RENORMALIZATION

As we have seen, the renormalization of scattering amplitudes is a physical necessity; what is measured is not the bare masses and the bare vertices[1], but the dressed propagators and vertices. This is pretty obvious; less obvious is the fact that the renormalization can cure a theory of its ultraviolet divergences. The miracle of unambiguous predictions extracted from divergent integrals is hard to swallow; the eagle from the land of Quefithe rejects it to this very day, and the crow is not too happy about it, either. The prevailing view today is pragmatic: what you cannot see, you cannot see. The field theories that we play with are phenomenological models valid over limited ranges of distances and energies. When we measure an electron spiralling in a weak magnetic field, we have no way of knowing what would happen to it at the Planck length. We measure a small shift in the electron's propagator; the contributions from the ultra-high energies are not affected by this shift, they are the same for the propagator and the renormalization constants, and they cancel.

Today we go even further, and turn this disease of the old field theory into a cornerstone of the new field theory. Instead of complaining about the renormalization of ultraviolet divergences, today we take the renormalizability, along with the locality and

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1. Unless there is a limit in which all radiative corrections decouple, as in the case of the QED Thompson limit.

unitarity, to be the starting line for model builders. One can even "derive", quite persuasively, the gauge theories as the only class of models of particle interactions that meets the requirements of locality, unitarity and the ultraviolet blindness.

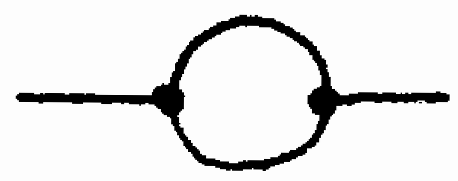
While the multiplicative renormalization of S-matrix elements is obvious, the proof that it cancels the UV divergences, and yields unique finite parts, is a longer story. It goes in (at least) four steps:

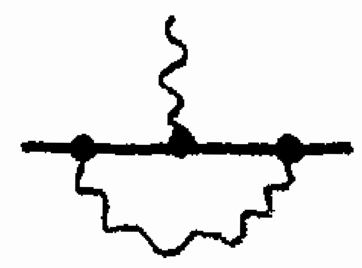
1. power counting identifies divergences in each Feynman diagram
2. subtractions remove divergences diagram by diagram
3. counterterms absorb divergences into the renormalization constants
4. finite renormalizations relate the counterterms for different renormalization conditions

The arguments are mostly combinatoric, and they require no details about the theory other than existence of a regularization scheme. The choice of the regularization scheme is a separate issue, of great practical importance, but no bearing on the arguments of this chapter.

A. Power counting

For a given graph G the degree of divergence deg(G) is the sum of the powers of loop momenta and vertex momenta, minus the sum of powers of the propagator momenta. A few examples suffice to illustrate the concept (the general rule is derived in exercise ......1):

ϕ^3:  $\approx \int d^d k \frac{1}{k^2}\frac{1}{k^2} \Rightarrow \mathrm{Deg}(G) = d-2-2 = d-4$ (.1)

QED:  $\approx \int d^d k \frac{1}{\not k}\frac{1}{\not k}\frac{1}{k^2} \Rightarrow \mathrm{Deg}(G) = d-2\cdot 1-2 = d-4$ (.2)

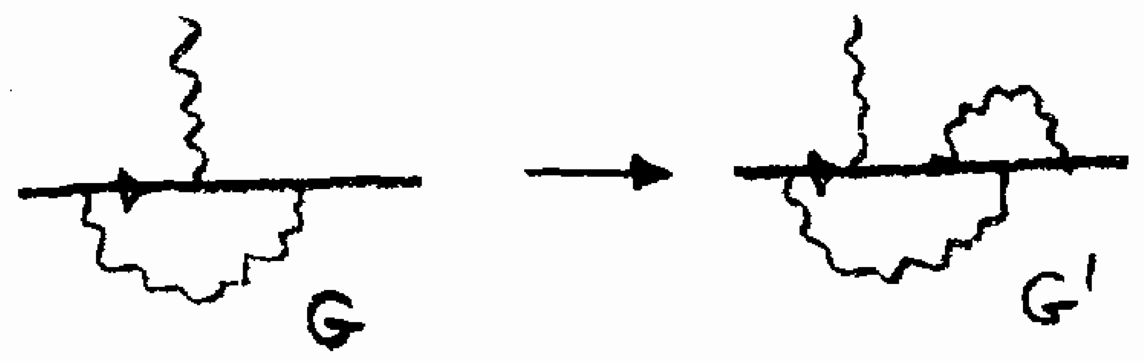
 $\approx \int d^d k\, d^d k \left(\frac{1}{\not k}\right)^4\left(\frac{1}{k^2}\right)^3 \Rightarrow \mathrm{Deg}(G) = 2d-4\cdot 1-3\cdot 2 = 2d-10$ (.3)

If deg(G)>=0 the graph is <u>overall</u> divergent. If deg(G)<0 the graph is (superficially) convergent; superficially, because it can still have divergent subdiagrams. For example, in four dimensions, the above box diagram is superficially overall, convergent, but contains a vertex subgraph which is divergent.

If a theory has only a finite number of 1PI Green functions whose Feynman diagrams are overall divergent, it is called <u>renormalizable</u>; otherwise the theory is called non-renormalizable and presumed hopeless.

To determine whether a theory is renormalizable, add an extra loop to a diagram.

If this does not change the degree of divergence, the theory is renormalizable. For example, if we add a photon correction to the QED vertex diagram (.2)



$$\mathrm{Deg}(G') - \mathrm{Deg}(G) = d - 2\cdot 1 - 2 \qquad (\ \ .4)$$

In four dimensions

the new loop integration is compensated by the extra propagators. Hence deg(G) depends only on the number of external legs of an 1PI diagram, and not on the order in perturbation theory (exercise .A.1). As QED has only a few overall divergent types of diagrams (exercise .A.3) it is renormalizable.

<u>Exercise .A.1</u> <u>Power counting</u>. Consider an arbitrary theory defined by a set of vertices. The <u>index of divergence</u> of a vertex is defined by

$$\delta = b + 3f/2 + k - d \qquad (\ \ .5)$$

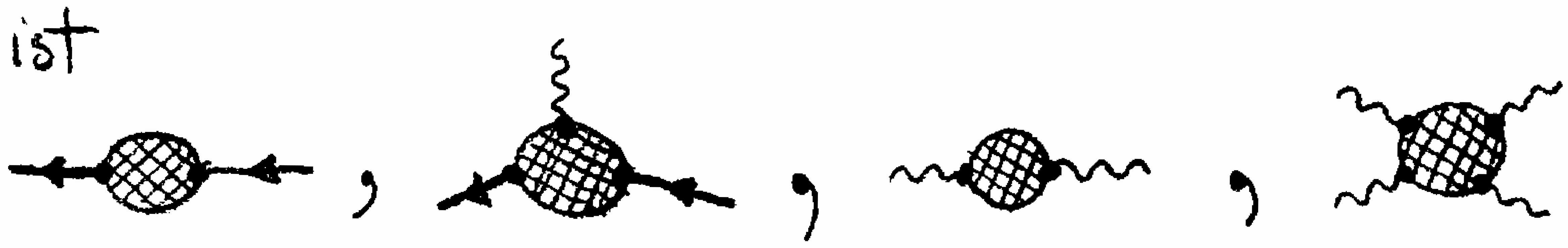
where k = number of derivatives, b = number of bosonic legs, f = number of fermionic legs. For renormalizable theories this index vanishes. Show that the degree of divergence of any diagram is given by

$$\mathrm{Deg}(G) = -B - 3F/2 + d + \sum_i n_i \delta_i \qquad (\ \ .6)$$

where B = number of external bosonic legs, F = number of external fermionic legs, and n_i = number of vertices of i-th type.

<u>Exercise .A.2</u> <u>Renormalizable actions</u>. Check that vanishing of (.5) is equivalent to requiring that the coupling constants be dimensionless.

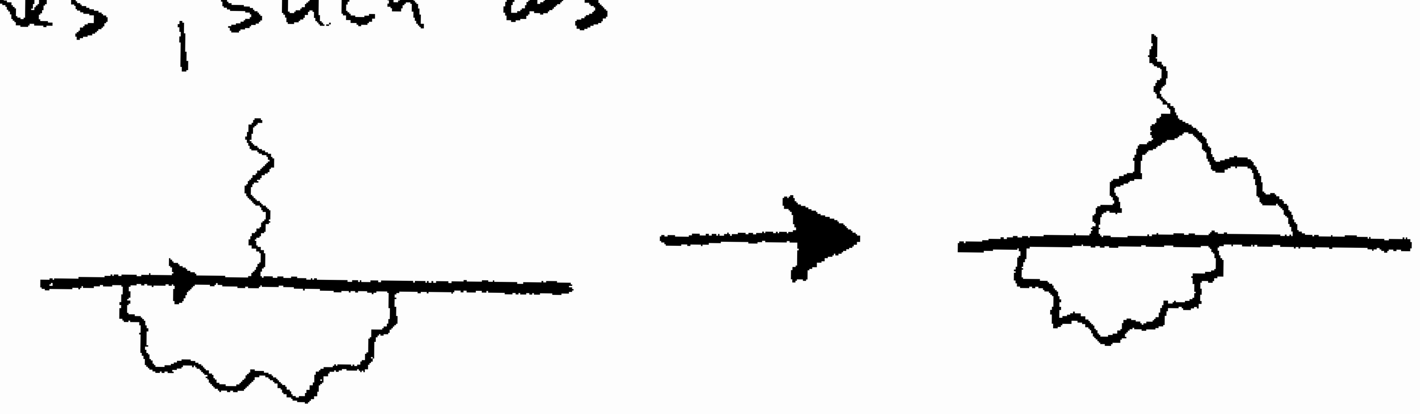
Exercise .A.3 QED divergences. Prove that the list



(.7)

exhausts all superficially overall-divergent diagrams of QED. We say "superficially", because the photon-photon scattering Green functions are actually convergent due to the gauge invariance (exercise ??). Gauge invariance also reduces the degree of divergence for photon self-energy diagrams to Deg(G) = 0, exercise ?.?.

Exercise .A.4 QCD divergences. Generalize the argument of (.4) to gluon insertions on gluon lines, such as



To do this, you only need to know that the 3-gluon vertex is proportional to momentum (appendix D). List the superficially divergent 1PI diagrams for QCD in four dimensions.

Exercise A.5 Prove that ϕ^3 is renormalizable in 6 dimensions.

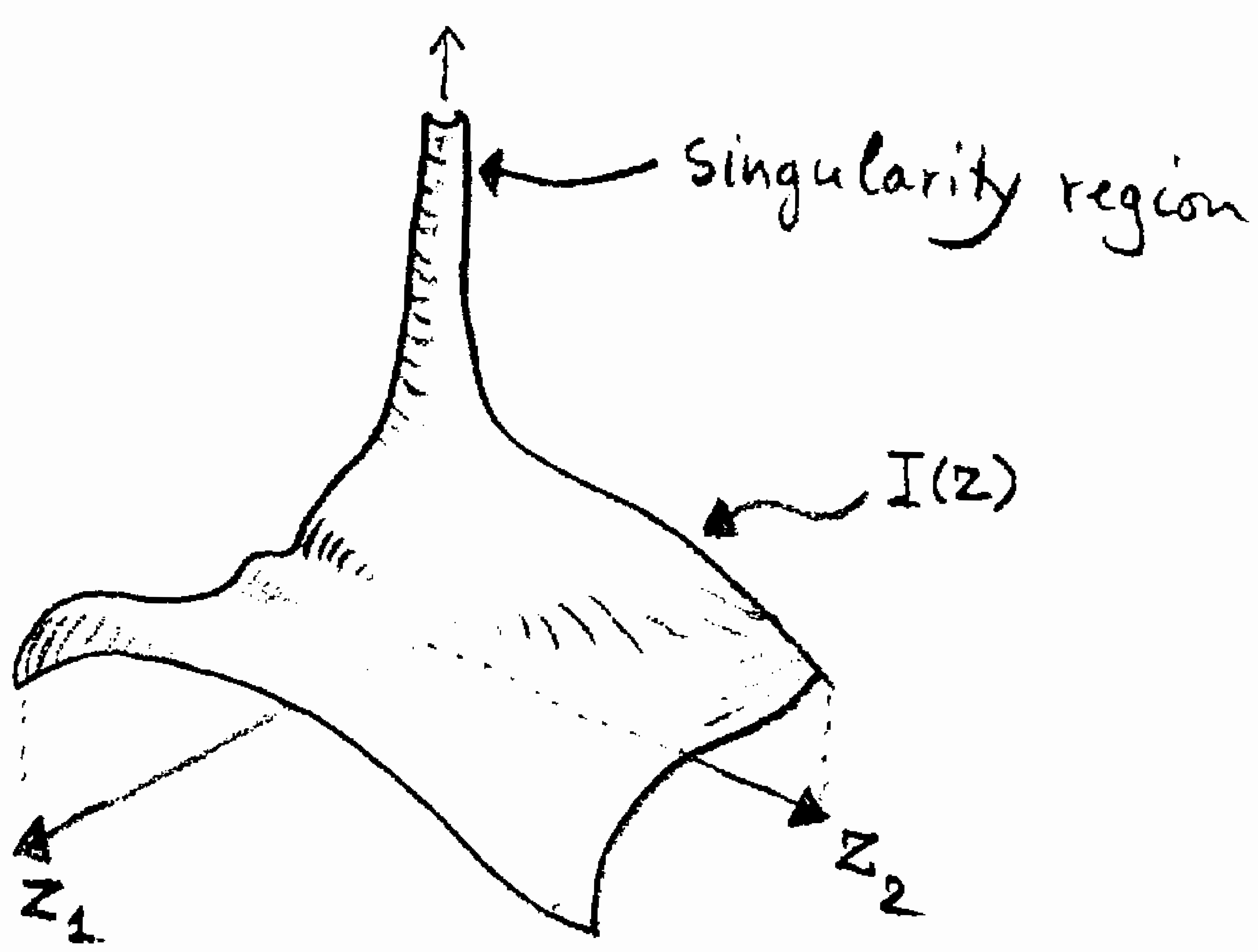
Exercise A.6. Sketch (without any index orgy) the power counting for gravity as perturbation theory in spin-2 graviton. Show that gravity is non-renormalizable above two dimensions (and non-existant in two dimensions).

B. Subtractions

Overall divergences. A typical Feynman integral is of form

$$M = \int [dz]\, I(z) \qquad (.8)$$

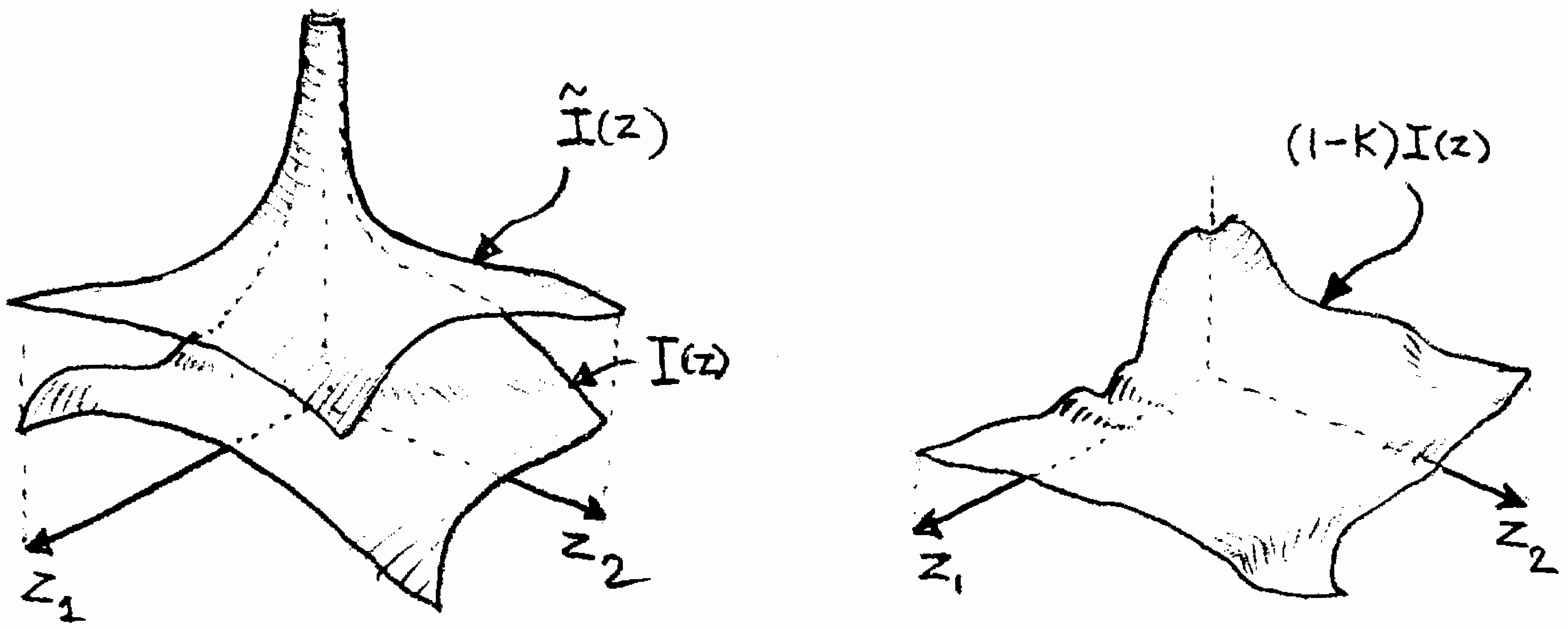
where the integration variables z vary over some range (they could be Feynman parameters, momenta, or whatever), and if the integral is divergent, the divergences arise from identifiable regions of the integration range:



(.9)

If a divergence arises from the region of the integration space which corresponds to all loop momenta very large and comparable, it is called an overall UV divergence. We can subtract this singularity by constructing any integrand $\tilde{I}(z)$ which coincides with I(z) in the singular region:

$$\mathbb{K} M \equiv \int [dz]\, \tilde{I}(z) \equiv \mathbb{L} \qquad (.10)$$



(.11)

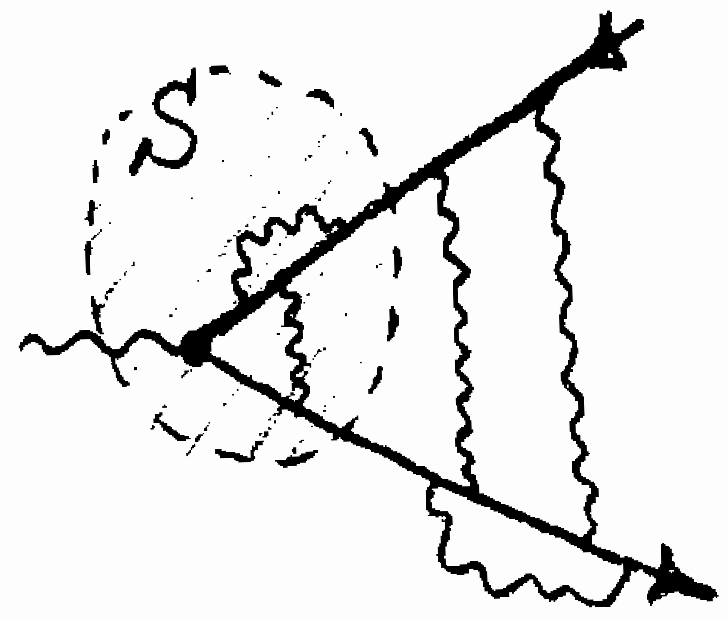
We call this construction K-operation.+ $\tilde{I}(z)$ should be sufficiently close to I(z) to ensure that the integral

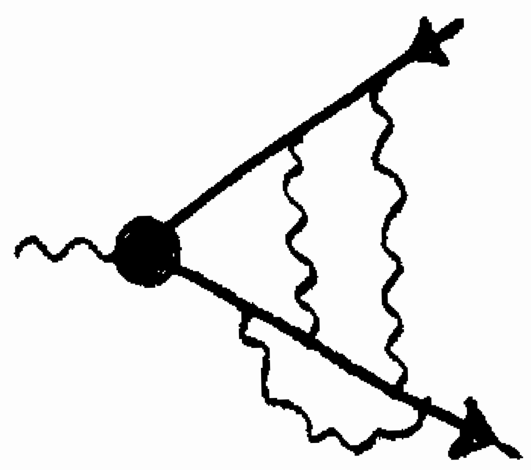
$$(1-K)M = \int [dz] \left(I(z) - \tilde{I}(z) \right) \tag{.12}$$

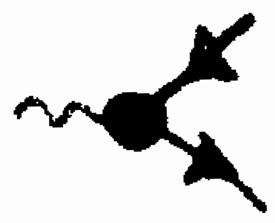
is finite. K-operation is typically a Taylor expansion of some sort (we shall give explicit examples later on). Its point is that it cancels the singularities point by point. That is especially important for numerical evaluation, as otherwise the integrals are dominated by cancelations of large contributions from different singularities, and the physically interesting finite parts get swamped by the numerical errors.

Subdivergences. Ultraviolet divergences can also arise from any subset of loop momenta. As loop momentum has to loop, it is by definition cosigned to a 1PI subdiagram. To subtract a subdivergence, we construct an integrand which approximates the divergent subdiagram around the singularity in the variables corresponding to high loop momenta, and which coincides with the integrand I(z) in the remaining variables:

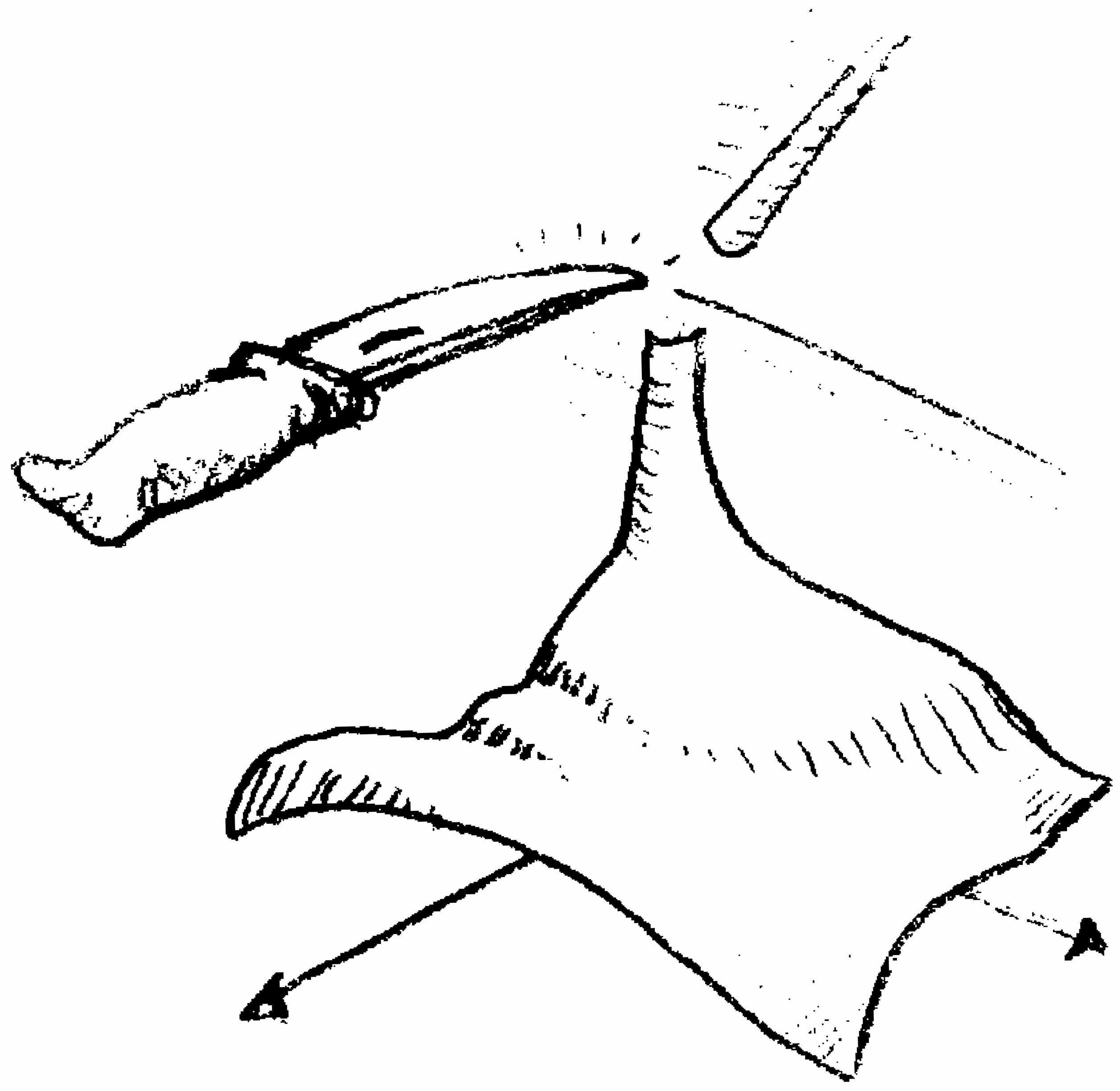
+ K for "kill".

$M_G =$ 

$K_S M_G = L_S * M_{G/S} =$ 

$L_S = \int [dz]\, \tilde{I}_S(z) =$  (.13)

L_S is a divergent vertex computed by the same K prescription as the overall divergent constant L; diagrammaticaly L is a new vertex of the theory, and * indicates all index contraction and momenta integrations implicit in the above graph . By construction, $(1-K_S)M$ has no subdivergence arising from S, . You can visualize K-operation as a knife that shaves off the UV singularity of the corresponding integrand:

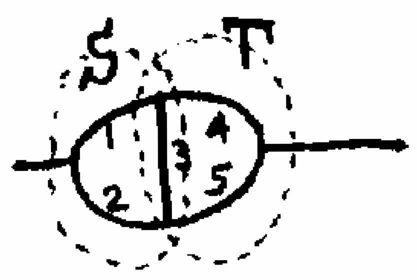
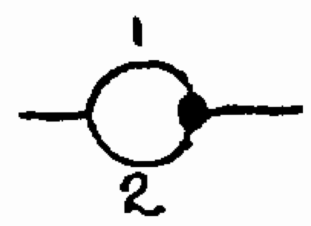


(.14)

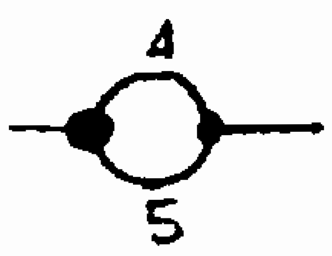
(1-K)M removes the overall UV divergence, $(1-K_S)(1-K)M$ then removes the subdiagram S divergence, and so forth, until the remaining integral is ultraviolet finite. The operation of removing all divergences is called the R-operation (ie., the renormalization operation):

$$RM = \prod_{S} (1-K_S) M \qquad (\ ,15)$$

Overlapping divergences. There is one potential problem with the above definition of the finite part of M. If two subdiagrams overlap, our prescription seems not unique, as the result of

$K_S K_T M = K_S K_T$  $=$ K_S  $= L_{12} * L_{345}$ $\qquad (\ ,15)$

is not the same as the result of isolating singularities in the other order

$K_T K_S M = K_T$  $= L_{123} * L_{45}$ $\qquad (\ ,16)$

If the value of RM depended on the order in which we constructed the subtractions, we would get quite confused. However, the overlap problem is only apparent. The reason is that if the momenta of both overlapping subdiagrams are high, then all the momenta are high, and K operation has no further effect, $KK_S K_T M = K_T K_S M$, so

$$(1-K) K_S K_T = 0 \qquad (\ ,17)$$

and the problematic overlap singularities do not exist in an overall-subtracted integral.

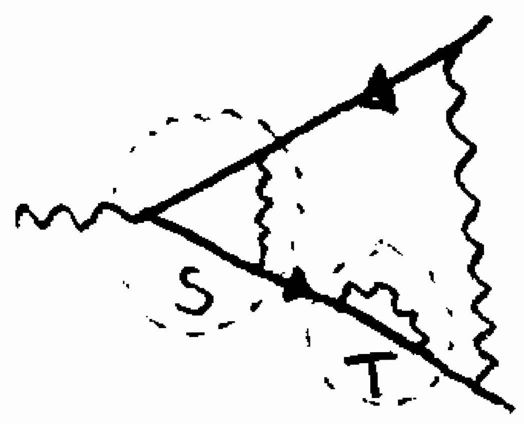
To summarize: given a prescription K for constructing integrand subtractions we can extract the unique finite part of any Feynman diagram.

C. Counterterms

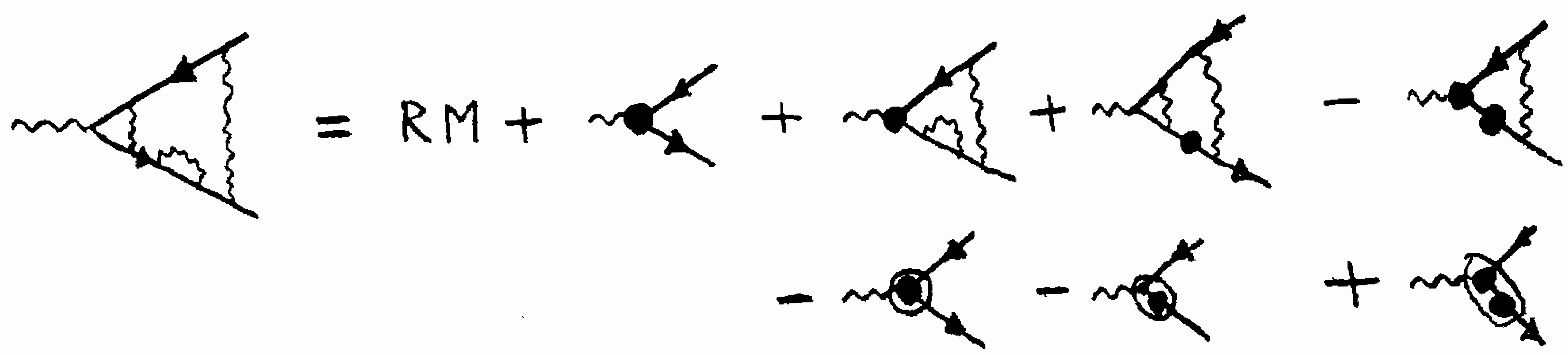
The R-operation rearranges a single Feynman diagram into a sum of a finite part plus miyard divergent terms:

$$M_G = \prod_S (1-K_S) M_G + \sum_S K_S M_G - \sum_{S,T} K_S K_T M_G + \cdots \qquad (\ .18)$$

For example



$$\longrightarrow RM = (1-K)(1-K_S)(1-K_T)M$$



(.19)

$$M = RM + L + L_S * M_{G/S} + L_T * M_{G/T} - L_S * L_T * M_{G/ST}$$
$$- L_S * L_{G/S} - L_T * L_{G/T} + L_S * L_T * L_{G/ST}$$

Our next task is to show that these divergent constants can be collected into counterterms and absorbed into renormalization constants. Unlike the R-operation, the counterterms do not subtract divergences graph by graph. Therefore one needs to prove that the combinatorics of R-subtractions is equivalent to subtractions generated by counterterms.

In a sense this is obvious. If you understand the diagrammatic derivations of the first few chapters, you'll see it immediately. If not, you will have to do some expansions and check the combinatorics.

K operation associates with each 1PI Green function either a divergent constant L, or nothing, depending on the degree of divergence. As 1PI Green functions are the generalized vertices of the theory, L's can be viewed as the additional vertices of the renormalized theory, ie. the theory that generates finite graphs RM instead of divergent M's. L's can be collected into a counterterm functional $L[\phi]$, and the action replaced by the renormalized action

$$S_R = S - L \qquad (\ .20)$$

$$L[\phi] = \sum L_{ij\cdots k} \frac{\phi_i \phi_j \cdots \phi_k}{m!} \qquad (\ .21)$$

Exercise .C.1 Convince yourself that (.20) generates all subtractions with correct combinatorics (and no overlap confusions)