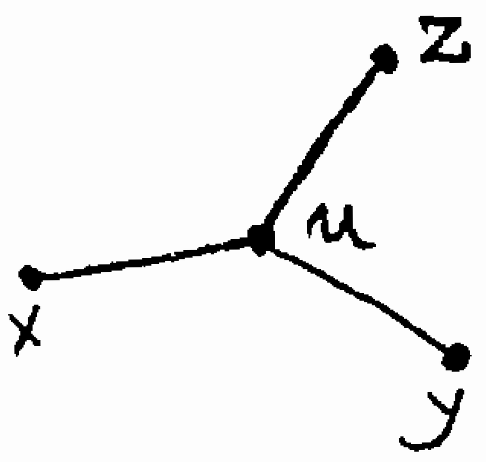
B. Momentum space Feynman integrals

The propagation of a free scalar particle is given by (5.6). Local interactions of a single scalar field are characterized by 3-vertex, 4-vertex, ... coupling strengths



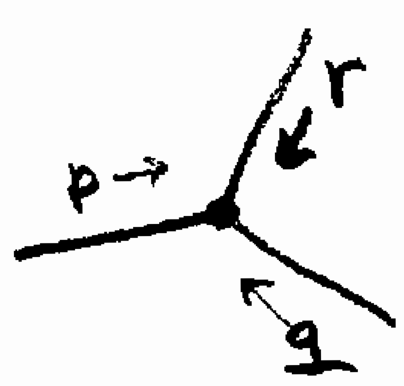
$$= g\int \frac{d^dp\,d^dq\,d^dr}{(2\pi)^{3d}}\,\frac{e^{ip\cdot x}}{p^2+m^2}\,\frac{e^{iq\cdot y}}{q^2+m^2}\,\frac{e^{ir\cdot z}}{r^2+m^2}\int d^dx\, e^{-iu\cdot(p+q+r)}$$

$\vdots$

The structure of a typical Feynman integral is already apparent. Each external leg carries a phase factor $e^{ip\cdot x}$. We can get rid of those by going to the momentum space

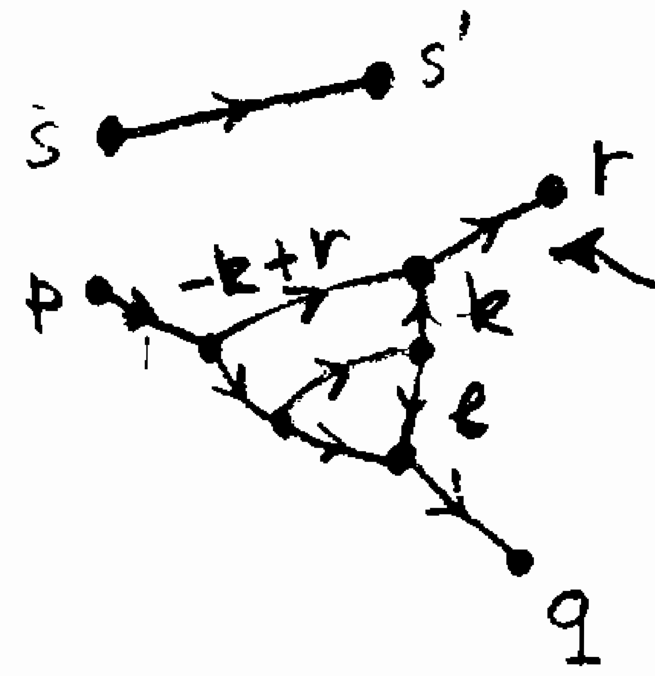
$$G(p,q,r,\dots) = \int d^dx\,d^dy\,d^dz\dots\, e^{-i(p\cdot x+q\cdot y+r\cdot z+\dots)}\, G(x,y,z,\dots)$$

Integration over the position of each vertex yields a momentum conservation δ-function



$$= (2\pi)^d\,\delta(p+q+r)\,,\ \dots$$

Most of the vertex δ-functions can be integrated; what remains is an overall momentum conservation δ-function per each connected component, and a loop-momentum integration per each independent loop:



the arrows indicate directions of momenta flows

$$= (2\pi)^d \delta(s-s') \frac{1}{s^2+m^2} (2\pi)^d \delta(p-q-r) \frac{1}{p^2+m^2} \frac{1}{q^2+m^2} \frac{1}{r^2+m^2}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{d^d\ell}{(2\pi)^d} \frac{1}{(q+k)^2+m^2} \frac{1}{(k+\ell)^2+m^2} \frac{1}{k^2+m^2} \frac{1}{(r-k)^2+m^2} \frac{1}{(q-\ell)^2+m^2} \frac{1}{\ell^2+m^2}$$

Feynman integrals for particles with spins, different masses, color, different couplings, etc., differ only by extra numerator factors.

Though easiest to write down, the momentum space integrals are not the most economical form of Feynman integrals for evaluations purposes.

C. PARAMETRIC_REPRESENTATION

It is intuitively clear that the value of a Feynman diagram cannot depend on [redacted] the choice of the origin of an internal loop momentum integration. The internal momenta are integrated over their entire range, and that is that. [redacted] In a momentum space Feynman integral there are many redundant variables which correspond to its invariance under momentum shifts, or momentum re-routings.

Removal of these redundant integration variables is attained by parametric_representation.

Feynman propagator for a scalar particle, $1/(p^2+m^2)$, is a factor that supresses large excursions in the momentum space. It does not care about the individual momentum components, it depends only on the magnitude of p2. The idea of the parametric representation is to replace the momentum variables by parameters z_i:

$$\frac{1}{k^2+m^2} = \int_0^\infty dz\, e^{-z(m^2+k^2)}$$

This has the virtue of turning momentum integrals into gaussian integrals, the only integrals that we know how to do. The result is of form

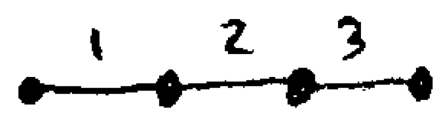
effective mass

effective momentum squared

$$\int \underbrace{\frac{dr_1}{(2\pi)^d}\frac{dr_2}{(2\pi)^d}\cdots}_{\text{loops}} \prod^{\text{lines}} \frac{1}{k_i^2+m^2} = \int \frac{dz_G}{U^{d/2}} \underbrace{e^{-\sum z_i m_i^2 - G(q_i\cdot q_j)}}_{\text{generalized "propagator" for the entire diagram}}$$

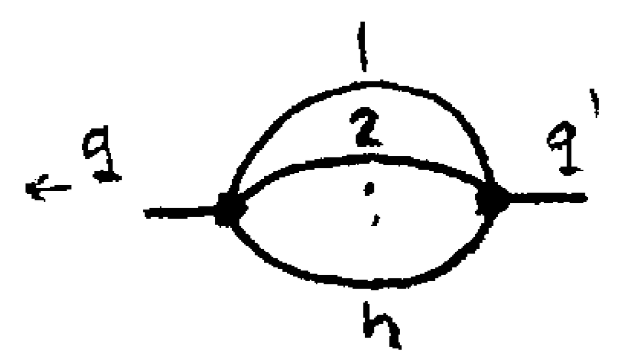
For large momenta, the dominant contribution is from $z \to 0$; the <u>ultraviolet</u> region of the momentum integral is transfered to the small z region of the parametric integral.

This behavior suggest an interpretation of Feynman diagrams which is very useful in developing intuition: <u>electric circuit</u> analogy. We think of k_i as an electric current, and z_i as the resistance of i-th line. It is easy to check that z's add up like resistances. If the current flows through a <u>series</u> of lines



$$\propto \quad e^{-z_1 k^2 - z_2 k^2 - z_3 k^2} = e^{-z_{123} k^2}$$

the resistances z_i <u>add</u>. If the current flows through a <u>parallel</u> network



$$= \int dx\,dy \int \prod \frac{dk_i}{(2\pi)^d}\, e^{-\sum_i \left(z_i k_i^2 + i(x-y)\cdot k_i\right)}\, e^{-ix\cdot q + iy\cdot q'}$$

we evaluate by going first to the configuration space

$$= \cdots \int \prod \frac{dk_i}{(2\pi)^d}\, e^{-\sum_i z_i \left[\left(k_i - \frac{i(x-y)^2}{2z_i}\right)^2 - \frac{(x-y)^2}{4z_i^2}\right]} \cdots$$

$$= \cdots \frac{1}{\prod (4\pi z_i)^{d/2}}\, e^{-\frac{(x-y)^2}{4}\sum_i \frac{1}{z_i}} \cdots$$

Going back to the momentum space we conclude that for a parallel network the resistances add as

$$\frac{1}{G} = \sum_i \frac{1}{Z_i} ,$$

just as in an ordinary electric circuit.

The parametric representation of Feynman integral has many virtues. It is the most economical representation; the functions U and G depend only on the topology of the diagram, and all arbitrariness due to $SO(d)$ invariances and freedom of momenta routings is eliminated. The divergences are neatly separated; ultraviolet divergences arise from U, and infrared divergences from the exponent.

D. Configuration space Feynman integrals

For massless particles the propagator takes a very simple form (exercise 5.A.3):

$$\Delta(x,y) = \frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}} \frac{1}{|x-y|^{d-2}}$$

The simplicity of the configuration space massless propagators is due to the conformal invariance of the theory,