periodic orbit!relative
relative!periodic orbit

Chapter 23

Continuous symmetries

Hard work builds character.
V.I. Warshavski, Private Investigator

Trace formulas relate short time dynamics (unstable periodic orbits) to long time invariant state space densities (natural measure). Higher dimensional dynamics requires inclusion of higher-dimensional compact invariant sets, such as partially hyperbolic invariant tori, into trace formulas. A trace formula for a partially hyperbolic $(N+1)$-dimensional compact manifold invariant under a global continuous symmetry is derived here. In this extension of "periodic orbit" theory there are no or very few periodic orbits - the relative periodic orbits that the trace formula has support on are almost never eventually periodic.

The classical trace formula for smooth continuous time flows relates the spectrum of the evolution operator

chapter 14

$$\mathcal{L}^t(x',x) = \delta\big(x' - f^t(x)\big)\, e^{\beta\cdot A^t(x)} \tag{23.1}$$

to the unstable periodic orbits p of the flow $f^t(x)$,

$$\sum_{\alpha=0}^{\infty} \frac{1}{s-s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta\cdot A_p - sT_p)}}{\left|\det\left(\mathbf{1}-M_p^r\right)\right|}\,.$$

This formula (and the associated spectral determinants and cycle expansions) is valid for fully hyperbolic flows.

chapter 15

Here we derive the corresponding formula for dynamics invariant under a compact group of symmetry transformations. In what follows, a familiarity with basic group-theoretic notions is assumed, with the definitions relegated to appendix J.1.

fast track:
chapter 22, p. 427

23.1 Compact groups

The group theory that we shall need here is summarized by the *Peter-Weyl theorem*, and its corrolaries: A compact Lie group $\mathcal{G}$ is completely reducible, its representations can be decomposed like the group algebra of the finite group representation theory, every Lie group is a closed subgroup of $U(n)$ for some n, and every continuous, unitary, irreducible representation of a compact Lie group is finite dimensional.

The theory of semisimple Lie groups is elegant, perhaps too elegant. In what follows, we serve group theoretic nuggets in need-to-know portions, offering a pedestrian route through a series of simple examples of familiar aspects of group theory and Fourier analysis, and a high, cyclist road in the text proper.

But main idea is this: the character $\chi_m(\theta)$ of the Frobenius-Weyl representations theory is a generalization to all compact continuous Lie groups of the weight $e^{i\theta m}$ in the Fourier transform of a smooth function on a circle. So a basis of for a d-dimensional representation $(m_1, m_2, \ldots, m_N)$ of an N-dimensional compact Lie group is a set of d linearly independent eigenfunctions on the N-dimensional compact group manifold, with m_1, m_2, $\ldots$, m_N "nodes" along the N directions needed to span the manifold. For a circle this is Fourier analysis; for a sphere these are spherical harmonics, and the Peter-Weyl theorem states that analogous expansion exists for every compact Lie group. We will not need to construct it explicitely.

23.1
page 476
23.2
page 476

23.1.1 Group representations

The vector space q_a.

Repeated indices summed throughout this chapter.

The symmetry group, i.e. the group of all linear transformations

$$q'_a = G_a{}^b q_b \qquad a, b = 1, 2, \ldots, d\,,$$

which leaves invariant the predictions of the theory. The $[d \times d]$ matrices G form a representation of the group $\mathcal{G}$. Vectors in the conjugate space $\overline{q}$ transform as

$$q'^a = G^a{}_b q^b\,.$$

Tensors transform as

$$h'_{ab}{}^c = G_a{}^f G_b{}^e G_c{}^d h_{fe}{}^d\,.$$

A function is an invariant function if (and only if) for any transformation $g \in \mathcal{G}$ and for any set of vectors $q, r, s, \ldots$

coordinate reflection
reflection, coordinate
group!integral—(textbf

$$H(\overline{Gq}, \overline{Gr}, \ldots Gs) = H(\overline{q}, \overline{r}, \ldots, s)\,. \tag{23.2}$$

The group transformations can be generated by sequences of infinitesimal transformations

$$G_a{}^b \simeq \delta^a_b + i\epsilon_i (T_i)^b_a \qquad \epsilon_i \in \mathbb{R}\,, T_i \text{ hermitian}\,.$$

In this chapter we consider only infinitesimal transformations of form $G = 1 + iD$, $|D^a_b| \ll 1$, i.e., the transformations connected to the identity. For example, we postpone study of discrete invariances under coordinate reflections to chapter 22.

Consider a multilinear invariant function

$$H(\overline{q}, \overline{r}, \ldots, s) = h_{ab\ldots}{}^{\ldots c}\, q^a r^b \ldots s_c$$

In terms of the generators T_i, H is invariant if

$$(T_i)^{a'}_a h_{a'b\ldots}{}^{c\ldots} + (T_i)^{b'}_b h_{ab'\ldots}{}^{c\ldots} - (T_i)^c_{c'} h_{ab\ldots}{}^{c'\ldots} + \ldots = 0\,. \tag{23.3}$$

For example, the generators T_i are invariant (one does not want the rules to change at every step)

$$(T_i)\,{}^a_b = G^a{}_{a'} G_b{}^{b'} G_{ii'}\,(T_{i'})\,{}^{a'}_{b'}\,, \tag{23.4}$$

where G_{ij} is the adjoint $[N \times N]$ matrix representation of $g \in \mathcal{G}$.

The $[d \times d]$ matrices T_i are in general non-commuting, and from (23.3) it follows that they close N-element *Lie algebra*

$$T_i T_j - T_j T_i = i C_{ijk} T_k \qquad i, j, k = 1, 2, ..., N$$

where C_{ijk} are the structure constants.

Vector space V is irreducible if the only invariant subspaces of V under the action of G are (0) and V. If every V on which G acts can be written as a direct sum of irreducible subspaces, then G is completely reducible.

Consider a group integral of form

$$\int dg\, G_a{}^b G^c{}_d\,, \tag{23.5}$$

where ${G_a}^b$ is a $[n\times n]$ matrix representation of $g \in \mathcal{G}$, $\mathcal{G}$ a compact Lie group, and ${G_a}^b$ is unitary. $G^\dagger$ is the matrix representation of the action of g on the conjugate vector space, which we write as

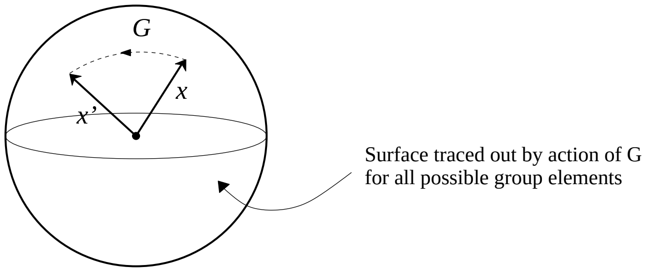
$$G^a{}_b = (G^\dagger)_b{}^a \,,$$

and the integration is over the entire range of g. The integral (23.5) is defined by two requirements:

(1) Normalization:

$$\int dg = 1\,. \tag{23.6}$$

(2) How do we define $\int dg\, {G_a}^b$? The action of $g \in \mathcal{G}$ is to rotate a vector x_a into $x'_a = {G_a}^b x_b$



The averaging smears x in all directions, hence the second integration rule

$$\int dg\, {G_a}^b = 0\,, \qquad G \text{ is a } \textit{non-trivial} \text{ representation of } g\,, \tag{23.7}$$

simply states that the average of a vector is zero.

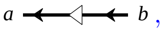
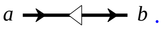
A representation is trivial if $G = 1$ for all group elements g. In this case no averaging is taking place, and the first integration rule (23.6) applies.

What happens if we average a pair of vectors x, y? There is no reason why a pair should average to zero; for example, we know that $|x|^2 = \sum_a x_a x_a^* = x_a x^a$ is invariant (we are considering only unitary representations), so it cannot have a vanishing average. Therefore, in general

$$\int dg\, {G_a}^b G^c{}_d \neq 0\,. \tag{23.8}$$

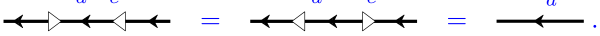
To get a feeling of what the right-hand side looks like, let us work out an example:

Example 23.1 A group integral for $SU(n)$: *Let $G_a{}^b$ be the defining $[n\times n]$ matrix representation of $SU(n)$. The defining representation is non-trivial, so it averages to zero by (23.7). The first non-vanishing average involves $G^\dagger$, the matrix representation of the action of g on the conjugate vector space. To avoid dealing with the multitude of dummy indices, we resort to diagrammatic notation:* singlet

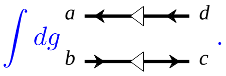
$G_a{}^b =$  $, \qquad G^a{}_b =$  . (23.9)

For G the arrows and the triangle point the same way, while for $G^\dagger$ they point the opposite way. Unitarity $G^\dagger G = 1$ is given by

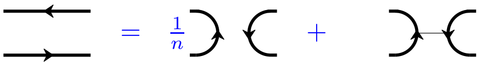
$$G^c{}_a G_c{}^b = G_a{}^c G^b{}_c = \delta_a^b$$

 (23.10)

In this notation, the $GG^\dagger$ integral to be evaluated is

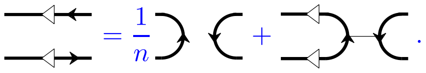
 (23.11)

For $SU(n)$ the $V\otimes\overline{V}$ tensors decompose into the singlet and the adjoint rep

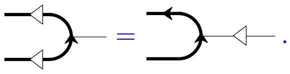


$$\delta_a^d\delta_c^b = \frac{1}{n}\delta_a^b\delta_c^d + \frac{1}{a}(T_i)_a^{\,b}(T_i)_c^{\,d}\,. \qquad (23.12)$$

We multiply (23.11) with the above decomposition of the identity. The unitarity relation (23.11) eliminates G's from the singlet:

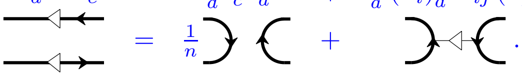
 (23.13)

The generators T_i are invariants, and transform under G according to (23.4). Multiplying by G_{ii}^{-1}, we obtain

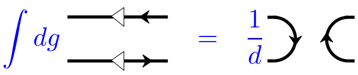
 (23.14)

Hence, the pair $GG^\dagger$ in the defining representation can be traded in for a single G in the adjoint rep

$$G_a{}^d G^b{}_c = \frac{1}{d}\delta_c^d\delta_a^b + \frac{1}{a}(T_i)_a^{\,b}\,G_{ij}\,(T_j)_c^{\,d}$$

 (23.15)

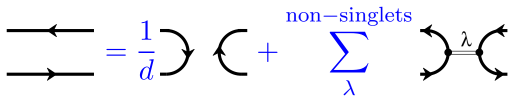
The adjoint representation G_{ij} is non-trivial, so it gets averaged to zero by (23.7). Only the singlet survives



$$\int dg\; G_a{}^d G^b{}_c = \frac{1}{d}\delta_c^d\delta_a^b\,. \qquad (23.16)$$

irreducible!representation
representation!irreducible
Schur's Lemma
Clebsch-Gordan!series
character—(textbf
representation!character
character!orthonormality
character—)
group!integral—)

Now let G be any irreducible $[d \times d]$ rep. Irreducibility (known in this context as "Schur's Lemma") means that any invariant tensor A^a_b is proportional to δ^a_b. As the only bilinear invariant is δ^a_b, the Clebsch-Gordan series



(23.17)

contains one and only one singlet. Only the singlet survives the group averaging, and (23.16) is true for any $[d \times d]$ *irreducible* rep. If we take $G^{(\mu)}{}_\alpha{}^\beta$ and $G^{(\lambda)}{}_d{}^c$ in inequivalent representations λ, μ (there is no matrix K such that $G^{(\lambda)} = KG^{(\mu)}K^{-1}$ for any $g \in \mathcal{G}$), then there is no way of forming a singlet, and

$$\int dg\, G^{(\lambda)}{}_a{}^d G_{(\mu)}{}^\beta{}_\alpha = 0 \qquad \text{if} \qquad \lambda \neq \mu\,. \tag{23.18}$$

The trace of an irreducible $[d \times d]$ matrix representation λ of g is called the *character* of the representation:

$$\chi_\lambda(g) = \mathrm{tr}\, G^{(\lambda)} = G^{(\lambda)}{}_a{}^a\,. \tag{23.19}$$

The character of the conjugate representation is

$$\chi^\lambda(g) = \mathrm{tr}\, G^{(\lambda)\dagger} = G^{(\lambda)a}{}_a = \chi_\lambda(g)^*\,. \tag{23.20}$$

Contracting (23.17) with two arbitrary invariant $[d \times d]$ tensors $h_d{}^a$ and $(f^\dagger)_b{}^c$, we obtain the *character orthonormality relation*

$$\int dg\ \chi_\lambda(hg)\, \chi^\mu(gf) = \delta^\mu_\lambda \frac{1}{d_\lambda} \chi_\lambda(hf^\dagger) \tag{23.21}$$

The character orthonormality tells us that if two group invariant quantities share a $GG^\dagger$ pair, the group averaging sews them into a single group invariant quantity. The replacement of $G_a{}^b$ by the character $\chi_\lambda(h^\dagger g)$ does not mean that any of the tensor index structure is lost; $G_a{}^b$ can be recovered by differentiating

$$G_a{}^b = \frac{d}{dh_b{}^a} \chi_\lambda(h^\dagger g)\,. \tag{23.22}$$

The diagrammatic notation and the characters are two equivalent notations for evaluating group integrals.

All the group theory we shall need here is most compactly summarized by the

group!representation
representation!matrix

Group Orthogonality Theorem: Let D_m, $D_{m'}$ be two irreducible matrix representations of a compact group G of dimensions d_m, $d_{m'}$,

$$\int_G dg\ D_m(g)_{ab} D^*_{m'}(g)_{a'b'} = \frac{1}{d_m}\delta_{m,m'}\delta_{aa'}\delta_{bb'}\,.$$

Here dg is the Haar measure, invariant under group transformations

$$\int_G dg \ = \int_G d\mu(x) \ = \int_G d\mu(gx)\,,$$

normalized

$$\int_G dg \ = 1\,,$$

so that the group average of an invariant scalar quantity is the quantity itself. For a finite group G with $|G|$ group elements the normalized Haar measure is a discrete sum.

$$d\mu(x) \ = \ \frac{1}{|G|}\sum_g \delta(gx)\,.$$

For compact groups left- and right-invariant Haar measures coincide. As all we shall do here is integrate over a Dirac delta function, we shall not need here any explicit global formula for $d\mu(x)$ (though it pays to work through a few examples, for one's own comfort).

The new trace formula follows from the full reducibility of representations of a compact group G acting linearly on a vector space V, with irreducible representations labeled by sets of integers $m = (m_1, \cdots, m_N)$, and the vector space V decomposed into invariant subspaces V_m. For a N-dimensional compact Lie group G the fundamental result is the Weyl full reducibility theorem, with projection operator onto the V_m irreducible subspace given by

Weyl reference

$$P_m = d_m \int_G dg\ \chi_m(g) D_m(g^{-1})\,. \tag{23.23}$$

The group elements $g = g(\theta_1, \ldots, \theta_N) = e^{i\theta\cdot T}$ are parametrized by N real numbers $\{\theta_1, \ldots, \theta_N\}$ of finite range, hence designation "compact." The N group generators T_a, $a = 1, \cdots, N$ close the Lie algebra of G.

23.1.2 Transformation operators, projection operators

credit [23.46]

[1] How does the group act on a function $f(x)$ of x? The transformed function should have the same value at gx as the initial function has at x, so, denoting by O_g the desired transformation:

$$O_g f(D(g)x) = f(x)\,.$$

Here the operator O_g acts on functions, whereas $D(g)$ is a matrix representation of the group element $D(g)$ that acts on position vectors $x \in \mathcal{M}$.

Let us replace x with $D(g^{-1})x$, giving

$$O_g f(x) = f(D(g^{-1})x)$$

This is the conventional, Wigner's definition of the effect of transformations on functions.

Now, consider the effect of two successive transformations $g_1 g_2 = g$. We have

$$O_{g_1} O_{g_2} f(x) = O_{g_1} f(g_2^{-1}x) = f(g_2^{-1}[g_1^{-1}x]) = f([g_1 g_2]^{-1}x) = O_g f(x)$$

Hence if $g_1 g_2 = g$, we have $O_{g_1} O_{g_2} = O_g$.

Except in special circumstances, we will not use the O_g notation, but just g when we operate on functions. But be aware of the difference between transformation of functions and of vectors.

Suppose we have an arbitrary function or set of functions. How do we obtain functions with desired symmetry properties? If f is an arbitrary function,

$$P^\alpha_{ij} f = \frac{d_\alpha}{|G|} \sum_G D^\alpha_{ij}(g)^* g f = F^\alpha_{ij}$$

which is either zero or a basis function for the ith row of irrep α: a function of symmetry species (α, i).

[23.46], p. 16 as a soluton to the problem of proving the projection property.

back reference to example 23.7

Example 23.2 Irreducible representations of the $SO(2)^N$ Abelian group: *(Example 23.7 continued) The irreducible representations of the $SO(2)^N$ Abelian group acting on torus T^N are labeled by N integers $m = (m_1, \cdots, m_N)$. All irreducible representations are 1-dimensional, $d_m = 1$. The character of m representation is*

$$\chi_m(g) = e^{-im\cdot\theta}$$

[1] Bartsch: I can't make sense out of the projector (23.23). P_m must operate on the big, reducible representation space. $D_m(g^{-1})$ is a matrix of dimension d_m, which operates on the irreducible representation. Should the formula have the total representation matrix $D(g^{-1})$ instead of D_m?

Example 23.3 Haar measure for $SO(2)$: *The normalized Haar measure is $dg = d\theta/(2\pi)$.*

group!integration
integration!group
Haar measure
symmetry!dynamical system
dynamics!symmetry
flow!generator of
generator!of flow

Example 23.4 Haar measure for $SO(3)$:

$$SO(3): \qquad dg = \frac{1}{2\pi^2}\sin^2(\phi/2) d\Omega_e d\phi$$

with $d\Omega_e$ solid angle surface element for unit vector e.

$$\int_{SO(3)} dg = 8\pi^2$$

see ref. [23.20].

The character χ is the trace $\chi_m(g) = \mathrm{tr}\, D_m(g) = \sum_{i=1}^{d_m} D_m(g)_{ii}$, where $D_m(g)$ is a $[d_m \times d_m]$-dimensional matrix representation of action of the group element g on the irreducible subspace V_m. We will sometimes employ notation g as a shorthand for $D(g)$, i.e., by $x' = gx$ we mean the matrix operation $x'_i = \sum_{j=1}^{d} D(g)_{ij} x_j$, and by $f'(x) = g\, f(x) = f(gx)$, $f(x)$ a smooth function over the state space $x \in \mathcal{M}$, we mean $f'(x) = f(D(g)x)$. For an invariant scalar quantity the average over the group in (23.23) must be the quantity itself, so the group integral is weighted by the normalized Haar measure, $\int_G dg = 1$, and d_m is the multiplicity of degenerate eigenvalues in representation m.

23.2 Continuous symmetries of dynamics

reintroduce shorthand g here

If action of every element g of a compact group G commutes with the flow $\dot{x} = v(x)$,

$$D(g)v(x) = v(D(g)x)\,, \; D(g)f^t(x) = f^t(D(g)x)\,,$$

G is a global symmetry of the dynamics. The finite time evolution operator (23.1) can be written as $\mathcal{L}^t = e^{t\mathcal{A}}$ in terms of the time evolution generator (9.26)

$$\mathcal{A}\rho(x) = \lim_{\delta\tau \to 0^+} \frac{1}{\delta\tau}\left(\mathcal{L}^{\delta\tau} - I\right)\rho(x) \; = \; -\partial_i(v_i(x)\rho(x))\,. \tag{23.24}$$

The operator $e^{t\mathcal{A}}$ commutes with all symmetry transformations $e^{i\theta\cdot T}$. For a given state space point x together they sweep out a $(N{+}1)$-dimensional manifold of equivalent orbits.

In other words, the time evolution itself is a (noncompact) 1-parameter Lie group. Thus all continuous symmetries can be considered as being on on the same footing.

stability!marginal
marginal!stability

A symmetry group element acts on $\mathcal{L}(x, y)$, the kernel of $\mathcal{L}^t$ in the state space representation (23.1), as

$$g^{-1}\mathcal{L}(y, x) = \mathcal{L}(D(g^{-1})y, x) = \mathcal{L}(y, D(g)x) \,. \tag{23.25}$$

Here we have used $|\det D(g)|$. The irreducible eigenspaces of G are also eigenspaces of the dynamical evolution operator $\mathcal{L}^t$, with the decomposition of the evolution operator to irreducible subspaces, $\mathcal{L} = \sum_m \mathcal{L}_m$, following immediately by application of the projection operator (23.23):

$$\mathcal{L}^t_m(y, x) = d_m \int_G dg \; \chi_m(g)\mathcal{L}^t(D_m(g^{-1})y, x) \,. \tag{23.26}$$

23.2.1 Relative periodic orbits

Relative periodic orbits are orbits $x(t)$ in state space $\mathcal{M}$ which exactly recur

$$x(t) = gx(t + T) \tag{23.27}$$

for a fixed *relative period* T and a fixed group action $g \in G$ of $\mathcal{M}$. This group action is sometimes referred to as a “phase,” or a “shift.” Relative periodic orbits are to periodic solutions what relative equilibria (traveling waves) are to equilibria (steady solutions).

For dynamical systems with continuous symmetries relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space. As almost any such orbit explores ergodically the manifold swept by action of G, they are sometimes referred to as “quasiperiodic”. However, an orbit can be periodic if it satisfies a special symmetry. If $g^k = 1$ is of finite order k, then the corresponding orbit is periodic with period kT. If g is not of finite order k, orbits can be periodic only after the action of g.

☞ chapter 22

In either case, we refer to the orbits in $\mathcal{M}$ satisfying (23.35) as *relative periodic orbits.*

⇓PRELIMINARY

As G commutes with f^t, all eigenfunctions ρ of $\mathcal{L}^t$ must be invariant under G, $\rho(x) = \rho(gx)$. Infinitesimally, in terms of Lie algebra generators $T_\theta\rho(x) = 0$.

⇑PRELIMINARY

23.2.2 Stability of relative periodic orbits

finish this section

A infinitesimal group transformation maps globally a trajectory in a nearby trajectory, so we expect the initial point perturbations along to group manifold to be marginal, growing at rates slower than exponential. The argument is akin to (4.7), the proof of marginality of perturbations along the trajectory. Consider two nearby initial points separated by an infinitesimal group rotation $\delta\theta$: $\delta x(0) = f^{\delta\theta}(x_0) - x_0 = v(x_0)\delta\theta$. By the commutativity of the group with the flow, $f^{t+\delta t} = f^{\delta t + t}$. Expanding both sides of $f^t(f^{\delta t}(x_0)) = f^{\delta t}(f^t(x_0))$, keeping the leading term in δt, and using the definition of the fundamental matrix (4.6), we observe that $J^t(x_0)$ transports the velocity vector at x_0 to the velocity vector at $x(t)$ at time t:

Lagrangian!coordinates
Eulerian!coordinates
cycle!marginal stability
marginal stability!cycle

$$v(x(t)) = J^t(x_0)\, v(x_0)\,. \tag{23.28}$$

In nomenclature of page 62, the fundamental matrix maps the initial, Lagrangian coordinate frame into the current, Eulerian coordinate frame.

For relative periodic orbits However, already at this stage we see that if the orbit is periodic, $g_p x(T_p) = x(0)$, at any point along cycle p the velocity v is an eigenvector of the fundamental matrix $J_p = J^{T_p}$ with an eigenvalue of unit magnitude,

$$J^{T_p}(x)\, v(x) = v(x)\,, \qquad x \in p\,. \tag{23.29}$$

Two successive points along the cycle separated by $\delta x(0)$ have the same separation after a completed period $\delta x(T_p) = g_p \delta x(0)$, hence eigenvalue of magnitude 1.

23.3 Symmetry reduced trace formula for flows

not quite true, fix

As any pair of nearby points on a periodic orbit returns to itself exactly at each cycle period, the eigenvalue of the fundamental matrix corresponding to the eigenvector along the flow necessarily equals unity for all periodic orbits. In presence of N-dimensional symmetry Lie group G, further N eigenvalues equal unity. Hence the trace integral $\mathrm{tr}\,\mathcal{L}^t$ requires a separate treatment for the direction along the flow and for the N group transformation directions.

Ole: clarify this — draw a picture.

To evaluate the contribution of a prime cycle p of period T_p, restrict the integration to an infinitesimally thin manifold $\mathcal{M}_p$ enveloping the cycle and all of its rotations by G, pick a point on the cycle, and choose a local coordinate system with a longitudinal coordinate $dx_\parallel$ along the direction of the flow, N coordinates dx_G along the invariant manifold swept by p under the action of the symmetry group G, and $(d{-}N{-}1)$ transverse coordinates $x_\perp$

draw figure

$$\mathrm{tr}_{\,p}\mathcal{L}^t_m = d_m \int_G dg\; \chi_m(g) \int_{\mathcal{M}_p} dx_\perp dx_\parallel dx_G\, \delta\big(x - D_m(g) f^t(x)\big)\ . \tag{23.30}$$

The integral along the longitudinal coordinate was computed in (14.18). Eliminating the time dependence by Laplace transform one obtains

$$\int_0^\infty e^{-st} \oint_p dx_\parallel \, \delta\big(x_\parallel - f^t(x_\parallel)\big) \;=\; T_p \sum_{r=1}^{\infty} e^{-sT_p r} \, . \tag{23.31}$$

Example 23.5 Trace group integral for $SO(2)$: *Parametrize rotations on a circle by $\theta \in [0, 2\pi)$. The normalized Haar measure is $dg = d\theta/(2\pi)$, and a trajectory point advanced by time t and shifted by θ can be denoted $x(t,\theta)$. The character is $e^{-im\theta}$. For a circle this is just Fourier analysis, for a general compact semisimple Lie group Weyl's generalization of Fourier analysis. Consider projection on the mth subspace of the integral along the rotational direction*

$$I_G = \int_G dg \; \chi_m(g) \oint dx_G \, \delta(x_G - D_m(g) x_G(t,0)) \, .$$

Coordinate x_G is the set of points swept by $[0, 2\pi]$ rotation of a point $x_0 = x_G(0,0)$, so it is natural to parametrize it by the rotation angle θ': $x_G = x(0,\theta')$, and rewrite the circle integral as

$$\oint dx_G \, \delta(x_G - D_m(g) x_G(t,0)) = \int_0^{2\pi} d\theta' \frac{dx}{d\theta}(0,\theta') \, \delta(x(0,\theta') - x(t, \theta' + \theta)) \, .$$

Inverting the order of integrations,

$$I_G = \int_0^{2\pi} d\theta' \int_0^{2\pi} \frac{d\theta}{2\pi} \, e^{-im\theta} \frac{dx}{d\theta}(0,\theta') \, \delta(x(0,\theta') - x(t, \theta' + \theta)) \, .$$

The integral is novanishing for smallest θ_p for which $x(0,\theta') = x(t, \theta' + \theta_p)$, and for all its repeats. Expand the argument of δ function in each such neighborhood $\theta' = \theta_p + \theta''$.

$$\begin{aligned} x(t, \theta' + \theta_p + \theta'') &= x(t, \theta' + \theta_p) + \theta'' \frac{dx}{d\theta}(t, \theta' + \theta_p) + \cdots \\ &= x(t, \theta') + \theta'' \frac{dx}{d\theta}(t, \theta') + \cdots \, . \end{aligned}$$

substituting back yields

$$\begin{aligned} I_G &= \int_0^{2\pi} \frac{d\theta'}{2\pi} \sum_{r=1}^{\infty} e^{-im\theta_p r} \frac{dx(0,\theta')}{d\theta} \int_{-\epsilon}^{\epsilon} d\theta'' \, e^{-im\theta''} \delta\left(\theta'' \frac{dx}{d\theta}(0,\theta')\right) \\ &= \sum_{r=1}^{\infty} e^{-im\theta_p r} \, . \end{aligned}$$

The m subspace group integral is simple:

$$\int_G dg \; \chi_m(g) \int_{\mathcal{M}_p} dx_G \, \delta\big(x_G - D_m(g) f^{rT_p}(x_G)\big) = \chi_m(g_p^r). \tag{23.32}$$

For the remaining transverse coordinates the fundamental matrix is defined in a (N+1)-dimensional surface of section $\mathcal{P}$ of constant $(x_\parallel, x_G)$. Linearization of the periodic flow transverse to the orbit yields

trace!formula!symmetry reduced
periodic orbit!relative
relative!periodic orbit

$$\int_{\mathcal{P}} dx_\perp \delta\big(x_\perp - D_m(g_p^r) f^{rT_p}(x_\perp)\big) = \frac{1}{\left|\det\left(\mathbf{1} - \tilde{M}_{m,p}^r\right)\right|} \,, \qquad (23.33)$$

ch from **J** to **M**

where $\tilde{M}_{m,p} = D_m(g_p)M_p$ is the p-cycle $[(d-1-N)\times(d-1-N)]$ symmetry reduced fundamental matrix, computed on the reduced surface of section and rotated by g_p. As in (14.5), we assume hyperbolicity, i.e., that the magnitudes of all transverse eigenvalues are bounded away from unity.

The *classical symmetry reduced trace formula for flows* follows by substituting (23.31) - (23.33) into (23.30):

$$\sum_{\beta=0}^{\infty} \frac{1}{s - s_{m,\beta}} = d_m \sum_p T_p \sum_{r=1}^{\infty} \chi_m(g_p^r) \frac{e^{r(\beta A_p - sT_p)}}{\left|\det\left(\mathbf{1} - \tilde{M}_{m,p}^r\right)\right|} \,. \qquad (23.34)$$

add the determinant

(we can restore $e^{\beta A_p}$ from (23.1) provided that the observable $a(x)$ also commutes with G.) The sum is over all prime *relative periodic orbits* p and their repeats, orbits in state space which satisfy

$$x(t) = D(g_p)x(t + T_p) \qquad (23.35)$$

for a fixed *relative period* T_p and a fixed *shift* g_p.

For example, for the full Navier-Stokes plane Couette flow defined in a box periodic in stream-wise and span-wise directions, a relative periodic solution is a solution that recurs at time T_p with exactly the same disposition of velocity fields over the entire box, but shifted by a 2-dimensional translation g_p.

The $m = (0, 0, \cdots, 0)$ subspace is the one of most relevance to chaotic dynamics, as its leading eigenfunction, with least nodes and the slowest decay rates, corresponds to the natural measure observed in the long time dynamics.

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Example 23.6 Translation group: *The generator $\mathcal{A}$ is reminiscent of the generator of translations. Indeed, for a constant velocity field dynamical evolution is nothing but a translation by (time $\times$ velocity):*

9.10 page 148

$$e^{-tv\frac{\partial}{\partial x}} a(x) = a(x - tv) \,. \qquad (23.36)$$

Example 23.7 $SO(2)$ **rotations:** *$SO(2)$ is the group of rotations in a plane, smoothly connected to the unit element (i.e. the inversion $x \to -x$ is excluded). A group element can be parameterized by angle θ', and its action on smooth periodic functions over θ is generated by*

$$g^\theta = e^{-i\theta T}\,, \quad T = \frac{d}{d\theta}$$

g^θ rotates a periodic function $u(\theta + 2\pi) = u(\theta)$ by θ mod 2π:

$$g^\theta u(\theta') = u(\theta' + \theta)$$

The multiplication law is $g^\theta g^{\theta'} = g^{\theta+\theta'}$. If the group G actions consists of N such rotations which commute, for example a N-dimensional box with periodic boundary conditions, the group G is an Abelian group that acts on a torus T^N.

For a compact Lie group the prefactor d_m in (23.23) counts degenerate eigenvalue multiplicities.

For example, the special case for $SO(3)$ where $m = \{j\}$, the degenerate eigenvalue multiplicity $d_m = (2j + 1)$ is familiar from quantum mechanics.

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23.3.1 Trace formula in presence of a discrete symmetry

☞ chapter 22

In the above, we have discussed only the continuous case (23.23), as (23.37) follows by replacing integration over Haar measure by a finite group discrete sum. For a finite group G with $|G|$ group elements, the sum in (23.37) is normalized, $|G|^{-1} \sum_{g \in G} = 1$, so that the group average of an invariant scalar quantity is the quantity itself.

Keep this in mind: Frobenious theory is a generalization of Fourier analysis for $|G|$ points on a circle to all finite groups, For a finite group $G = \{e, g_2, \ldots, g_{|G|}\}$, the fundamental result is Frobenius reduction formula in terms of group characters, with the projection operator onto the irreducible subspace V_m given by

$$P_m = \frac{d_m}{|G|} \sum_{g \in G} \chi_m(g) D_m(g^{-1})\,. \tag{23.37}$$

For a finite group the prefactor d_m in (23.37) reflects the fact that a d_m-dimensional representation occurs d_m times.

In contrast to the case of continuous symmetries, where relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space [23.28], for discrete symmetries all relative periodic orbits are eventually periodic.

We turn to the case of discrete symmetry group: $G = \{e, g_2, \ldots, g_{|G|}\}$, in great detail in chapter 22.

add here parts of PER's Appendix B.

Commentary

Remark 23.1 Semisimple Lie groups: Here we need only basic results, on the level of any standard group theory textbook [23.8, 23.9]. Chapter 2. of ref. [23.10] offers a clear and pedagogical introduction to Lie groups of transformations. For the Group Orthogonality Theorem see, for example, [23.17, 23.19], or google.

find good reference on Peter-Weyl theorem

Remark 23.2 Full reducibility of semisimple Lie groups:

The study of integrals over compact Lie groups with respect to Haar measure is important in many areas of mathematics and physics, see Mehta [23.21]. In 1896-1897 Frobenius introduced notions of "characters" and group "representations," and proved the full reducibility of representations of finite groups. The characters $\chi_m(g)$ for all compact semisimple Lie groups were constructed and the full reducibility proven by Weyl, extending Cartan's local Lie algebra classification to a global theory of group representations. For the history of this period, see the excellent essay by Hawkins [23.16].

Diagrammatic notation for group theory is explained in the `ChaosBook.orgGroupTheory` webbook.

Remark 23.3 A brief history of relativity: The relative equilibria and relative periodic solutions are related to equilibria and periodic solutions respectively of the Hamiltonian reduced by the symmetries. They appear in many physical situations, such as motion of rigid bodies, gravitational N-body problems, molecules and nonlinear waves. Chenciner [23.37, 23.42] says the first attempt to find (relative) periodic solutions of the N-body problem by was the 1896 short note by Poincaré [23.39], in the context of the 3-body problem. Relative equilibria of the N-body problem (known in this context as the Lagrange points, stationary in the co-rotating frame) are circular motions in the inertial frame, and relative periodic orbits correspond to quasiperiodic motions in the inertial frame. In the planar N-body problem the equations are unchanged by rotations. Relative equilibria can exist in a rotating frame, and are called central configurations. For relative periodic orbits in celestial mechanics see also ref. [23.23] .

Lan has some relative equilibria (traveling waves) for KS in his thesis [25.8], and for complex LG in a paper on "MAWs". Viswanath [23.6] found them in the plane Couette problem. Siminos and Davidchack have for box size $L = 22$ some equilibria. Striking application of relative periodic orbits has been the discovery of "choreographies" of N-body problems [23.38, 23.40, 23.41].

In context of semiclassical quantization Creagh and Littlejohn [23.25, 23.26] concentrate on the case when the continuous symmetry family of orbits includes a true periodic orbit (they use infinitesimal variation around true periodic orbit), not the symmetry reduced case considered here (where almost every relative periodic orbit of the symmetry-reduced dynamics is not a periodic orbit in the full space). They emphasize generalized surface of section dynamics. They refer to relative periodic orbits as "generalized periodic orbits", with "generalized period" $\mathbf{T}_p = (T_p, m_p)$. They mention, but do not go to irreducible reps of the symmetry groups, hence no equivariant trace formula in these papers. Instead, the explicitly compute group volumes. In addition to the reduced dynamics $|\det(1 - M_\perp)|$ they get $\partial\theta/\partial J$ which we do not have. The Berkeley group did it right for discrete symmetries [22.14, 22.4].

Probably suffices to refer to Creagh [23.5] only

This is done in ref. [23.5] which we follow here, and in ref. [23.28] for axially-symmetric case. Creagh [23.5] refers to relative periodic orbits as "pseudoperiodic" orbits. Ref. [23.28] refers to relative periodic orbits as "reduced periodic" orbits, and to the corresponding orbits in the full state space as "quasiperiodic". Creagh remarks at the very end of his paper to his formula (6.4) as the "pleasing result that the quantally reduced spectrum is determined by the classically reduced periodic orbits in the usual way". Ref. [23.36] discusses a trace formula in symmetry-reduced space. Muratore-Ginanneschi [23.29] gives an elegant discussion of "zero-modes" in the path integral formulation, but does not go to irreps either.

use word quasiperiodic

but no character weight in (6.4)?

Ref. [23.27] applies the method to the problems of noninteracting identical particles,

give references: [?] [?] [23.33] [23.34] [23.35]

Résumé

One of the goals of nonlinear dynamics is to describe the long time evolution of ensembles of trajectories, when individual trajectories are exponentially unstable. The main tool in this effort have been trace formulas because they make explicit the duality between individual short time trajectories, and long time invariant densities (natural measures, eigenfunctions of evolution operators). So far, the main successes have been in applications to low dimensional flows and iterated mappings, where the compact invariant sets of short-time dynamics are equilibria, periodic points and periodic orbits. Dynamics in higher dimensions requires extension of trace formulas to higher-dimensional compact invariant sets, such as partially hyperbolic invariant tori.

Here we have used a particularly simple direct product structure of a global symmetry that commutes with the flow to reduce the dynamics to a symmetry reduced $(d-N-1)$-dimensional state space $\mathcal{M}/G$. The trace formulas do not require explicit construction (in general difficult) neither of the reduced state space, nor of the Haar measures.

Amusingly, in this extension of "periodic orbit" theory from unstable 1-dimensional closed orbits to unstable $(N+1)$-dimensional compact manifolds invariant under continuous symmetries, there are no or very few periodic orbits. Relative periodic orbits are almost never eventually periodic, i.e., they almost never lie on periodic trajectories in the full state space [23.28], unless forced to do so by a discrete symmetry, so looking for periodic orbits in systems with continuous symmetries is a fool's errand.

The symmetry reduced trace formula for a *finite* symmetry group $G = \{e, g_2, \ldots, g_{|G|}\}$ with $|G|$ group elements, where the integral over Haar measure in projection (23.23) is replaced by a finite group discrete sum $|G|^{-1}\sum_{g\in G} = 1$, was derived in ref. [23.13, 23.14]. In contrast to the continuous symmetry case, for a discrete symmetry all relative periodic orbits are eventually periodic.

Restriction to compact Lie groups in derivation of the trace formula (23.34) was a matter of convenience, as the general case is more transparent than particular implementations (such as $SO(2)$ and $SO(3)$ rotations, with their explicit Haar measures and characters). This can be relaxed as the need arises - much powerful group theory developed since Cartan-Weyl era is at our disposal. For example, the time evolution is in general non-compact (a generic trajectory is an orbit of infinite length). Nevertheless, the trace formulas have support on compact invariant sets in $\mathcal{M}$, such as periodic orbits and $(N+1)$-dimensional manifolds generated from them by action of the global symmetry groups. Just as existence of a periodic orbit is a consequence of given dynamics, not any global symmetry, higher-dimensional flows beckon us on with nontrivial higher-dimensional compact invariant sets (for example, partially hyperbolic invariant tori) for whom the trace formulas are still to be written.

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[23.4] See chapter "Spectral determinants," ChaosBook.org [23.2].

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In the appendices Harter summarizes clearly projections operators and their relation to characters for fintie groups

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Flotsam

Look for *rpo s* in the pipe flow, relative both in the shift down the pipe and rotation around the pipe axis.

Likelyhood of finding a periodic orbit is *zero*. One expects some only if in addition to a continuous symmetry one has a discrete symmetry which is not a subgroup of the continuous symmetry. THer are none such for the pipe flow. Duct flow would have them.

However, Kuramoto-Sivashinsky and plane Coutte do have discrete symmetries in addition - that is why there are some equilibria (as opposed to relative equilibria) and some periodic orbits in these cases. They belong to discrete symmetry subspaces. Atypical as they are (no turbulent solution will be confined to these subspaces) they are important for periodic orbit theory, as there the shortest orbits dominate.

I believe some but not all of Kawahara's periodic orbits (for the reason explained above).

I have not looked at the isotropic turbulence orbits paper in detail (as I do not think feel that there is interesting physics in such models) and do not know what discrete symmetries they have; if any periodic orbits are found, they should be in the discrete symmetry reduced subspaces.

Thomas comments

[2] The Liouville operator can only be decomposed into symmetry subspaces of the observable A is G-invariant. You should make a pertinent remark somewhere.

The transformation (5) of the Liouville operator is not clear to me. I transform densities according to $rho(x) - > rho_g(x) = rho(gx)$, and covariance of the Liouville operator requires that $q_g = L[rho_g]$ if $q = L[rho]$, with two densities q, rho. To achieve that, the kernel must satisfy $L(gy, gx) = L(y, x)$ (if the representation matrices have unit determinant). I would therefore define the transformed kernel by $gL(y, x) = L(gy, gx)$.

When deriving the trace formula, you neglect the possibility that the PO might be invariant under a subgroup of G. In this case, the manifold swept out by the PO and its images has dimension less than $N + 1$. This is discussed in Creagh's paper (but I haven't had a look at it for a while).

<u>Thomas:</u> In your example of the flow with periodic boundary conditions, the group of symmetries induced by the boundary conditions is a two-dimensional lattice, which is discrete and non-compact. The derivations of the paper do not apply.

<u>Predrag:</u> No - both plane Couette and KS are compactified on periodic domain and have compact continuous symmetries: S_1 (circle) or $SO(2)$ for Kuramoto-Sivashinsky, T^2 (torus) or $SO(2)^2$ for plane Couette. There is no two-dimensional Z^2 non-compact discrete lattice.

<u>Remember to check this paper:</u> ref. [23.32].

The symmetry is spatial translation by an arbitrary amount, and you are looking for eigenfunctions with the space denpendence exp(i k x), with integer k, which belongs to a representation space of S1. Relative POs are then orbits that return to the initial configuration up to an arbitrary shift. Right?

Right - except shift in not arbitrary, it is as fixed as the time period is. Currently shifts are mod L (system size) as with both Kuramoto-Sivashinsky and plane Couette we turn translational directions into finite boxes with periodic b.c.

Eventual goal is to find these relative periodic orbit for spatially infinite systems (just like we find unsable periodic orbits of fintie period along the infinite time direction). But we have not done it, except for equilibria.

[2]Bartsch: Predrag uses "Liouville" operator only for Hamiltonian flows, "evolution" operator for any dynamical flow

Paolo's comments

Paolo Muratore-Ginanneschi 24 Oct 2006

1)In my opinion difference between your calculation and the one I present in the phys. rep. (which reproduces by other means a result by Littlejohn and Creagh J. Phys A 25 1992) consists only in the fact that localization for classical trace formule is exact.

2) This (obvious) observation has though an important consequence. The result for classical trace formulae must be interpreted as distribution valued as you explained long ago in your paper with Eckhard.

Finite results may be obtained by integrating the trace over smooth test functions as you did in the subspace associated to the zero mode associated with the time derivative of the flow.

Rather than doing this, it seems to me that in the subspace spanned the zero modes generated by the "Jacobi fields" induced by the symmetry group you used the "completeness relation" for the matrix elements of irreducible representations to express the "delta-function" otherwise emerging from the calculation of the trace.

This choice is at the origin of the discrepancy with my calculation. There localization is not exact and the trace of the unitary operator I am computing can be easily regularised for example by defining it as the limit of an euclidean continuation. This latter choice provide in general the simplest way to make mathematical sense of Feynman path integrals. So I do not need to decompose the result in its irreducible components. This difference accounts in my opinion for the difference in the weights. It seems to me that a formal resummation over irreducible representation in your formula provides a positive check of my claim.

3) What the two calculations have in common is the need to separate fluctuations around the periodic orbit into longitudinal and transversal ones. In my report I tried to show as clearly as I could that this change of variable is the core of any degenerate by continuos symmetry stationary point calculation disregarding whether localization is exact or not.

Such technique is in the path-integral context generally referred to as the Faddeev-Popov trick. Over last weekend I became aware of two further references where the above claim is proved for finite dimensional systems and formally for infinite dimensional ones in full generality. It enjoyed the reading of them so I thought to advertise them

Fainberg and Semikhatok, Nuclear Physics B Volume 226, Issue 1 , 26 September 1983, Pages 253-268

this is the pedagogic version of Appendix 2 of

A.S. Schwarts Comm. Math. Phys 64 233

4) Coming to the detail of the calculation, I was assuming that the symmetry group acts freely on the periodic orbit manifold. So the integral over the transversal fluctuation trivially coincided with the group volume time period $T{\times}G\,.\}MaybeIammissingsomethingitbutitseemstomethatasimilarassumptio$ over the group orbit space may not include a non-trivial isotropy subgroup. The weight would be in such a case

$$\mu(G/H)\,, \qquad \mu \text{ is the measure and } H \text{ the stabilizer}$$

The outer integral involving the group characteristic may become then non-trivial. A partial example of this situation would be in my opinion an ODE with SO(3) symmetry. In such a case the x_G variables would be described by two angles rather than three.

Historical notes

(On group characters) presented to the Berlin Academy on July 16 1896 is of fundamental importance. The following year that representations of groups began to enter the picture, and again it was a concept due to Frobenius. Hence 1897 is the year in which the representation theory of groups was born. Only in 1897 did Frobenius learn of Molien's work which he described in a letter to Dedekind as "very beautiful but difficult". He reformulated Molien's work in terms of matrices and then showed that his characters are the traces of the irreducible representations. This work was published in 1897. Frobenius's character theory was used with great effect by Burnside and was beautifully written up in Burnside's 1911 edition of his Theory of Groups of Finite Order.

Consider a smooth continuous time flow

$$x(t) = f^t(x_0) = x_0 + \int_0^t d\tau\, v(x(\tau))\,, \qquad x(0) = x_0\,. \tag{23.38}$$

obtained by integrating a d-dimensional set of ODEs $\dot{x} = v(x)$.

$$\text{Spectrum }(\mathcal{L}^t) = \{e^{s_0 t}, e^{s_1 t}, e^{s_2 t}, \cdots\}\,, \qquad \mathrm{Re}\ s_\alpha \geq \mathrm{Re}\, s_{\alpha+1} \tag{23.39}$$

(assumed discrete)

Definition: A Lie group is a topological group G such that (1) G has the structure of a smooth differential manifold. (2) The composition map $G \times G \to G : (g,h) \to gh^{-1}$ is smooth.

By "smooth" we here always mean $\mathbb{C}^\infty$ differentiable.

Definition. Let $\pi_V : G \to GL(V)$ be a representation, then the character of V is $\chi_V = \mathrm{tr}\,\pi_V$.

Character of the identity group element is $\chi(e) = d$, since $D_{ij}(e) = \delta_{ij}$ for $1 \le i, j \le d$.

Essentially, this is a special case of what we have discussed above. The projection operator onto the m irreducible subspace of dimension d_m is given by $P_m = (d_m/|G|) \sum \chi_m(h) h^{-1}$, where $\chi_m(h) = \mathrm{tr}\, D_m(h)$ are the group characters, and the transfer operator $\mathcal{L}$ splits into a sum of inequivalent irreducible subspace contributions $\sum_m \mathrm{tr}\,\mathcal{L}_m$,

where is D_m defined

why h and not g?

$$\mathcal{L}_m(y,x) = \frac{d_m}{|G|} \sum_{g \in G} \chi_m(g) \mathcal{L}(h^{-1}y, x) \ . \tag{23.40}$$

The prefactor d_m in the above reflects the fact that a d_m-dimensional representation occurs d_m times.

A dynamical system $(\mathcal{M}, f)$ is invariant under a symmetry group G if the map $f : \mathcal{M} \to \mathcal{M}$ (or the continuous flow f^t) from the d-dimensional manifold $\mathcal{M}$ into itself (with d finite) is invariant

$$f(gx) = gf(x)$$

for any coordinate $x \in \mathcal{M}$ and any finite non-singular linear representation (i.e., a non-singular $[d \times d]$ matrix) g of any element $g \in G$.

in terms the time evolution generator (9.26)

$$\mathcal{A} \,=\, -(\partial \cdot v) - \sum_{i=1}^{d} v_i(x) \partial_i \, . \tag{23.41}$$

We ignore for the time being the question of what function space the eigenfunctions belong to, as we shall compute the eigenvalue spectrum without constructing any explicit eigenfunctions.

The reduced state space is the quotient $\mathcal{M}/G$. This amounts to ignoring "angles" θ and fixing the conjugate variables p_θ to irreducible representation values θ.

Duality between t and s, θ and m endows this extended space with a natural symplectic structure: In deriving trace formulas, we switch from (x, t, g) to $(x_\perp, s, m)$ representation.

The summation is over orbits that start at x with "angular momentum" m and end at y after time t and "rotation" by g.

character!orthonormalitity need to evaluate Jacobian for transformation

$$dx_{\|}\left|\frac{\partial(\cdots,s,m)}{\partial(\cdots,t,\theta)}\right| = dt d\theta\,.$$

Need to evaluate

$$W_p = \int_p dt dg \; = T_p V_p\,.$$

Or do we? Isn't $V_p \int_G dg \; = 1$ for any relative periodic orbit? Creagh [23.5] gets to that point (modulo discrete symmetry) by the end of his paper.

Consider motion of a test particle of mass $\mu \ll 1$ in the restricted three-body problem [23.22], under the influence of the gravitational force of two heavy bodies with masses 1 and $\mu \ll 1$ fixed at $(-\mu, 0)$ and $(1-\mu, 0)$.

A reduction of the translation symmetry is obtained by fixing the center of mass at the origin or, considering a configuration as an element of the disposition space $\mathbb{R}^D/(1,...,1)\mathbb{R}$ introduced in [2] in the spirit of Jacobi.

Berry and Tabor should be re-derived here in this approach.

Example 23.8 Haar measure for $SU(2)$: *$SU(2)$ acts on vectors in $\mathbb{C}^2$, and preserves their absolute value, hence its action can be parameterized by a 3-sphere S_3, and multiplication can be viewed as an orthogonal transformation of S_3. This is a special case of the formula $N = n^2 - 1$ for the dimension of $SU(n)$. An invariant Haar measure on $SU(2)$ is a normalized surface measure on S_3, such as*

$$\int_{SU(2)} f(g)dg = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_0^{\pi}\int_0^{\pi} \sin^2\phi_1 \sin\phi_2 d\theta d\phi_1 d\phi_2 f(\theta,\phi_1,\phi_2)$$

Note the following laws

$$\int_G dg \; D(g) = 0,$$

Furthermore, if we consider all non-equivalent irreducible representations
recheck of a group G, then the quantities $D^m_{ij}(g)$ for fixed m, i and j

$$\int_G dg \; \chi_m(hg)\chi_m(g^{-1}k^{-1}) = \chi_m(hk^{-1}) \quad \text{with } h,k \in G \text{ fixed}$$

Characters are orthonormal.

$$P_m = d_m \int_G dg \; \chi^*_m(g) D^m(g)$$

Orthonormality of the set $\{P_m\}$.

Completeness of the set $\{P_m\}$. If operator X commutes with the symmetry, the reduced operator X_m acts on V_m. Such operators:

time evolution $\mathcal{L}^t = e^{t\mathcal{A}}$

resolvent $1/(s - \mathcal{A})$

Study eigenvalues $s_{m,\alpha}$ restricted to subspace V_m.

Symmetries can be used to reduce dynamics to a system with fewer degrees of freedom, elimination of ignorable coordinates.

Consider the dynamics on the extended state space $\mathcal{M} \times \mathbb{R} \times G$ where $\mathbb{R}$ corresponds to time.

$(x', t, g') \to (gf^t(x'), t + t', gg')$

in generalized "time" (t, g).

Let $\mathcal{G}$ be a group, X a set, and $\cdot : G \times X \longrightarrow X$ a group action. For any $x \in X$, the *orbit* of x under the group action is the set

$$\{g \cdot x \mid g \in \mathcal{G}\} \subset X.$$

⇑PRELIMINARY