Symmetry is your friend - overview. The power of thinking.*
- Applied math version: how to discretize derivatives:
  ChaosBook Appendix A24 Deterministic diffusion
  Sects. A24.1 to A24.1.1 *Lattice Laplacian*.

The fastest way to watch this week's lecture videos is by letting YouTube run the course playlist (click here):

- *Lattice discretization, lattice state*
- *Lattice derivative*
- *Shift operator: the generator of discrete translations*
  - *Discussion: Shift matrix must have the periodic b.c.; Derivative being non-local is easiest to grasp on discrete lattice. It's so easy to make errors in the continuum formulation.*
- *Derivative is a linear operator*
- *Lattice Laplacian*
- *Derivative is a non-local operator*
  - *Discussion: Lattice discretization; What if geometry is not flat in all directions, but spherical? What about General Relativity? Life's persistent questions, skated around.*
  - *Discussion: What is a derivative? Hypercubic lattice is a graph, with nodes connected by links. Every graph has a notion of derivative associated with it; in particular a Laplacian. I was not allowed to say "Laplacian" here, as I have not gotten to defining it in my lecture at that point...*
- A periodic lattice as the simplest example of the theory of finite groups:
  ChaosBook Sects. A24.1.2 to A24.3.1.
  ChaosBook Example A24.2 *Projection operators for discrete Fourier representation.*
  ChaosBook Example A24.3 *'Configuration-momentum' Fourier space duality.*
- *Have symmetry? Use it!*
  - *Rant: Symmetrize you must. Karl Schwarzschild found his exact solution in 1915, a month after the publication of Einstein's theory of general relativity, while serving on a World War I front.*

Have symmetry? Go to "eigen"subspace! Fourier decomposition of a 2-sites periodic lattice.

Periodic lattices

Fourier eigenvalues

Discrete Fourier representation

Laplacian in Fourier representation

Propagator in Fourier representation

A meta truth; We live in The Matrix; Fourier transformation is just a matrix

Optional reading

A theoretical physicist's version of the above notes: *Quantum Field Theory - a cyclist tour*, Chapter 1 *Lattice field theory* motivates discrete Fourier representations by computing a free propagator on a lattice.

Quantum Mechanics in a box: Sometimes it is simplest to impose the periodic b.c. on a localized solution, than relax it towards the correct (infinite extent) continuum solution.

Group theory voodoo

Rocket science needs complex numbers; Why Fourier? Digital image processing!

Roger Penrose gets Nobel Prize. How David Ritz Finkelstein and Roger Penrose met, and exchanged their lives' paths.

Exercises

8.1. **Laplacian is a non-local operator.**

While the Laplacian is a simple tri-diagonal difference operator, its inverse (the "free" propagator of statistical mechanics and quantum field theory) is a messier object. A way to compute is to start expanding propagator as a power series in the Laplacian

$$\frac{1}{m^2\mathbf{1} - \Delta} = \frac{1}{m^2}\sum_{n=0}^{\infty}\frac{1}{m^{2n}}\Delta^n\,. \tag{8.1}$$

As Δ is a finite matrix, the expansion is convergent for sufficiently large m^2. To get a feeling for what is involved in evaluating such series, show that Δ^2 is:

$$\Delta^2 = \frac{1}{a^4}\begin{bmatrix} 6 & -4 & 1 & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & 1 & -4 & \ddots & & & \\ & & & & & 6 & -4 \\ -4 & 1 & & & 1 & -4 & 6 \end{bmatrix} . \tag{8.2}$$

What Δ^3, Δ^4, $\cdots$ contributions look like is now clear; as we include higher and higher powers of the Laplacian, the propagator matrix fills up; while the *inverse* propagator is differential operator connecting only the nearest neighbors, the propagator is integral operator, connecting every lattice site to any other lattice site.

This matrix can be evaluated as is, on the lattice, and sometime it is evaluated this way, but in case at hand a wonderful simplification follows from the observation that the lattice action is translationally invariant, exercise 8.2.

8.2. **Lattice Laplacian diagonalized.** Insert the identity $\sum P^{(k)} = \mathbf{1}$ wherever you profitably can, and use the shift matrix eigenvalue equation to convert shift σ matrices into scalars. If $\mathbf{M}$ *commutes* with σ, then $(\varphi_k^\dagger \cdot \mathbf{M} \cdot \varphi_{k'}) = \tilde{M}^{(k)}\delta_{kk'}$, and the matrix $\mathbf{M}$ acts as a multiplication by the scalar $\tilde{M}^{(k)}$ on the kth subspace. Show that for the 1-dimensional lattice, the projection on the kth subspace is

$$(\varphi_k^\dagger \cdot \Delta \cdot \varphi_{k'}) = \frac{2}{a^2}\left(\cos\left(\frac{2\pi}{N}k\right) - 1\right)\delta_{kk'} \, . \tag{8.3}$$

In the kth subspace the propagator is simply a number, and, in contrast to the mess generated by (8.1), there is nothing to evaluating it:

$$\varphi_k^\dagger \cdot \frac{1}{m^2\mathbf{1} - \Delta} \cdot \varphi_{k'} = \frac{\delta_{kk'}}{m^2 - \frac{2}{(ma)^2}\left(\cos 2\pi k/N - 1\right)} \, , \tag{8.4}$$

where k is a site in the N-dimensional dual lattice, and $a = L/N$ is the lattice spacing.