H.2 Deterministic diffusion, sawtooth map (13.9)

To illustrate the main idea of chapter 13, tracking of a globally diffusing orbit by the associated confined orbit restricted to the fundamental cell, we consider in more detail the class of simple 1-d dynamical systems, chains of piecewise linear maps (13.9). The translational symmetry (13.10) relates the unbounded dynamics on the real line to the dynamics restricted to a "fundamental cell" - in the present example the unit interval curled up into a circle. The corresponding circle map $f(x)$ is obtained by modulo the integer part. The elementary cell map $f(x)$ is sketched in fig. 13.2. The map has the symmetry property

$$\hat{f}(\hat{x}) \;=\; -\hat{f}(-\hat{x})\,, \tag{H.18}$$

so that the dynamics has no drift, and all odd derivatives of the generating function (13.3) with respect to β evaluated at $\beta = 0$ vanish.

The cycle weights are given by

$$t_p = z^{n_p}\frac{e^{\beta\hat{n}_p}}{|\Lambda_p|}\,. \tag{H.19}$$

The diffusion constant formula for 1-d maps is

$$D = \frac{1}{2}\frac{\langle\hat{n}^2\rangle_\zeta}{\langle n\rangle_\zeta} \tag{H.20}$$

where the "mean cycle time" is given by

$$\langle n\rangle_\zeta = z\frac{\partial}{\partial z}\frac{1}{\zeta(0,z)}\bigg|_{z=1} = -\sum{}'(-1)^k\frac{n_{p_1}+\cdots+n_{p_k}}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|}\,, \tag{H.21}$$

the mean cycle displacement squared by

$$\langle\hat{n}^2\rangle_\zeta = \frac{\partial^2}{\partial\beta^2}\frac{1}{\zeta(\beta,1)}\bigg|_{\beta=0} = -\sum{}'(-1)^k\frac{(\hat{n}_{p_1}+\cdots+\hat{n}_{p_k})^2}{|\Lambda_{p_1}\cdots\Lambda_{p_k}|}\,, \tag{H.22}$$

and the sum is over all distinct non-repeating combinations of prime cycles. Most of results expected in this projects require no more than pencil and paper computations.

H.2.1 The full shift

Reproduce the formulas of sect. 13.2.1 for the diffusion constant D for Λ both even and odd integer.

fig. 13.3	Λ	D
	4	$\frac{1}{4}$
(a)	$2+\sqrt{6}$	$1-\frac{3}{4}\sqrt{6}$
(b)	$2\sqrt{2}+2$	$\frac{15+2\sqrt{2}}{16+4\sqrt{2}}$
(c)	5	1
(d)	$3+\sqrt{5}$	$\frac{5}{2}\frac{\Lambda-1}{3\Lambda-4}$
(e)	$\frac{5+\sqrt{41}}{2}$	$\frac{5\Lambda+1}{5\Lambda+8}$
	6	$\frac{5}{6}$

Table H.2: The diffusion constant as function of the slope Λ for the $\Lambda = 4, 6$ values of (13.19) and the 5 Markov partitions like the one indicated in fig. 13.3.

H.2.2 Subshifts of finite type

We now work out examples when the partition is Markov, although the slope is not an integer number. The key step is that of having a partition where intervals are mapped *onto* unions of intervals.

Start by reproducing the formula (13.27) of sect. 13.2.3 for the diffusion constant D for the Markov partition, the case where the critical point is mapped onto the right border of I_{1_+}.

Think up other non-integer values of the parameter Λ for which the symbolic dynamics is given in terms of Markov partitions: in particular consider the remaing four cases for which the critical point is mapped onto a border of a partion in one iteration. Work out the Markov graph symmetrization factorization and the diffusion constant, and check the material flow conservation for each case. Fill in the diffusion constants missing in table H.2. It is not clear why the final answers tend to be so simple. What value of Λ appears to yield the maximal diffusion constant?

The 7 cases considered here (see table H.2 and fig. 13.3) are the 7 simplest complete Markov partitions in the $4 \leq \Lambda \leq 6$ interval, the criterion being that the critical points map onto partition boundary points. In case at hand, the "critical" point is the highest point of the left branch of the map (symmetry automatically takes care of the other critical point, the lowest point of the left branch), and for the cases considered the critical point maps into the endpoint of each of the seven branches.

One can fill out parameter a axis arbitrarily densely with such points - each of the 6 primary intervals can be subdivided into 6 intervals obtained by 2-nd iterate of the map, and for the critical point mapping into any of those in 2 steps the grammar (and the corresponding cycle expansion) is finite, and so on. Some details of how this is accomplished are given in appendix B.2 for a related problem, the pruned Bernulli shift.

H.2.3 Diffusion coefficient, numerically

(optional:)
Attempt a numerical evaluation of

$$D = \frac{1}{2} \lim_{n\to\infty} \frac{1}{n} \left\langle \hat{x}_n^2 \right\rangle \ . \tag{H.23}$$

Study the convergence by comparing your numerical results to the exact answers derived above. Is it better to use few initial $\hat{x}$ and average for long times, or to use many initial $\hat{x}$ for shorter times? Or should one fit the distribution of $\hat{x}^2$ with a gaussian and get the D this way? Try to plot dependence of D on Λ; perhaps blow up a small region to show that the dependance of D on the parameter Λ is fractal. Compare with figures in refs. [H.1, H.2].

H.2.4 D is a nonuniform function of the parameters

(optional:)
The dependence of D on the map parameter Λ is rather unexpected - even though for larger Λ more points are mapped outside the unit cell in one iteration, the diffusion constant does not necessarily grow. Fig. 13.4 taken from ref. [H.1] illustrates the fractal dependence of diffusion constant on the map parameter. An interpretation of this lack of monotonicity would be interesting.

You can also try applying periodic orbit theory to the sawtooth map (13.9) for a random "generic" value of the parameter Λ, for example $\Lambda = 4.5$. The idea is to bracket this value of Λ by the nearby ones, for which higher and higher iterates of the critical value $a = \Lambda/2$ fall onto the partition boundaries, compute the exact diffusion constant for each such approximate Markov partition, and study their conergence toward the value of D for $\Lambda = 4.5$. Judging how difficult such problem is already for a tent map (see sect. 3.6 and appendix B.1), this is too ambitious for a week-long exam.

References

[H.1] R. Klages and J.R. Dorfman, "Simple Maps with Fractal Diffusion Coefficients", *Phys. Rev. Lett.* **74**, 387-390 (1995)

[H.2] R. Klages, *Deterministic diffusion in one-dimensional chaotic dynamical systems*, (W & T, Berlin, 1996)