mathematical methods - week 11

Continuous symmetries

Georgia Tech PHYS-6124

Homework HW #11 due Thursday, November 5, 2020

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 11.1 *Decompose a representation of* S_3
(a) 2; (b) 2; (c) 3; and (d) 3 points
(e) 2 and (f) 3 points bonus points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

edited October 30, 2020

Week 11 syllabus

Tuesday, October 27, 2020

I have given up Twitter in exchange for Tacitus & Thucydides, for Newton & Euclid; & I find myself much the happier.
— Thomas Jefferson to John Adams, 21 January 1812

Clip 1 - They still do not get it!

This week's lectures are related to AWH Chapter 17 *Group Theory, Sect. 17.7 Continuous groups* (click here). The fastest way to watch any week's lecture videos is by letting YouTube run the course playlist (click here).

- Lie groups, sect. 11.2
  - Definition of a Lie group
  - Cyclic group $C_N \to$ continuous $SO(2)$ plane rotations
  - Infinitesimal transformations, $SO(2)$ generator of rotations
  - $SO(2)$ (group element) = $\exp$(generator)

Clip 2 - What is a symmetry?

Clip 3 - Group element; transformation generator

Clip 4 - What is a symmetry group?

Clip 5 - What is a group orbit?

Clip 6 - What is dynamics?

Clip 7 - Group SO(2)

- The $N \to \infty$ limit of C_N gets you to the continuous Fourier transform as a representation of $SO(2)$, but from then on this way of thinking about continuous symmetries gets to be increasingly awkward. A fresh restart is afforded by matrix groups, and in particular the mother unitary group $U(n) = U(1) \otimes SU(n)$, which contains all other compact groups, finite or continuous, as subgroups.

Clip 10 - Unitary groups are mothers of all finite / compact symmetries. (1 h 4 min)

Discussion 1 - How did we get the Lie algebra? Why is (almost) every symmetry we care about a subgroup of an unitary group? (9 min)

Discussion 2 - How did we get the SO(2) generator? (2 min)

Optional viewing and reading

Clip 8 - Infinitesimal symmetries: Lie derivative

Clip 9 - Symmetries of solutions. (18 min)

Clip 11 Special orthogonal group SO(n). (9 min)

Clip 12 Symplectic group Sp(n). (9 min)

Discussion 3 - Orthogonal and unitary transformations

Rant 1 - Is beauty symmetry? The first piece of art found in China is a perfect disk carved out of jade. All of Bach is symmetries. (9 min)

Rant 2 - students find letter A beautifully symmetric, but Predrag finds zero 'O' the most beautiful grade. (1 min)

Rant 3 - $SO(3)$ $SU(2)$ preview and a long rant - listen to it at your own risk. Roger Penrose thoughts on quantum spacetime and quantum brain. Are laws of physics time invariant? Waiting for dark energy to go away. Arrow of time. (17 min)

Rant 4 - $SO(3)$ $SU(2)$ preview and a long rant - listen to it at your own risk. Get this: math uses 2d complex vectors (spinors) to build our real 3d space. And all we see - starlight, graphene, greenhouse effect, helioseismography, gravitational wave detectors - it is all irreps! (12 min)

Rant 5 - Help me, I'm bullied by a mathematician. (3 min)

Rant 6 - you can always count on Prof. Z. (1/2 min)

Week 10 stragglers

Discussion 1 - There might be many examples of it, but a 'group' itself is an abstract notion. (3 min)

Discussion 2 - Fourier modes are so simple, that no one calls them irreps. But add more symmetries, and there have to be fewer irreps. (11 min)

Discussion 3 - what are these "characters"? And why is there a Journal of Linear Algebra*, today? Inconclusive blah blah. (12 min)*

Discussion 4 - Homework. (3 min)

Question 11.1. Henriette Roux, pondering exercise 11.1, writes

Q I want to make sure I understand the concept of irreducible representations. In the last homework, we saw that

1. if a representation (which can be thought of as a sort of basis) is reducible, all group element matrices can be simultaneously diagonalized. I want to be able to see how this definition of reducibility matches with the notion of block diagonalizability of an overall representation D(g).
2. AWH p. 822-823 has a discussion of this, but I'm wondering if there's an intuitive way to connect these two definitions or if it's just linear algebra.

3. We familiarized ourselves with the concept of (conjugacy) classes in the last homework. Here, we now add in the concept of character, which, according to AWH, is just the trace of any matrix in a given class (and every matrix of the same class will have the same trace b/c of the properties of classes/traces).
4. So to find the characters for a given representation (part c), we just need to find the classes and then take the trace of a matrix representation in each class?
5. My next and related question then concerns what character means conceptually. Does it relate classes to other classes within a given representation, or different representations (whether reducible or not), or both? AWH says that "the set of characters for all elements and irreducible representations of a finite group defines an orthogonal finite-dimensional vector space."
6. How does a vector space come about from a set of traces, each of which I normally think of as just a number, like the determinant? And finally,
7. how can we use our knowledge of classes/character to find irreducible representations, since that seems to be an important goal in examining a group.
8. exercise 11.1 (c) says to find the characters for this representation, which seems to imply that character depends on representation. But I would've thought that character, which is a trace of a matrix, is invariant under any similarity transform, which is how you get from a reducible representation to an irreducible representation.
9. Also, this is more of a guess than anything, but do the multiplicities of irreducible representations correspond to the multiplicity of characters (i.e. the number of elements in each class)? If so, why? (Or if not, why not?)
10. Same thing for classes, correct?. Classes shouldn't depend on representation b/c they can be thought of as corresponding to a physical operation (e.g. transposition or cyclic permutation), something which is independent of basis.

.

A Great framing for a discussion, thanks! I'll probably reedit this post several times, everybody's input is very welcome. Items numbered as in above:

(2) My favorite step-by-step, pedagogical exposition are the chapters 2 *Representation Theory and Basic Theorems* and 3 *Character of a Representation* of Dresselhaus *et al.* [2]. There is too much material for our course, but if you want to understand it once for all times, it's worth your time.

(3) Correct.

(4) Correct. Note, however, that while every matrix representation has a trace, and thus a character, you want to decompose this character into the sum of irrep characters, as it is obvious after the block diagonalization has been attained.

(5) The unitary diagonalization matrix, whose entries are characters, takes character-weighted sums of classes in order to project them onto irreps, just like what the Fourier representation does. The result, as we know from projection operators of weeks 1 & 2, are mutually orthogonal sub-spaces.

(6) Whenever you do not understand something about finite groups, ask yourself - how does it work for finite lattice Fourier representation?

There the vector space comes via a unitary transformation from the configuration coordinates (where each group element is represented by a full matrix) to the diagonalized, irreducible subspaces coordinates (Fourier modes).

The unitary $\mathcal{F}$ matrix is full of ω^{ij}, ie, characters of the cyclic group C_n. That's where the characters come from.

Now mess up C_3 by adding a reflection. Dihedral group D_3, the group of rotations and reflections, has more symmetry constrains, it cannot have 6 irreps, as reflection invariance mixes together the two senses of rotation. Now there are 3 classes, ie, kinds of things the group does: nothing, flip, rotate. The unitary transformation that diagonalizes group element matrices is now morally a smaller unitary $[3 \times 3]$ matrix from 'classes' in configuration space to 'irreps' in the diagonalized representation, where some sub-spaces must have dimension higher than one.

The surprise, for me, is that the entries in the unitary diagonalization matrix can still be written as traces of irreps, ie, characters. For me it is a calculation, a beautiful example of mathematics leading us somewhere where our intuition falls short. If you find a good intuitive explanation somewhere, please let us all know.

(7) That's automatic, now. Each irrep has a projection operator associated with it. In weeks 1 & 2 we constructed it as a sub-product of factors in Hamilton-Cayley formula. Now we know we can write it -just as we did with the Fourier representation- as sum over all class group actions, each weighted by a the irrep's character.

(8) Characters are elements of the unitary matrix with one index running over classes, the other over irreps. So you expect character to differ from representation to representation; very clear from D_3 character table. As always, you already know that from the Fourier representation example.

(9) Good question. The do not. Dresselhaus *et al.* [2] has the answer - enter it here once you understand it.

(10) Correct.

11.1 Lie groups

In week 1 we introduced projection operators (1.20). How are they related to the character projection operators constructed in the group theory lectures? While the character orthogonality might be wonderful, it is not very intuitive - it's a set of solutions to a set of symmetry-consistent orthogonality relations. You can learn a set of rules that enables you to construct a character table, but it does not tell you what it means. Similar thing will happen again when we turn to the study of continuous groups: all semisimple Lie groups will be classified by Killing and Cartan by a more complex set of orthogonality and integer-dimensionality (Diophantine) constraints. You obtain all possible Lie algebras, but have no idea what their geometrical significance is.

In my own Group Theory book [1] I (almost) get all simple Lie algebras using projection operators constructed from invariant tensors. What that means is easier to understand for finite groups, and here I like the Harter's exposition [4] best. Harter constructs 'class operators', shows that they form a basis for the algebra of 'central' or 'all-commuting' operators, and uses their characteristic equations to construct the projection operators (1.21) from the 'structure constants' of the finite group, i.e., its class multiplication tables. Expanded, these projection operators are indeed the same as the ones obtained from character orthogonality.

11.2 Continuous symmetries : unitary and orthogonal

This week's lectures are not taken from any particular book, they are about basic ideas of how one goes from finite groups to the continuous ones that any physicist should know. We have worked one example out earlier, in week 9 and ChaosBook Sect. A24.4. It gets you to the continuous Fourier transform as a representation of $\mathrm{U}(1) \simeq \mathrm{SO}(2)$, but from then on this way of thinking about continuous symmetries gets to be increasingly awkward. So we need a fresh restart; that is afforded by matrix groups, and in particular the unitary group $\mathrm{U}(n) = \mathrm{U}(1) \otimes \mathrm{SU}(n)$, which contains all other compact groups, finite or continuous, as subgroups.

The main idea in a way comes from discrete groups: the cyclic group C_N is generated by the powers of the smallest rotation by $\Delta\theta = 2\pi/N$, and in the $N \to \infty$ limit one only needs to understand the commutation relations among T_ℓ, generators of infinitesimal transformations,

$$D(\Delta\theta) = 1 + i \sum_{\ell} \Delta\theta_\ell T_\ell + O(\Delta\theta^2) \,. \tag{11.1}$$

These thoughts are spread over chapters of my book *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [1] that you can steal from my website, but the book itself is too sophisticated for this course. If you ever want to learn some group theory in depth, you'll have to petition the School to offer it.

11.2.1 Lie groups for pedestrians

> [...] which is an expression of consecration of angular momentum.
>
> — Mason A. Porter's student

Definition: A Lie group is a topological group G such that (i) G has the structure of a smooth differential manifold, and (ii) the composition map $G \times G \to G : (g, h) \to gh^{-1}$ is smooth, i.e., $\mathbb{C}^\infty$ differentiable.

Do not be mystified by this definition. Mathematicians also have to make a living. The compact Lie groups that we will deploy here are a generalization of the theory of $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ rotations, i.e., Fourier analysis. By a 'smooth differential manifold' one means objects like the circle of angles that parameterize continuous rotations in a plane, figure 11.1, or the manifold swept by the three Euler angles that parameterize $\mathrm{SO}(3)$ rotations.

By 'compact' one means that these parameters run over finite ranges, as opposed to parameters in hyperbolic geometries, such as Minkowsky $\mathrm{SO}(3,1)$. The groups we focus on here are compact by default, as their representations are linear, finite-dimensional matrix subgroups of the unitary matrix group $\mathrm{U}(d)$.

Example 1. Circle group. A circle with a direction, figure 11.1, is invariant under rotation by any angle $\theta \in [0, 2\pi)$, and the group multiplication corresponds to composition of two rotations $\theta_1 + \theta_2 \mod 2\pi$. The natural representation of the group action

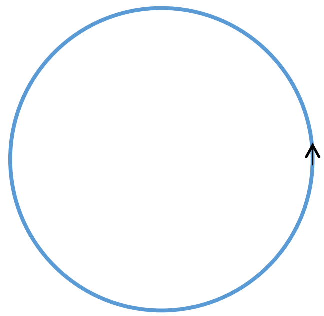

Figure 11.1: Circle group $S^1 = \text{SO}(2)$, the symmetry group of a circle with directed rotations, is a compact group, as its natural parametrization is either the angle $\phi \in [0, 2\pi)$, or the perimeter $x \in [0, L)$.

is by a complex numbers of absolute value 1, i.e., the exponential $e^{i\theta}$. The composition rule is then the complex multiplication $e^{i\theta_2}e^{i\theta_1} = e^{i(\theta_1+\theta_2)}$. The circle group is a *continuous group*, with infinite number of elements, parametrized by the continuous parameter $\theta \in [0, 2\pi)$. It can be thought of as the $n \to \infty$ limit of the cyclic group C_n. Note that the circle divided into n segments is *compact*, in distinction to the infinite lattice of integers $\mathbb{Z}$, whose limit is a *line* (noncompact, of infinite length).

An element of a $[d\times d]$ -dimensional matrix representation of a *Lie group* continuously connected to identity can be written as

$$g(\boldsymbol{\phi}) = e^{i\boldsymbol{\phi}\cdot T}\,, \qquad \boldsymbol{\phi}\cdot T = \sum_{a=1}^{N} \phi_a T_a\,, \tag{11.2}$$

where $\boldsymbol{\phi}\cdot T$ is a *Lie algebra* element, T_a are matrices called 'generators', and $\boldsymbol{\phi} = (\phi_1, \phi_2, \cdots, \phi_N)$ are the parameters of the transformation. Repeated indices are summed throughout, and the dot product refers to a sum over Lie algebra generators. Sometimes it is convenient to use the Dirac bra-ket notation for the Euclidean product of two real vectors $x, y \in \mathbb{R}^d$, or the product of two complex vectors $x, y \in \mathbb{C}^d$, i.e., indicate complex x-transpose times y by

$$\langle x|y\rangle = x^\dagger y = \sum_i^d x_i^* y_i\,. \tag{11.3}$$

Finite unitary transformations $\exp(i\boldsymbol{\phi}\cdot T)$ are generated by sequences of infinitesimal steps of form

$$g(\boldsymbol{\delta\phi}) \simeq 1 + i\boldsymbol{\delta\phi}\cdot T\,, \quad \boldsymbol{\delta\phi} \in \mathbb{R}^N\,, \quad |\boldsymbol{\delta\phi}| \ll 1\,, \tag{11.4}$$

where T_a, the *generators* of infinitesimal transformations, are a set of linearly independent $[d\times d]$ hermitian matrices (see figure 11.2 (b)).

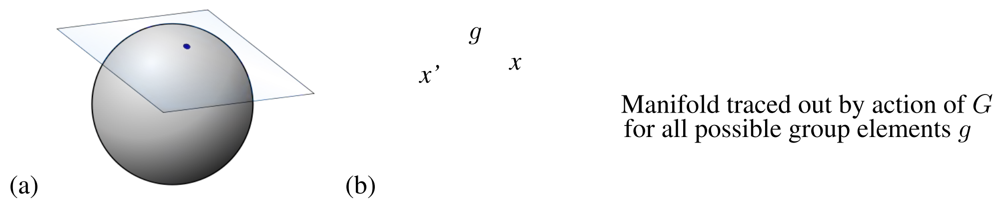

Figure 11.2: (a) Lie algebra fields $\{t_1, \cdots, t_N\}$ span the tangent space of the group orbit $\mathcal{M}_x$ at state space point x, see (11.6) (figure from WikiMedia.org). (b) A global group transformation $g : x \to x'$ can be pieced together from a series of infinitesimal steps along a continuous trajectory connecting the two points. The group orbit of state space point $x \in \mathbb{R}^d$ is the N-dimensional manifold of all actions of the elements of group G on x.

The reason why one can piece a global transformation from infinitesimal steps is that the choice of the "origin" in coordinatization of the group manifold sketched in figure 11.2 (a) is arbitrary. The coordinatization of the tangent space at one point on the group manifold suffices to have it everywhere, by a coordinate transformation g, i.e., the new origin y is related to the old origin x by conjugation $y = g^{-1}xg$, so all tangent spaces belong the same class, they are geometrically equivalent.

Unitary and orthogonal groups are defined as groups that preserve 'length' norms, $\langle gx|gx\rangle = \langle x|x\rangle$, and infinitesimally their generators (11.4) induce no change in the norm, $\langle T_a x|x\rangle + \langle x|T_a x\rangle = 0$, hence the Lie algebra generators T_a are hermitian for,

$$T_a^\dagger = T_a \,. \tag{11.5}$$

The flow field at the state space point x induced by the action of the group is given by the set of N tangent fields

$$t_a(x)_i = (T_a)_{ij} x_j \,, \tag{11.6}$$

which span the d-dimensional *group tangent space* at state space point x, parametrized by $\boldsymbol{\delta\phi}$.

For continuous groups the Lie algebra, i.e., the algebra spanned by the set of N generators T_a of infinitesimal transformations, takes the role that the $|G|$ group elements play in the theory of discrete groups (see figure 11.2).

References

[1] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2004).

[2] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[3] M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Dover, New York, 1962).

[4] W. G. Harter, *Principles of Symmetry, Dynamics, and Spectroscopy* (Wiley, New York, 1993).

[5] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

Exercises

11.1. **Decompose a representation of S_3.** Consider a reducible representation $D(g)$, i.e., a representation of group element g that after a suitable similarity transformation takes form

$$D(g) = \begin{pmatrix} D^{(a)}(g) & 0 & 0 & 0 \\ 0 & D^{(b)}(g) & 0 & 0 \\ 0 & 0 & D^{(c)}(g) & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix} ,$$

with character for class $\mathcal{C}$ given by

$$\chi(\mathcal{C}) = c_a \, \chi^{(a)}(\mathcal{C}) + c_b \, \chi^{(b)}(\mathcal{C}) + c_c \, \chi^{(c)}(\mathcal{C}) + \cdots ,$$

where c_a, the multiplicity of the ath irreducible representation (colloquially called "irrep"), is determined by the character orthonormality relations,

$$c_a = \overline{\chi^{(a)*} \, \chi} = \frac{1}{h} \sum_k^{class} N_k \chi^{(a)}(\mathcal{C}_k^{-1}) \, \chi(\mathcal{C}_k) . \tag{11.7}$$

Knowing characters is all that is needed to figure out what any reducible representation decomposes into!

As an example, let's work out the reduction of the matrix representation of S_3 permutations. The identity element acting on the three objects $(a, b, c)^\top$, arranged as components of a 3-vector, is a $[3\times3]$ identity matrix,

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transposing the first and second object yields $(b, a, c)^\top$, represented by the matrix

$$D(A) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

since

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}$$

a) Find all six matrices for this representation.

b) Split this representation into its conjugacy classes.

c) Evaluate the characters $\chi(\mathcal{C}_j)$ for this representation.

d) Determine multiplicities c_a of irreps contained in this representation.

e) Construct explicitly all irreps.

f) Explain whether any irreps are missing in this decomposition, and why.

11.2. **Invariance under fractional rotations.** Argue that if the discrete cyclic subgroup

$$\mathrm{C}_N = \{e, C^{1/N}, C^{2/N}, \cdots, (C^{1/N})^{N-1}\}\,, \qquad (C^{1/N})^N = e$$

of $\mathrm{SO}(2)$ rotations about an axis (let's say the 'z-axis') is a symmetry group of the 'equations of motion' $\dot{x} = v(x)$,

$$C^{1/N} v(x) = v(C^{1/N} x) = v(x)\,,$$

the only non-zero components of Fourier-transformed equations of motion are a_{jN} for $j = 1, 2, \cdots$. Argue that the Fourier representation is then the 'quotient map' of the dynamics, $\mathcal{M}/\mathrm{C}_N$. (Hint: this sounds much fancier than what is - think first of how it applies to the 2- and 3-disk pinballs.)

11.3. **Characters of $\mathbf{D}_3$.** (continued from exercise 10.3) $\mathrm{D}_3 \cong \mathrm{C}_{3v}$, the group of symmetries of an equilateral triangle: has three irreducible representations, two one-dimensional and the other one of multiplicity 2.

(a) All finite discrete groups are isomorphic to a permutation group or one of its subgroups, and elements of the permutation group can be expressed as cycles. Express the elements of the group D_3 as cycles. For example, one of the rotations is (123), meaning that vertex 1 maps to 2, $2 \to 3$, and $3 \to 1$.

(b) Use your representation from exercise 10.3 to compute the D_3 character table.

(c) Use a more elegant method from the group-theory literature to verify your D_3 character table.

(d) Two D_3 irreducible representations are one dimensional and the third one of multiplicity 2 is formed by $[2\times 2]$ matrices. Find the matrices for all six group elements in this representation.

(Hint: get yourself a good textbook, like Dresselhaus *et al.* [2], Tinkham [5] or Hamermesh [3], and read up on classes and characters.)